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*A numerical solution of the dissipative wave equation
by means of the cubic B-spline method*

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E-mail: zaki.alaofi@live.vu.edu.au, zaleawfi@kku.edu.sa, tdanaf@taibahu.edu.sa and sever.dragomir@vu.edu.au**Keywords:** nonlinear dissipative wave equation, cubic B-spline polynomial, Von Neumann stability analysis**Abstract**

In the past few decades, partial differential equations have drawn considerable attention, owing to their ability to model certain physical phenomena. The aim of this paper is to investigate a cubic B-spline polynomial to obtain a numerical solution of a nonlinear dissipative wave equation. For the numerical procedure, the time derivative is obtained by the usual finite difference scheme. The approximate solution and its principal derivatives over the subinterval is approximated by the combination of the cubic B-spline and unknown element parameters. The accuracy of the proposed method will be shown by computing L_∞ error norms for different time levels. By applying Von Neumann stability analysis, the developed method is shown to be conditionally stable for given values of specified parameters. A numerical example is given to illustrate the accuracy of the cubic B-spline polynomial method. The obtained numerical results show that our proposed method maintains good accuracy.

1. Introduction

The nonlinear partial differential equations arise in a wide variety of physical phenomena in several different aspects of physics, such as water wave theory, fluid dynamics, plasma physics, solid mechanics, and nonlinear optics, etc. There are many methods for solving partial differential equations via numerical solutions. One of these is numerically solving a nonlinear dissipative wave equation by using the Adomian decomposition method [1, 2]. The cubic B-spline, used for solving nonlinear partial differential equations, has been employed by many researchers. The most known and well-focused results are those presented by Dağ *et al* (2004) who presented a way to solve the Regularised Long Wave (RLW) equation. The numerical results obtained in this paper demonstrate that the method is capable of solving the RLW equation accurately and reliably [3]. Dağ *et al* published a paper that described a numerical solution for the one-dimensional Burger's equation in 2005. The comparison of the calculations with the analytic solution shows that a cubic B-spline collocation method is capable of solving Burgers' equation accurately. The proposed method is easy to implement and does not require any inner iteration or corrector to deal with the nonlinear term of Burgers' equation [4]. Khalifa *et al* (2008) discussed the Modified Regularised Long Wave (MRLW) equation. The collocation method using cubic B-splines was applied to study the solitary waves of the MRLW equation, and it is shown that the scheme is marginally stable. Moreover, despite the fact that the wave does not change, results show that the interaction results in a tail of small amplitude in two, and clearly in three, soliton interactions, and the conservation laws were reasonably satisfied. The appearance of such a tail can be beneficial in further study [5]. In 2008, El Danaf and E I Abdel Alaal constructed a non-polynomial spline-based method to obtain numerical solutions of a dissipative wave equation. The obtained numerical results show that their proposed method maintains good accuracy [6]. Later, Mittal and Jain (2012) argued that some numerical method should be proposed to approximate the solution of the nonlinear parabolic partial differential equation with Neumann's boundary

conditions. The numerical results produced by the present method are quite satisfactory and in good agreement with the exact solutions. The computed results justify the advantage of this method. The proposed method can be extended to solve multi-dimensional parabolic equations [7]. In 2015, Zaki developed a new numerical method based on quadratic non-polynomial spline functions, which has three coefficients in each sub interval for solving a dissipative wave equation. The results obtained by the proposed technique show that the approach is easy to implement and computationally attractive. The proposed method is shown to be robust, efficient, and easy to implement for linear and nonlinear problems arising in science and engineering [8]. A year later, El-Danaf *et al* addressed methods for solving the Generalised Regularised Long Wave (GRLW) equation. The cubic B-splines used to study solitary waves of GRLW equation show that the scheme is unconditionally stable. Also, the obtained approximate numerical solutions maintain good accuracy when compared with the exact solutions [9]. Hepson and Dağ, in their 2017 research, implemented a numerical technique to obtain approximate solutions of Fisher's equation. The method is capable of producing solutions for Fisher's equation fairly and can be used as an alternative to the method's accompanied B-spline functions [10]. In 2017, Iqbal *et al*'s proposed numerical technique was based on the cubic B-spline collocation method. Their version used a new approximation for the second order derivative. The proposed scheme is based on the cubic B-spline collocation method equipped with a new approximation for second order derivative and produces fifth order accurate results. The proposed method also generates a piecewise spline solution in the presence of the singularity, which can be used to obtain a numerical solution at any point in the domain and is not restricted to the values at the selected knots, unlike existing finite difference methods [11]. A year later, Başhan (2018) studied the numerical solutions of the third-order nonlinear Korteweg–de Vries (KdV) equation by using modified cubic B-splines in five different test problems. The performance and accuracy of the modified cubic B-splines method was shown by calculating and comparing the L_2 and L_∞ error norms with earlier works. The stability analysis has been performed for all of the test problems, and all of the eigenvalues are in convenience with stability criteria. So, MCBC-DQM may be useful in obtaining the numerical solutions of other important nonlinear problems [12]. In research conducted in 2019 by Bashan, a modified cubic B-spline differential quadrature method has successfully been implemented for the numerical solution of nonlinear Kawahara equation. To obtain the first, third, and fifth-order derivative approximation, a modified cubic B-spline differential quadrature method was utilised. Four different test problems have then been investigated separately. These newly obtained results obviously indicate that a modified cubic B-spline differential quadrature method can be used to produce numerical solutions of the Kawahara equation with high accuracy [13]. More recently, Iqbal *et al* studied the Galerkin method, based on a cubic B-spline function, where the shape and weight functions are applied for the numerical solution of the one-dimensional coupled nonlinear Schrödinger equation. The use of the cubic B-spline Galerkin method produced smooth solutions without numerical smearing in 2020 [14]. In the same year, by Ahmed *et al* (2020) a Non-polynomial spline function was used to get numerical solutions of a Dissipative Wave equation at middle points for lattice in space direction and at the same time, a finite difference method was used in time direction. The presented method is shown to be conditionally stable. The approximating results showed to have well agreement compared with the true solutions, hence it can be used to set approximate solutions for such type of problems [15]. In the current work, we propose a mathematical treatment for the nonlinear dissipative wave equation, utilising the collocation technique with cubic B-spline shape functions. For the mathematical methodology, the time derivatives will be achieved through the typical finite difference method. The technique will be shown to be conditionally stable by applying the Von Neumann stability investigation procedure. We will test the precision of the proposed strategy by conducting an examination of the mathematical outcomes and the specific arrangement of the condition.

