Stable Sharing

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Antonio Nicolò[†], Pietro Salmaso[‡], Arunava Sen[§], Sonal Yadav[¶]

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Abstract

We propose a simple model in which agents are matched in pairs in order to complete a task of unit size. The preferences of agents are single-peaked and continuous on the amount of time they devote to it. Our model combines features of two models: assignment games (Shapley and Shubik (1971)) and the division problem (Sprumont (1991)). We provide an algorithm (Select-Allocate-Match) that generates a stable and Pareto efficient allocation. We show that stable allocations may fail to exist if either the single-peakedness or the continuity assumption fail.

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1 Introduction

Consider a setting where an organization such as a firm, a governmental agency or an NGO, is required to perform multiple tasks. Each task has to be performed by a team consisting of a pair of agents.[1] In order for a task to be completed successfully, the paired agents

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[†]Department of Economics, University of Padova, via del Santo 33, 35123 Padova, Italy and School of Social Sciences, The University of Manchester, Manchester M13 9PL, United Kingdom.

[‡]Department of Economics, University of Padova, via del Santo 33, 35123 Padova, Italy

[§]Economics and Planning Unit, Indian Statistical Institute, 7 S.J.S Sansanwal Marg, New Delhi-110016, India.

[¶]Umeå University, Umeå, Sweden. Email address of the corresponding author: sonal.yadav@umu.se.

[1]We will also consider the multi-agent team problem briefly in Section 7.

must jointly supply effort levels that sum to one. Each agent dislikes effort. However, they receive "credit" (or equivalently, acquire a reputation) for their work which depends on their observable effort level. The payoff to agent i for working an amount t_i is given by $U_i(t_i) = a_i c_i(t_i) - e_i t_i$ where $c_i(\cdot)$ is a strictly concave function that specifies the credit received by i as a function of her effort level, a_i is i's valuation of credit and e_i is i's per-unit disutility of effort. A slightly different but equivalent formulation is one where each agent is paid according to a fixed wage rate but has a strictly convex cost of effort, i.e. i's payoff for working is $V_i(t_i) = wt_i - h_i(t_i)$ where w is the wage rate and $h_i(\cdot)$ is the strictly increasing convex cost of effort function. An important feature of both models is that agent i has an induced preference ordering over effort levels that is single-peaked.

There are several other situations where our model is applicable. Roommates who are assigned together have to share household responsibilities. Some agents clearly prefer to spend more time in activities such as cooking than others and it is reasonable to assume that preferences over time spent on these duties is single-peaked. Pairs of police officers on patrol duty have to share driving and other duties. Sales agents often work in teams to serve a common pool of customers and typically have different preferences over the proportion of customers they would like to serve. In each case, the assumption of single-peaked preferences is natural. All the situations we have described have a sharing problem embedded in a matching problem. Our goal in this paper is to investigate this problem from the perspectives of stability and efficiency.

A pair of agents can block an allocation by proposing contributions summing to one that make them strictly better off than they were in the allocation. An allocation is stable if it cannot be blocked by any pair of agents. Stability is an important requirement in decentralized market design because it can be interpreted as a form of envy-freeness. The motivation for Pareto efficiency is, of course, evident.

Our main result is that a stable and Pareto efficient allocation exists for every profile of single-peaked and continuous preferences. Our existence proof is constructive - we provide an algorithm, the Select-Allocate-Match (SAM) algorithm that identifies a stable and Pareto efficient allocation at every preference profile. Stability fails if either of the assumptions on preferences, single-peakedness and continuity are violated.

The SAM algorithm proceeds by partitioning agents into a set of high type agents (denoted by H) and a set of low type agents (denoted by L) depending upon whether their peaks are greater than or less than 0.5. The algorithm relies on the key notion of an *improvement set* which is defined with respect to an agent's contribution. Roughly speaking, the improvement set for a low type agent is the set of her contributions in the interval $[0, 0.5]$ that would make her strictly better-off. The improvement set for a high type agent is the set of contributions of her partner in $[0, 0.5]$ that would make her strictly better-off. An important result that undergirds our algorithm is that an allocation is stable if the following two conditions are satisfied: (i) the intersection of the improvement sets for each pair of low and high type agents is empty and (ii) 0.5 does not belong to the improvement set of any agent.

In an initializing step, an equal pool of high and low type agents is created by removing the "excess" agents of one type. This is done by choosing agents whose "equivalent contribution" to 0.5 is closest to 0.5.[2] These agents are matched to each other with each contributing 0.5. In subsequent steps, an agent of one type called the primary agent is selected and her contribution chosen. This is done in a manner such that the primary agent does not want to block with any of the agents who have been assigned in a previous step, by considering their improvement sets. This may involve the primary agent being given a contribution equal to her peak. Then an agent of the opposite type called the secondary agent is selected based on the equivalent of the chosen contribution. We continue in this fashion till all agents are matched. It is worth emphasizing that the partitioning of agents into high and low types is artificial in the sense that blocking pairs can be formed by agents of the same type. In our procedure, we first *select* a primary agent, *allocate* a contribution to her and *match* her with an appropriate secondary agent. This motivates our use of the SAM term. The SAM algorithm works in polynomial time and its time complexity is $O(n^2)$ where n is the number of the agents.

The apparent simplicity of our model may suggest that more naive approaches to finding stable and Pareto efficient allocations exist. The examples in Section 3 show that several natural procedures fail. These include procedures based on rewarding agents on "the short side of the market", "top peak-bottom peak" matching and on the uniform rule (Sprumont (1991)).

The existence of a stable allocation in our problem is far from obvious for several reasons. Unlike the celebrated Shapley-Shubik assignment game (Shapley and Shubik (1971)), we do not have a bipartite structure on the set of agents. Instead, we have a roommate type problem (Gale and Shapley (1962)) where every pair of agents can potentially be matched. It is well-known that non-fractional stable matchings in the roommate model do not exist in general (see Teo and Sethuraman (1998) and Eriksson and Karlander (2001)). There are two additional features of our model that are absent in the Shapley-Shubik model. The first is that the free-disposal assumption is violated in ours. Agents cannot always be made better-off by giving them "more" - making an agent contribute more than her peak results in the agent being worse-off. The second feature is that the assumption of single-peaked preferences implies that our model cannot be represented by a transferable utility game. The underlying non-transferable utility game is hard to analyze using standard techniques because of satiation in preferences, the roommate type structure and the non-convexity of stable allocations (see Section 7.5).

[2] For an L agent, this is the contribution in $[0, 0.5]$ that is indifferent to 0.5. The equivalent for a H type agent can be suitably defined. The notion of an equivalent contribution is used extensively in the algorithm. We are glossing over important details here which can be found in Section 4.

Our model is also related to the class of division problems first studied in Sprumont (1991). The uniform rule plays a central role in this setting and has been a characterized in a variety of ways (see Sprumont (1991), Sönmez (1994), Ching (1994), Schummer and Thomson (1997) etc.). This model and the uniform rule have been generalized in a variety of ways, for example in Barberà et al. (1997), Bochet et al. (2012), Bochet et al. (2013) and Klaus et al. (1998). However it does not appear to be important for the analysis of stability in our model. We provide an example where there is no agent matching that generates a stable allocation if the uniform rule or its asymmetric variants are used to determine the contribution of the paired agents.[3]

In our model, agents do not have preferences over their partner as in the classical roommate problem. Nicolò et al. (2019) study a model where agents are matched in pairs and have preferences over both the partner and the project they are assigned to. They show that the existence of stable allocations cannot be guaranteed except when specific assumptions are made on an agent's ranking of partners and projects. The assumption that agents only care about the amount of time (or effort) they devote to the task, and not about their partner or the specific portion of the day or the week they work, is indeed a simplification.[4]

An interesting open question is whether the existence of stable allocations in our model can be derived from existing stability results such as Scarf (1967) and Shapley and Vohra (1991). Proving balancedness of the game and dealing with the absence of free-disposal appears to be non-trivial. In any case, we believe that our approach is more direct and illuminating since we provide an algorithm which generates a stable and Pareto efficient allocation.

The rest of the paper is organized as follows. In Section 2 we introduce the model and basic definitions. Section 3 contains some illustrative examples. Section 4 introduces the concept of improvement sets while Section 6 presents the algorithm and the main result. Section 7 discusses various aspects of our model and results. The proof of stability is contained in Appendix A. The proof of Pareto efficiency is contained in Appendix B.

2 The Model

The set of agents is $N = \{1, \dots, n\}$ where n is even. Agents have to be assigned in pairs and each pair has to complete a task of unit value. No agent can remain on her own[5] and each agent has only one partner.

[3] A more detailed discussion of the literature on stability in division problems can be found in Section 7.3.

[4] The model nevertheless captures relevant features of job sharing such as the demand for reduced working time and the need to find a compatible match. The assumption that workers are indifferent towards their matched partner is likely to be satisfied in routine jobs, or jobs in which the contribution of each worker is fully verifiable.

[5] For further discussion on this assumption, see Section 5.3.

An allocation σ is a collection of triples, (i, j, t_i) where $i, j \in N$, $t_i \in [0, 1]$ and each agent belongs to exactly one triple. We interpret t_i as the contribution of agent i. The contribution of agent i's partner j is $t_j = 1 - t_i$. We refer to (t_i, t_j) as the contribution vector associated with the matched pair (i, j). We say $(i, j, t_i) \in \sigma$ if the pair (i, j) has the contribution vector (t_i, t_j) in σ. Let Σ denote the set of all feasible allocations.

Each agent i has a preference ordering $\succsim_i$ over her contribution.[6] We assume $\succsim_i$ is *single-peaked* and *continuous*. The ordering $\succsim_i$ is single-peaked if there exists a unique contribution $p_i \in [0, 1]$ such that for all $x, y \in [0, 1]$, if $x < y \leq p_i$ or $x > y \geq p_i$ then $y \succ_i x$.[7] The contribution p_i will be referred to as the peak of agent i in $\succsim_i$. A special instance of a single-peaked preference is a *symmetric* or *Euclidean* preference: $x \succsim_i y$ if and only if $|x - p_i| \leq |y - p_i|$. The ordering $\succsim_i$ is continuous if the sets $\{y : y \succsim_i x\}$ and $\{y : x \succsim_i y\}$ are closed for all $x \in [0, 1]$. A preference profile $\succsim$ is an n-tuple of preferences $(\succsim_1, \ldots, \succsim_n)$.

The fundamental property that an allocation should satisfy is *stability*.

Definition 1 *Fix a preference profile $\succsim$. Let σ be an allocation and $i, j \in N$ be agents with contributions t_i and t_j respectively in σ. Then the pair (i, j) blocks σ at $\succsim$ if there exists a contribution vector (t'_i, t'_j) with $t'_i + t'_j = 1$, $t'_i \succ_i t_i$ and $t'_j \succ_j t_j$. An allocation is stable at $\succsim$ if it cannot be blocked by any pair of agents.*

Blocking can occur in two ways. It is possible that i and j are matched together in σ, but can propose an alternative contribution vector which makes both of them better-off.[8] The other possibility is that i and j are not matched together in σ, but can abandon their partners and come together with a contribution vector which makes both better-off.

A more permissive notion of blocking is weak blocking where only one of the blocking agents is better-off and the other one no worse-off.

Definition 2 *Fix a preference profile $\succsim$. Let σ be an allocation and $i, j \in N$ be agents with contributions t_i and t_j respectively in σ. Then the pair (i, j) weakly blocks σ at $\succsim$ if there exists a contribution vector (t'_i, t'_j) with $t'_i + t'_j = 1$, $t'_i \succsim_i t_i$ and $t'_j \succsim_j t_j$ with either $t'_i \succ_i t_i$ or $t'_j \succ_j t_j$. An allocation is strongly stable at $\succsim$ if it cannot be weakly blocked by any pair of agents.*

We are also interested in *Pareto efficient* allocations. Note that we are using the stronger notion of Pareto efficiency.

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[6] The asymmetric and symmetric components of $\succsim_i$ are denoted by $\succ_i$ and $\sim_i$ respectively.

[7] The notion of single-peaked preferences is standard - see Mas-Colell et al. (1995). It is used extensively in a variety of contexts such as political economy and axiomatic allocation theory.

[8] See Appendix B for a discussion of blocking by a pair of agents who are matched together in an allocation.

Definition 3 *Let σ be an allocation where the contribution of agent i is t_i^σ. The allocation τ Pareto dominates σ at preference profile $\succsim$ if $t_i^\tau \succsim_i t_i^\sigma$ for all $i \in N$ and $t_i^\tau \succ_i t_i^\sigma$ for some $i \in N$. The allocation σ is Pareto efficient at $\succsim$ if there does not exist $\tau \in \Sigma$ that Pareto dominates it.*

Stability and Pareto efficiency are independent properties in our model. Consider a problem with four agents all of whom have symmetric preferences with their peak at 0.3. An allocation where one agent in each pair receives her peak, is Pareto efficient. However it is not stable because their partners contribute 0.7 and can strictly improve by forming a pair with each contributing 0.5.

To show that stability does not imply Pareto efficiency, consider the problem with four agents 1, 2, 3 and 4 who have symmetric preferences with peaks 0.1, 0.2, 0.8 and 0.9 respectively. The allocation $(1, 3, 0.1)$ and $(2, 4, 0.2)$ is stable because 1 and 2 are receiving their peaks. It is not Pareto efficient because the allocation $(1, 4, 0.1)$ and $(2, 3, 0.2)$ dominates it.

A characterization of stable allocations is not straightforward. However the two propositions below identify some of their important features.

Fix a preference profile $\succsim$. Partition the set of agents into high type and low type agents depending upon whether their peaks are greater than or equal to or less than 0.5. Formally, $H = \{i \in N : p_i \geq 0.5\}$ and $L = \{i \in N : p_i < 0.5\}$. Furthermore, the set of strictly high type agents is $\hat{H} = \{i \in N : p_i > 0.5\}$.[9] A mixed pair in an allocation is a pair consisting of an agent from L and an agent from $\hat{H}$. The next proposition shows that the number of mixed pairs in any stable allocation is maximal.

Proposition 1 *In any stable allocation, the number of mixed pairs must be equal to* $\min\{|\hat{H}|, |L|\}$.

Proof: Assume for contradiction that there exists an agent $i_1 \in L$ who is matched to an agent $i_2 \notin \hat{H}$ and an agent $j_1 \in \hat{H}$ who is matched to an agent $j_2 \notin L$. There are two cases to consider.

The first case is when each of the agents i_1, i_2, j_1, j_2 contribute 0.5 in σ. There exists $\epsilon > 0$ small enough such that $p_{i_1} \leq 0.5 - \epsilon$ and $p_{j_1} \geq 0.5 + \epsilon$. Thus the pair (i_1, j_1) blocks σ with $(0.5 - \epsilon, 0.5 + \epsilon)$.

Suppose the first case does not hold. Then there exists at least one agent whose contribution is not 0.5. Suppose i_1 is one of these agents. We must also have $t_{i_2} \neq 0.5$. Clearly either t_{i_1} or t_{i_2} is greater than 0.5. Suppose $t_{i_2} > 0.5$. Consider the pair (j_1, j_2). There are two possibilities: $t_{j_1} \leq 0.5$ and $t_{j_1} > 0.5$. If $t_{j_1} \leq 0.5$, then there exists $\epsilon > 0$ small enough such that $0.5 + \epsilon \leq p_{j_1}$ and $0.5 - \epsilon \succ_{i_2} t_{i_2}$. Thus σ is blocked by (i_2, j_1) with $(0.5 - \epsilon, 0.5 + \epsilon)$. If $t_{j_1} > 0.5$, then $t_{j_2} < 0.5$ and the pair (i_2, j_2) blocks σ with $(0.5, 0.5)$.

[9] Note that agents with peak 0.5 are included in the set H. The sets H and L are used throughout the paper except in Propositions 1 and 2 where the sets $\hat{H}$ and L are used.

The remaining case is where i_1 and i_2 contribute 0.5 in σ. Then j_1, j_2 have contributions not equal to 0.5. This case can be dealt with in a manner similar to the earlier case. ■

Suppose the difference in the cardinalities of the sets $\hat{H}$ and L exceeds two. According to Proposition 2, the surplus agents who do not belong to mixed pairs must contribute 0.5 in any stable allocation.

PROPOSITION **2** *In any stable allocation, the following must hold:*

(a). If $|\hat{H}| - |L| > 2$, then every agent $i \in \hat{H}$ who is not matched to an agent in L must contribute 0.5.

(b). If $|L| - |\hat{H}| > 2$, then every agent $i \in L$ who is not matched to an agent in $\hat{H}$ must contribute 0.5.

Proof: We only prove Part (a) since the proof of Part (b) is the symmetric analogue. Let σ be a stable allocation. Assume for contradiction that there exists $(i_1, j_1, t_{i_1}) \in \sigma$ where $i_1 \in \hat{H}$, $j_1 \notin L$ and $t_{i_1} \neq 0.5$. By hypothesis, there exists at least another triple, say $(i_2, j_2, t_{i_2}) \in \sigma$ where $i_2 \in \hat{H}, j_2 \notin L$.

Since $t_{i_1} \neq 0.5$, either $t_{i_1} < 0.5$ or $t_{i_1} > 0.5$ (this implies $t_{j_1} < 0.5$) must hold. Suppose $t_{i_1} < 0.5$. There are two subcases to consider. If $t_{i_2} \leq 0.5$, there exists $\epsilon > 0$ and small enough such that $t_{i_1} < 0.5 - \epsilon$ and $0.5 + \epsilon \succ_{i_2} t_{i_2}$. Then the pair (i_1, i_2) blocks σ with $(0.5 - \epsilon, 0.5 + \epsilon)$. Otherwise, $t_{j_2} < 0.5$. In this case, there exists $\epsilon > 0$ and small enough such that $t_{i_1} < 0.5 - \epsilon$ and $0.5 + \epsilon \succ_{j_2} t_{j_2}$. Then (i_1, j_2) blocks σ with $(0.5 - \epsilon, 0.5 + \epsilon)$.

Suppose $t_{j_1} < 0.5$. Note that $p_{j_1} \geq 0.5$ whereas $p_{i_1} > 0.5$. However, it is easily verified that the argument in the previous paragraph works in this case as well. ■

3 Illustrative Examples

The purpose of this section is to highlight important features of our model with simple examples. The first example shows that strongly stable allocations may not exist.

EXAMPLE **1** Let $N = \{1, 2, 3, 4\}$. Agents' preferences are symmetric and the peaks are summarized in Table 1.

p_1	p_2	p_3	p_4
0.8	0.3	0.3	0.3

Table 1: Peaks of agents in Example 1.

