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Constructing integer-magic graphs via the combinatorial nullstellensatz*

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Abstract

Let A be a nontrivial additive abelian group and $A^* = A \setminus \{0\}$. A graph is A-magic if there exists an edge labeling f using elements of A^* which induces a constant vertex labeling of the graph. Such a labeling f is called an A-magic labeling and the constant value of the induced vertex labeling is called an A-magic value. In this paper, we use the Combinatorial Nullstellensatz to construct nontrivial classes of $\mathbb{Z}_p$-magic graphs, prime $p \geq 3$. For these graphs, some lower bounds on the number of distinct $\mathbb{Z}_p$-magic labelings are also established.

Keywords: integer-magic graph, integer-magic labeling, Combinatorial Nullstellensatz

Math. Subj. Class.: 05C78

1 Introduction

Let $G = (V, E)$ be a graph, where G might be disconnected and/or a multigraph. For any nontrivial additive abelian group A, let $A^* = A \setminus \{0\}$. A mapping $f : E(G) \to A^*$ is called an *edge labeling* of G. Any such edge labeling induces a *vertex labeling* $f^+ : V(G) \to A$, where the label at a vertex is the sum of the edge labels incident to that vertex. Here, a loop label is counted only once. An edge labeling f whose induced mapping f^+ on $V(G)$

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†Corresponding author. Although Stephen G. Hartke was unable to join us on this project, the first author wishes to thank him for his contribution in formulating the original Hartke polynomial.

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is a constant is called an A-*magic labeling* of G. In this case, the constant is called the A-*magic value* of f and G is called an A-*magic* graph. If G has a $\mathbb{Z}_k$-magic labeling (for some $k \geq 2$), then G is an *integer-magic* graph. The *integer-magic spectrum* of a graph G is the set $\mathrm{IM}(G) = \{k \geq 2 : G \text{ is } \mathbb{Z}_k\text{-magic}\}$. Generally speaking, it is quite difficult to determine the integer-magic spectrum of a graph. Note that the integer-magic spectrum of a graph is not to be confused with the set of achievable magic values.

The concept of an A-magic graph was first introduced in [12]. Since then, A-magic graph labelings have been studied in [15, 20, 22, 37, 39, 41] and $\mathbb{Z}_k$-magic graphs were investigated in [11, 13, 14, 16, 17, 18, 19, 21, 24, 25, 30, 31, 32, 33, 42, 38, 40]. $\mathbb{Z}$-magic graphs were considered by Stanley [43, 44], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous Diophantine equations. They were also considered in [2, 34].

Labelings form a large and important area of study in graph theory. First formally introduced by Rosa [29] in the 1960s, graph labelings have captivated the interest of many mathematicians in the ensuing decades. In addition to the intrinsic beauty of the subject matter, graph labelings have applications (discussed in papers by Bloom and Golomb [4, 3]) in graph factorization problems, X-ray crystallography, radar pulse code design, and addressing systems in communication networks. The interested reader is directed to Gallian's [6] dynamic survey, which contains 2900+ references to research papers and books on the topic of graph labelings.

2 Preliminaries

All graph-theoretic terms (which are not explicitly defined) are standard ones and can be found in [7]. Throughout this paper, we consider general graphs which might be disconnected and/or a multigraph. We first note a few important facts which are known about $\mathbb{Z}_k$-magic labelings. Lemmas 2.2 and 2.4 are found in [20], whereas Lemma 2.1 is a slight generalization of a lemma found in [20].

Lemma 2.1. *For a graph G, let $i(v)$ denote the number of edges, multiedges and loops incident to $v \in V(G)$. Then, G is $\mathbb{Z}_2$-magic $\iff$ $i(v)$ are of the same parity, for all $v \in V(G)$.*

Lemma 2.2. *If G is $\mathbb{Z}_k$-magic and $k|n$, then G is $\mathbb{Z}_n$-magic.*

Remark 2.3. The converse of Lemma 2.2 is not true, in general. For example, it was shown in [13] that $\mathrm{IM}(K_4 - \{uv\}) = \{4, 6, 8, \dots\}$. In particular, $K_4 - \{uv\}$ is $\mathbb{Z}_6$-magic. However, $K_4 - \{uv\}$ is not $\mathbb{Z}_3$-magic.