2. The governing equation and the derivation of the proposed method

This paper is worried about applying the cubic B-spline method to build up a mathematical strategy for approximating the specific arrangement of the nonlinear dissipative wave equation [1] of the structure:

$$u_{tt} - u_{xx} + 2u_t u = \eta(x, t), \quad \eta(x, t) = -2 \sin^2 x \sin t \cos t. \quad (2.1)$$

Under the boundary and initial conditions:

$$u_{xx}(a, t) = 0, \quad u_{xx}(b, t) = 0, \quad u(x, 0) = \sin x, \quad u_t(x, 0) = 0. \quad (2.2)$$

The interval $[a, b]$ can be divided into equal subintervals $[x_{i-1}, x_i]$, $i = 0, 1, \dots, N + 1$, where $x_i = a + ih$, and $h = \frac{b-a}{n}$.

Let the cubic B-spline basis functions $\varphi_i(x)$ given as:

$$\varphi_i(x) = \frac{1}{h^3} \begin{cases} (x - x_{i-2})^3 & x \in [x_{i-2}, x_{i-1}] \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3 & x \in [x_{i-1}, x_i] \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3 & x \in [x_i, x_{i+1}] \\ (x_{i+2} - x)^3 & x \in [x_{i+1}, x_{i+2}] \\ 0 & otherwise, \end{cases}$$

where $\{\varphi_i\}$ for $i = 0, 1, \dots, N + 1$ are the basis for the function defined over the interval $[a, b]$, this implies that the estimations of the cubic B-spline $\varphi_i(x)$ and its derivatives vanish outside the interval $[x_{i-2}, x_{i+2}]$, $i = 0, 1, \dots, N$.

The mathematical treatments for equation (2.1) by the collocation method with cubic B-splines is to track down an inexact arrangement $U_N(x, t)$ to the exact solution $u(x, t)$.

Set the approximate solution $U_N(x, t)$ as follows:

$$U_N(x, t) = \sum_{i=-1}^{N+1} \omega_i(t)\varphi_i(x), \tag{2.3}$$

where $\omega_i(t)$ are the time dependent parameters which can be resolved utilizing the boundary conditions:

$$(U_{xx})_N(a, t) = 0, \quad (U_{xx})_N(b, t) = 0, \tag{2.4}$$

and the collocation form of equation (2.1)

$$(U_{tt})_N(x_j, t) - (U_{xx})_N(x_j, t) + 2(U)_N(x_j, t)(U_t)_N(x_j, t) = \eta(x_j, t). \tag{2.5}$$

By subbing equations (2.3) into (2.5), we get:

$$\begin{aligned} \sum_{i=-1}^{N+1} \frac{d^2\omega_i(t)}{dt^2} \varphi_i(x) - \sum_{i=-1}^{N+1} \omega_i(t) \varphi_i''(x_j) + 2\omega_i(t) \varphi_i(x_j) \\ \sum_{\delta=-1}^{N+1} \frac{d\omega_\delta(t)}{dt} \varphi_\delta(x_j) = \eta_j^n(x, t). \end{aligned} \tag{2.6}$$

Applying the finite difference method, we have:

$$\omega_i^n = \frac{\omega_i^{n+1} + \omega_i^{n-1}}{2}, \quad \frac{d^2\omega}{dt^2} = \frac{\omega_i^{n-1} - 2\omega_i^n + \omega_i^{n+1}}{k^2}, \quad \text{where } k = \Delta t. \tag{2.7}$$

Substituting equations (2.7) into (2.6) and simplifying the results, we get:

$$\begin{aligned} \sum_{i=-1}^{N+1} [\omega_i^{n-1} - 2\omega_i^n + \omega_i^{n+1}] \varphi_i(x_j) - k^2 \sum_{i=-1}^{N+1} \omega_i(t) \varphi_i''(x_j) + k^2[\omega_i^{n+1} + \omega_i^{n-1}] \varphi_i(x_j) \\ \sum_{\delta=-1}^{N+1} \frac{d\omega_\delta(t)}{dt} \varphi_\delta(x_j) = k^2 \eta_j^n(x, t). \end{aligned} \tag{2.8}$$

Equation (2.8) can be determined at $x_j, j = 0, 1, 2, \dots, N$, so that

$$\begin{aligned} a_i \omega_{i-1}^{n+1} + b_i \omega_i^{n+1} + c_i \omega_{i+1}^{n+1} = -d_i \omega_{i-1}^n - e_i \omega_i^n - f_i \omega_{i+1}^n \\ -n_i \omega_{i-1}^{n-1} - s_i \omega_i^{n-1} - l_i \omega_{i+1}^{n-1} + k^2 \eta_j^n(x, t), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} a_i = 1 + k^2 z_{i-1}, \quad d_i = -2 - \frac{6}{h^2} k^2, \quad n_i = 1 + k^2 z_{i-1} b_i = 4 + 4k^2 z_{i-1}, \\ e_i = -8 + 12 \frac{k^2}{h^2}, \quad s_i = 4 + 4k^2 z_{i-1}, \\ c_i = 1 + k^2 z_{i-1}, \quad f_i = -2 - \frac{6}{h^2} k^2, \quad l_i = 2 + k^2 z_{i-1} \end{aligned} \tag{2.10}$$

with $z_{i-1} = \frac{\partial U_N(x_j, t_n)}{\partial t} = \left[\frac{\omega_{i-1}^n - \omega_{i-1}^{n-1}}{k} + 4 \frac{\omega_i^n - \omega_i^{n-1}}{k} + \frac{\omega_{i+1}^n - \omega_{i+1}^{n-1}}{k} \right]$, for all $i = 0, 1, 2, \dots, N$. The nonlinear logarithmic system (2.9) contains $(N + 1)$ equations of $(N + 3)$ unknowns. To find the solution for this system, we need two additional conditions which are gotten from the conditions (2.4) as follows:

$$\frac{6}{h^2} \omega_{-1} - \frac{12}{h^2} \omega_0 + \frac{6}{h^2} \omega_1 = 0, \quad \frac{6}{h^2} \omega_{N-1} - \frac{12}{h^2} \omega_N + \frac{6}{h^2} \omega_{N+1} = 0. \tag{2.11}$$