Consider an arbitrary allocation. Since agents 2, 3 and 4 have identical preferences, we can assume w.l.o.g. that $(1, 2)$ and $(3, 4)$ are the matched pairs. At least one of the agents in $\{3, 4\}$ must have a contribution of at least 0.5. Suppose this agent is 3. The pair $(1, 3)$ blocks with the contribution vector $(t_1, t_3) = (0.8, 0.2)$. Agent 1 is at least as well-off as before while agent 3 is strictly better-off. Clearly there are no strongly stable allocations. □

A stable allocation does exist in Example 1, for instance, $(1, 2, 0.8)$ and $(3, 4, 0.5)$. In fact in any stable allocation, agent 1 must receive her peak. Otherwise agent 1 together with the agent who contributes at least 0.5 will block.

Agents whose peaks sum exactly to 1 are obviously perfect matches. This suggests the following procedure for generating stable allocations. Order all pairs of agents by the distance between the sum of their peaks and one, then greedily create pairs of agents using that ordering. Unfortunately this algorithm does not produce a stable matching as the next example shows.

Example **2** Let $N = \{1, 2, 3, 4, 5, 6\}$. Agents' preferences are symmetric and the peaks are summarized in Table 2.

p_1	p_2	p_3	p_4	p_5	p_6
0	0.4	0.41	0.42	0.43	0.74

Table 2: Peaks of agents in Example 2

The procedure outlined earlier generates the following matching: $(2, 6)$,$(4, 5)$,$(1, 3)$. By Proposition 2, we know that agents $1, 3, 4, 5$ must contribute 0.5 in any stable allocation. Also agent 6 must receives her peak, otherwise $(1, 6)$ can block the allocation with $(0.26, 0.74)$. So agent 2's contribution is 0.26. Then the pair $(2, 3)$ can block with $(0.51, 0.49)$. Thus no stable allocation can be obtained using this procedure. Observe that $(1, 6, 0.26)$, $(2, 3, 0.5)(4, 5, 0.5)$ is a stable allocation.

The following example shows that giving the peaks to either side of the market when the market is balanced (the number of high type agents is equal to the number of low type agents) may not generate a stable allocation. We consider a procedure where agents are partitioned into high and low type agents as before. An allocation is constructed by giving the peaks of the agents on one side of the market and matching them with agents of the other type.

Example **3** Let $N = \{1, 2, 3, 4, 5, 6\}$. Agents' preferences are symmetric and the peaks are summarized in Table 3.

p_1	p_2	p_3	p_4	p_5	p_6
0	0.45	0.45	0.65	0.65	0.65

Table 3: Peaks of agents in Example 3

The set of low and high type agents are $\{1,2,3\}$ and $\{4,5,6\}$ respectively. Assume w.l.o.g. that the pairs in the allocation are $(1,4)$, $(2,5)$ and $(3,6)$. The allocation where all high type agents get their peaks is not stable. Consider the allocation with the triples $(1,4,0.35)$, $(2,5,0.35)$ and $(3,6,0.35)$. The same side pair $(2,3)$ blocks with the contribution vector $(0.5,0.5)$. Similarly the allocation where all low type agents get their peaks is not stable. For instance, the allocation $(1,4,0)$, $(2,5,0.45)$ and $(3,6,0.45)$ is blocked by the same side pair $(4,5)$ using the contribution vector $(0.35,0.65)$.

Stable allocations exist in this case as well. One such allocation is $(1,4,0.2)$, $(2,5,0.45)$ and $(3,6,0.45)$. □

In the model, despite the fact that Proposition 1 suggests a natural partitioning of agents into low and high types, naively following a Shapley and Shubik (1971) type procedure will not work because of the possibility of same-side blocking.

Another "obvious" procedure would be to arrange the agents in order of their peaks from the highest to the lowest. The highest agent would then be matched with the lowest, the second highest with the second lowest and so on. The next example shows that this procedure does not generate stable allocations irrespective of the way contributions are specified.

Example **4** Let $N = \{1,2,3,4\}$. The peaks of the agents are summarized in Table 4. Agents 2, 3 and 4 have symmetric preferences while 1 has single-peaked but non-symmetric preferences with the following restriction: $0.35 \sim_1 0.51$.

p_1	p_2	p_3	p_4
0.39	0.4	0.4	0.9

Table 4: Peaks of agents in Example 4.

The pairs formed by the procedure are $(1,4)$ and $(2,3)$. Let (t_1,t_4) and (t_2,t_3) be their contribution vectors in a stable allocation.

By feasibility, one of the agents $i \in \{2,3\}$ will have a contribution $t_i \geq 0.5$. Assume w.l.o.g. $i = 2$. We claim $t_1 \geq 0.35$. If $t_1 < 0.35$, then the pair $(1,2)$ can block by proposing the contribution vector $(0.51,0.49)$. Agent 2 strictly improves as $0.49 \succ_2 t_2$. For agent 1, single-peakedness implies $0.35 \succ_1 t_1$. Since $0.35 \sim_1 0.51$, we have $0.51 \succ_1 t_1$ and agent 1 strictly improves.

We also claim $t_4 \geq 0.68$. If $t_4 < 0.68$, then $(2,4)$ can block by proposing $(0.31, 0.69)$. Agent 4 strictly improves by blocking as she moves closer to her peak. For agent 2, symmetry and $t_2 \geq 0.5$ implies $0.31 \succ_2 t_2$. Thus agent 2 also strictly improves.

We have argued that $t_1 \geq 0.35$ and $t_4 \geq 0.68$. Since 1 and 4 are paired together, we have a violation of feasibility. Hence there are no stable allocations with the pairs $(1,4)$ and $(2,3)$. □

The previous procedure first specified a way to match agents in pairs and then attempted to find suitable contributions. An alternative approach would be to first choose a rule for determining the contributions of agents and then finding a way to form pairs. A natural candidate for such a rule would be one from the class of strategy-proof and Pareto-efficient rules for two agent allotment problems characterized by Moulin (1980) and Barberà et al. (1997). We describe this class below.

Suppose i and j are matched together. Pick real numbers p_a, p_b with $0 \leq p_a \leq p_b \leq 1$. There exists an agent in the matched pair, say i such that her contribution at every profile is given by:[10]

$$t_i = \begin{cases} p_i, & \text{if } p_a \leq p_i \leq p_b \\ \min\{\max\{p_i, 1-p_j\}, p_a\}, & \text{if } p_i < p_a \\ \max\{\min\{p_i, 1-p_j\}, p_b\}, & \text{if } p_i > p_b. \end{cases} \tag{1}$$

The uniform rule characterized by Sprumont (1991) is obtained when $p_a = p_b = 0.5$. In this case, the rule is anonymous, i.e. agents are treated symmetrically. On the other hand, the agents are treated asymmetrically whenever $p_a \neq p_b$. For instance, 1 is a dictator if $p_a = 0$ and $p_b = 1$. An important observation pertaining to these rules is that agent i either contributes her peak or one of the numbers p_a, p_b.

We consider rules of this type where two arbitrary numbers $(p_a^{i,j}, p_b^{i,j})$ that do not depend on agents' preferences, are assigned to each pair (i,j). Agent i's contribution is computed as in Equation 1. We claim that rules of this type do not generate stable outcomes at every preference profile, irrespective of how the matching is formed.

Example 5 Let $N = \{1, 2, \ldots, 6\}$. Agents' preferences are symmetric and Table 5 summarizes the peaks of the agents.

Here ϵ is chosen to be less than 0.1. Applying Proposition 2 we can conclude that all agents not matched to agent 1 must contribute 0.5. Suppose agent 1 is matched with agent $l \in \{2,3,4,5,6\}$. In order to prevent 1 and l from blocking with one of the remaining agents, it must be the case that 1 and l must contribute $0.5 - 2\epsilon$ and $0.5 + 2\epsilon$ respectively, i.e. stable allocations are of the form $(i,j,0.5)$,$(h,k,0.5)$,$(1,l,0.5-2\epsilon)$ where (i,j,h,k,l) is a permutation of the agents $\{2,3,4,5,6\}$. According to our earlier observation, the contributions

[10] The contribution of j is simply the complement of t_i.

p_1	p_2	p_3	p_4	p_5	p_6
0	$0.5+\epsilon$	$0.5+\epsilon$	$0.5+\epsilon$	$0.5+\epsilon$	$0.5+\epsilon$

Table 5: Peaks of agents in Example 5.

of either 1 and l must coincide either with her peak or with one of the numbers $p_a^{1,l}, p_b^{1,l}$, $l \in \{2, 3, 4, 5, 6\}$ for all ϵ. Observe that the contributions of 1 and l depend on ϵ. Moreover their peaks are distinct from their contributions for all ϵ. Since ϵ can vary continuously while the number of parameters $p_a^{1,l}, p_b^{1,l}$ is finite, it is clearly not possible to find the parameters that generate stable allocations for all profiles. □

We have shown that several naive procedures fail. In the following sections, we describe the SAM algorithm which always delivers a stable and Pareto efficient allocation.

4 Improvement sets and stability

We introduce the notion of improvement sets which play a key role in our algorithm.

We partition agents into sets H and L as defined in Section 2. We represent the peaks and the contributions of agents in the interval $[0, 0.5]$. The peak of a low type agent p_i is represented by $p_i \in [0, 0.5]$. The peak of a high type agent p_i is represented by the point $1 - p_i$ in $[0, 0.5]$.[11] We will also represent contributions in $[0, 0.5]$. Every point in the interval $[0.0.5]$ also represents a contribution vector for a pair of agents. The distance of the point from 0 is the contribution of one agent and the distance of the point from 0.5 is the "excess" over 0.5 of the contribution of the other agent. See Example 6 for an illustration.

Consider agent $i \in L$ with preference $\succsim_i$ (with peak p_i) and contribution t_i. We define the *improvement set* for i at t_i as follows:

$$I_{i,t_i} = \{x \in [0, 0.5] : x \succ_i t_i\}.$$

Consider agent $i \in H$ with preference $\succsim_i$ (with peak p_i) and contribution t_i. We define the *improvement set* for i at t_i as follows:

$$I_{i,t_i} = \{x \in [0, 0.5] : 1 - x \succ_i t_i\}.$$

Observe that the improvement set of an agent i, I_{i,t_i}, is empty when $t_i = p_i$.

We make a brief remark about the asymmetry in the definitions of improvement sets for low and high type agents. For an agent $i \in L$, the improvement set consists of contributions

[11] The peak of a low type agent is measured from left to right starting at 0, while the "excess" of a high type agent is measured from right to left starting from 0.5.

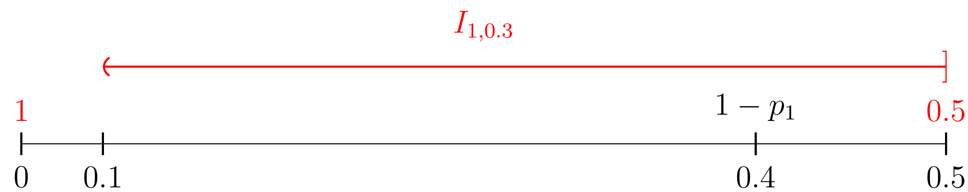

Figure 1: Improvement set for agent 1 in Example 6

in $[0, 0.5]$ which she strictly prefers to t_i. For an agent $i \in H$, the improvement set consists of contributions made by a potential partner in $[0, 0.5]$ which would make agent i strictly better-off relative to t_i.

The assumptions of single-peakedness and continuity on $\succsim_i$ imposes structure on the improvement sets which we record below as an observation.

Observation **1** The improvement set of an agent is a connected open subset of $[0, 0.5]$ or equivalently an open interval in $[0, 0.5]$.

Example 6 illustrates improvement sets for both low and high type agents.

Example **6** Let $N = \{1, 2, 3, 4\}$. Agents' preferences are symmetric. Table 6 summarizes their peaks and contributions. Agents 1 and 2 are matched together as are 3 and 4.

Agent	1	2	3	4
p_i	0.6	0.75	0.10	0.75
t_i	0.3	0.7	0.25	0.75

Table 6: Peaks of agents in Example 6.

Figure 1 shows the improvement set of agent 1 while Figure 2 shows the improvement sets of agents 2 and 3. The improvement set of agent 4 is empty. Improvement sets of high (low) types are indicated above (below) the $[0, 0.5]$ line. □

It is useful to define the notion of an *equivalent* contribution. We denote the equivalent contribution for agent i at t_i by $e_i(t_i)$ when $i \in L$ and $e_i(1 - t_i)$ when $i \in H$.[12]

Consider $i \in L$. If there exists a contribution $x \in [0, 0.5]$ such that $x \sim_i t_i$ and $x \neq t_i$, then $e_i(t_i) = x$. The equivalent $e_i(t_i)$ is an end-point of the improvement set but not included in it. Here the improvement set is one of the following: $(t_i, e_i(t_i))$ if $t_i < p_i$, $(e_i(t_i), t_i)$ if $p_i < t_i \leq 0.5$ or $(e_i(t_i), 0.5]$ if $p_i < 0.5 < t_i$.

[12] We suppress the dependence of the equivalent of agent i on $\succsim_i$ since we keep the latter constant throughout the analysis.

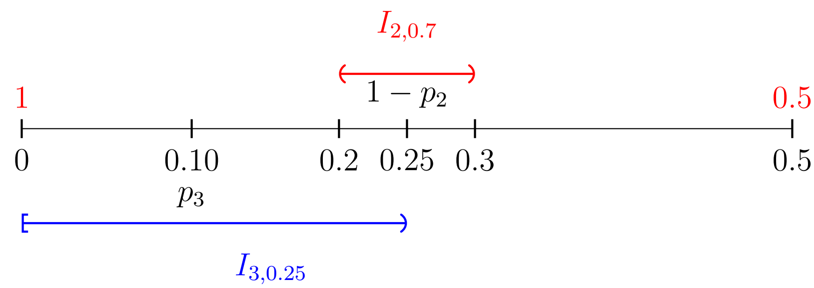

Figure 2: Improvement sets in Example 6

If such an x does not exist, then $e_i(t_i)$ is defined as follows:

$$e_i(t_i) = \begin{cases} -\epsilon & \text{if } I_{i,t_i} = [0, t_i) \text{ or } I_{i,t_i} = [0, 0.5] \\ 0.5 + \epsilon & \text{if } I_{i,t_i} = (t_i, 0.5] \\ t_i & \text{if } I_{i,t_i} = \emptyset \end{cases}$$

where ϵ is any small positive number.

In Example 6, 3 is a low type agent with $p_3 = 0.10$ and contribution $t_3 = 0.25$. Since $0 \succ_3 0.25$, $e_3(t_3) = -\epsilon$. In fact $e_3(t_3) = -\epsilon$ for all $t_3 > 0.25$. If $t_3 = 0.05$ then $e_3(t_3) = 0.15$.

Consider $i \in H$. If there exists a contribution $x \in [0, 0.5]$ such that $1 - x \sim_i t_i$ and $1 - x \neq t_i$, then $e_i(1 - t_i) = x$. As before, the equivalent $e_i(1 - t_i)$ is an end-point of the improvement set but not included in it. Here the improvement set is one of the following: $(1 - t_i, e_i(t_i))$, $(e_i(t_i), 1 - t_i)$ or $(e_i(t_i), 0.5]$ if $t_i < 0.5$.

If such an x does not exist, then $e_i(1 - t_i)$ is defined as follows:

$$e_i(1-t_i) = \begin{cases} -\epsilon & \text{if } I_{i,t_i} = [0, 1-t_i) \text{ or } I_{i,t_i} = [0, 0.5] \\ 0.5 + \epsilon & \text{if } I_{i,t_i} = (1-t_i, 0.5] \\ 1-t_i & \text{if } I_{i,t_i} = \emptyset \end{cases}$$

where ϵ is any small positive number.

In Example 6, 1 is a high type agent with $p_1 = 0.6$ and $t_1 = 0.3$. Observe $0.9 \sim_1 0.3$. Thus there exists $x = 0.1 \in [0, 0.5]$ such that $1 - x \sim_1 0.3$. Consequently $e_1(1 - t_1) = 0.1$. If $t_1 = 1$, the improvement set for agent 1 is $(0, 0.5]$. Thus $e_1(1 - t_1) = 0.5 + \epsilon$.

Improvement sets provide a natural way to check for the existence of stable allocations. For instance, in Example 6 agents 2 and 3 who receive contributions 0.7 and 0.25 respectively in an allocation will block. This is evident from the fact that their improvement sets have a non-empty intersection.

Definition 4 *A allocation satisfies Condition S if the associated improvement sets satisfy the following:*

1. *For every $h \in H$ and $l \in L$, $I_{h,t_h} \cap I_{l,t_l} = \emptyset$.*

2. For all $i \in N$, $0.5 \notin I_{i,t_i}$.

PROPOSITION **3** *If an allocation satisfies Condition S, it is stable. Moreover if an allocation is stable, it satisfies Part 1 of Condition S.*

Proof: Consider an allocation that satisfies Condition S but is not stable, i.e. there exists a pair of agents who block. There are two cases to consider.

Case 1: The blocking pair is (l, h) where $l \in L$ and $h \in H$. Let t_l and t_h be the contributions of l and h respectively in the allocation. Suppose they block with the contribution vector (t'_l, t'_h). If $t'_l \geq 0.5$, single-peakedness implies $0.5 \in I_{l,t_l}$ which would contradict Part 2 of Condition S. Therefore $t'_l < 0.5$, i.e. $1 - t'_l = t'_h > 0.5$. Since $t'_l \succ_l t_l$ and $t'_l < 0.5$, we have $t'_l \in I_{l,t_l}$. Since $t'_h \succ_h t_h$ and $t'_h = 1 - t'_l > 0.5$, we have $t'_l \in I_{h,t_h}$. Therefore $t'_l \in I_{l,t_l} \cap I_{h,t_h}$ contradicting Part 1 of Condition S.

Case 2: Both agents in the blocking pair are of the same type. Suppose (l_1, l_2) is the blocking pair where $l_1, l_2 \in L$. Let t_{l_1} and t_{l_2} be the contributions of agents l_1 and l_2 respectively in the allocation. In the contribution vector used to block, at least one of the agents in the pair, say l_1 has a contribution of at least 0.5. Single-peakedness implies $0.5 \in I_{l_1,t_{l_1}}$ contradicting Part 2 of Condition S. The argument in the case where both agents are high type is virtually identical.

We now show that any stable allocation must satisfy Part 1 of Condition S.