Lemma 2.4. *Let p be prime. If G is $\mathbb{Z}_p$-magic for some magic value $t \neq 0$, then G is $\mathbb{Z}_p$-magic with magic value t' for any nonzero $t' \in \mathbb{Z}_p$.*

Proof. Let $b = t't^{-1}$. Multiply all of the edge labels by b. Since $\mathbb{Z}_p$ is a field, this gives edge labels which are non-zero. Hence, we have the desired $\mathbb{Z}_p$-magic labeling. □

Lemmas 2.1 and 2.2 allow us to focus on primes $p \geq 3$. Because of Lemma 2.4, it suffices to look at $\mathbb{Z}_p$-magic labelings with magic values equal to 0 and 1.

3 The Combinatorial Nullstellensatz

In [1], Alon proved the following result and successfully applied it to problems in additive number theory and graph theory.

Theorem 3.1 (Combinatorial Nullstellensatz)**.** *Let $f = f(x_1, \ldots, x_m)$ be a polynomial of degree d over a field $\mathbb{F}$. Suppose that the coefficient of the monomial $x_1^{t_1} \cdots x_m^{t_m}$ in f is nonzero and $t_1 + \cdots + t_m = d$. If $S_1, \ldots, S_m$ are subsets of $\mathbb{F}$ with $|S_i| \geq t_i + 1$, then there exists an $\underline{x}' = (x_1', x_2', \ldots, x_m') \in S_1 \times \cdots \times S_m$ for which $f(\underline{x}') \neq 0$.*

Example 3.2. Let $f(x_1, x_2, x_3, x_4) = x_1^4 x_2 x_3 - 2x_1^5 + x_1^2 x_2^2 x_3^2 + x_4^2 \in \mathbb{Z}_3[x_1, x_2, x_3, x_4]$. We will apply Theorem 3.1 on the term $x_1^2 x_2^2 x_3^2$ in f. Note that $\deg(f) = 6 = \deg(x_1^2 x_2^2 x_3^2)$. We choose $S_1 = \{0, 1, 2\}$, $S_2 = \{0, 1, 2\}$, $S_3 = \{0, 1, 2\}$ and $S_4 = \{2\}$. Then, Theorem 3.1 implies that there exist $s_i \in S_i$, where $1 \leq i \leq 4$, such that $f(s_1, s_2, s_3, s_4) \neq 0$. Note that the Combinatorial Nullstellensatz cannot be applied to any of the other monomial terms in f.

After its discovery, the Combinatorial Nullstellensatz would soon become a powerful tool in extremal combinatorics [10]. With regards to graph labeling and coloring problems, it has been used to prove theorems on anti-magic labelings, neighbor sum distinguishing total colorings, and list colorings [27, 36, 8]. For a recent research monograph on the Combinatorial Nullstellensatz and graph coloring problems, the reader is directed to [45].

4 The Hartke polynomials

Let $G = (V, E)$, where $|V(G)| = n$, $|E(G)| = m$, and the edges, multiedges and loops of G are identified with variables $x_1, x_2, \ldots, x_m$. As mentioned previously, we will focus on $\mathbb{Z}_p$-magic labelings (prime $p \geq 3$) and magic values equal to 0 and 1. For fixed prime $p \geq 3$ and $t \in \{0, 1\}$, define the polynomials f_t in $\mathbb{Z}_p[x_1, \ldots, x_m]$ as

$$f_t(\underline{x}) = f_t(x_1, \ldots, x_m) = \prod_{v \in V(G)} \left[1 - \left(t - \sum_{v \in x_j} x_j \right)^{p-1} \right], \tag{4.1}$$

where the addition and multiplication are taken modulo p. The given factorization of (4.1) and its factors are called the *canonical factorization* and *canonical factors* of f_t, respectively. The f_t are called *Hartke polynomials* and were introduced in [23]. Note that each Hartke polynomial describes a unique graph G up to isomorphism.

In this section, we recall the basic properties of f_t (see [23]) and give additional analysis of these polynomials. This is used in conjunction with Theorem 3.1 in subsequent sections of this paper, where we construct $\mathbb{Z}_p$-magic graphs.

Remark 4.1. Note that $\deg(f_t(\underline{x})) = (p - 1) \cdot |V(G)|$. This follows from:

1. There are $|V(G)|$ canonical factors of $f_t(\underline{x})$.
2. Each of the canonical factors is of degree $p - 1$.
3. Theorem in [9]: Let R be a commutative ring with unity and $g, h \in R[x_1, x_2, \ldots, x_m]$. If R has no zero divisors, then $\deg(gh) = \deg(g) + \deg(h)$.