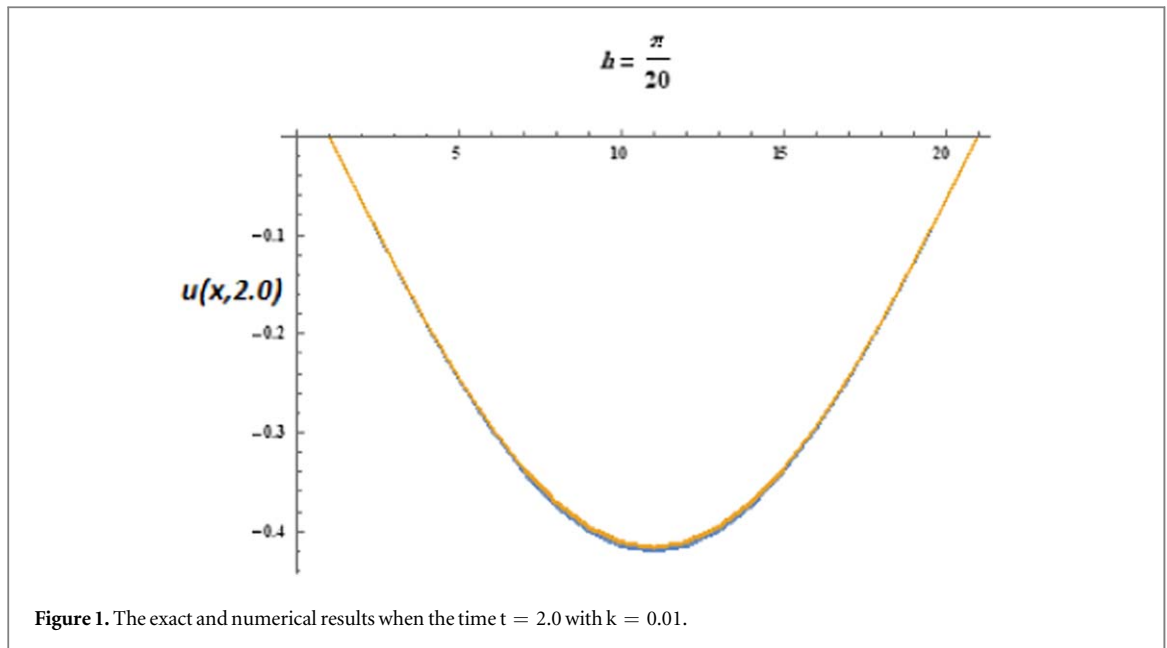


Figure 1. The exact and numerical results when the time $t = 2.0$ with $k = 0.01$.

Table 1. The values of $\varphi_i(x)$ and their derivative within the interval $[x_{i-2}, x_{i+2}]$.

x	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$\varphi_i(x)$	0	1	4	1	0
$\varphi_i'(x)$	0	$3/h$	0	$-3/h$	0
$\varphi_i''(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

System (2.9) and the additional equation (2.11) has $(N + 3)$ equations with $(N + 3)$ unknowns, so we can determine the time dependent variables ω_i in the matrix form:

$$A\omega^{n+1} = -B\omega^n - C\omega^{n-1} + k^2\eta_i^n(x, t), \tag{2.12}$$

where

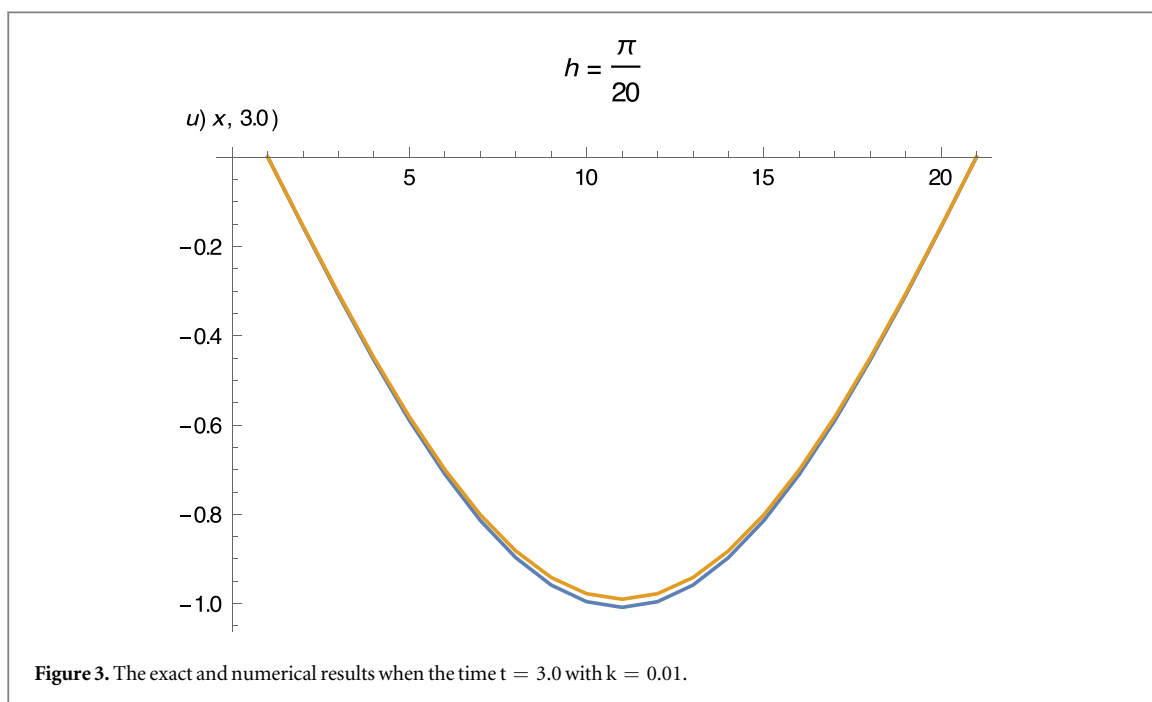
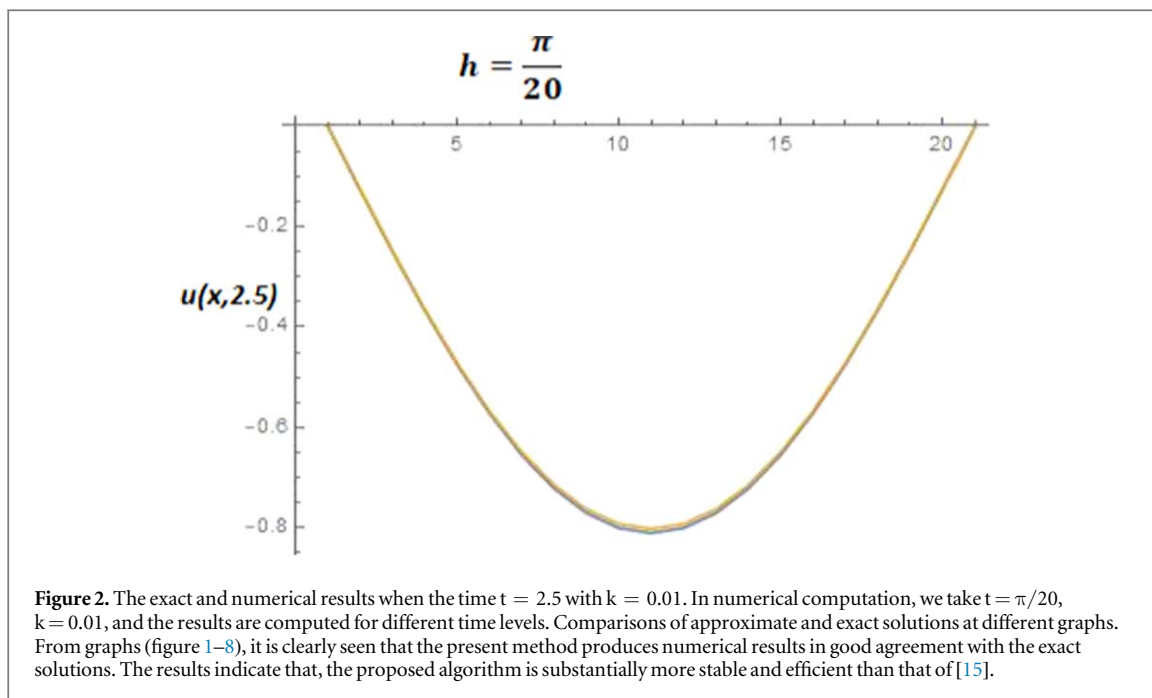
$$A = \begin{bmatrix} \frac{6}{h^2} & \frac{-12}{h^2} & \frac{6}{h^2} & 0 & \dots & 0 \\ a_0 & b_0 & c_0 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot & \dots & 0 \\ 0 & \dots & 0 & a_N & b_N & c_N \\ 0 & \dots & 0 & \frac{6}{h^2} & \frac{-12}{h^2} & \frac{6}{h^2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ d_0 & e_0 & f_0 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot & \dots & 0 \\ 0 & \dots & 0 & d_N & e_N & f_N \\ 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ n_0 & s_0 & l_0 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot & \dots & 0 \\ 0 & \dots & 0 & n_N & s_N & l_N \\ 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. The initial state

In this section, we apply the first initial condition:

$$u(x, 0) = \sin x \tag{3.1}$$

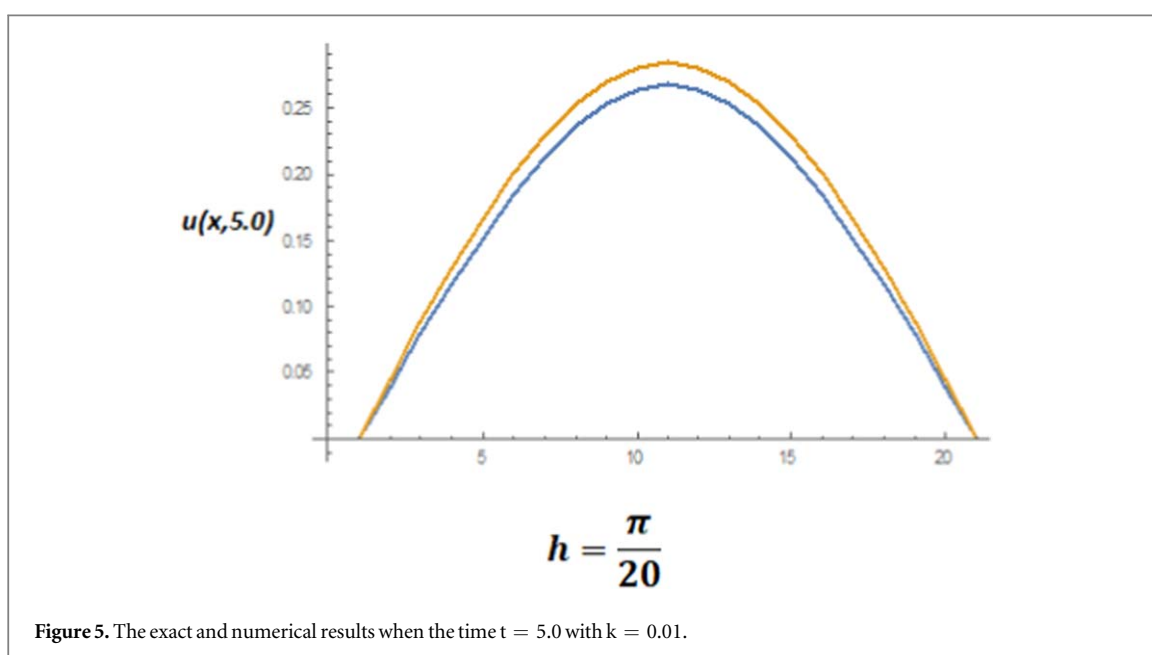
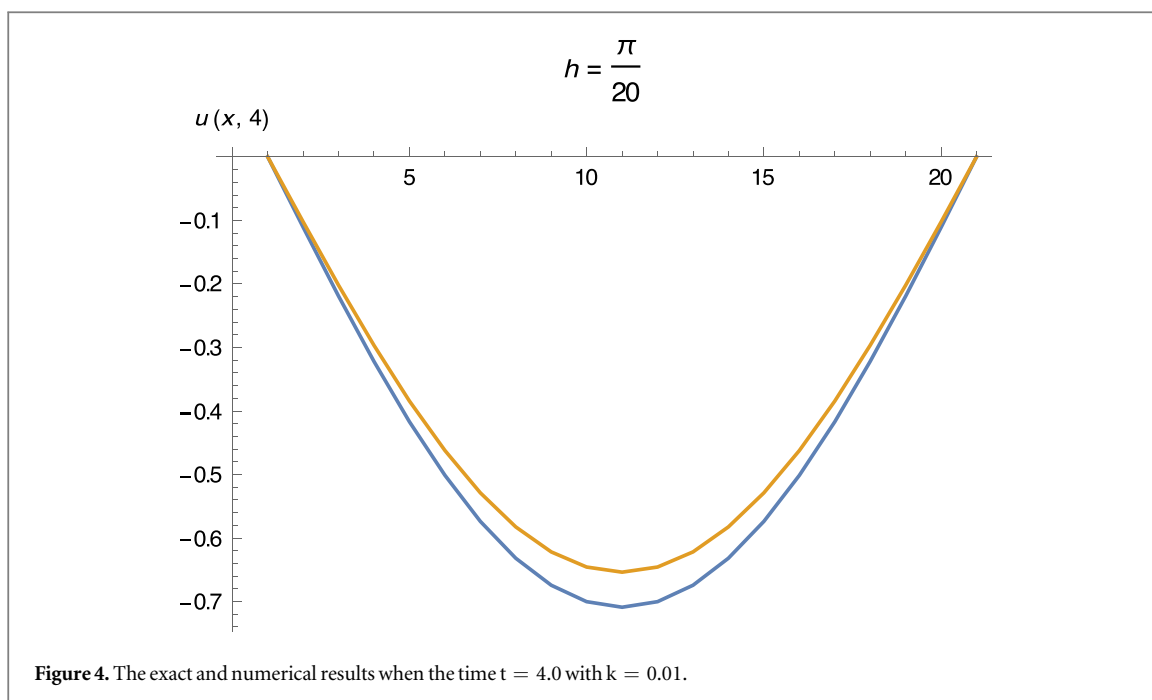