Consider a stable allocation. Assume for contradiction that there exist $i \in L$ and $j \in H$ such that $I_{i,t_i} \cap I_{j,t_j} \neq \emptyset$ where t_i and t_j are contributions of agents i and j respectively in the allocation. Consider $x \in I_{i,t_i} \cap I_{j,t_j}$. By the definition of improvement sets, we have $x \succ_i t_i$ and $1 - x \succ_j t_j$. Thus the pair (i, j) blocks the allocation with the contribution vector $(x, 1 - x)$. ■

Part 2 of the Condition S can be interpreted as a fairness requirement such that no agent is worse off than the case where she equally shares the contribution with her partner. Note that Part 2 of Condition S is not necessary for the existence of stable allocations. Consider the case where there are four agents l_1, l_2, h_1, h_2 with symmetric preferences. The peaks of l_1 and l_2 are 0.4 and 0.3 while the peaks of h_1 and h_2 are 0.9 and 0.7. The allocation with the triples $(l_1, h_1, 0.1)$ and $(l_2, h_2, 0.3)$ is stable because all agents except l_1 are satiated. However $0.5 \in I_{l_1,0.1}$.

5 The SAM Algorithm: A broad overview

The main contribution of this paper is the Select-Allocate-Match (SAM) algorithm that generates a stable and Pareto efficient allocation for every instance of the sharing problem.[13]

[13]For stability, we show that the allocation generated by SAM algorithm satisfies Condition S.

In this section we informally discuss key ideas in the algorithm and illustrate them with examples. For expositional convenience, we shall mainly confine our discussion to the case of symmetric preferences with a few remarks at the end regarding the general single-peaked case.

The SAM algorithm proceeds as follows. In the initial step (Step 0), the number of high and low type agents are counted and "excess agents" from the larger side, identified. The excess agents are arbitrarily paired with each other and assigned contributions of 0.5. The improvement sets of the paired agents are then calculated. Inputs to the algorithm at Step q are the set of agents paired upto (and including) Step $q-1$ and the improvement sets of these agents. We let D_q and U_q denote the union of the improvement sets of the low and high type agents respectively who have been paired upto (and including) Step $q-1$. It is clear that D_q and U_q are unions of open intervals. In Step q, a low type agent (say l) and a high type agent (say h) are chosen with the property that their peaks are closest to 0.5 among the set of unmatched low and high type agents respectively. These agents are paired together and their contribution vector decided using the D_q and U_q sets and the peaks p_l, p_h. Three considerations are involved in choosing this contribution vector (i) the contribution of l lies in between $1-p_h$ and p_l (ii) l's improvement set does not intersect with U_q and (iii) h's improvement set does not intersect with D_q. The manner in which agents are paired ensures that all three requirements can be reconciled. The contribution of agent l belongs to the set $\{p_l, 1-p_h, \inf D_q, 1-\inf U_q\}$. An important feature of the algorithm is that contributions vectors are specified in a manner such that 0.5 does not belong to the improvement set of any agent. This ensures that "same-side" blocking does not occur. We illustrate the working of the algorithm in several examples below.

In Example 1, there are two excess low type agents. Since all low type agents are identical, two of them, say 2 and 3 are picked arbitrarily in Step 0. They are paired together with the contribution vector $(0.5, 0.5)$. Their improvement sets are $(0.1, 0.5)$ so that $U_1 = (0.1, 0.5)$. In Step 1, the remaining agents 1 and 4 are paired. If 1's contribution differs from 0.8, her improvement set will be an open interval with 0.2 as the mid-point. Since 0.2 belongs to U_1, the algorithm fixes 1's contribution as her peak, i.e the triple $(1, 4, 0.8)$ is generated.

In Example 2, there are five low type agents and one high type agent. The excess agents are the four low type agents whose peaks are closest to 0.5, i.e. agents $2, 3, 4$ and 5. In Step 0, the algorithm pairs agents $(2, 3)$ and $(4, 5)$ with each agent contributing 0.5. The improvement sets for agents $2, 3, 4$ and 5 are $(0.3, 0.5)$, $(0.32, 0.5)$, $(0.33, 0.5)$ and $(0.34, 0.5)$ respectively. Thus $D_1 = (0.3, 0.5)$ and U_1 is empty. In Step 1, agents 1 and 6 will be paired. Here $1 - p_6 = 0.26 < \inf D_1$. The algorithm forms the triple $(1, 6, 0.26)$ and agent 6 gets her peak. Observe that the requirements (i), (ii) and (iii) stated earlier are satisfied.

Example **7** Let $N = \{1, 2, 3, 4, 5, 6\}$. Agents' preferences are symmetric and the peaks are summarized in Table 7.

p_1	p_2	p_3	p_4	p_5	p_6
0.18	0.2	0.45	0.65	0.66	0.9

Table 7: Peaks of agents in Example 7

Consider Example 7 above. Here there are no surplus agents and no agents are paired in Step 0. Thus D_1 and U_1 are empty. In Step 1, the highest low type agent is paired with the lowest high type agent, with the agent with peak closest to 0.5 getting her peak - the triple $(3, 4, 0.45)$ is formed. The improvement sets for agents 3 and 4 are the null set and $(0.25, 0.45)$ respectively. Thus D_2 is empty and $U_2 = (0.25, 0.45)$. In Step 2, agents 2 and 5 are paired. Since $0.2 = p_2 < \inf U_2 < 1 - p_5 = 0.34$, the algorithm forms the triple $(2, 5, 0.25)$ where agent 2's contribution is $\min\{\inf U_2, 1 - p_5\}$. None of the agents get their peak in Step 2 and thus have non-empty improvement sets. The improvement sets for agents 2 and 5 are $(0.15, 0.25)$ and $(0.25, 0.43)$. Note that 2's improvement set does not intersect with U_2, fulfilling requirement (ii). Requirement (iii) is satisfied trivially since D_2 is empty while (i) is satisfied by construction. The inputs into Step 3 are $D_3 = (0.15, 0.25)$ and $U_3 = (0.25, 0.43) \cup U_2 = (0.25, 0.45)$.

In Step 3, the pair $(1, 6)$ is formed. Observe that $1 - p_6 < \inf D_3 < p_1 < \inf U_3$. The algorithm forms the triple $(1, 6, 0.15)$. In particular, the contribution of agent 1 is $\min\{\inf D_3, p_1\}$. The improvement set of agent 1 is $(0.15, 0.21)$ and does not intersect U_3. The improvement set of agent 6 is $(0.05, 0.15)$ and does not intersect D_3. Once again, all three requirements are satisfied.

Example 8 is a variant of Example 7 where only the peak of agent 1 has changed.

Example **8** Let $N = \{1, 2, 3, 4, 5, 6\}$. Agents' preferences are symmetric and the peaks are summarized in Table 8.

p_1	p_2	p_3	p_4	p_5	p_6
0.05	0.2	0.45	0.65	0.66	0.9

Table 8: Peaks of agents in Example 8

Steps 0, 1 and 2 are identical to their counterparts in Example 7 as all other peaks remain unchanged. In Step 3, the pair $(1, 6)$ is formed. Note that $p_1 < 1 - p_6 < \inf D_3 < \inf U_3$. The algorithm forms the triple $(1, 6, 0.1)$ where the contribution of agent 1 is $\min\{1 - p_6, \inf U_3\}$ and agent 6 gets her peak. The improvement set of agent 1 is $(0, 0.1)$ and does not intersect U_3. Since 6 gets her peak, she does not participate in any blocking coalition.

Example 9 is a variant of Example 8 where only the peak of agent 6 has changed.

EXAMPLE **9** Let $N = \{1, 2, 3, 4, 5, 6\}$. Agents' preferences are symmetric and the peaks are summarized in Table 9.

p_1	p_2	p_3	p_4	p_5	p_6
0.05	0.2	0.45	0.65	0.66	0.8

Table 9: Peaks of agents in Example 9

Steps 0,1 and 2 are identical to their counterparts in Examples 7 and 8. In Step 3, the pair $(1, 6)$ is formed. Note that $\inf D_3 = 0.15 < 0.2 = 1 - p_6$ and $1 - p_6$ belongs to D_3. Thus agent 6 must be given her peak to prevent her from blocking with the already matched low type agents. The algorithm forms the triple $(1, 6, 0.2)$. The improvement set of agent 1 is $(0.16, 0.2) \subset D_3$ and does not intersect U_3.[14]

A possible issue of concern regarding the algorithm is that requirements (ii) and (iii) for the assignment of contributions in some Step q may be in conflict with each other. In particular, a situation may arise where $p_l \in U_q$ and $1 - p_h \in D_q$. Then both agents l and h must be assigned their peaks which would be impossible if $p_l + p_h \neq 1$. In the proof of Theorem 1, we show that this situation can never occur. For an insight into this claim, consider the following variant of Example 9. Recall that $1 - p_6 \in D_3$. Suppose p_1 is adjusted so that $p_1 \in U_3 = (0.25, 0.45)$. Then $p_2 = 0.2 < 0.25 \leq p_1$. However this cannot occur because the algorithm would then have paired agent 1 (with agent 5) in Step 2, instead of agent 2.

The algorithm works on the same principle for the general single-peaked case. Improvement sets are no longer based on distances from the peak, but still remain open intervals. The selection of low and high type agents to be paired in any step is now more delicate and depends generally on agent preferences, and not simply on the distance of peaks from 0.5. These details are fully specified in Section 6 - here we briefly summarize the general procedure that motivates the name of the algorithm. In every Step $q \geq 1$, we perform the following operations.[15]

- **S**elect an agent i amongst the set of unmatched agents. We call this agent as the primary agent. The following three cases[16] identify the primary agent:

[14] The fact that the improvement set of agent 1 is a subset of D_3 is not a coincidence. This follows from the fact that the low type agents matched so far have higher peaks than agent 1 and preferences are symmetric.

[15] As described earlier, in Step 0, we match in pairs the "excess" agents of one type so that in Step 1 there are an equal number of unmatched high and low type agents.

[16] In the Appendix we prove that these cases are mutually exclusive.

(i) If there is a high type agent whose peak is less than $1 - \inf D_q$, then the primary agent is a high type agent. If there are several such high type agents, the primary agent is the one in this set whose peak is closest to 0.5.

(ii) If there is a low type agent whose peak is greater than $\inf U_q$, then the primary agent is a low type agent. If there are several such low type agents, the primary agent is the one in this set whose peak is closest to 0.5.

(iii) If neither (i) nor (ii) hold, the primary agent is the agent whose peak is closest to 0.5.

- **A**llocate the primary agent, her contribution t_i. If the primary agent is a low type agent, she receives the minimum of her peak and $\inf D_q$. If the primary agent is a high type agent, she receives the maximum of her peak and $1 - \inf U_q$.
- **M**atch the primary agent with a secondary agent. If the primary agent i is a high type agent, the secondary agent is the low type agent with the highest equivalent of $(1 - t_i)$. If the primary agent i is a low type agent, the secondary agent is the high type agent with the highest equivalent of t_i.

In the special case where all agents have symmetric preferences, the primary and secondary agents are the agents of each type amongst the set of unmatched agents whose peaks are closest to 0.5.

6 The Select-Allocate-Match (SAM) Algorithm

In this section, we provide a formal description of our algorithm and state our main result.

In the rest of the paper, we adopt the following convention: whenever we write a triple (i, j, t_i) in the description of an allocation, we assume $p_i \leq p_j$.[17]

Let $\succ^N$ be a linear ordering of the set N. This ordering will serve as a tie-breaking rule. Fix an arbitrary preference profile $\succsim$. The peaks of the agents at $\succsim$ are $p_1, p_2, \ldots, p_n$. We begin by partitioning the set of agents N into the sets H and L.

Step 0: We remove excess agents from either the set H or the set L to ensure that the cardinality of the adjusted H and L sets is equal. If $|H| > |L|$, we remove $|H| - |L|$ agents (chosen in a specific way) from the set H. We denote the set of agents removed from H by $\bar{H}$. Similarly if $|L| > |H|$, we remove $|L| - |H|$ agents from L. The set of agents removed from L is denoted by $\bar{L}$. In addition, we define two sets U_1 and D_1 with $U_1, D_1 \subseteq [0, 0.5]$.

There are three possibilities to consider.

[17] Suppose agents 1 and 2 are paired in an allocation. Let $p_1 = 0.4$, $p_2 = 0.7$ and their contributions in the allocation be $t_1 = 0.1$, $t_2 = 0.9$. We shall write the triple as $(1, 2, 0.1)$.

1. $|H| = |L|$. Here $\bar{H} = \emptyset$ and $\bar{L} = \emptyset$. Also $U_1 = \emptyset$ and $D_1 = \emptyset$.

2. $|H| > |L|$. Compute $e_i(0.5)$ for all $i \in H$. Pick the $|H| - |L|$ agents whose equivalents $e_i(0.5)$ are closest to 0.5. Ties are broken using the ordering $\succ^N$. The set of these agents is $\bar{H}$. Pair the agents in $\bar{H}$ arbitrarily and the contribution of all agents is 0.5. Here $\bar{L} = \emptyset$, $U_1 = \cup_{i \in \bar{H}} I_{i,0.5}$ and $D_1 = \emptyset$.

3. $|H| < |L|$. Compute $e_i(0.5)$ for all $i \in L$. Pick the $|L| - |H|$ agents whose equivalents $e_i(0.5)$ are closest to 0.5. Ties are broken using the ordering $\succ^N$. The set of these agents is $\bar{L}$. Pair the agents in $\bar{L}$ arbitrarily and the contribution for all agents is 0.5. Here $\bar{H} = \emptyset$, $D_1 = \cup_{i \in \bar{L}} I_{i,0.5}$ and $U_1 = \emptyset$.

The adjusted partition of N is $H_1 = H \setminus \bar{H}$ and $L_1 = L \setminus \bar{L}$. By construction, $|H_1| = |L_1| = K$. The algorithm has $K + 1$ steps including Step 0. In each Step q (where $1 \le q \le K$), we form a pair consisting of a high type agent and a low type agent. We denote these agents by h_q and l_q respectively. At the start of the generic step q where $q \in \{1, \ldots, K\}$, the algorithm is provided three inputs: (i) Preferences of the agents in H_q and L_q (ii) U_q where $U_q \subseteq [0, 0.5]$ and (iii) D_q where $D_q \subseteq [0, 0.5]$. The set D_q is the union of the improvement sets of the L type agents who have been matched till Step q. A similar comment holds for U_q and H type agents.

Step q: Each step q is divided into four substeps, referred to as Substep $q.s$ where $s \in \{1, 2, 3, 4\}$. We will determine the agents $l_q \in L_q$ and $h_q \in H_q$ who will be matched to each other and their contribution vector (t_{l_q}, t_{h_q}). At the end of the step, we will determine L_{q+1}, H_{q+1}, D_{q+1}, and U_{q+1}. In case D_q or U_q is empty, we adopt the convention that infimum of D_q and U_q is ∞.

Step $q.1$: Consider the set $\{h \in H_q : 1 - p_h > \inf D_q\}$. If it is empty, proceed to Step $q.2$. Otherwise, choose h_q to be the agent with the lowest peak (or the highest $1 - p_h$) in this set. The agent h_q is the primary agent in this substep. The contribution of agent h_q is $t_{h_q} = \max\{p_{h_q}, 1 - \inf U_q\}$. Choose l_q to be the low type agent in L_q who has the highest $e(1 - t_{h_q})$ (using the tie-breaking ordering $\succ^N$ on agents if necessary). The agent l_q is the secondary agent in this substep. We add the triple $(l_q, h_q, 1 - t_{h_q})$ to the allocation and proceed to Step $q.4$.

Step $q.2$: Consider the set $\{l \in L_q : p_l > \inf U_q\}$. If it is empty, proceed to Step $q.3$. Otherwise, choose l_q to be the agent with the highest peak in this set. The contribution of agent l_q is $t_{l_q} = \min\{p_{l_q}, \inf D_q\}$. Choose h_q to be the high type agent in H_q who has the highest $e(t_{l_q})$ (using the tie-breaking ordering $\succ^N$ on agents in case of ties). We add the triple (l_q, h_q, t_{l_q}) to the allocation and proceed to Step $q.4$. In this substep, l_q is the primary agent while h_q is the secondary agent.

Step $q.3$: If $1 - p_h \leq \inf D_q$ for all $h \in H_q$ and $p_l \leq \inf U_q$ for all $l \in L_q$, we identify the following agents.

1. The high type agent with the lowest peak in H_q. Denote this agent by $\tilde{h}_q$. In case there is more than one such agent, use $\succ^N$ as the tie breaker.

2. The low type agent with the highest peak in L_q. Denote this agent by $\tilde{l}_q$. Use $\succ^N$ as the tie breaker if required.

There are two possibilities leading to Steps $q.3.1$ and $q.3.2$.

Step $q.3.1$: If $p_{\tilde{l}_q} \leq 1 - p_{\tilde{h}_q}$, choose $h_q = \tilde{h}_q$. The contribution of agent h_q is $t_{h_q} = \max\{p_{h_q}, 1 - \inf U_q\}$. Choose l_q to be the low type agent in L_q who has the highest $e(1 - t_{h_q})$ (using the tie-breaking ordering $\succ^N$ on agents if necessary). We add the triple $(l_q, h_q, 1 - t_{h_q})$ to the allocation and proceed to Step $q.4$. In this substep, h_q is the primary agent while l_q is the secondary agent.

Step $q.3.2$: If $p_{\tilde{l}_q} > 1 - p_{\tilde{h}_q}$, choose $l_q = \tilde{l}_q$. The contribution of agent l_q is $t_{l_q} = \min\{p_{l_q}, \inf D_q\}$. Choose h_q to be the high type agent in H_q who has the highest $e(t_{l_q})$ (using the tie-breaking ordering $\succ^N$ on agents in case of ties). We add the triple (l_q, h_q, t_{l_q}) to the allocation. Proceed to Step $q.4$. In this substep, l_q is the primary agent while h_q is the secondary agent.

Step $q.4$: Sets $D_{q+1} = D_q \cup I_{l_q, t_{l_q}}$ and $U_{q+1} = U_q \cup I_{h_q, t_{h_q}}$. Also sets $H_{q+1} = H_q \setminus \{h_q\}$ and $L_{q+1} = L_q \setminus \{l_q\}$. The set H_{q+1} (or L_{q+1}) contains the high type (or low type) agents who are unmatched after Step q. Proceed to Step $q + 1$.

⋮

⋮
⋮

Step K: Note that $|L_K| = |H_K| = 1$. After the completion of this step, all agents in N have been matched and the algorithm terminates.

We state our result below.

Theorem 1 *The SAM algorithm generates a stable and Pareto efficient allocation.*

The proof of Theorem 1 is in the Appendices. Appendix A contains the proof of stability and Appendix B contains the proof of Pareto efficiency. The allocation generated by the SAM algorithm is stable since it satisfies Condition S. The key step in the proof is to show that the sets D_q and U_q do not intersect and do not contain 0.5 at any step q of the algorithm. The proof of Pareto efficiency requires several steps.