Observations 4.2. Let $\underline{x}'$ be an m-tuple in $\mathbb{Z}_p^* \times \mathbb{Z}_p^* \times \cdots \times \mathbb{Z}_p^*$. Then, we note the following:

1. $f_\iota(\underline{x})$ is defined for all connected multigraphs G.
2. The range of $f_\iota(\underline{x})$ is $\{0, 1\}$. This follows from the fact that each canonical factor of f_ι takes on a value of 0 or 1, due to Fermat's Little Theorem [9]: If p is prime, then $a^p = a$ for all $a \in \mathbb{Z}_p$.
3. $f_0(\underline{x}') = 1 \Rightarrow \underline{x}'$ is a $\mathbb{Z}_p$-magic labeling of G with magic value 0.
4. $f_1(\underline{x}') = 1 \Rightarrow \underline{x}'$ is a $\mathbb{Z}_p$-magic labeling of G with magic value 1.
5. $f_0(\underline{x}') = 0$ and $f_1(\underline{x}') = 0 \Rightarrow \underline{x}'$ is not a $\mathbb{Z}_p$-magic labeling of G with magic value 0 or 1.
6. $f_0(\underline{x}') = 1 \Rightarrow f_1(\underline{x}') = 0$. If $f_0(\underline{x}') = 1$, then $\underline{x}'$ is a $\mathbb{Z}_p$-magic labeling of G with magic value 0. Thus, $\underline{x}'$ is not a $\mathbb{Z}_p$-magic labeling of G with magic value 1.
7. $f_1(\underline{x}') = 1 \Rightarrow f_0(\underline{x}') = 0$. This is the contrapositive of Observation 6.

Two techniques are often used to establish results in graph labeling problems. Either a construction of a desired labeling is obtained through ingenuity, or one shows the non-existence of the labeling (via proof by contradiction). In practice, these methods can be time-consuming and difficult to use.

In [23], the Combinatorial Nullstellensatz and Hartke polynomials were used to prove that certain graphs were $\mathbb{Z}_p$-magic (prime $p \geq 3$), without having to construct an actual $\mathbb{Z}_p$-magic labeling. As far as the authors know, it was the first time that a nonconstructive method was used to analyze integer-magic graph labelings. The focus of this paper is to use the Combinatorial Nullstellensatz to construct $\mathbb{Z}_p$-magic graphs, for prime $p \geq 3$. In particular, we construct *Hartke* $\mathbb{Z}_p$-magic graphs.

Definition 4.3. Let $p \geq 3$ be prime. A graph G is called *Hartke* $\mathbb{Z}_p$-magic if Theorem 3.1 can be used on a Hartke polynomial f_ι of G to prove that G is $\mathbb{Z}_p$-magic. In this case, a nonvanishing monomial term M of degree $(p-1) \cdot |V(G)|$ in the expansion of f_ι (where Theorem 3.1 is applied in such a manner) is called a *Hartke term*.

Example 4.4. Let $p = 3$ and G_7 be the graph illustrated in Figure 1. Note that G_7 is the graph F4 in [28]. Using Mathematica 12, we see that $\deg(f_0(\underline{x})) = 16$ and that $f_0(\underline{x})$ contains the monomial term $14336 x_5 x_6 \cdots x_{20} \equiv 2 x_5 x_6 \cdots x_{20} \pmod 3$. Let $S_i = \{1, 2\}$ for $i = 5, 6, \ldots, 20$, and $S_i = \{1\}$ for $i = 1, 2, 3$ and 4. So by Theorem 3.1, we have that $f_0(\underline{x}') \neq 0$, for some $\underline{x}' \in S_1 \times S_2 \times \cdots \times S_{20}$. Thus, $f_0(\underline{x}') = 1$ and we conclude that G_7 is a Hartke $\mathbb{Z}_3$-magic graph with magic value 0.

Proposition 4.5. *Let $p \geq 3$ be prime and G be a graph with Hartke polynomial f_ι. Then, G is Hartke $\mathbb{Z}_p$-magic with magic value $t \iff f_\iota$ has a nonvanishing monomial term M of degree $(p-1) \cdot |V(G)|$, where all the exponents t_i satisfy $0 \leq t_i \leq p-2$.*

Proof. ($\Longrightarrow$). Suppose that G is a Hartke $\mathbb{Z}_p$-magic graph with Hartke polynomial f_ι. Then, there exists a Hartke term M of degree $(p-1) \cdot |V(G)|$ in f_ι. Since G is a Hartke $\mathbb{Z}_p$-magic graph, there exist nonempty subsets $S_1, S_2, \ldots, S_m$ of $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\}$ corresponding to the m variables in f_ι, which satisfy the hypothesis of Theorem 3.1 (when applied to M). In particular, all of the exponents t_i of M satisfy $0 \leq t_i \leq p-2$.