The initial conditions can be communicated as:

$$\begin{aligned}
 (U_x)_N(a, 0) &= u_x(a, 0), \quad U_N(x_j, 0) = \sum_{i=-1}^{N+1} \varphi_i(x_j) \omega_i^0, \quad j = 0, 1, 2, \dots, N, \\
 (U_x)_N(b, 0) &= u_x(b, 0).
 \end{aligned}
 \tag{3.2}$$

By using the values of φ_i and their derivatives in table 1, the system (3.2) takes the structure

$$\begin{aligned}
 -3\omega_{-1}^0 + 3\omega_1^0 &= hu_x(a, 0), \quad \omega_{i-1}^0 + 4\omega_i^0 + \omega_{i+1}^0 = u(x_j, 0), \quad j = 0, 1, 2, \dots, N, \\
 -3\omega_{N-1}^0 + 3\omega_{N+1}^0 &= hu_x(b, 0).
 \end{aligned}
 \tag{3.3}$$



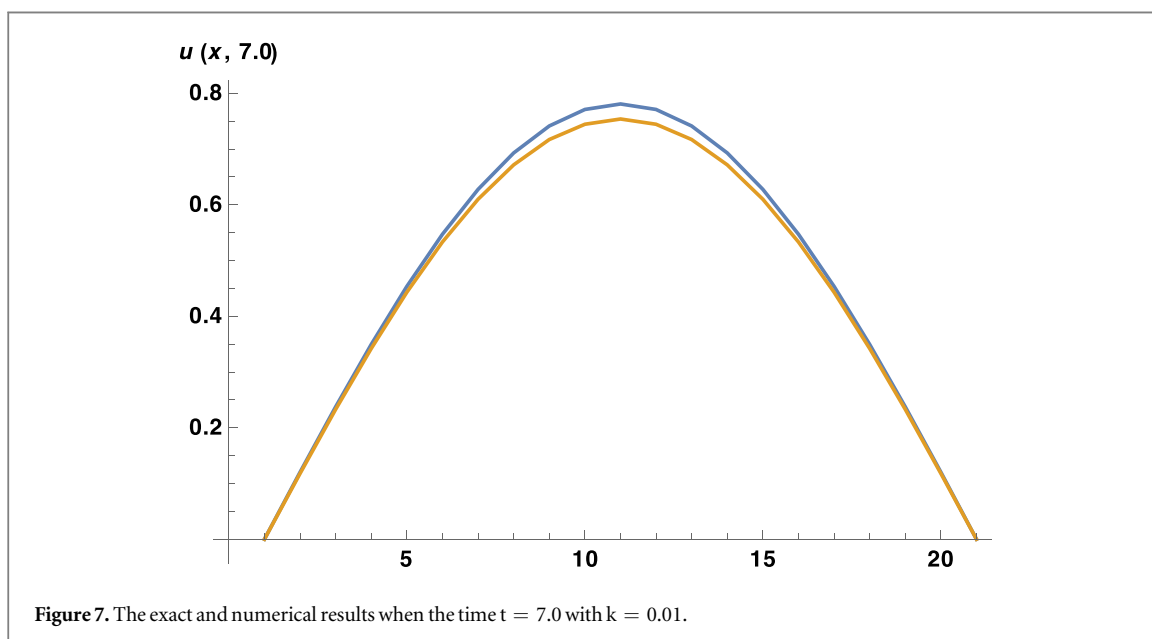
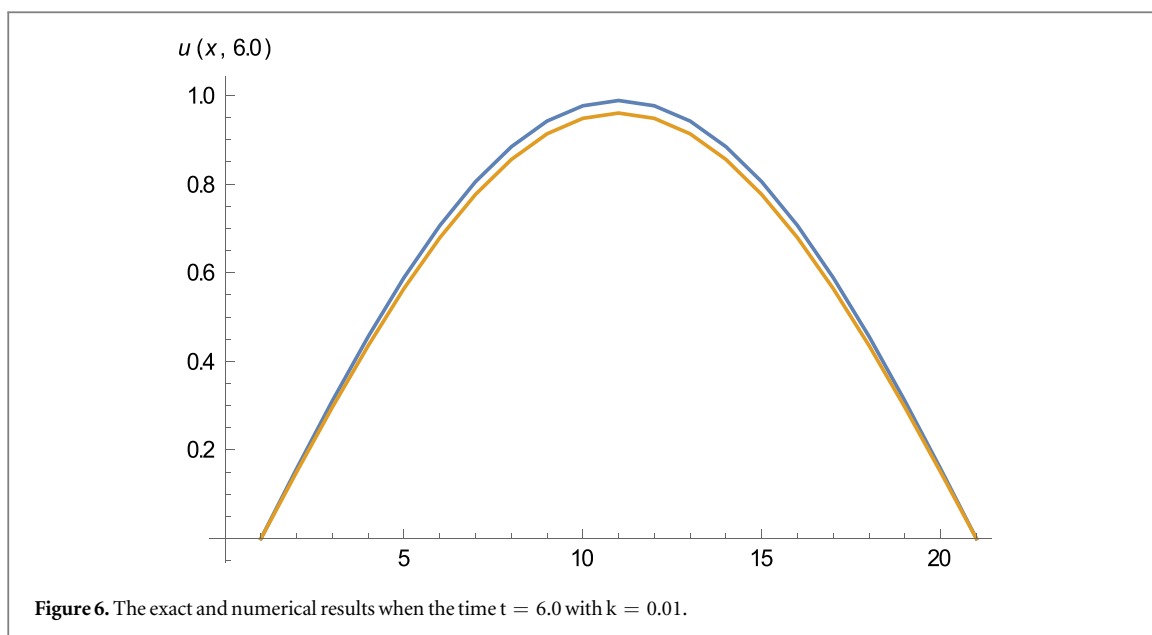
Rewrite the system (3.3) in a matrix form:

$$Mv = q \tag{3.4}$$

where

$$M = \begin{bmatrix} -3 & 0 & 3 & 0 & \dots & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & 1 & 4 & 1 \\ 0 & \dots & 0 & -3 & 0 & 3 \end{bmatrix},$$

and $v = (\omega_{-1}^0, \omega_0^0, \dots, \omega_N^0, \omega_{N+1}^0)^T$, $q = (hu_x(a, 0), u(x_0, 0), \dots, u(x_N, 0), hu_x(b, 0))^T$.



To find the second initial condition using Taylor expansion to $U_N(x, t_i)$ at $t = t_0$

$$U_N(x, t_1) = U_N(x, t_0) + k \frac{\partial U_N(x, t_0)}{\partial t} + \frac{k^2}{2!} \frac{\partial^2 U_N(x, t_0)}{\partial t^2} + O(k^3), \quad k = t_1 - t_0. \tag{3.5}$$

Set $t_0 = 0$, we get:

$$U_N(x, t_1) = U_N(x, 0) + k \frac{\partial U_N(x, 0)}{\partial t} + \frac{k^2}{2!} \frac{\partial^2 U_N(x, 0)}{\partial t^2} + O(k^3). \tag{3.6}$$

Subbing equations (2.1) into (3.6) we get:

$$U_N(x, t_1) \approx U_N(x, 0) + \frac{k^2}{2!} \left(\frac{\partial^2 U_N(x, 0)}{\partial x^2} - 2u \frac{\partial U_N(x, 0)}{\partial t} + \eta(x, 0) \right). \tag{3.7}$$

After simplifying, equation (3.7) becomes:

$$U_N(x, t_1) \approx U_N(x, 0) + \frac{k^2}{2!} \frac{\partial^2 U_N(x, 0)}{\partial x^2}, \tag{3.8}$$

where $\eta(x, 0) = 0$.

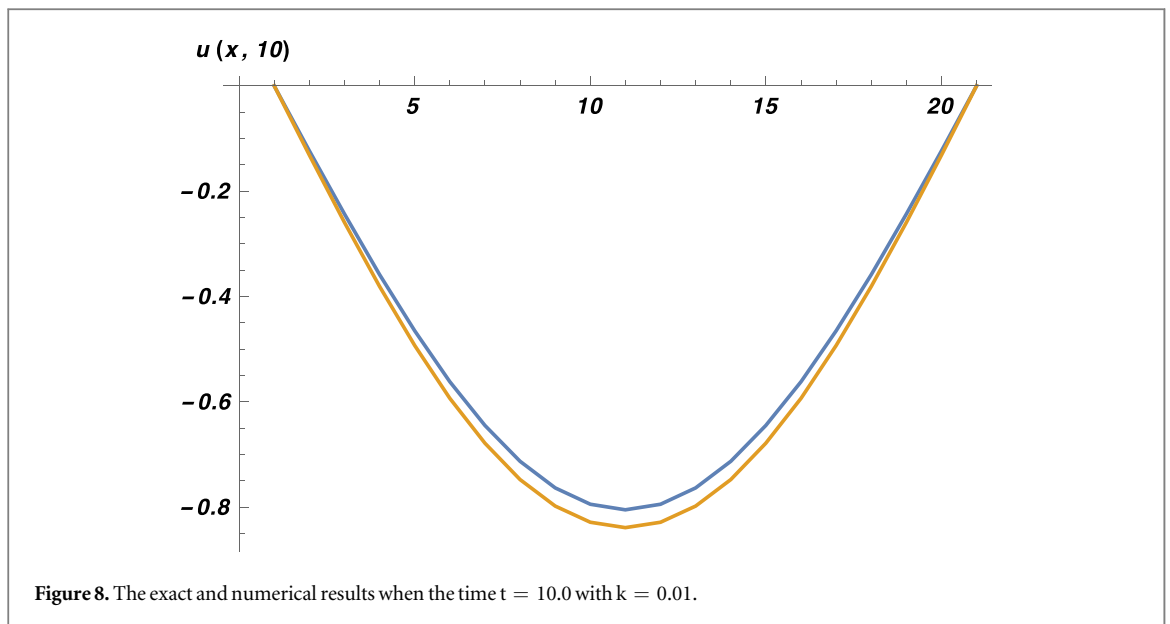


Figure 8. The exact and numerical results when the time $t = 10.0$ with $k = 0.01$.