7 Discussion

In this section, we discuss various aspects of our model.

7.1 Single-peaked preferences

In Example 10 we show that the single-peakedness assumption on preferences is vital for the existence of stable allocations.

Example **10** Let $N = \{1, 2, \ldots, 6\}$. Table 10 summarizes the peaks of agents 1 to 5 and the dip of agent 6. All agents in $\{1, \ldots, 5\}$ have symmetric single-peaked preferences. Agent 6 has symmetric single-dipped preferences with 0.5 as the dip. This means that 0.5 is her worst contribution and she is progressively better-off as she moves farther away from 0.5. As a result, 0 and 1 are her most preferred contributions.

Agent	1	2	3	4	5	6
Peak	0.49	0.49	0.49	0.01	0.98	-
Dip	-	-	-	-	-	0.5

Table 10: Peaks/dip of agents in Example 10.

We argue that there are no stable allocations. Notice that one of the agents in $\{1, 2, 3\}$ must be paired with an agent from $\{4, 5, 6\}$. Assume w.l.o.g. that agent 3 is paired with an agent from $\{4, 5, 6\}$. We consider each case in turn.

Case A: Agent 3 is paired with agent 4. Let their contribution vector be (t_3, t_4). If the allocation is stable, it must be the case that $t_3 \geq 0.49$ and $t_4 \geq 0.01$.[18]

One of the agents $3, 4$ must be at a distance of $\max\{t_3 - 0.49, t_4 - 0.01\}$ for any (t_3, t_4). Minimising over (t_3, t_4), we infer that one of the agents must be a distance of at least 0.25 from her peak. Suppose this agent is 3. An immediate consequence is that agents 1 and 2 must be receiving their peaks in the allocation. Otherwise agent 3 can block with the non-satiated agent by offering her 0.49 and being only 0.02 away from her own peak. This implies that agents 1 and 2 are not paired together but are paired with 5 and 6. Moreover agents 5 and 6 will each get a contribution of 0.51. Then the pair $(5, 6)$ blocks with the contribution vector $(0.98, 0.02)$.

Suppose agent 4 is the agent who is at a distance of at least 0.25 from her peak. Then agent 5 must get her peak and agent 6 must be getting either 0 or 1. If not, agent 4 can

[18]If $t_3 > 0.49$ and $t_4 < 0.01$, then the pair $(3, 4)$ blocks by proposing a contribution vector (t'_3, t'_4) such that $t'_3 < t_3$ and $t'_4 > t_4$. Similarly the case $t_3 < 0.49$ and $t_4 > 0.01$ is ruled out. Of course $t_3 < 0.49$ and $t_4 < 0.01$ is infeasible.

block with 5 by offering her 0.98 and being 0.01 away from her peak. Agent 4 can block with agent 6 by offering her 1 and being 0.01 away from her own peak. It follows that 5 and 6 cannot be paired together but are paired with 1 and 2. The agent paired with 5, say agent 1 gets 0.02. While agent 2 (paired with 6) gets either 0 or 1. In either case, the pair $(1, 2)$ blocks with the vector $(0.5, 0.5)$. These arguments establish that Case A cannot occur.

Case B: Agent 3 is paired with agent 5. Let (t_3, t_5) be the contribution vector. Using arguments similar to those in Case A, we can argue that $t_3 \leq 0.49$ and $t_5 \geq 0.98$. One of the agents $3, 5$ must be at a distance of $\max\{0.49 - t_3, 0.98 - t_5\}$ for any (t_3, t_5). Therefore either 3 or 5 must be at a distance of at least 0.235 from her peak.

Suppose agent 3 is this agent. Like in Case A, agents 1 and 2 must receive their peaks. Thus they are not paired together and are paired with 4 and 6. Moreover 4 and 6 are each receiving 0.51. The pair $(4, 6)$ blocks with $(0, 1)$.

Suppose agent 5 is at a distance of at least of 0.235 from her peak. Like in Case A, agents 4 and 6 must get their peaks. So they cannot be paired together and are paired with 1 and 2. Once again, 1 and 2 will form a blocking coalition.

Case C: Agent 3 is paired with agent 6. Since 1, 2 and 3 have the same preferences, we can apply Cases A and B to argue that neither 1 nor 2 can be matched to an agent in $\{4, 5\}$. Consequently the pairs in this allocation are $(1, 2)$ and $(4, 5)$. Note that there must be an agent in each pair who does not get her peak. Assume w.l.o.g. that 1 is not getting her peak. To ensure that $(1, 3)$ does not block with $(0.49, 0.51)$, it must be the case that $0.47 \leq t_3 \leq 0.51$. Thus $0.49 \leq t_6 \leq 0.53$. If 4 is the agent in the pair $(4, 5)$ who is not getting her peak, then $(4, 6)$ blocks with $(0.01, 0.99)$. If 5 is the agent not getting her peak, then $(5, 6)$ blocks with $(0.98, 0.02)$.

Therefore Case C cannot occur and there are no stable allocations. □

Example 10 illustrates the key role played by the "complementarity of preferences" in the existence of stable allocations in our model. For simplicity, suppose all agents have symmetric (single-peaked) preferences. Consider two agents of very high type (with peaks close to one) and one of a very low type (with a peak close to zero). Each of the two high type agents are a "good fit" for the low type agent but are not well-suited to be paired together. This prevents the cyclical pattern of blocking which typically underlies the non-existence of stable allocations. This is exactly what occurs in Example 10 - agents 4, 5 and 6 are mutually "good fits" for each other.

7.2 Continuity of preferences

The following example shows that stable allocations may not exist if preferences are single-peaked but not continuous.

EXAMPLE **11** Let $N = \{1, 2, \ldots, 6\}$. Table 11 summarizes the peaks of the agents. Preferences of agents 1 and 2 are symmetric and continuous. For any agent $i \in \{3, 4, 5, 6\}$, $\succsim_i$ is single-peaked but not continuous at 0.3. In particular, $\succsim_i$ satisfies: (i) for any z such that $0.3 < z \leq 0.41$, $z \succ_i 0.5$ and (ii) $\exists\, \bar{\epsilon} > 0$ such that $0.5 + \bar{\epsilon} \succ_i 0.3$. Continuity of $\succsim_i$ will imply $0.3 \succsim_i 0.5$. Single-peakedness implies $0.5 \succ_i 0.5 + \bar{\epsilon}$. Thus $0.3 \succ_i 0.5 + \bar{\epsilon}$ contradicting (ii).

Agent	1	2	3	4	5	6
Peak	0.7	0.7	0.41	0.41	0.41	0.41

Table 11: Peaks of agents in Example 11.

We claim that stable allocations do not exist.

Consider an arbitrary allocation. If it is stable, agents 1 and 2 are not paired together. If they are, one of them (say 1) gets a contribution of at most 0.5. One of the agents in $\{3, 4, 5, 6\}$, say 3 has a contribution of at least 0.5. Then the pair $(1, 3)$ blocks with $(0.51, 0.49)$.

We can therefore assume w.l.o.g. that the pairs $(1, 3)$, $(2, 4)$ and $(5, 6)$ belong to the allocation. By feasibility, one of the agents in $\{5, 6\}$, say 5 has a contribution $t_5 \geq 0.5$.

Let (t_1, t_3) be the contribution vector for the pair $(1, 3)$. In order for $(1, 3)$ not to block, we must have $0.59 \leq t_1 \leq 0.7$ and $0.3 \leq t_3 \leq 0.41$. There are two cases to consider.

The first is when $t_3 > 0.3$. Since $t_3 \leq 0.41$, we can find $\delta > 0$ small enough such that $t_3 - \delta \in (0.3, 0.41)$ and $t_1 + \delta < 0.7$. By assumption (i), $t_3 - \delta \succ_5 t_5$ as $t_5 \geq 0.5$. Therefore the pair $(1, 5)$ can block with $(t_1 + \delta, t_3 - \delta)$.

The remaining case is $t_3 = 0.3$. If $t_5 > 0.5$, the pair $(3, 5)$ blocks with $(0.5, 0.5)$. Agent 3 strictly improves as $0.5 \succ_3 0.3$ (Assumption (*ii*) and single-peakedness). Agent 5 strictly improves as she moves closer to her peak. Suppose $t_5 = 0.5$. Pick $0 < \epsilon < \bar{\epsilon}$ where $\bar{\epsilon}$ is specified in Assumption (ii). By single-peakedness and Assumption (ii), $0.5 + \epsilon \succ_3 0.5 + \bar{\epsilon} \succ_3 0.3$. Hence $(3, 5)$ blocks with $(0.5 + \epsilon, 0.5 - \epsilon)$. □

7.3 Coalitions of arbitrary size

The example below shows that stable allocations may not exist if coalitions of arbitrary size are permitted. Agents in a coalition have to make an aggregate contribution of 1.

EXAMPLE **12** Let $N = \{1, 2, 3, 4\}$. Agents' preferences are symmetric. Table 12 summarizes the peaks of the agents.

In any stable allocation, agent 4 must have a contribution of 1. Otherwise the allocation will be blocked by the singleton coalition $\{4\}$ where she contributes 1.

p_1	p_2	p_3	p_4
0.55	0.55	0.55	1

Table 12: Peaks of agents in Example 12.

Consider an arbitrary allocation σ. We have just shown that $t_4 = 1$ if σ is stable. There are two cases to consider. The first case is where agent 4 belongs to a coalition C with some other agents in σ. All agents in $C \setminus \{4\}$ will have a contribution of 0. If $|C| = 4$ or $|C| = 3$, then any two agents from $C \setminus \{4\}$ will block with the contribution vector $(0.5, 0.5)$. Assume w.l.o.g. $C = \{1, 4\}$. One of the agents in the set $\{2, 3\}$ (say 2) does not get her peak. The pair $(1, 2)$ blocks with $(0.45, 0.55)$.

The second case is where agent 4 is on her own in σ. If 1, 2 and 3 belong to the same coalition, there exists an agent $i \in \{1, 2, 3\}$ with $t_i \leq \frac{1}{3}$. Also there is at most one agent who receives her peak, i.e there exists $j \neq i$ with $t_j \neq 0.55$. The pair (i, j) blocks with $(0.45, 0.55)$. In all other remaining cases, there exists an agent i who is on her own (her contribution is 1) and another agent j who does not get her peak. Then (i, j) can block with $(0.45, 0.55)$. □

Our negative result in the case of arbitrary coalitions bears a resemblance to some earlier results on stability in division problems. Gensemer et al. (1996) consider the problem of allocating agents with single-peaked preferences across a set of islands. Each island has a unit amount of resource and operates with a fixed division rule. It can also accommodate an arbitrary number of agents. The paper formulates a notion of a migration equilibrium according to which no agent can benefit by migrating to another island. No island has the right to refuse an entrant. The paper provides a number of negative results about the existence of migration equilibria.

Bergantiños et al. (2015) consider a related model where all islands use the same division rule. They also weaken the equilibrium condition to a stability notion - in order for successful blocking to take place, the migrant's well-being must strictly improve while no member of the receiving island is made strictly worse-off by the move. The paper shows that stable allocations exist for some special division rules such as the proportional rule and the sequential dictatorship rule, provided agents' preferences are symmetric.

7.4 Strategy-proofness

An allocation rule is strategy-proof if no agent can strictly improve by misrepresenting her preferences. This property ensures that the mechanism designer can achieve the allocation specified at a preference profile by relying on the reports of the agents themselves. We show in Example 13 below that the SAM algorithm is not strategy-proof.[19]

We begin with an important observation regarding the SAM algorithm. It first selects a primary agent, then determines her contribution and finally chooses her partner. The contribution of the primary agent i depends whether she is a high or a low type agent: namely if $i \in H$ she gets $t_i = \max\{p_i, 1 - \inf U_q\}$ and is matched with the unmatched low type agent j with the largest $e_j(1-t_i)$; if $i \in L$ she gets $t_i = \min\{p_i, \inf D_q\}$ and is matched with the unmatched high type agent j with the largest $e_j(t_i)$. Suppose agents (i, j) are matched in Step q. Suppose also that $\inf U_q \leq \inf D_q$ and agent i is l_q. By letting $p_a = \inf U_q$ and $p_b = \inf D_q$, we observe that the allocation to agent l_q is exactly as specified in Equation 1, i.e. *once a matching is formed*, the SAM algorithm specifies the contributions of the matched pair according to one of the rules described in Barberà et al. (1997). However, the two numbers p_a, p_b depends on the allocation and the preferences of the agents that are already matched by the algorithm. This observation suggests that an agent's preference misreport can be profitable only if it induces a change in her matching. The following example shows that this can indeed happen in the SAM algorithm - by misreporting her preferences, an agent can be matched to a different partner and obtain a higher utility.

Example **13** Let $N = \{1, 2, 3, 4\}$. Agents' preferences $\succsim_i$ are symmetric and Table 13 summarizes their peaks. The ordering of the agents is $1 \succ^N 2 \succ^N 3 \succ^N 4$.

p_1	p_2	p_3	p_4
0.59	0.58	0.57	0.4

Table 13: Peaks of agents in Example 13.

In Step 0, agents 2 and 3 are removed from H and paired together. This is because $e_3(0.5) > e_2(0.5) > e_1(0.5)$. The triple $(2, 3, 0.5)$ is formed and $U_1 = (0.34, 0.5)$ and $D_1 = \emptyset$. Since $p_4 = 0.4 > \inf U_1 = 0.34$, Substep 1.2 applies. Agent $4 \in L$ is the primary agent and $t_4 = \min\{p_4, \inf D_1\} = 0.4$. The triple $(4, 1, 0.4)$ is formed. The allocation generated by the SAM algorithm is $(2, 3, 0.5)$, $(4, 1, 0.4)$.

Suppose agent 2 reports a peak of 0.6 and symmetric preferences $\succsim_2'$. Now in Step 0, the triple $(1, 3, 0.5)$ is formed and $U_1 = (0.32, 0.5)$. Substep 1.2 applies as $p_4 > \inf U_1 = 0.32$. The triple $(2, 4, 0.4)$ is formed. Agent 2 strictly improves at $\succsim_2$ by misreporting since $0.6 \succ_2 0.5$. $\square$

The existence of a strategy-proof, stable and Pareto-efficient rule in our model remains an open question. However, it is easy to show the existence of rules that are Pareto-efficient

[19] According to the definition, the algorithm specifies an allocation at a preference profile. We are slightly abusing terms here by regarding the algorithm as an allocation rule.

and strategy-proof. We describe such a rule in the case of four agents. It can be extended to an arbitrary even number of agents but we omit the details since they are cumbersome.

1. Agent 1 is always assigned her peak.

2. If the sum of the peaks of 1 and 2 is one, they are matched together and contribute their respective peaks. Agents 3 is matched with 4 and 3's contribution is her peak.

3. If case 2 above does not occur, 3 is matched with either 1 or 2, and her contribution is the complement of her partner's peak. Specifically, 3 is assigned to the triple she strictly prefers between $(1, 3, p_1)$ and $(2, 3, p_2)$.

4. In case 3 above, if 3 is indifferent between$(1, 3, p_1)$ and $(2, 3, p_2)$, then 4 chooses between the triples $(1, 4, p_1)$ and $(2, 4, p_2)$. Agent 3 is matched with the remaining agent and her contribution is still equal to the complement of her partner's peak.

5. If both 3 and 4 are both indifferent in cases 3 and 4, a tie-breaking rule determines the agent to who they are matched.

We claim that the rule is strategy-proof. Note that 1 and 2 always receive their peak so they cannot gain by misreporting. Agent 3 chooses the preferred alternative between two alternatives that do not depend on her reporting. The same holds for 4 in case 3 is indifferent. The rule also satisfies Pareto-efficiency holds because the rule is a sequential dictatorship that takes into account the potential efficiency losses due to mismatches (when it is efficient to match together agents 1 and 2) or due to indifferences.

The rule does not however, generate stable allocations. Consider the case where $p_1 = 0.9, p_2 = 0.9, p_3 = 0.49, p_4 = 0.49$. We can assume w.l.o.g that the allocation is $(1, 3, 0.9)$ and $(2, 4, 0.9)$. Then 3 and 4 will block with the triple $(3, 4, 0.5)$.

7.5 Non-convexity of the set of stable allocations

The next example shows that a convex combination of two stable allocations with the same set of matched pairs may not be stable.

Example **14** Let $N = \{1, 2, 3, 4\}$. The peaks of the agents are summarized in Table 14. All agents have symmetric preferences.

It is easy to verify that the allocations $\sigma^1 = \{(1, 4, 0.3), (2, 3, 0.3)\}$ and $\sigma^2 = \{(1, 4, 0.2), (2, 3, 0.1)\}$ are both stable. However the allocation $\sigma^3 = \{(1, 4, 0.25), (2, 3, 0.2)\}$ is not stable because the pair $(2, 4)$ can block with $(0.21, 0.79)$. Note that σ^3 has the same matched pairs as σ^1 and σ^2 but the contribution vector of each pair is a convex combination of the respective contributions in σ^1 and σ^2 with weights $(0.5, 0.5)$. □

p_1	p_2	p_3	p_4
0.3	0.3	0.8	0.9

Table 14: Peaks of agents in Example 14.

8 Appendix A

In this section, we provide a proof of the stability part of Theorem 1, i.e. the SAM algorithm generates a stable allocation. We begin with a few key observations.

Recall that $D_{q+1} = D_q \cup I_{l_q,t_{l_q}}$ and $U_{q+1} = U_q \cup I_{h_q,t_{h_q}}$ for $q \in \{0, 1, \ldots, K\}$. For every step q of the algorithm where $q \in \{1, \ldots, K\}$, we have $D_q = \cup_{r<q} I_{l_r,t_{l_r}}$ and $U_q = \cup_{r<q} I_{h_r,t_{h_r}}$. Since the improvement sets are open (see Observation 1), it follows that D_q and U_q are also open in $[0, 0.5]$.

The sets D_q and U_q can be written as the disjoint union of their connected components. Since D_q and U_q are open sets, none of their connected components are singletons - thus each connected component of D_q and U_q is an interval in $[0, 0.5]$. Moreover the connected components of D_q and U_q can be ordered from "left" to "right". Let D_q^1 and U_q^1 denote the "leftmost" connected components of D_q and U_q respectively. By definition, $\inf D_q = \inf D_q^1$ and $\sup D_q^1 \le \inf D_q^r$ for any component D_q^r other than D_q^1. Similar inequalities hold for U_q^1. Incase D_q or U_q is empty (then D_q^1 or U_q^1 do not exist), we adopt the convention that the infimum and supremum of D_q^1 and U_q^1 is $+\infty$.