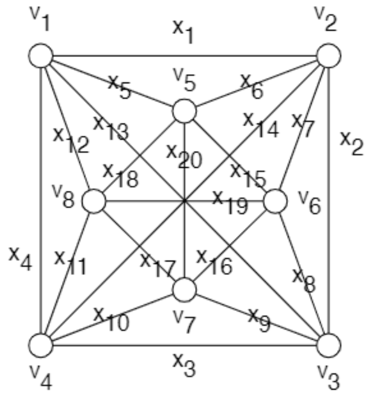

Figure 1: G_7 has a $\mathbb{Z}_3$-magic labeling with magic value 0.

($\Longleftarrow$). Suppose that f_t has a nonvanishing monomial term M of degree $(p-1)\cdot|V(G)|$, where all of the exponents t_i satisfy $0 \leq t_i \leq p-2$. For each exponent t_i (associated with variable x_i) appearing in M, choose a nonempty subset S_i of $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\}$ where $|S_i| \geq t_i + 1$. Thus by Theorem 3.1, G has a $\mathbb{Z}_p$-magic labeling. In particular, G is Hartke $\mathbb{Z}_p$-magic with magic value t. □

Proposition 4.6. *Let $p \geq 3$ be prime. If G is a Hartke $\mathbb{Z}_p$-magic graph, then* $|E(G)| \geq \frac{p-1}{p-2} \cdot |V(G)|$.

Proof. We prove the contrapositive. If $|E(G)| < \frac{p-1}{p-2} \cdot |V(G)|$, then a straightforward counting argument shows that every nonvanishing monomial of degree $(p-1)\cdot|V(G)|$ in f_t has an exponent $t_i \geq p-1$. Thus by Proposition 4.5, G is not a Hartke $\mathbb{Z}_p$-magic graph. □

Remark 4.7. The converse of Proposition 4.6 is not true, in general. For example, let $p = 3$ and consider the graph G comprised of P_3 with K_6 attached at an end-vertex. Then, G has 8 vertices and 17 edges. Thus, the inequality $|E(G)| \geq \frac{p-1}{p-2} \cdot |V(G)|$ is satisfied. However, G is not $\mathbb{Z}_3$-magic since P_3 is not $\mathbb{Z}_3$-magic; hence, G is not Hartke $\mathbb{Z}_3$-magic.

Theorem 4.8. *Suppose $p \geq 3$ is prime and G is a graph. Let $\mathcal{M}_0$ and $\mathcal{M}_1$ denote the sets of nonvanishing monomial terms of degree $(p-1)\cdot|V(G)|$, of $f_0(\underline{x})$ and $f_1(\underline{x})$, respectively. Then, $\mathcal{M}_0 = \mathcal{M}_1$.*

Proof. Let $M \in \mathcal{M}_0$. For each vertex $v \in V(G)$, let b_v denote the sum inside the corresponding canonical factor in Equation (4.1). Observe that every term in the expansion of $(0-b_v)^{p-1} = b_v^{p-1}$ is of degree $p-1$. Thus in the expansion of $f_0(\underline{x})$, M arises from the product of terms $\Pi_{v\in V(G)} b_v^{p-1}$. More specifically, M is equal to a product consisting of one term from each b_v^{p-1}.

Now, let us examine $f_1(\underline{x})$ carefully. First, we note that every term in the expansion of $(1-b_v)^{p-1}$ is of the form $\_1^k b_v^{p-1-k}$, where $0 \leq k \leq p-1$. In the expansion of the $f_1(\underline{x})$, every nonvanishing monomial term of degree $(p-1)\cdot|V(G)|$ will arise from a product of terms, one from each of the b_v^{p-1}. Monomial terms of $f_1(\underline{x})$ which do not arise in this manner have degree at most $(p-1)\cdot(|V(G)|-1)+(p-2) = (p-1)\cdot|V(G)|-1$. In particular, $M \in \mathcal{M}_1$.