Substituting equation (3.1) and initial condition (2.2) into equation (3.8), we obtain:

$$\sum_{i=-1}^{N+1} \varphi(x_j) \omega_i^1 \approx \eta(x_j), \quad j = 0, 1, \dots, N, \tag{3.9}$$

where

$$\eta(x_j) = \sin x_j - \frac{k^2}{2!} \sin x_j.$$

To complete this system, differentiate (3.9) with respect to x , and compute its value at the ends of the range, which gives us the following system:

$$-3\omega_{-1}^1 + 3\omega_1^1 = h\eta'(x_0), \quad \omega_{i-1}^1 + 4\omega_i^1 + \omega_{i+1}^1 = h\eta(x_j), \quad -3\omega_{N-1}^1 + 3\omega_{N+1}^1 = h\eta'(x_N). \tag{3.10}$$

The system (3.10) in a matrix equation form as follows: $My = H$

where:

$$M = \begin{bmatrix} -3 & 0 & 3 & 0 & \dots & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & 1 & 4 & 1 \\ 0 & \dots & 0 & -3 & 0 & 3 \end{bmatrix},$$

and

$$y = (\omega_{-1}^1, \omega_0^1, \dots, \omega_N^1, \omega_{N+1}^1)^T,$$

while H has the form:

$$H = (h\eta'(x_0), h\eta(x_0), \dots, h\eta(x_N), h\eta'(x_N))^T.$$

4. Stability analysis

The Von Neumann stability analysis for system (2.9) takes effect after linearizing the nonlinear term as:

$$z_{i-1} = d + 4d + d = (6d), \quad m = 6d.$$

Then the Von Neumann stability analysis takes the form:

$$\omega_j^n = \varepsilon^n \exp(q\sigma jh), \quad q = \sqrt{-1}, \tag{4.1}$$

where σ is the wave number and h is the element size. At $x = x_i$, equation (2.9) can be written as:

$$a_i \omega_{i-1}^{n+1} + b_i \omega_i^{n+1} + c_i \omega_{i+1}^{n+1} = -d_i \omega_{i-1}^n - e_i \omega_i^n - f_i \omega_{i+1}^n - n_i \omega_{i-1}^{n-1} - s_i \omega_i^{n-1} - l_i \omega_{i+1}^{n-1} + k^2 \eta_i^n. \tag{4.2}$$

Substituting equation (4.1) into the recurrence relation (4.2) and dividing both sides by $\varepsilon^n \exp(jq\phi h)$, we obtain the equation:

$$\varepsilon^2[(c_j + a_j) \cos \phi h + b_j + q(c_j - a_j) \sin \phi h] + \varepsilon[(f_j + d_j) \cos \phi h + e_j + q(f_j - d_j) \sin \phi h] + [(l_j + n_j) \cos \phi h + s_j + q(l_j - n_j) \sin \phi h] = 0. \quad (4.3)$$

So we have:

$$\varepsilon^2[(2 + 2k^2m) \cos \phi h + (4 + 4k^2m)] + \varepsilon[(-4 - 2r_1) \cos \phi h + (-8 + 2r_1)] + [(2 + 2k^2m) \cos \phi h + (4 + 4k^2m)] = 0, \quad m = 6d, \quad (4.4)$$

where $r_1 = \left[\frac{6}{h^2} k^2 \right]$.

Dividing equation (4.4) by $[(2 + 2k^2m) \cos \phi h + (4 + 4k^2m)]$, we obtain the equation:

$$\varepsilon^2 + \varepsilon \frac{[(-4 - 2r_1) \cos \phi h + (-8 + 2r_1)]}{[(2 + 2k^2m) \cos \phi h + (4 + 4k^2m)]} + 1 = 0. \quad (4.5)$$

Equation (4.5) written as:

$$\varepsilon^2 + 2\beta\varepsilon + 1 = 0, \quad (4.6)$$

where: $\beta = \frac{[(-2 - r_1) \cos \phi h + (-4 + r_1)]}{[(2 + 2k^2m) \cos \phi h + (4 + 4k^2m)]}$.

Equation (4.9) is a quadratic in ε and hence will have two roots, that is $\varepsilon = -\beta \pm \sqrt{\beta^2 - 1}$.

For stability, then $|\varepsilon| \leq 1$. Now, from equation (4.6) we see that the result of the two estimations of ε should rise to solidarity, which emerge three cases as follows:

Case 1. On the off chance that the two roots are equivalent to solidarity, which infers that the segregate of the equation (4.6) is zero.

Case 2. One of the two roots is more prominent than solidarity. At that point, the separate is more noteworthy than nothing. This implies that the steadiness condition, ($|\varepsilon| \leq 1$), isn't fulfilled.

Case 3. The discriminate is less than zero, that is $\beta^2 - 1 < 0$.

Thus for stability:

$$-1 \leq \beta \leq 1. \quad (4.7)$$

Using equation (4.7), the above inequality becomes

$$-1 \leq \frac{[(-2 - r_1) \cos \phi h + (-4 + r_1)]}{[(2 + 2k^2m) \cos \phi h + (4 + 4k^2m)]} \leq 1. \quad (4.8)$$

The right inequality (4.8) takes the form:

$$\frac{[(-2 - r_1) \cos \phi h + (-4 + r_1)]}{[(2 + 2k^2m) \cos \phi h + (4 + 4k^2m)]} \leq 1. \quad (4.9)$$

After simplifying inequality (4.9), we obtain:

$$\begin{aligned} \frac{6}{h^2} k^2 &\leq \left[\left(\frac{6}{h^2} k^2 + 4 + 2k^2m \right) \cos \phi h + (8 + 4k^2m) \right], \\ \text{or} \\ 6 &\leq \left[\left(6 + 4\frac{h^2}{k^2} + 2h^2m \right) \cos \phi h + \left(8\frac{h^2}{k^2} + 4h^2m \right) \right]. \end{aligned} \quad (4.10)$$