Observation **2** Consider step q where $q \in \{1, \ldots, K\}$. Recall that the triple formed in this step is (l_q, h_q, t_{l_q}). If $p_{l_q} \ge \inf D_q^1$ then $\sup D_q^1 \le \sup D_{q+1}^1$. Similarly, if $1 - p_{h_q} \ge \inf U_q^1$ then $\sup U_q^1 \le \sup U_{q+1}^1$. This is an immediate consequence of the definition of improvement sets.

We now establish a series of results that are loop invariants of the algorithm.

Lemma **1** *Fix $q \in \{1, \ldots, K\}$ and assume $D_q \cap U_q = \emptyset$. Then for all $q \in \{1, \ldots, K\}$, we have*

$$[\forall h \in H_q,\ 1 - p_h < \sup U_q^1] \text{ and } [\forall l \in L_q,\ p_l < \sup D_q^1].$$

Proof: We will prove the lemma by induction on q.

Base Case ($q = 1$): There are two cases to consider - $U_1 = \emptyset$ and $U_1 \neq \emptyset$. If the former holds, then $[\forall h \in H_1,\ 1 - p_h < \sup U_1^1]$ is true since $\sup U_1^1 = +\infty$. Suppose $U_1 \neq \emptyset$. All agents allocated in Step 0 have a contribution of 0.5. Hence $\sup U_1^1 = 0.5$ and $1 - p_h \le 0.5 = \sup U_1^1$ for all $h \in H$. Suppose there exists an agent $h' \in H_1$ and $1 - p_{h'} = 0.5$, i.e. $e(p_{h'}) = 0.5$. Since $U_1 \neq \emptyset$, there exists an agent $\bar{h}$ allocated in Step 0 for whom $p_{\bar{h}} > 0.5$, i.e. $e_{\bar{h}}(0.5) < 0.5$. But then h' has higher priority than $\bar{h}$ in H and should have been allocated in Step 0. Therefore $[\forall h \in H_1,\ 1-p_h < \sup U_1^1]$ holds. The argument for L_1 is identical and omitted.

Inductive step: Consider $q \in \{1,\ldots,K\}$. Assume $D_q \cap U_q = \emptyset$, $[\forall h \in H_q,\ 1-p_h < \sup U_q^1]$ and $[\forall l \in L_q,\ p_l < \sup D_q^1]$. We have to show

$$[\forall h \in H_{q+1},\ 1-p_h < \sup U_{q+1}^1] \text{ and } [\forall l \in L_{q+1},\ p_l < \sup D_{q+1}^1].$$

We refer to $[\forall h \in H_{q+1},\ 1-p_h < \sup U_{q+1}^1]$ and $[\forall l \in L_{q+1},\ p_l < \sup D_{q+1}^1]$ as Statements A and B respectively. There are two cases to consider depending on whether D_q^1 lies to the left or to the right of U_q^1.

Case I: $\sup D_q^1 \le \inf U_q^1$ (see Figure 3). There are two sub-cases to consider depending on the selection of the primary agent.

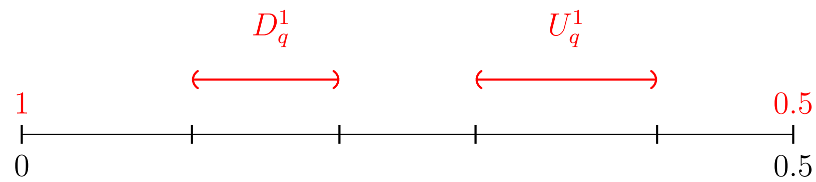

Figure 3: Case I in the proof of Lemma 1

Case I.a: The primary agent is h_q. If $1-p_{h_q} > \inf U_q^1$, Substep $q.1$ applies and the contribution of h_q is $1-\inf U_q^1$. By Observation 2, $\sup U_q^1 \le \sup U_{q+1}^1$. Statement A follows from the induction hypothesis and $H_{q+1} \subset H_q$. If $1-p_{h_q} \le \inf U_q^1$, then the contribution of h_q is p_{h_q}. Consequently the improvement set of h_q is empty and $\sup U_q^1 = \sup U_{q+1}^1$. Statement A once again follows from the induction hypothesis and $H_{q+1} \subset H_q$.

We now prove Statement B for the secondary agent l_q. By the induction hypothesis, $p_{l_q} < \sup D_q^1$. If $p_{l_q} \ge \inf D_q^1$, then $p_{l_q} \in D_q^1$. Observation 2 implies $\sup D_q^1 \le \sup D_{q+1}^1$ and Statement B holds following the earlier argument. Suppose $p_{l_q} < \inf D_q$. There are two possibilities depending on the location of the peak of h_q.

(i) If $1-p_{h_q} > \inf D_q^1$, then Substep $q.1$ is applicable. Here $t_{l_q} \ge \inf D_q^1$ as $t_{l_q} = \min\{1-p_{h_q}, \inf U_q\}$. This implies $\sup D_q^1 \le \sup D_{q+1}^1$ and Statement B follows from the induction hypothesis.

(ii) The remaining case is when $1-p_{h_q} \le \inf D_q^1$. This can happen only if Substep $q.3.1$ occurs. In particular, we have $p_l \le 1-p_{h_q}$ for all $l \in L_q$. So $p_{l_q} \le 1-p_{h_q}$. The contribution of agent l_q is $t_{l_q} = 1-p_{h_q}$. If $p_{l_q} = 1-p_{h_q}$, then the improvement set of l_q is empty and Statement B follows from the induction hypothesis and the fact that $D_{q+1}^1 = D_q^1$. Suppose $p_{l_q} < 1-p_{h_q} = t_{l_q}$. Then $\sup D_{q+1}^1 = t_{l_q}$ and $p_l \le \sup D_{q+1}^1$ for all $l \in L_q$. Suppose there exists $\bar{l} \in L_q$ with $p_{\bar{l}} = \sup D_{q+1}^1 = t_{l_q}$. Then $e_{\bar{l}}(t_{l_q}) = t_{l_q}$. However $e_{l_q}(t_{l_q}) < t_{l_q}$ as $p_{l_q} < t_{l_q}$.

Then $\bar{l}$ should have been matched with h_q in Step q instead of l_q. This establishes Statement B in this case.

Case I.b: The primary agent is l_q. This case is symmetric to the case when h_q is the primary agent. If $p_{l_q} \geq \inf D_q^1$, then the result follows from Observation 2. If $p_{l_q} < \inf D_q^1$, then l_q receives her peak and the improvement set is empty. Statement B again follows immediately.

Here h_q is the secondary agent. This can happen only if Step q.3.2 occurs. In particular, $1 - p_h < p_{l_q}$ and $1 - p_h \leq \inf D_q$ for all $h \in H_q$. Note that $t_{l_q} = \min\{p_{l_q}, \inf D_q\}$. Thus $1 - p_h \leq t_{l_q}$ for all $h \in H_q$ and $\sup U_{q+1}^1 = t_{l_q}$. Using arguments like those in Case I.a.(ii) above, we can show $1 - p_h < t_{l_q}$ for all $h \in H_q$. This establishes Statement A.

Case II: $\sup U_q^1 \leq \inf D_q^1$ (see Figure 4). Once again there are two cases depending upon the selection of the primary agent.

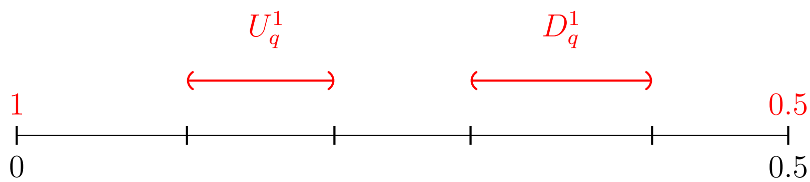

Figure 4: Case II in the proof of Lemma 1

The arguments for the primary agent are symmetric to those for the primary agent in Case I. This is also true for the secondary agent. For instance, consider the case where l_q is the secondary agent. This occurs only if Step q.3.1 occurs. In particular, $p_l \leq 1 - p_{h_q}$ and $p_l \leq \inf U_q$ for all $l \in L_q$. So $p_l \leq t_{l_q}$ for all $l \in L_q$. Hence $\sup D_{q+1}^1 = t_{l_q}$. Using arguments similar to those in Case I.b.(ii), we can verify Statement B.

We omit the other arguments. ■

Lemma **2** *Consider Step q where $q \in \{1, \ldots, K\}$. Assume $D_q \cap U_q = \emptyset$. Then the triple formed in Step q satisfies the following:*

1. *if it is formed in Substep q.1, then either $p_{l_q} \leq t_{l_q} \leq 1 - p_{h_q}$ or $t_{l_q} = 1 - p_{h_q} < p_{l_q}$.*
2. *if it is formed in Substep q.2, then either $1 - p_{h_q} \leq t_{l_q} \leq p_{l_q}$ or $t_{l_q} = p_{l_q} < 1 - p_{h_q}$.*
3. *if it is formed in Substep q.3.1, then $p_{l_q} \leq t_{l_q} \leq 1 - p_{h_q}$.*
4. *if it is formed in Substep q.3.2, then $1 - p_{h_q} \leq t_{l_q} \leq p_{l_q}$.*

Proof: Since $D_q \cap U_q = \emptyset$, it must be the case that D_q^1 either lies entirely to the "left" of U_q^1 or entirely to the "right" of U_q^1. We now consider each of the four cases in turn.

1. Suppose the triple (l_q, h_q, t_{l_q}) is formed in Substep q.1. Lemma 1 rules out the case where D_q^1 lies entirely to the right of U_q^1.

There are now two possibilities. The first is $1 - p_{h_q} < p_{l_q}$. By Lemma 1, we know $p_{l_q} < \sup D_q^1$. Since $t_{l_q} = \min\{1 - p_{h_q}, \inf U_q\}$ and $\sup D_q^1 \leq \inf U_q$, we have $t_{l_q} = 1 - p_{h_q}$. Thus $t_{l_q} = 1 - p_{h_q} < p_{l_q}$.

The second possibility is $1 - p_{h_q} \geq p_{l_q}$. If $1 - p_{h_q} \geq \inf U_q$, we have $t_{l_q} = \inf U_q$. By Lemma 1, $p_{l_q} < \sup D_q^1$. Thus $p_{l_q} \leq t_{l_q} \leq 1 - p_{h_q}$. If $1 - p_{h_q} < \inf U_q$, we have $t_{l_q} = 1 - p_{h_q}$ and once again $p_{l_q} \leq t_{l_q} = 1 - p_{h_q}$.

2. Suppose the triple (l_q, h_q, t_{l_q}) is formed in Substep q.2. In this case, Lemma 1 implies D_q^1 must lie to the right of U_q^1. We can use the symmetric counterparts of the arguments in Case 1 to derive the necessary conclusion.

3. Suppose the triple (l_q, h_q, t_{l_q}) is formed in Substep q.3.1. Here $t_{l_q} = \min\{\inf U_q, 1 - p_{h_q}\}$. By the hypothesis of Substep q.3.1, we know $p_l \leq \inf U_q$ for all $l \in L_q$. Also $p_l \leq 1 - p_{\tilde{h}_q} = 1 - p_{h_q}$ for all $l \in L_q$.[20] In particular, $p_{l_q} \leq \inf U_q$ and $p_{l_q} \leq 1 - p_{h_q}$. Thus $p_{l_q} \leq \min\{\inf U_q, 1 - p_{h_q}\} = t_{l_q} \leq 1 - p_{h_q}$.

4. Suppose the triple (l_q, h_q, t_{l_q}) is formed in Substep q.3.2. Here $t_{l_q} = \min\{\inf D_q, p_{l_q}\}$. We can use the symmetric counterparts of the arguments in Case 3 to derive the necessary conclusion. ■

Lemma 2 immediately leads to the following corollary.

Corollary **1** *Consider Step q where $q \in \{1, \ldots, K\}$. Assume $D_q \cap U_q = \emptyset$. Then the contribution of agent l_q, t_{l_q} lies in the closed interval with the end points p_{l_q} and $1 - p_{h_q}$.*

Lemma **3** *Consider Step q where $q \in \{1, \ldots, K\}$. Assume $D_q \cap U_q = \emptyset$.*

1. If $t_{l_q} \leq p_{l_q}$, then $I_{l_q, t_{l_q}} \subseteq D_q$.

2. If $t_{l_q} \leq 1 - p_{h_q}$, then $I_{h_q, t_{h_q}} \subseteq U_q$.

Proof: We only consider Part 1 - a symmetric argument applies for Part 2. Consider Step q where the triple (l_q, h_q, t_{l_q}) is formed.

If $p_{l_q} = t_{l_q}$, the improvement set of l_q is empty and the result follows immediately. Assume therefore that $t_{l_q} < p_{l_q}$. By Corollary 1, we have $1 - p_{h_q} \leq t_{l_q} < p_{l_q}$.

We will argue that D_q is non-empty when $t_{l_q} < p_{l_q}$. By Lemma 2, we know Substep q.3.1 cannot occur as $t_{l_q} < p_{l_q}$. We establish the claim for Substeps q.1, q.2 and q.3.2. If Step q.1 occurs, we know $1 - p_{h_q} > \inf D_q$. Thus $\inf D_q$ is not $+\infty$ and D_q is non-empty. If Step q.2 or Step q.3.2 occurs, we know $t_{l_q} = \min\{p_{l_q}, \inf D_q\}$. Since $t_{l_q} < p_{l_q}$, we have $t_{l_q} = \inf D_q$ and thus D_q is non-empty.

Claim 1: If $t_{l_q} < p_{l_q}$, then $t_{l_q} \geq \inf D_q$.

[20] Recall $\tilde{h}_q$ is the high type agent with the lowest peak in h_q and $h_q = \tilde{h}_q$ in Substep q.3.1.

Proof: Lemma 2 implies that the triple cannot be formed in Substep q.3.1. We establish the claim for Substeps q.1, q.2 and q.3.2.

Suppose the triple is formed in Substep q.1. Since $D_q \cap U_q = \emptyset$ by assumption, Lemma 1 implies D_q^1 lies entirely to the left of U_q^1 and $\sup D_q^1 \leq \inf U_q^1$. In Step q.1, we know $1 - p_{h_q} > \inf D_q$ and $t_{l_q} = \min\{1 - p_{h_q}, \inf U_q\}$. Thus $t_{l_q} > \inf D_q$.

Suppose the triple is formed in Substep q.2. Here $t_{l_q} = \min\{p_{l_q}, \inf D_q\}$. Since $t_{l_q} < p_{l_q}$ by assumption, it must be the case that $t_{l_q} = \inf D_q$.

Suppose the triple is formed in Substep q.3.2. Here $t_{l_q} = \min\{p_{l_q}, \inf D_q\}$. Since $t_{l_q} < p_{l_q}$ by assumption, it follows that $t_{l_q} = \inf D_q$. This completes the proof of the claim.

■

Since D_q is a finite union of intervals, $\inf D_q$ is the infimum of at least one of these intervals. Thus $\inf D_q$ is attained in a step before q. Let i be the smallest integer such that $\inf I_{l_i, t_{l_i}} = \inf D_q$ where l_i is a low type agent matched in Step i.[21] Note that $\inf D_j > \inf D_q$ for $j \in \{1, \ldots, i\}$ and $\inf D_{i+1} = \inf D_q$ when $i \geq 1$. If $i = 0$, then $\inf D_1 = \inf D_q$.

Consider Step i where agent l_i is matched and her contribution is t_{l_i}. We claim that $t_{l_i} > p_{l_i}$. Suppose not. If $t_{l_i} = p_{l_i}$, the improvement set of l_i is empty, contradicting the assumption that $\inf I_{l_i, t_{l_i}} = \inf D_q$. Suppose $t_{l_i} < p_{l_i}$. Here $\inf I_{l_i, t_{l_i}} = t_{l_i}$. Arguing as we did to establish Claim 1, it follows $t_{l_i} \geq \inf D_i$ in Step i. Therefore $t_{l_i} = \inf D_q \geq \inf D_i$. This leads to a contradiction since $\inf D_i > \inf D_q$.

Since $t_{l_i} > p_{l_i}$, we have $e_{l_i}(t_{l_i}) \leq \inf I_{l_i, t_{l_i}} = \inf D_q$.[22]

Claim 2: For agents l_i and l_q, we have (i) $e_{l_q}(t_{l_i}) \leq e_{l_i}(t_{l_i}) \leq \inf D_q$ and (ii) $p_{l_q} < t_{l_i}$.

Proof: Recall that agent l_i is matched in Step i.

Suppose $i = 0$. Then it must be the case that $|L| > |H|$ and $D_1 \neq \emptyset$. Every agent matched in this step has a contribution of 0.5. There exists $l_0 \in \bar{L}$ such that $\inf I_{l_0, 0.5} = \inf D_1$. Since the improvement set of l_0 is non-empty, $p_{l_0} < 0.5$ and $e_{l_0}(0.5) < 0.5$. As l_q is not matched in Step 0, we have $e_{l_q}(0.5) \leq e_{l_0}(0.5) < 0.5$. Hence $p_{l_q} < 0.5$. This establishes Claim 2 for $i = 0$.

Suppose $i \geq 1$. Recall that $t_{l_i} > p_{l_i}$. By Lemma 2, we know that l_i is matched in Substep i.1 or Substep i.3.1. In both cases, l_i is the secondary agent. Also $l_q \in L_i$ as $i < q$. Thus $e_{l_q}(t_{l_i}) \leq e_{l_i}(t_{l_i})$. This establishes Part (i).

Now consider Part (ii). If $p_{l_q} = t_{l_i}$, then $e_{l_q}(t_{l_i}) = t_{l_i}$. Since $p_{l_i} < t_{l_i}$, then $e_{l_i}(t_{l_i}) < t_{l_i}$ and l_i should not have been chosen as the secondary agent in Step i. If $p_{l_q} > t_{l_i}$, we know $e_{l_q}(t_{l_i}) > t_{l_i}$. Once again l_i should not have been chosen as the secondary agent in Step i. This establishes Part (ii). ■

We now return to the proof of the lemma. Claims 1, 2 and the assumption $t_{l_q} < p_{l_q}$ imply

[21] If $i = 0$, note that l_i may not be unique. If $i \geq 1$, then l_i is unique.