This argument is reversible. Thus, the claim is established. □

Corollary 4.9. *Let $p \geq 3$ be prime. Then, G is a Hartke $\mathbb{Z}_p$-magic graph with magic value $0 \iff G$ is a Hartke $\mathbb{Z}_p$-magic graph with magic value 1.*

Proof. This follows immediately from Theorems 3.1 and 4.8. □

Example 4.10. This example illustrates the proof of Corollary 4.9. Let $p = 3$. Consider the graph G_7 (from Example 4.4) illustrated in Figure 1. In that example, the monomial term $14336x_5x_6\cdots x_{20} \equiv 2x_5x_6\cdots x_{20} \pmod 3$ of degree $8\cdot 2 = 16$ was found in $f_0(\underline{x})$. This was then used to show that G_7 is a Hartke $\mathbb{Z}_3$-magic graph with magic value 0. By Theorem 4.8, $f_1(\underline{x})$ must also contain this particular Hartke term. This is easily verified by using Mathematica 12. Hence, we conclude that G_7 is a Hartke $\mathbb{Z}_3$-magic graph with magic value 1.

5 Constructing $\mathbb{Z}_p$-magic graphs

Definition 5.1. Let $p \geq 2$ be prime, $t \in \{0, 1\}$ and G have a $\mathbb{Z}_p$-magic labeling with magic value t. Then, G is called an *edge-stable* $\mathbb{Z}_p$-magic graph if the addition of any number of simple edges, multiedges and/or loops to G results in a $\mathbb{Z}_p$-magic graph with magic value t.

Example 5.2. The 1-vertex loop graph is an edge-stable $\mathbb{Z}_p$-magic graph. In [13], it was shown that $\mathrm{IM}(K_4 - \{e\}) = \{4, 6, 8, \dots\}$. Thus, C_4 is $\mathbb{Z}_p$-magic but not edge-stable, for all primes p.

Theorem 5.3. *Let $p \geq 3$ be prime. Adding simple edges, multiedges and/or loops to a Hartke $\mathbb{Z}_p$-magic graph results in a new Hartke $\mathbb{Z}_p$-magic graph. In particular, every Hartke $\mathbb{Z}_p$-magic graph is edge-stable.*

Proof. Suppose that G is a Hartke $\mathbb{Z}_p$-magic graph with Hartke polynomial f_t. Let G^* (with Hartke polynomial f_t^*) be obtained by adding simple edges, multiedges and/or loops to G. First, note that $\deg(f_t) = \deg(f_t^*)$. Since G is Hartke $\mathbb{Z}_p$-magic, there exists a Hartke term M in f_t. By Proposition 4.5, all of the exponents t_i of M satisfy $0 \leq t_i \leq p-2$. Furthermore, M also appears in the expansion of f_t^*. By Proposition 4.5, G^* is a Hartke $\mathbb{Z}_p$-magic graph with magic value t. □

Example 5.4. Let $p = 5$ and G_5 be the first graph illustrated in Figure 2. Then, $f_1(\underline{x}) \in \mathbb{Z}_5[x_1, x_2, \dots, x_8]$, where

$$\begin{aligned} f_1(\underline{x}) = {} & [1-(1-(x_1+x_3))^4]\cdot[1-(1-(x_1+x_2+x_6+x_7))^4]\cdot \\ & [1-(1-(x_2+x_8))^4]\cdot[1-(1-(x_3+x_4))^4]\cdot \\ & [1-(1-(x_4+x_5+x_7+x_8))^4]\cdot[1-(1-(x_5+x_6))^4]. \end{aligned}$$

Using Mathematica 12, we see that $\deg(f_1(\underline{x})) = 24$ and that $f_1(\underline{x})$ contains the monomial term $1069056x_1^3x_2^3\cdots x_8^3 \equiv x_1^3x_2^3\cdots x_8^3 \pmod 5$. Let $S_i = \{1, 2, 3, 4\}$, for $i = 1, 2, \dots, 8$. By Theorem 3.1, we have that $f_1(\underline{x}') \neq 0$, for some $\underline{x}' \in S_1 \times S_2 \times \cdots \times S_8$. Thus, $f_1(\underline{x}') = 1$ and we conclude that G_5 is a Hartke $\mathbb{Z}_5$-magic graph with magic value 1.