And by using the relation $\cos \phi h = 1 - 2 \sin^2 \frac{\phi h}{2}$, inequality (4.10) reduces to:

$$6 \leq \left[\left(6 + 12\frac{h^2}{k^2} + 6h^2m \right) - \left(12 + 8\frac{h^2}{k^2} + 4h^2m \right) \sin^2 \frac{\phi h}{2} \right]. \quad (4.11)$$

After simplifying inequality (4.11), we get:

$$12\frac{h^2}{k^2} + 6h^2m \geq \left(12 + 8\frac{h^2}{k^2} + 4h^2m \right) \sin^2 \frac{\phi h}{2}.$$

Table 2. Comparison between the numerical and exact solutions at $t = 0.2$, $k = 0.002$, $h = \frac{\pi}{20}$.

x	Numerical Solution	Exact Solution
0.1π	0.302 857	0.303 116
0.2π	0.576 069	0.576 509
0.3π	0.792 891	0.793 442
0.4π	0.932 099	0.932 707
0.5π	0.980 067	0.980 692
0.6π	0.932 099	0.932 707
0.7π	0.792 891	0.793 442
0.8π	0.576 069	0.576 509
0.9π	0.302 857	0.303 116

Table 3. The L_∞ error when $k = 0.001$, $h = \frac{\pi}{20}$ from $t = 0.05$ to $t = 0.2$.

Time	0.05	0.1	0.15	0.20
L_∞ error [Our]	2.0097×10^{-5}	3.1051×10^{-5}	3.3695×10^{-5}	2.8898×10^{-5}
L_∞ error [15]	2.5236×10^{-4}	9.8616×10^{-4}	2.1532×10^{-3}	3.7090×10^{-3}

Table 4. The L_∞ error for the numerical and exact solutions when $k = 0.01$, $h = \frac{\pi}{20}$ from $t = 0.5$ to $t = 2.0$.

Time	0.5	1.0	1.5	2.0
L_∞ error [Our]	6.2131×10^{-4}	6.2454×10^{-4}	$1.929 96 \times 10^{-3}$	$3.874 06 \times 10^{-3}$
L_∞ error [15]	8.7356×10^{-4}	2.5274×10^{-3}	4.4852×10^{-3}	7.5875×10^{-3}

Satisfied for $k \ll h$, where h is small enough. But the left inequality (4.8) becomes:

$$-1 \leq \frac{[(-2 - r_1) \cos \phi h + (-4 + r_1)]}{[(2 + 2k^2m) \cos \phi h + (4 + 4k^2m)]}. \quad (4.12)$$

After simplifying inequality (4.12), we get:

$$\left(-2k^2m + \frac{6}{h^2}k^2\right) \cos \phi h \leq \frac{6}{h^2}k^2 + 4k^2m,$$

or

$$\left(\frac{-mh^2}{3} + 1\right) \cos \phi h \leq 1 + \frac{2mh^2}{3}. \quad (4.13)$$

Using the relation $\cos \phi h = 1 - 2 \sin^2 \frac{\phi h}{2}$, inequality (4.13) becomes:

$$\left(\frac{-mh^2}{3} + 1\right) + \left(\frac{2mh^2}{3} - 2\right) \sin^2 \frac{\phi h}{2} \leq \left(1 + \frac{2mh^2}{3}\right), \quad (4.14)$$

if h is small enough, thus the method is conditionally stable.

5. Numerical illustration

We apply cubic B-spline method to obtain numerical solution of the dissipative equation for one standard issue. The precision of our proposed mathematical technique estimated by registering the L_∞ error norm. The exact solution of the dissipative equation (2.1) which obtained in [1] given by:

$$u(x, t) = \cos t \sin x, \quad 0 \leq x \leq \pi, \quad t \geq 0.$$

We use the following conditions:

$$u(x, 0) = \sin x, \quad u_{xx}(0, t) = 0, \quad u_{xx}(\pi, t) = 0.$$

We put the acquired mathematical outcomes in the accompanying tables 2–7.

From tables 3–7, we observe that the smaller the $\Delta t = k$ (than the value of h), the better the accuracy. The numerical approximations is still acceptable within the large time.

Table 5. The L_∞ -error for the numerical and exact solutions when $k = 0.01$, $h = \frac{\pi}{20}$ from $t = 6.0$ to $t = 9.0$.

Time	6.0	7.0	8.0	9.0
$L_\infty error$	2.9063×10^{-2}	2.7145×10^{-2}	1.7337×10^{-2}	3.1307×10^{-2}

Table 6. The $L_\infty error$ for the numerical and exact solutions for a big time when $k = 0.01$, $h = \frac{\pi}{20}$.

Time	10.0	20.0	30.0	40.0
$L_\infty error$	3.4557×10^{-2}	3.3574×10^{-2}	1.4119×10^{-2}	2.3355×10^{-2}

Table 7. Comparison between the numerical and exact solutions at $t = 2$, $k = 0.002$, $h = \frac{\pi}{20}$.

x	Numerical solution	Exact solution
0.1π	-0.128 596	-0.129 169
0.2π	-0.244 605	-0.245 756
0.3π	-0.336 67	-0.338 348
0.4π	-0.395 779	-0.397 833
0.5π	-0.416 147	-0.418 337
0.6π	-0.395 779	-0.397 833
0.7π	-0.336 67	-0.338 348
0.8π	-0.244 605	-0.245 756
0.9π	-0.128 596	-0.129 169

6. Conclusion

In this paper, a numerical solution for the nonlinear dissipative wave equation, utilising a collocation strategy with the cubic B-splines, is proposed. To illustrate our method and to demonstrate its convergence and applicability of our presented methods computationally, we will apply the Von Neumann stability method. The stability analysis investigation will show that the method is conditionally stable. The performance and accuracy of the present method have been shown by calculating and comparing the L_∞ error norms with earlier works. The obtained invariants are considered acceptable when compared with some earlier works. The numerical results produced by the present method are quite satisfactory and show good agreement with the exact solutions. The computed results justify the advantage of this method. As seen in tables 3 and 4, the present results are better than [15]. The estimated mathematical arrangements that achieve great precision with the specific arrangements, particularly when Δt is more modest than the estimation of h .

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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