[22] Note that $e_{l_i}(t_{l_i}) < \inf I_{l_i, t_{l_i}}$ if and only if $e_{l_i}(t_{l_i}) = -\epsilon$. Here $I_{l_i, t_{l_i}} = [0, t_{l_i})$.

$$e_{l_q}(t_{l_i}) \le e_{l_i}(t_{l_i}) \le \inf D_q \le t_{l_q} < p_{l_q} < t_{l_i}.$$

Since $\succsim_{l_q}$ is single-peaked, we have $t_{l_q} \succsim_{l_q} \max\{0, e_{l_q}(t_{l_i})\}$. Also $\max\{0, e_{l_q}(t_{l_i})\} \succsim_{l_q} t_{l_i}$.[23] Thus $t_{l_q} \succsim_{l_q} t_{l_i}$ and $t_{l_i} \notin I_{l_q,t_{l_q}}$. Consequently $\sup I_{l_q,t_{l_q}} \le t_{l_i} = \sup I_{l_i,t_{l_i}}$. We have already shown $\inf I_{l_i,t_{l_i}} = \inf D_q \le t_{l_q} = \inf I_{l_q,t_{l_q}}$. Therefore $I_{l_q,t_{l_q}} \subseteq I_{l_i,t_{l_i}} \subseteq D_q$.

■

We complete the proof of the theorem by showing that the SAM algorithm generates an allocation which satisfies Condition S (Proposition 3).

Proof: Recall that the algorithm terminates in K steps, generating the sets D_{K+1} and U_{K+1}. We will show that $D_{K+1} \cap U_{K+1} = \emptyset$ and $0.5 \notin D_{K+1} \cup U_{K+1}$. This guarantees that Condition S is satisfied by the allocation generated.[24]

In fact, we will show that $D_q \cap U_q = \emptyset$ and $0.5 \notin D_q \cup U_q$ holds for all $q \in \{1, \ldots, K+1\}$. We will use induction on q for this purpose.

We claim that $D_1 \cap U_1 = \emptyset$ and $0.5 \notin D_1 \cup U_1$. By construction, at least one of the sets D_1 and U_1 is empty. Also the contribution of any agent matched in Step 0 is 0.5. Thus the improvement sets of agents matched in this step do not contain 0.5.

Induction Hypothesis (IH): Fix $q \in \{1, \ldots, K\}$ and assume (i) $D_q \cap U_q = \emptyset$ and (ii) $0.5 \notin D_q \cup U_q$.

The IH implies that the antecedents of Lemmas 1, 2, 3 and Corollary 1 are satisfied. We show that $D_{q+1} \cap U_{q+1} = \emptyset$ and $0.5 \notin D_{q+1} \cup U_{q+1}$. We have,

$$\begin{aligned} D_{q+1} \cap U_{q+1} &= (D_q \cup I_{l_q,t_{l_q}}) \cap (U_q \cup I_{h_q,t_{h_q}}) \\ &= (D_q \cap U_q) \cup (D_q \cap I_{h_q,t_{h_q}}) \cup (I_{l_q,t_{l_q}} \cap U_q) \cup (I_{l_q,t_{l_q}} \cap I_{h_q,t_{h_q}}) \end{aligned}$$

From IH, it follows $D_q \cap U_q = \emptyset$. By Corollary 1 and the fact that the improvement sets are open intervals, we have $I_{l_q,t_{l_q}} \cap I_{h_q,t_{h_q}} = \emptyset$. We will show (A) $I_{l_q,t_{l_q}} \cap U_q = \emptyset$, $0.5 \notin I_{l_q,t_{l_q}}$ and (B) $I_{h_q,t_{h_q}} \cap D_q = \emptyset$, $0.5 \notin I_{h_q,t_{h_q}}$.

We first prove (A). There are two cases to consider depending on the contribution of agent l_q. The first case is when $t_{l_q} \le p_{l_q}$. Part 1 of Lemma 3 implies $I_{l_q,t_{l_q}} \subseteq D_q$. Thus $I_{l_q,t_{l_q}} \cap U_q = \emptyset$ and $0.5 \notin I_{l_q,t_{l_q}}$ since $D_q \cap U_q = \emptyset$ and $0.5 \notin D_q$ by IH.

In the second case, $p_{l_q} < t_{l_q}$. Here $t_{l_q} = \sup I_{l_q,t_{l_q}}$. Lemma 2 implies only Substeps $q.1$ and $q.3.1$ can occur. In both steps, we have $t_{l_q} = \min\{1 - p_{h_q}, \inf U_q\}$. Since $1 - p_{h_q} \le 0.5$, it follows that $t_{l_q} \le \min\{\inf U_q, 0.5\}$. Furthermore, $t_{l_q} = \sup I_{l_q,t_{l_q}}$ and $I_{l_q,t_{l_q}}$ is an open set. Therefore $x < t_{l_q} \le \min\{\inf U_q, 0.5\}$ for all $x \in I_{l_q,t_{l_q}}$. Thus $I_{l_q,t_{l_q}} \cap U_q = \emptyset$ and $0.5 \notin I_{l_q,t_{l_q}}$.

[23] If $\max\{0, e_{l_q}(t_{l_i})\} = 0$, then $e_{l_q}(t_{l_i}) = -\epsilon$ and $I_{l_i,t_{l_i}} = [0, t_{l_i})$. Here $0 \succ_{l_q} t_{l_i}$.

[24] Recall D_{K+1} and U_{K+1} are the unions of the improvement sets of all L and H type agents respectively.

The proof of (B) is virtually identical to the arguments for (A), but uses Part 2 of Lemma 2. We omit the details. This completes the proof of the theorem. ■

We have shown above that the sets D_q and U_q do not intersect for any $q \in \{1, \ldots, K\}$. Thus the antecedents of Lemmas 1, 2, 3 and Corollary 1 are true. We will use these facts in the proof of Pareto efficiency in Appendix B.

9 Appendix B

This section contains the proof of the Pareto efficiency part of Theorem 1.

We define some notation which will be used in the proof of Pareto efficiency. For any allocation σ and agent $i \in N$, we shall denote the contribution of agent i in σ by t_i^σ. The improvement set for agent i at t_i^σ is I_{i,t_i^σ} and it's closure is $\overline{I_{i,t_i^\sigma}}$. Note that $\overline{I_{i,t_i^\sigma}} = \{x \in [0, 0.5] : x \succsim_i t_i^\sigma\}$ when $i \in L$ and $\overline{I_{i,t_i^\sigma}} = \{x \in [0, 0.5] : 1 - x \succsim_i t_i^\sigma\}$ when $i \in H$. For the allocation σ, we define sets $D^\sigma = \cup_{i \in L} I_{l,t_l^\sigma}$ and $U^\sigma = \cup_{i \in H} I_{l,t_h^\sigma}$.

We first establish several key observations and lemmas.

Observation **3** In any stable allocation, agents who are paired together must be given contributions on the same side of the peak. We refer to this property as *internal stability* for the pair of agents who are matched together in the allocation. Internal stability is a necessary condition for both stability and Pareto efficiency.

Our next step is to establish a monotonicity lemma.

Lemma **4** *For any Step q where $q \in \{1, \ldots, K-1\}$, we have (i) $t_{l_{q+1}} \le t_{l_q} \le 0.5$ and (ii) $0.5 \le t_{h_q} \le t_{h_{q+1}}$.*

Proof: We first prove Part (i). Suppose $q = 1$. The allocation to agent l_1 in Step 1 is either $\min\{\inf D_1, p_{l_1}\}$ or $\min\{\inf U_1, 1 - p_{h_1}\}$. Both these values are smaller or equal to 0.5.

Suppose $q \geq 1$. The triple (l_q, h_q, t_{l_q}) is formed in Step q. Next we show that $t_{l_q+1} \le t_{l_q}$. We have to consider four cases based on which substep occurs in Step q.

Case 1: Suppose the allocation is made in Substep q.1.

Here $t_{l_q} = \min\{\inf U_q, 1 - p_{h_q}\} \le 1 - p_{h_q}$. By Lemma 3, we know $U_{q+1} = U_q$ and $\inf U_{q+1} = \inf U_q$.

Lemma 1 and the hypothesis of Substep q.1 imply $\inf D_q < 1 - p_{h_q} < \sup U_q^1$. Since $D_q \cap U_q = \emptyset$, it must be the case that D_q^1 lies to the left of U_q^1, i.e. $\inf D_q < \inf U_q$. Thus $\inf D_q < \min\{\inf U_q, 1 - p_{h_q}\} = t_{l_q}$.

Now we consider the triple $(l_{q+1}, h_{q+1}, t_{l_{q+1}})$ formed in Step $q + 1$. There are four possibilities to consider.

1. The allocation is made in Substep $q+1.1$. By hypothesis of the substep, we have $\inf D_{q+1} < 1 - p_{h_{q+1}}$. If $1 - p_{h_{q+1}} \leq \inf D_q$, then $1 - p_{h_q} < \inf U_q = \inf U_{q+1}$ and $t_{l_{q+1}} = 1 - p_{h_{q+1}}$. Thus $t_{l_{q+1}} = 1 - p_{h_{q+1}} \leq \inf D_q < t_{l_q}$.

   If $1 - p_{h_{q+1}} > \inf D_q$, then $h_{q+1} \in \{h \in H_q : 1 - p_h > \inf D_q\}$. Since h_q is matched in Step q, it must be the case that $1 - p_{h_{q+1}} \leq 1 - p_{h_q}$. Recall $\inf U_{q+1} = \inf U_q$. Thus $t_{l_{q+1}} = \min\{\inf U_{q+1}, 1 - p_{h_{q+1}}\} \leq \min\{\inf U_q, 1 - p_{h_q}\} = t_{l_q}$.

2. The allocation is made in Substep $q+1.2$. Note than in Step q, D_q^1 lies to the left of U_q^1. Since $D_q \cap U_q = \emptyset$, we have $\sup D_q^1 \leq \inf U_q^1$. Thus $p_l < \sup D_q^1 \leq \inf U_q^1$ for all $l \in L_q$ (Lemma 1). Since $U_{q+1} = U_q$ and $L_{q+1} \subset L_q$, there does not exist $l \in L_{q+1}$ such that $p_l > \inf U_{q+1}$. So Substep $q+1.2$ is not possible.

3. The allocation is made in Substep $q+1.3.1$. By the hypothesis of the substep, we know $1 - p_{h_{q+1}} \leq \inf D_{q+1}$. Note that $\inf D_{q+1} \leq \inf D_q$.[25] Thus

$$t_{l_q+1} = \min\{\inf U_{q+1}, 1 - p_{h_{q+1}}\} \leq \inf D_{q+1} \leq \inf D_q < t_{l_q}.$$

4. The allocation is made in Substep $q+1.3.2$. Here $t_{l_{q+1}} = \min\{\inf D_{q+1}, p_{l_{q+1}}\}$. Also $\inf D_{q+1} \leq \inf D_q$. Thus

$$t_{l_{q+1}} \leq \inf D_{q+1} \leq \inf D_q < t_{l_q}.$$

Case 2: The allocation is made in Substep $q.3.1$.

Here $t_{l_q} = \min\{\inf U_q, 1 - p_{h_q}\} \leq 1 - p_{h_q}$. By Lemma 3, we have $U_{q+1} = U_q$ and $\inf U_{q+1} = \inf U_q$.

Consider the allocation is Step $q+1$. There are four possibilities.

1. The allocation is done in Substep $q+1.1$. Here $t_{l_{q+1}} = \min\{\inf U_{q+1}, 1 - p_{h_{q+1}}\} \leq 1 - p_{h_{q+1}}$. Since agent h_q is matched in Step q and $h_{q+1} \in H_q$, it must be the case that $1 - p_{h_{q+1}} \leq 1 - p_{h_q}$. Thus

$$t_{l_{q+1}} = \min\{\inf U_{q+1}, 1 - p_{h_{q+1}}\} \leq \{\inf U_q, 1 - p_{h_q}\} = t_{l_q}.$$

2. The allocation is made in Subtep $q+1.2$. We will show that this is not possible. Since the first allocation is done in Substep $q.3.1$, we know $p_l \leq \inf U_q$ for all $l \in L_q$. Note that $\inf U_{q+1} = \inf U_q$ and $L_{q+1} \subset L_q$. Thus there does not exist $l \in L_{q+1}$ such that $p_l > \inf U_{q+1}$.

[25] Recall $D_q \subseteq D_{q+1}$. Thus $\inf D_{q+1} \leq D_q$.

3. The allocation is made in Substep $q+1.3.1$. Here $t_{l_{q+1}} = \min\{\inf U_{q+1}, 1-p_{h_{q+1}}\}$. Since $h_{q+1} \in H_q$ and agent h_q is matched in Substep $q.3.1$, we have $1-p_{h_{q+1}} \leq 1-p_{h_q}$. Thus

$$t_{l_{q+1}} = \min\{\inf U_{q+1}, 1-p_{h_{q+1}}\} \leq \{\inf U_q, 1-p_{h_q}\} = t_{l_q}.$$

4. The allocation is made in Substep $q+1.3.2$. Here $t_{l_{q+1}} = \min\{\inf D_{q+1}, p_{l_{q+1}}\}$. Since the first allocation was done in Substep $q.3.1$ and $l_{q+1} \in L_q$, we know $p_{l_{q+1}} \leq p_{\tilde{l_q}} \leq 1-p_{h_q}$ and $p_{l_{q+1}} \leq \inf U_q$.[26] Thus

$$t_{l_{q+1}} = \min\{\inf D_{q+1}, p_{l_{q+1}}\} \leq p_{l_{q+1}} \leq \min\{\inf U_q, 1-p_{h_q}\} = t_{l_q}.$$

Case 3 occurs when the allocation is made in Substep $q.2$. Case 4 occurs when the allocation is made in Substep $q.3.2$. These cases can be argued similarly by making the appropriate changes. We omit the details. This completes the proof of Part (i) of the lemma.

Note that $t_{h_q} = 1 - t_{l_q}$ for any $q \in \{1, \ldots, K-1\}$. Thus Part (i) of the lemma implies Part (ii). ■

OBSERVATION **4** Consider an allocation σ and an $x \in [0,1]$. Suppose there exists $i \in N$ with $t_i^\sigma = x$. Then agent i's partner in σ, say agent j has contribution $t_j^\sigma = 1-x$. Thus $|\{i \in N : t_i^\sigma = x\}| = |\{j \in N : t_j^\sigma = 1-x\}|$.

LEMMA **5** *Consider any preference profile $\succsim$. Let σ and τ be stable allocations at the preference profile $\succsim$. For all $x \in [0, 0.5]$ such that σ and τ satisfy*

(1) $t_i^\sigma \succsim_i x$ and $t_i^\sigma \succsim_i 1-x$ for all $i \in N$ and

(2) $t_i^\tau \succsim_i x$ and $t_i^\tau \succsim_i 1-x$ for all $i \in N$

we have,

(a) $|\{i \in N : t_i^\sigma = x\}| = |\{i \in N : t_i^\tau = x\}|$.

(b) Consider agents $i, j \in N$ such that $p_i < 0.5$ and $p_j > 0.5$. If 0.5 satisfies (1) and (2), then $t_i^\sigma = t_j^\tau = 0.5$ is not possible.

Proof: Consider an arbitrary preference profile $\succsim$. Let σ and τ be stable allocations at $\succsim$. Also consider an $x \in [0, 0.5]$ such that σ satisfies Condition (1) and τ satisfies Condition (2) in the antecedent of the lemma.

We partition the agents in N into three groups depending on where their peaks lie with respect to the points x and $1-x$.[27] Define,

[26] If $p_{l_{q+1}} > \inf U_q$, then the allocation is Step q would be done in Substep $q.2$.

[27] Note that it is possible that $x = 1-x = 0.5$. Then S_x^2 is the set of agents whose peaks are exactly 0.5.

1. $S_x^1 = \{i \in N | p_i < x\}$.

2. $S_x^2 = \{i \in N | x \leq p_i \leq 1 - x\}$.

3. $S_x^3 = \{i \in N | p_i > 1 - x\}$.

We will prove several claims about the allocation σ.

Claim 1: For the allocation σ, we have

1. $t_i^\sigma \leq x$ for all $i \in S_x^1$.

2. $x \leq t_i^\sigma \leq 1 - x$ for all $i \in S_x^2$.

3. $1 - x \leq t_i^\sigma$ for all $i \in S_x^3$.

Proof: We will prove the claim by contradiction. We first prove Part 1. Consider an agent $i \in S_x^1$ such that $t_i^\sigma > x$. Here $p_i < x < t_i^\sigma$ and single-peakedness implies $x \succ_i t_i^\sigma$. This contradicts the fact that σ satisfies Condition (1) in the antecedent of the lemma.

For Part 2, consider $i \in S_x^2$ such that $t_i^\sigma < x$ or $t_i^\sigma > 1-x$. If $t_i^\sigma < x$, we have $t_i^\sigma < x \leq p_i$. Thus $x \succ_i t_i^\sigma$ by single-peakedness and we have a contradiction. If $t_i^\sigma > 1 - x$, we have $p_i \leq 1 - x < t_i^\sigma$ and $1 - x \succ_i t_i^\sigma$. Once again we have a contradiction.

Part 3 can be proved using similar arguments. ■

Claim 2: For the allocation σ, we have

(a) Consider $i \in S_x^1$. Let (i, j, t_i^σ) be the triple that agent i belongs to in σ. If $t_i^\sigma \neq x$, then $j \in S_x^3$.

(b) Consider agents $i \in S_x^1$, $j \in S_x^3$ such that $(i, j, t_i^\sigma) \in \sigma$. Then $t_i^\sigma < x$.

Proof: (a) Consider $i \in S_x^1$. By assumption, $t_i^\sigma \neq x$. By Part 1 of Claim 1, we have $t_i^\sigma < x$. Thus $t_j^\sigma = 1 - t_i^\sigma > 1 - x$. From Claim 1, we know that all agents in S_x^1 receive a contribution of contribution of at most x in σ (Part 1). Similarly, all agents in S_x^2 have a contribution of at most $1 - x$ (Part 2). Finally all agents in S_x^3 have a contribution of at least $1 - x$ in σ (Part 3). Thus $j \in S_x^3$.

(b) Consider a triple $(i, j, t_i^\sigma) \in \sigma$ where $i \in S_x^1$ and $j \in S_x^3$. By internal stability for the pair (i, j), we have

$$\min\{p_i, 1 - p_j\} \leq t_i^\sigma \leq \max\{p_i, 1 - p_j\}.$$

Since $\max\{p_i, 1 - p_j\} < x$ (follows from the definition of S_x^1 and S_x^3), we have $t_i^\sigma < x$. ■

Claim 3: For the allocation σ, we have

(a) Consider $i \in S_x^3$. Let (i, j, t_i^σ) be the triple that agent i belongs to in σ. If $t_i^\sigma \neq 1 - x$, then $j \in S_x^1$.

(b) Consider agents $i \in S_x^3$, $j \in S_x^1$ such that $(i, j, t_i^\sigma) \in \sigma$. Then $t_i^\sigma > 1 - x$.