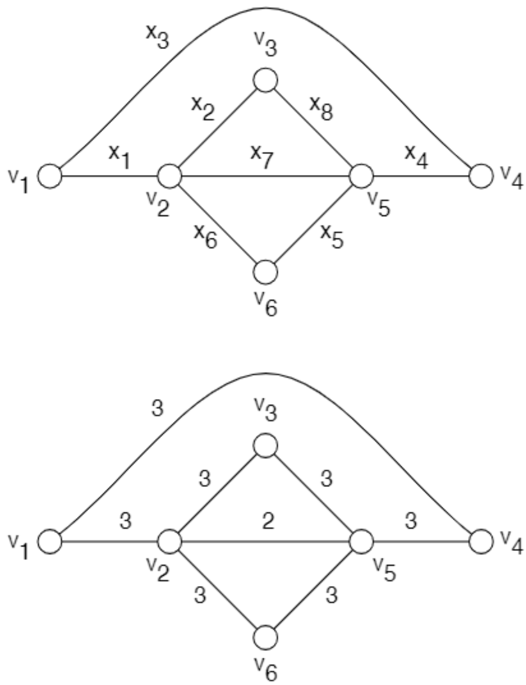

Figure 2: A $\mathbb{Z}_5$-magic labeling of G_5 with magic value 1.

With some considerable effort (by hand), one can obtain a $\mathbb{Z}_5$-magic labeling of G_5 with magic value 1, as illustrated in the second graph of Figure 2.

Suppose we add a loop or an additional edge to G_5. Then, there exist $\mathbb{Z}_5$-magic labelings with magic value 1, for these new graphs. Figures 3, 4 and 5 illustrate Theorem 5.3.

Remark 5.5. Theorem 5.3 says that any graph which contains a Hartke $\mathbb{Z}_p$-magic graph as a spanning subgraph is also a Hartke $\mathbb{Z}_p$-magic graph. Note that the converse of Theorem 5.3 is not necessarily true. The 1-vertex loop graph in Example 5.2 illustrates an edge-stable $\mathbb{Z}_p$-magic graph which is not Hartke $\mathbb{Z}_p$-magic.

We now establish some lower bounds for the number of $\mathbb{Z}_p$-magic labelings with magic value t for a given graph. Symmetry is ignored when counting these labelings. For example, if one labeling can be obtained by "rotating" another labeling, then these two labelings are counted separately.

Theorem 5.6. *Let $p \geq 3$ be prime and G be a Hartke $\mathbb{Z}_p$-magic graph. Suppose that G^* is obtained by adding z simple edges, multiedges and/or loops to G. Then, the number of different (ignoring symmetry) $\mathbb{Z}_p$-magic labelings of G^* (with magic value t) is greater than or equal to $(p-1)^z$.*

Proof. Let $G, G^*, f_\iota, f_\iota^*$ and M be defined as in the proof of Theorem 5.3. There, we saw that M is also a term in the expansion of f_ι^*. The variables in f_ι^* corresponding to the

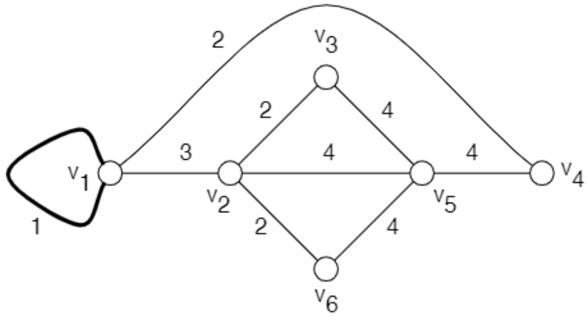

Figure 3: A $\mathbb{Z}_5$-magic labeling (magic value 1) of G_5 with a loop (labeled 1) at v_1.

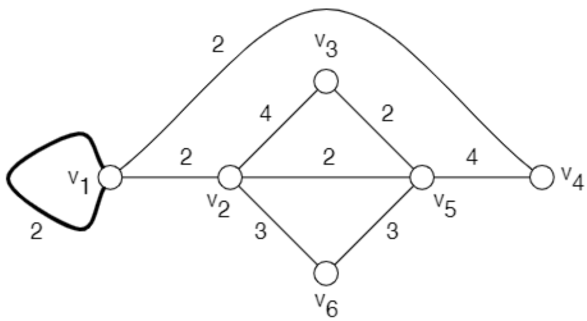

Figure 4: A $\mathbb{Z}_5$-magic labeling (magic value 1) of G_5 with a loop (labeled 2) at v_1.

additional z simple edges, multiedges, and/or loops do not appear in M. Hence we can apply Theorem 3.1 to M in f_t^*, where each of the z new variables can take on any of the $p-1$ non-zero elements from $\mathbb{Z}_p$. Thus for each $t \in \{0,1\}$, there are at least $(p-1)^z$ different $\mathbb{Z}_p$-magic labelings of G^* with magic value t. □