Proof: (a) Consider $i \in S_x^3$. By assumption, $t_i^\sigma \neq 1 - x$. By Part 3 of Claim 1, we have $t_i^\sigma > 1 - x$. Thus $t_j^\sigma = 1 - t_i^\sigma < x$. From Claim 1, all agents in S_x^2 must have a contribution of at least x in σ (Part 2). Also all agents in S_x^3 have a contribution of at least $1 - x$ in σ (Part3). Finally all agents in S_x^1 have a contribution of at most x in σ (Part 1). Thus $j \in S_x^1$.

(b) This follows from Part (b) of Claim 2. ■

Claim 4: For the allocation σ, there does not exist a pair of agents $i \in S_x^1$, $j \in S_x^3$ such that $t_i^\sigma = x$ and $t_j^\sigma = 1 - x$.

Proof: Suppose not. Then there exist $i \in S_x^1$, $j \in S_x^3$ such that $t_i^\sigma = x$ and $t_j^\sigma = 1 - x$. By the definition of S_x^1 and S_x^3, we know $\max\{p_i, 1 - p_j\} < x$. Thus there exists $\epsilon > 0$ such that $x > x - \epsilon \geq \max\{p_i, 1 - p_j\}$. The pair (i, j) blocks σ with the contribution vector $(x - \epsilon, 1 - x + \epsilon)$. This results in a contradiction as σ is stable by assumption. ■

Claim 5: For the allocation σ,

1. For any $i \in S_x^1$ and $j \in N \setminus S_x^3$ such that $(i, j, t_i^\sigma) \in \sigma$, we have $t_i^\sigma = x$.
2. For any $i \in N \setminus S_x^1$ and $j \in S_x^3$ such that $(i, j, t_i^\sigma) \in \sigma$, we have $t_i^\sigma = x$ and $t_j^\sigma = 1 - x$.

Proof: (1) Consider agents $i \in S_x^1$ and $j \in N \setminus S_x^3$ such that $(i, j, t_i^\sigma) \in \sigma$. By Claim 1 and $i \in S_x^1$, we know $t_i^\sigma \leq x$. Agent j belongs to S_x^1 or S_x^2. If $j \in S_x^2$, Claim 1 implies $x \leq t_j^\sigma \leq 1 - x$. These together imply $t_i^\sigma = x$. If $j \in S_x^1$, we have $t_j^\sigma \leq x$. By feasibility, it must be the case that $x = 0.5$ and $t_i^\sigma = x$.

(2) Consider agents $i \in N \setminus S_x^1$ and $j \in S_x^3$ such that $(i, j, t_i^\sigma) \in \sigma$. By Claim 1 and $j \in S_x^3$, we know $t_j^\sigma \geq 1 - x$. Agent i either belongs to S_x^3 or S_x^2. If $i \in S_x^3$, then $t_i^\sigma \geq 1 - x$ (by Claim 1). Also since $j \in S_x^3$, $t_j^\sigma \geq 1 - x$. Feasibility implies $1 - x = 0.5$. Thus $t_i^\sigma = 0.5 = x$ and $t_j^\sigma = 1 - x$. If $i \in S_x^2$, Claim 1 implies $x \leq t_i^\sigma \leq 1 - x$. We know $t_j^\sigma \geq 1 - x$. Feasibility implies $t_i^\sigma = x$ and $t_j^\sigma = 1 - x$. ■

Claim 6: Consider the allocation σ. If $x \neq 0.5$, we have

1. There does not exist a triple $(i, j, t_i^\sigma) \in \sigma$ such that $i, j \in S_x^1$.
2. There does not exist a triple $(i, j, t_i^\sigma) \in \sigma$ such that $i, j \in S_x^3$.

Proof: Part 1 follows immediately from Claim 1 and the definition of S_x^1. Similarly Part 2 follows from Claim 1 and the definition of S_x^3. ■

Claim 7: Consider agents $i, j \in S_x^2$ such that $p_i, p_j \in (x, 1-x)$. Then in the allocation σ, we have $\neg[t_i^\sigma = x \text{ and } t_j^\sigma = 1-x]$.

Proof: We assume for contradiction that there exist $i, j \in S_x^2$ such that $p_i, p_j \in (x, 1-x)$, $t_i^\sigma = x$ and $t_j^\sigma = 1-x$. Here $x < \min\{p_i, 1-p_j\}$. Thus there exists $\epsilon > 0$ such that $x + \epsilon \succ_i t_i^\sigma$ and $1 - x - \epsilon \succ_j t_j^\sigma$. The pair (i, j) blocks σ. This is a contradiction as σ is stable by assumption. ■

We will first prove Part (a) of the lemma. Our aim is to calculate the cardinality of the set $\{i \in N : t_i^\sigma = x\}$. In order to do this, we will first deduce how agents are matched across the three groups. There are two cases to consider depending on the cardinalities of the sets S_x^1 and S_x^3: (I) $|S_x^1| \geq |S_x^3|$ and (II) $|S_x^1| < |S_x^3|$.

(I) Consider the first case where $|S_x^1| \geq |S_x^3|$. We claim that in this case, all agents in S_x^3 are matched to agents in S_x^1 in σ. We assume for contradiction that there exists $i \in S_x^3$ who is not matched to an agent in S_x^1 in σ. By Claim 5 (Part 2), we know $t_i^\sigma = 1 - x$. Also there exists $j \in S_x^1$ such that j is not matched to an agent in S_x^3. This is because $|S_x^1| \geq |S_x^3|$ and the assumption that an agent in S_x^3 is not matched to an agent in S_x^1. By Claim 5 (Part 1), we know $t_j^\sigma = x$. We have shown that there exists $i \in S_x^3$ with $t_i^\sigma = 1 - x$ and $j \in S_x^1$ with $t_j^\sigma = x$. This is not possible by Claim 4.

Also for any $i \in S_x^3$, we have $t_i^\sigma \neq 1 - x$. Suppose not, i.e. $t_i^\sigma = 1 - x$. We have shown above that agent i is matched with some agent $j \in S_x^1$. Thus $t_j^\sigma = x$. This is not possible by Claim 4.

Now we will calculate the cardinality of the set $\{i \in N : t_i^\sigma = x\}$. There are two possibilities to consider.

1. If $x = 0.5$, then
$$|\{i \in N : t_i^\sigma = x\}| = |S_x^1| + |S_x^2| - |S_x^3|.$$
We have shown that all agents in S_x^3 are matched to agents in S_x^1. Thus $|S_x^3|$ agents in S_x^1 are matched to agents in S_x^3. Also for all $i \in S_x^3$, we have $t_i^\sigma \neq 1 - x$. Thus the agents in S_x^1 matched to agents in S_x^3 do not get a contribution of x. The remaining agents in S_x^1 (the cardinality of this set is $|S_x^1| - |S_x^3|$) are matched to agents in $N \setminus S_x^3$. By Claim 5 (Part 1), we know all such agents get a contribution of x. Thus $|S_x^1| - |S_x^3|$ agents in S_x^1 have a contribution of x in σ. Since $x = 0.5$, we know $p_i = 0.5$ for all $i \in S_x^2$. By Claim 1, we have $t_i^\sigma = 0.5$ for all $i \in S_x^2$. So all agents in S_x^2 get a contribution of x in σ. Thus $|S_x^1| - |S_x^3|$ agents in S_x^1, all agents in S_x^2 and none of the agents in S_x^3 receive x in σ.

2. If $x \neq 0.5$, then
$$|\{i \in N | t_i^\sigma = x\}| = \max\{|S_x^1| - |S_x^3| + |\{i \in S_x^2 : p_i = x\}|, |\{j \in S_x^2 : p_j = 1 - x\}|\}.$$

Note that $|S_x^1| - |S_x^3|$ agents in S_x^1 receive x in σ. Consider the set S_x^2. For all $i \in S_x^2$ with $p_i = x$, we have $t_i^\sigma = x$. This is because $t_i^\sigma \succsim_i x$ (recall σ satisfies Condition (1) in the antecedent of the lemma). Similarly for all $i \in S_x^2$ with $p_i = 1 - x$, we have $t_i^\sigma = 1 - x$. This is because $t_i^\sigma \succsim_i 1 - x$ by Condition (1) of the lemma. Note that their partners in σ get x (recall Observation 4). Also none of the agents in S_x^3 receive x in σ. This follows from Claim 1 (all agents in S_x^3 get at least $1 - x$) and the fact that $x < 0.5$. Thus $\max\{|S_x^1| - |S_x^3| + |\{i \in S_x^2 : p_i = x\}|, |\{j \in S_x^2 : p_j = 1 - x\}|\}$ is the minimum number of agents in σ who receive x. We have,

$$|\{i \in N : t_i^\sigma = x\}| \geq \max\{|S_x^1| - |S_x^3| + |\{i \in S_x^2 : p_i = x\}|, |\{j \in S_x^2 : p_j = 1 - x\}|\}.$$

Suppose the above condition holds with strict inequality. We know that none of the agents in S_x^3 get x. Also exactly $|S_x^1| - |S_x^3|$ in S_x^1 receive x. Thus in the case of strict inequality, there exists an agent $i \in S_x^2$ with $p_i \in (x, 1 - x)$ and $t_i^\sigma = x$.

Note that $|\{i \in N : t_i^\sigma = x\}| = \{j \in N : t_j^\sigma = 1 - x\}$. Since the condition holds with strict inequality, we have

$$|\{j \in N : t_i^\sigma = 1 - x\}| > |\{j \in S_x^2 : p_j = 1 - x\}|.$$

Thus there exists $j \in N$ such that $p_j \neq 1 - x$ and $t_j^\sigma = 1 - x$. Note that j cannot belong to S_x^3 (recall we have shown that all agents in S_x^3 do not receive $1 - x$). Also all agents in S_x^1 get at most x. So $j \in S_x^2$.[28]

So there exists $i, j \in S_x^2$ such that $p_i, p_j \in (x, 1 - x)$, $t_i^\sigma = x$ and $t_j^\sigma = 1 - x$. This is not possible by Claim 7.

Thus,

$$|\{i \in N : t_i^\sigma = x\}| = \max\{|S_x^1| - |S_x^3| + |\{i \in S_x^2 : p_i = x\}|, |\{j \in S_x^2 : p_j = 1 - x\}|\}.$$

(II) The second case is $|S_x^1| < |S_x^3|$. We claim that all agents in S_x^1 are matched to agents in S_x^3 in σ. Also for all $i \in S_x^1$, we have $t_i^\sigma \neq x$.

In order to compute the cardinality of the set $\{i \in N | t_i^\sigma = x\}$, we compute the minimum number of agents that must receive $1 - x$ in σ.[29] There are two possibilities.

1. If $x = 0.5$, then
$$\{i \in N : t_i^\sigma = x\} = |S_x^3| - |S_x^1| + |S_x^2|.$$

[28] Note that $p_j \neq x$. We have shown above that if $j \in S_x^2$ and $p_j = x$, then $t_j^\sigma = x$.

[29] This number is useful as for each agent who gets $1 - x$ in σ, her partner gets x in σ (Observation 4).

All agents in S_x^1 are matched to agents in S_x^3. Also none of the agents in S_x^1 get $1-x$ in σ.[30] So $|S_x^1|$ agents in S_x^3 do not receive x in σ. By Claim 5 (Part 2), the remaining agents in S_x^3 (who are matched to agents in $N \setminus S_x^1$) must get $1-x$ in σ. Thus $|S_x^3|-|S_x^1|$ agents in S_x^3 have a contribution of $1-x$ in σ.

Since $x=0.5$, we have $p_i = 0.5$ for all $i \in S_x^2$. By Claim 2, we know that all agents in S_x^2 have a contribution of $x = 1-x = 0.5$ in σ.

Thus $|S_x^3| - |S_x^1|$ agents in S_x^1, all agents in S_x^2 and none of the agents in S_x^1 receive $1-x$ in σ.

2. If $x \neq 0.5$, then

$$|\{i \in N | t_i^\sigma = x\}| = \max\{|S_x^3| - |S_x^1| + |\{i \in S_x^2 : p_i = 1-x\}|, |\{i \in S_x^2 : p_i = x\}|\}.$$

We can prove this case using similar arguments to Part 2 in Case I.

We have shown that $|\{i \in N | t_i^\sigma = x\}|$ only depends on the preference profile $\succ$ and x. In particular, it does not depend on the allocation σ.

Note that the cardinality of the set $\{i \in N : t_i^\tau = x\}$ can be computed in exactly the same manner as we did for the allocation σ.[31] Thus we conclude $|\{i \in N : t_i^\sigma = x\}| = |\{i \in N : t_i^\tau = x\}|$.

We now prove Part (b) of the lemma. Consider agents $i, j \in N$ such that $p_i < 0.5$ and $p_j > 0.5$. There are two possibilities based on the cardinalities of $S_{0.5}^1|$ and $S_{0.5}^3$.

In Case I where $|S_{0.5}^1| \geq |S_{0.5}^3|$, we have shown above that all agents in $S_{0.5}^3$ are matched to agents in $S_{0.5}^1$. Also each agent in $S_{0.5}^3$ has a contribution strictly greater than 0.5 (by Claim 2 (b)) in both σ and τ. Since $p_j > 0.5$, we know $j \in S_{0.5}^3$. Thus $t_j^\tau = 0.5$ is not possible.

In Case II where $|S_{0.5}^1| < |S_{0.5}^3|$, all agents in $S_{0.5}^1$ are matched to agents in $S_{0.5}^3$. Each agent in $S_{0.5}^1$ has a contribution strictly smaller than 0.5 (by Claim 2 (b)) in both σ and τ. Since $p_i < 0.5$, we know $i \in S_{0.5}^1$. Thus it is not possible that $t_i^\sigma = 0.5$. ■

OBSERVATION **5** Consider a preference profile $\succsim$ and allocations σ and τ. If τ Pareto dominates σ at $\succsim$, then $I_{i,t_i^\tau} \subseteq I_{i,t_i^\sigma}$ for all $i \in N$. This implies $D^\tau \subseteq D^\sigma$ and $U^\tau \subseteq D^\sigma$.

LEMMA **6** *Consider a preference profile $\succsim$ and allocations σ and τ. If σ satisfies Condition S and τ Pareto dominates σ, then τ satisfies Condition S.*

[30] This is because none of the agents in S_x^1 get $x = 0.5$.

[31] All the claims proved are true for the allocation τ as well. The only change required in the arguments is to replace Condition (1) of the lemma by Condition (2).

Proof: We assume for contradiction that τ does not satisfy Condition S. There are two cases. The first case is when τ violates Part 1 of Condition S, i.e. there exists $l \in L, h \in H$ such that $I_{h,t_h^\tau} \cap I_{l,t_l^\tau} \neq \emptyset$. Since $I_{l,t_l^\tau} \subseteq I_{l,t_l^\sigma}$ and $I_{h,t_h^\tau} \subseteq I_{h,t_h^\sigma}$ (by Observation 5), we have $I_{l,t_l^\sigma} \cap I_{h,t_h^\sigma} \neq \emptyset$. Thus σ violates Condition S. The second case is when τ violates Part 2 of Condition S, i.e. there exists $i \in N$ such that $0.5 \in I_{i,t_i^\tau}$. By Observation 5, $0.5 \in I_{i,t_i^\sigma}$. Thus σ violates Condition S. ■

Observation **6** Consider an allocation σ which satisfies Condition S. For any $x \in [0, 0.5]$, if $x \notin D^\sigma \cup U^\sigma$, then $[t_i^\sigma \succsim_i x$ and $t_i^\sigma \succsim_i 1 - x]$ for all $i \in N$.

Lemma **7** *Consider a preference profile $\succsim$. If σ satisfies Condition S and τ Pareto dominates σ at $\succsim$, then for all $x \in [0, 1]$,*

$$|\{i \in N | t_i^\tau = x\}| = |\{i \in N | t_i^\sigma = x\}|.$$

Proof: Consider a preference profile $\succsim$ and allocations σ and τ. Assume σ satisfies Condition S and τ Pareto dominates σ at $\succsim$. Consider an $x \in [0, 1]$. There are two possibilities.

(A) Let $x \in [0, 0.5]$. We consider three subcases.

1. $x \in [0, 0.5] \setminus [U^\sigma \cup D^\sigma]$.

   Since τ Pareto dominates σ, we have $U^\tau \subseteq U^\sigma$ and $D^\tau \subseteq D^\sigma$ (Observation 5). Thus $x \in [0, 0.5] \setminus [U^\tau \cup D^\tau]$. We know $x \notin D^\sigma \cup U^\sigma$. Observation 6 implies $t_i^\sigma \succsim_i x$ and $t_i^\sigma \succsim_i 1 - x$ for all $i \in N$. Similarly for the allocation τ, we have $t_i^\tau \succsim_i x$ and $t_i^\tau \succsim_i 1 - x$ for all $i \in N$.

   Applying Lemma 5, we have

   $$|\{i \in N : t_i^\sigma = x\}| = |\{i \in N : t_i^\tau = x\}|.$$

2. $x \in U^\sigma$. Then there exists $h \in H$ such that $x \in I_{h,t_h^\sigma}$ and $1 - x \succ_h t_h^\sigma$. Since σ satisfies Condition S, we have $0.5 \notin U^\sigma$. Thus $x < 0.5$.

   For any $i \in N$ with $p_i = x$, it must be the case that $t_i^\sigma = x$. If not, then (i, h) will block σ with $(x, 1 - x)$.

   Also for any $i \in N$ with $t_i^\sigma = x$, it must be the case that $p_i = x$. If not, the pair (i, h) will block σ.[32] This implies

   $$|\{i \in N : t_i^\sigma = x\}| = |\{i \in N : p_i = x\}|.$$

[32]Suppose $p_i < x$. Then there exists $\epsilon > 0$ such that $p_i < x - \epsilon < x$ and $x - \epsilon \in I_{h,t_h^\sigma}$. The pair (i, h) blocks σ with $(x - \epsilon, 1 - x + \epsilon)$. We can give a similar argument for the case $p_i > x$.

Note that all these agents must receive their peaks in τ as τ Pareto dominates σ. Thus,

$$|\{i \in N : t_i^\tau = x\}| \geq |\{i \in N : p_i = x\}| = |\{i \in N : t_i^\sigma = x\}|.$$

3. $x \in D^\sigma$. Then there exists $l \in L$ such that $x \in I_{l,t_l^\sigma}$.