Example 5.7. In Example 4.4, we saw that graph G_7 (Figure 1) is a Hartke $\mathbb{Z}_3$-magic graph with magic value 0. By Theorem 4.8 and Corollary 4.9, G_7 is also a Hartke $\mathbb{Z}_3$-magic graph with magic value 1. Let G_7^* denote the graph obtained by adding loop x_{21} and multiple edge x_{22} to G_7, as illustrated in Figure 6. Since the monomial term $14336x_5x_6\cdots x_{20} \equiv 2x_5x_6\cdots x_{20} \pmod 3$ is a Hartke term in the f_t of G_7^*, there exist $\mathbb{Z}_3$-magic labelings of G_7^* (with magic value t), with x_{21} and x_{22} having labels 1 or 2. Thus, there are at least $(3-1)^2 = 4$ different $\mathbb{Z}_3$-magic labelings of G_7^* with magic value t. For this particular example, even more can be said. The Hartke term $2x_5x_6\cdots x_{20} \pmod 3$ does not involve x_1, x_2, x_3 and x_4. Each of these particular edges can be labeled with 1 or 2. Hence for each $t \in \{0,1\}$, there are at least $(3-1)^6 = 64$ different $\mathbb{Z}_3$-magic labelings of G_7^* with magic value t.

Definition 5.8. Let $p \geq 3$ be prime, G be a Hartke $\mathbb{Z}_p$-magic graph, and M be a Hartke term of f_t. Then, the *excess set* of M (denoted by $\mathcal{E}_M$) is the set of variables that have

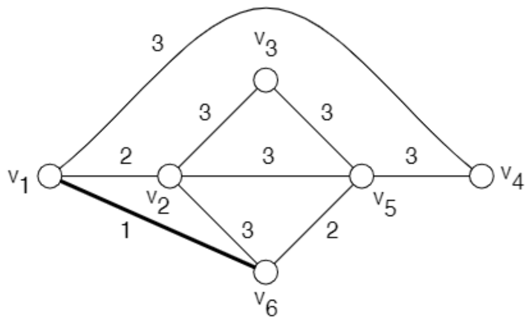

Figure 5: A $\mathbb{Z}_5$-magic labeling (magic value 1) of G_5 with edge v_1v_6 (labeled 1).

exponent zero in M.

Theorem 5.9. *Let $p \geq 3$ be prime, G be a Hartke $\mathbb{Z}_p$-magic graph, and M be a Hartke term of f_t. Then for each $t \in \{0,1\}$, G has at least $(p-1)^{|\mathcal{E}_M|}$ different $\mathbb{Z}_p$-magic labelings with magic value t. Furthermore if $G - E$ is connected, where E is any subset of edges corresponding to variables in $\mathcal{E}_M$, then $G - E$ has a $\mathbb{Z}_p$-magic labeling with magic value t.*

Proof. Suppose that $p \geq 3$ is prime, G is a Hartke $\mathbb{Z}_p$-magic graph, and M is a Hartke term of f_t. The variables in $\mathcal{E}_M$ do not appear in M. Thus, we can apply Theorem 3.1 to M in f_t, where each variable in $\mathcal{E}_M$ can take on any of the $p-1$ non-zero elements from $\mathbb{Z}_p$. Thus for each $t \in \{0,1\}$, there are at least $(p-1)^{|\mathcal{E}_M|}$ different $\mathbb{Z}_p$-magic labelings of G with magic value t. Finally, M will still be a Hartke term of the Hartke polynomials of $G - E$, where E is any subset of edges (corresponding to variables in $\mathcal{E}_M$). Hence, Theorem 3.1 can be applied to the Hartke polynomials of $G - E$ and we conclude that $G - E$ has a $\mathbb{Z}_p$-magic labeling with magic value t. □

Example 5.10. Consider the graph G_7 in Example 4.4. Then $G_7 - E$, where E is any subset of edges (corresponding to x_1, x_2, x_3, x_4), has a $\mathbb{Z}_3$-magic labeling with magic value t. This is because its associated Hartke polynomial contains the same Hartke term $14336x_5x_6 \cdots x_{20} \equiv 2x_5x_6 \cdots x_{20} \pmod 3$.