   We will look at the agents who receive $1-x$ in σ. For all $i \in N$ such that $p_i^\sigma = 1-x$, we have $t_i^\sigma = 1-x$. Also for all $i \in N$ with $t_i^\sigma = 1-x$, it must be the case that $p_i = 1-x$. Thus,

$$|\{i \in N : t_i^\sigma = 1-x\}| = |\{i \in N : p_i = 1-x\}|.$$

   Since τ Pareto dominates σ, we have

$$|\{i \in N : t_i^\tau = 1-x\}| \geq |\{i \in N : p_i = 1-x\}| = |\{i \in N : t_i^\sigma = 1-x\}|.$$

   Note that $|\{i \in N : t_i^\tau = 1-x\}| = |\{i \in N : t_i^\tau = x\}|$ and $|\{i \in N : t_i^\sigma = 1-x\}| = |\{i \in N : t_i^\sigma = x\}|$ (Observation 4).

   Therefore,

$$|\{i \in N : t_i^\tau = x\}| \geq |\{i \in N | p_i = 1-x\}| = |\{i \in N | t_i^\sigma = 1-x\}| = |\{i \in N | t_i^\sigma = x\}|.$$

We have shown that for every $x \in [0, 0.5]$,

$$|\{i \in N : t_i^\tau = x\}| \geq |\{i \in N : t_i^\sigma = x\}|.$$

(B) $x \in (0.5, 1]$.

Note that $|\{i \in N : t_i^\tau = x\}| = |\{i \in N : t_i^\tau = 1-x\}|$ and $|\{i \in N : t_i^\sigma = x\}| = |\{i \in N : t_i^\sigma = x\}|$ (by Observation 4). Since $1-x \in [0, 0.5]$, Case (A) is applicable. So $|\{i \in N : t_i^\tau = 1-x\}| \geq |\{i \in N : t_i^\sigma = 1-x\}|$. Combining the facts above, we have $|\{i \in N : t_i^\tau = x\}| \geq |\{i \in N | t_i^\sigma = x\}|$.

From (A) and (B), we know

$$|\{i \in N : t_i^\tau = x\}| \geq |\{i \in N | t_i^\sigma = x\}| \text{ for all } x \in [0,1] \tag{2}$$

The sum of both the LHS and RHS of Equation 2 over all $x \in [0,1]$ is $|N|$. Suppose there exists some x for which Equation 2 holds with strict inequality. Then there must exist another x for which the LHS of Equation 2 will be strictly less than the RHS. This contradicts 2. Thus Equation 2 holds with equality.

This completes the proof of the lemma. ∎

Observation 7 Consider an allocation σ that satisfies Condition S. For any agent $i \in N$, we have (a) if $t_i^\sigma < 0.5$ then $p_i < 0.5$ and (b) if $t_i^\sigma > 0.5$ then $p_i > 0.5$. To prove (a), assume $t_i^\sigma < 0.5$ and $p_i \geq 0.5$ for some agent i. Then 0.5 belongs to I_{i,t_i^σ} and σ violates Condition S. Similarly we can prove (b). Also if $p_i = 0.5$ then $t_i^\sigma = 0.5$.

Definition 5 *Consider a preference profile $\succsim$ and two allocations $\sigma, \tau \in \Sigma$. Assume τ Pareto dominates σ at $\succsim$. We say τ is minimal if for all $\gamma \in \Sigma$ such that γ Pareto dominates σ at $\succsim$, we have*

$$|\{i \in N : t_i^\tau \neq t_i^\sigma\}| \leq |\{i \in N : t_i^\gamma \neq t_i^\sigma\}|.$$

Observation 8 Consider a preference profile $\succsim$ and allocations $\sigma, \gamma \in \Sigma$ such that γ Pareto dominates σ at $\succsim$. Since $|N|$ is finite, there exists an allocation $\tau \in \Sigma$ such that τ Pareto dominates σ at $\succsim$ and is minimal. This allocation may not be unique.

We now complete the proof of Pareto efficiency.

Proof: Let σ be the allocation generated by the algorithm. We will prove the theorem by contradiction. Suppose there exists an allocation $\gamma \in \Sigma$ such that γ Pareto dominates σ. By Observation 8, we know there exists $\tau \in \Sigma$ such that τ Pareto dominates σ and is minimal. We will now work with the allocation τ and use it to show a contradiction.

Note that σ satisfies Condition S as it is generated by the algorithm. Since τ Pareto dominates σ, we know τ also satisfies Condition S (Lemma 6).

There are two cases to consider: (I) there exists an agent $l \in L$ such that $t_l^\tau \neq t_l^\sigma$ and (II) $t_l^\tau = t_l^\sigma$ for all $l \in L$.

Case I: There exists an agent $l \in L$ such that $t_l^\tau \neq t_l^\sigma$.

The SAM algorithm generates σ in K steps.[33] There are two possibilities. The first possibility is that there exists a low type agent l who is matched in Step 0 and $t_l^\sigma \neq t_l^\tau$.[34] We denote agent l as $l_{\bar{i}}$ where $\bar{i} = 0$. The second possibility is that all low type agents matched in Step 0 (if any) have the same contribution values in both σ and τ. Then there exists a low type agent who is matched in some Step $q \geq 1$ and has different contribution values in σ and τ. Let $\bar{i} \in \{1, \ldots, K\}$ be the smallest integer such that (i) $t_{l_{\bar{i}}}^\sigma \neq t_{l_{\bar{i}}}^\tau$ and (ii) $t_{l_i}^\sigma = t_{l_i}^\tau$ for all $i < \bar{i}$. Note that $l_{\bar{i}}$ is the agent matched in Step $\bar{i}$ of the algorithm.

There are three cases based on the contribution value of agent $l_{\bar{i}}$ in σ.

[33] Recall in each step q (where $1 \leq q \leq K$), a low type agent l_q is matched. In Step 0, low type agents are matched if $|L| > |H|$.

[34] If there are several such agents, we choose an agent arbitrarily from this set.

1. $t^{\sigma}_{l_{\bar{i}}} = p_{l_{\bar{i}}}$.

   Since τ Pareto dominates σ, we have $t^{\tau}_{l_{\bar{i}}} = p_{l_{\bar{i}}}$. This is a contradiction as by assumption the contribution values of agent $l_{\bar{i}}$ are different in σ and τ.

2. $t^{\sigma}_{l_{\bar{i}}} < p_{l_{\bar{i}}}$.

   Here $t^{\sigma}_{l_{\bar{i}}}$ is the infimum of the improvement set of $l_{\bar{i}}$ in σ. Since τ Pareto dominates σ and $t^{\sigma}_{l_{\bar{i}}} \neq t^{\tau}_{l_{\bar{i}}}$, we have $t^{\tau}_{l_{\bar{i}}} \in \overline{I_{l_{\bar{i}},t^{\sigma}_{l_{\bar{i}}}}} \setminus \{t^{\sigma}_{l_{\bar{i}}}\}$. Note that $t^{\sigma}_{l_{\bar{i}}} < t^{\tau}_{l_{\bar{i}}} \leq 0.5$.[35]

   Let $t^{\tau}_{l_{\bar{i}}} = m^*$. Since σ satisfies Condition S and τ Pareto dominates it, applying Lemma 7 at $x = m^*$, we have

   $$|\{i \in N : t^{\sigma}_i = m^*\}| = |\{i \in N | t^{\tau}_i = m^*\}|.$$

   By assumption, $l_{\bar{i}}$ does not belong to the former set but belongs to the latter. Thus there exists an agent j who belongs to the former set and not to the latter. Note that $t^{\sigma}_j = m^* \neq t^{\tau}_j$.

   We claim that $j \in L$. To show this, we consider two cases. The first case is when $m^* < 0.5$. Suppose $j \notin L$. Then $p_j \geq 0.5$ and $0.5 \in I_{j,t^{\sigma}_j}$. This results in a contradiction as σ satisfies Condition S. Thus when $m^* < 0.5$, it must be the case that $p_j < 0.5$ and $j \in L$. The remaining case is when $m^* = 0.5$. We know $t^{\tau}_{l_{\bar{i}}} = m^* = 0.5$, $p_{l_{\bar{i}}} < 0.5$ and $t^{\sigma}_j = m^* = 0.5$. Applying Lemma 5 (Part (b)), we have $p_j \leq 0.5$. We will show that $p_j \neq 0.5$ if τ satisfies Condition S. Suppose $p_j = 0.5$. We know $t^{\tau}_j \neq t^{\sigma}_j = m^* = 0.5$. Then $p_j = 0.5 \in I_{j,t^{\tau}_j}$ and τ violates Condition S. Thus $p_j < 0.5$ and $j \in L$.

   We know $l_{\bar{i}}, j \in L$. Agent j has different contribution values in σ and τ. By assumption, $l_{\bar{i}}$ is the first low type agent who has different contribution values in σ and τ. We first argue that $\bar{i} \neq 0$. If $\bar{i} = 0$, we have $t^{\sigma}_{l_{\bar{i}}} = 0.5$. Since $p_{l_{\bar{i}}} < 0.5$, we have a contradiction to the assumption $t^{\sigma}_{l_{\bar{i}}} < p_{l_{\bar{i}}}$. Thus $\bar{i} \geq 1$ and agent j is matched in a step strictly greater than Step $\bar{i}$ in the algorithm. By Lemma 4, we have $t^{\sigma}_j = m^* \leq t^{\sigma}_{l_{\bar{i}}}$. We have a contradiction as $t^{\sigma}_{l_{\bar{i}}} < t^{\tau}_j = m^* = t^{\sigma}_j$.

3. $t^{\sigma}_{l_{\bar{i}}} > p_{l_{\bar{i}}}$.

   Let $t^{\sigma}_{l_{\bar{i}}} = s^*$. So $p_{l_{\bar{i}}} < t^{\sigma}_{l_{\bar{i}}} = s^* \leq 0.5$ (by Observation 7 and the fact that σ satisfies Condition S).

   Since σ satisfies Condition S and τ Pareto dominates it, applying Lemma 7 at $x = s^*$, we have

   $$|\{i \in N : t^{\sigma}_i = s^*\}| = |\{i \in N : t^{\tau}_i = s^*\}|.$$

[35] We know τ satisfies Condition S. By Observation 7 and $p_{l_{\bar{i}}} < 0.5$, we have $t^{\tau}_{l_{\bar{i}}} \leq 0.5$.

Agent $l_{\bar{i}}$ belongs to the first set and not to the latter. Thus there exists an agent j who belongs to the latter set and not to the first. Note that $t^\tau_j = s^* = t^\sigma_{l_{\bar{i}}} \neq t^\sigma_j$.

We claim that $j \in L$. To show this, we consider two cases. The first case is when $s^* < 0.5$. Suppose $j \notin L$ and $p_j \geq 0.5$. Then $0.5 \in I_{j,t^\tau_j}$ and τ violates Condition S (Observation 7). Hence $p_j < 0.5$ and $j \in L$. The second case is when $s^* = 0.5$. We know $t^\sigma_{l_{\bar{i}}} = s^* = 0.5$, $p_{l_{\bar{i}}} < 0.5$ and $t^\tau_j = s^* = 0.5$. Applying Lemma 5 (Part (b)), we have $p_j \leq 0.5$. We will show that $p_j \neq 0.5$ if σ satisfies Condition S. Suppose $p_j = 0.5$. We know $t^\sigma_j \neq s^* = 0.5$. Then $p_j = 0.5 \in I_{j,t^\sigma_j}$ and σ violates Condition S. Thus $p_j < 0.5$ and $j \in L$. This establishes the claim that $j \in L$.

We now show that $e_j(s^*) \leq e_{l_{\bar{i}}}(s^*)$. To prove this claim, we consider two cases.

The first case is when $\bar{i} \geq 1$. Recall that both agents $l_{\bar{i}}$ and j have different contribution values in σ and τ. Since $\bar{i} \geq 1$ and agent $l_{\bar{i}}$ is the first low type agent to have different contribution values in σ and τ, agent j is matched in a step strictly greater than Step $\bar{i}$. Thus $j \in L_{\bar{i}}$.

Since $p_{l_{\bar{i}}} < t^\sigma_{l_{\bar{i}}}$, we know this is only possible in Substep $\bar{i}.1$ or Substep $\bar{i}.3.1$ of the algorithm (by Lemma 2). In both cases, $l_{\bar{i}}$ is the secondary agent. Thus $e_l(s^*) \leq e_{l_{\bar{i}}}(s^*)$ for all $l \in L_{\bar{i}} \setminus \{l_{\bar{i}}\}$. Since $j \in L_{\bar{i}}$, we have $e_j(s^*) \leq e_{l_{\bar{i}}}(s^*)$.

The second case is when $\bar{i} = 0$. Note that $t^\sigma_{l_{\bar{i}}} = s^* = 0.5 \neq t^\sigma_j$. This means that j is not matched in Step 0, when $l_{\bar{i}}$ is matched. Thus $e_j(0.5) \leq e_{l_{\bar{i}}}(0.5)$. This completes the proof of the claim.

Thus,

$$\overline{I_{l_{\bar{i}},t^\sigma_{l_{\bar{i}}}}} = [e_{l_{\bar{i}}}(s^*), s^*] \cap [0, 0.5] \subseteq [e_j(s^*), s^*] \cap [0, 0.5] = \overline{I_{j,t^\tau_j}} \tag{3}$$

$$I_{l_{\bar{i}},t^\sigma_{l_{\bar{i}}}} = (e_{l_{\bar{i}}}(s^*), s^*) \cap [0, 0.5] \subseteq (e_j(s^*), s^*) \cap [0, 0.5] = I_{j,t^\tau_j} \tag{4}$$

Since τ Pareto dominates σ, we have $t^\tau_{l_{\bar{i}}} \in \overline{I_{l_{\bar{i}},t^\sigma_{l_{\bar{i}}}}}$. This together with Fact 3 implies $t^\tau_{l_{\bar{i}}} \in \overline{I_{j,t^\tau_j}}$. So $t^\tau_{l_{\bar{i}}} \succsim_j t^\tau_j$. As τ Pareto dominates σ, we know $t^\tau_j \succsim_j t^\sigma_j$. Hence for agent j,

$$t^\tau_{l_{\bar{i}}} \succsim_j t^\tau_j \succsim_j t^\sigma_j \tag{5}$$

Note that if $t^\tau_{l_{\bar{i}}} \in I_{l_{\bar{i}},t^\sigma_{l_{\bar{i}}}}$, then $t^\tau_{l_{\bar{i}}} \in I_{j,t^\tau_j}$ (by Fact 4). Thus $t^\tau_{l_{\bar{i}}} \succ_j t^\tau_j \succsim_j t^\sigma_j$ (as τ Pareto dominates σ). This implies $t^\tau_{l_{\bar{i}}} \succ_j t^\sigma_j$. Thus

$$\text{If } t^\tau_{l_{\bar{i}}} \succ_{l_{\bar{i}}} t^\sigma_{l_{\bar{i}}} \text{ then } t^\tau_{l_{\bar{i}}} \succ_j t^\sigma_j \tag{6}$$

We will now construct an allocation $\delta \in \Sigma$ such that

(a) δ Pareto dominates σ and

(b) $|\{i \in N : t_i^\delta \neq t_i^\sigma\}| < |\{i \in N : t_i^\tau \neq t_i^\sigma\}|$.

This will contradict the assumption that τ Pareto dominates σ and is minimal.

We construct δ as follows. The pairs in δ are defined as,

- The partner of $l_{\bar{i}}$ in δ is the partner of j in τ.
- The partner of j in δ is the partner of $l_{\bar{i}}$ in τ.
- For any agent s (different from $l_{\bar{i}}, j$ and their partners' in τ), the partner of s in δ is the same as her partner in τ.

To obtain the pairs in δ, we interchange the partners of agents $l_{\bar{i}}$ and j in τ and the partners of all other agents remain unchanged.

The contributions of the agents in δ are defined as,

- $t_s^\delta = t_s^\tau$ for all $s \in N \setminus \{l_{\bar{i}}, j\}$. The contribution of all such agents in δ is equal to their contribution in τ.
- $t_{l_{\bar{i}}}^\delta = t_{l_{\bar{i}}}^\sigma = s^*$. The contribution of $l_{\bar{i}}$ in δ is equal to her contribution in σ.
- $t_j^\delta = t_{l_{\bar{i}}}^\tau$. The contribution of agent j in δ is equal to the contribution of agent $l_{\bar{i}}$ in τ.

Note that for all $s \in N \setminus \{l_{\bar{i}}, j\}$, we have $t_s^\delta = t_s^\tau \succsim_s t_s^\sigma$ as τ Pareto dominates σ. The contribution of agent $l_{\bar{i}}$ is the same in δ and σ. For agent j, we have $t_j^\delta = t_{l_{\bar{i}}}^\tau \succsim_j t_j^\sigma$ (Fact 5). Thus all agents in N weakly prefer their contributions in δ to their contributions in σ.

In order to show that δ Pareto dominates σ, we need to show that there exists an agent who strictly improves in δ with respect to σ.[36] We consider three cases based on the agent who strictly prefers her contribution in τ to that in σ. Let $k \in N$ be the agent who strictly improves from σ to τ.

(a) $k \in N \setminus \{l_{\bar{i}}, j\}$. Here $t_k^\delta = t_k^\tau \succ_k t_k^\sigma$. Thus k also strictly improves in δ in comparison to σ.

(b) $k = l_{\bar{i}}$. Here $t_{l_{\bar{i}}}^\tau \succ_{l_{\bar{i}}} t_{l_{\bar{i}}}^\sigma$. Then Fact 6 implies $t_j^\delta = t_{l_{\bar{i}}}^\tau \succ_j t_j^\sigma$. Thus j strictly improves in δ in comparison to σ.

(c) $k = j$. Here $t_j^\tau \succ_j t_j^\sigma$. By Fact 5, $t_j^\delta = t_{l_{\bar{i}}}^\tau \succsim_j t_j^\tau$. Thus $t_j^\delta \succ_j t_j^\sigma$ and j strictly improves in δ in comparison to σ.

[36] Note that this will also establish that σ and δ are distinct.

We have established that there exists an agent who strictly improves in δ with respect to σ.

We now show that τ is not minimal. Let K be the number of agents in $N \setminus \{l_{\bar{i}}, j\}$ who have different contribution values in σ and τ. By contruction, K is also the number of agents in $N \setminus \{l_{\bar{i}}, j\}$ who have different contribution values in σ and δ.

Note that $|\{i \in N : t_i^\tau \neq t_i^\sigma\}| = K + 2$ and $|\{i \in N : t_i^\delta \neq t_i^\sigma\}| \leq K + 1$. So τ is not minimal and we have a contradiction.

Case II: For all $l \in L$, $t_l^\tau = t_l^\sigma$ Then there exists $h \in H$ such that $t_h^\tau \neq t_h^\sigma$.

The proof of Case II is virtually identical to that of Case I. We omit the details.

This completes the proof of the theorem. ■

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