Corollary 5.11. *Let $p \geq 3$ be prime and G be a Hartke $\mathbb{Z}_p$-magic graph. Suppose that $|E(G)| \geq (p-1) \cdot |V(G)|$. Then, G has at least $(p-1)^{[|E(G)|-(p-1)\cdot|V(G)|]}$ different $\mathbb{Z}_p$-magic labelings with magic value t, for each $t \in \{0,1\}$.*

Proof. Suppose that $p \geq 3$ is prime, G is a Hartke $\mathbb{Z}_p$-magic graph, and $|E(G)| \geq (p-1) \cdot |V(G)|$. Let M be a Hartke term of f_t. Note that when $|E(G)| = \frac{p-1}{p-2} \cdot |V(G)|$, the corollary follows from Theorem 5.9. When $|E(G)| > \frac{p-1}{p-2} \cdot |V(G)|$, M includes at most $(p-1) \cdot |V(G)|$ distinct variables. Since $|E(G)| \geq (p-1) \cdot |V(G)|$, we have that $|\mathcal{E}_M| \geq |E(G)| - (p-1) \cdot |V(G)|$. By Theorem 5.9, the result follows. □

Theorem 5.12. *Let $p \geq 3$ be prime. Then, the disjoint union of Hartke $\mathbb{Z}_p$-magic graphs is a Hartke $\mathbb{Z}_p$-magic graph.*

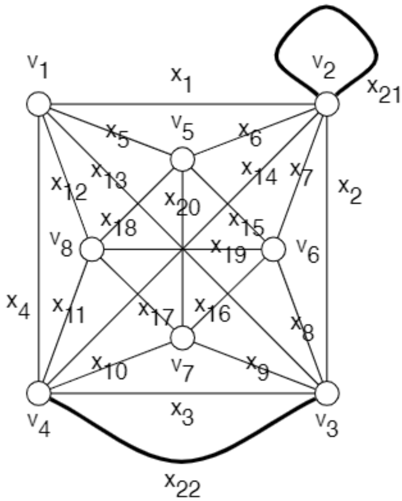

Figure 6: For each $t \in \{0, 1\}$, G_7^* has at least $(3-1)^6 = 64$ different $\mathbb{Z}_3$-magic labelings with magic value t.

Proof. It suffices to show the claim is true for the disjoint union of two Hartke $\mathbb{Z}_p$-magic graphs H_1 and H_2, with magic value $t \in \{0, 1\}$. Note that the degrees of the Hartke polynomials of H_1, H_2, and the disjoint union of H_1 and H_2 are $(p-1) \cdot |V(H_1)|$, $(p-1) \cdot |V(H_2)|$, and $(p-1) \cdot (|V(H_1)| + |V(H_2)|)$, respectively. Let M_1 and M_2 be Hartke terms in the Hartke polynomials of H_1 and H_2, respectively. Note that $M_1 \cdot M_2$ does not vanish, since the coefficients come from a field. Thus, the degree of $M_1 \cdot M_2$ is $(p-1) \cdot (|V(H_1)| + |V(H_2)|)$ and $M_1 \cdot M_2$ appears in the expansion of the Hartke polynomial of the disjoint union of H_1 and H_2. We also see that $M_1 \cdot M_2$ is a Hartke term, since M_1 and M_2 individually are Hartke terms. Therefore, the disjoint union of Hartke $\mathbb{Z}_p$-magic graphs is a Hartke $\mathbb{Z}_p$-magic graph. □

Definition 5.13. A *weak join* of graphs $H_1, H_2, \ldots, H_r$ is defined to be a connected graph with vertex set $\bigcup_{i=1}^{r} V(H_i)$ and edge set $Z \cup (\bigcup_{i=1}^{r} E(H_i))$, where Z is a set of simple and/or multiedges of the form uv with $u \in V(H_i)$ and $v \in V(H_j)$, where $i \neq j$.

Example 5.14. Figure 7 illustrates a weak join of C_6, W_6 (of order six) and P_2.

Theorem 5.15. *Let $p \geq 3$ be prime. Then, a weak join of Hartke $\mathbb{Z}_p$-magic graphs is a Hartke $\mathbb{Z}_p$-magic graph.*

Proof. Let $H_1, H_2, \ldots, H_r$ be Hartke $\mathbb{Z}_p$-magic graphs. By Theorem 5.12, the disjoint union $\bigcup_{i=1}^{r} H_i$ is a Hartke $\mathbb{Z}_p$-magic graph. Since a weak join of $H_1, H_2, \ldots, H_r$ is formed by adding simple edges and/or multiedges between the H_i in $\bigcup_{i=1}^{r} H_i$, the claim is established by Theorem 5.3. □

Example 5.16. Let $p = 3$ and G_3 be the top half of the graph in Figure 8. Then,

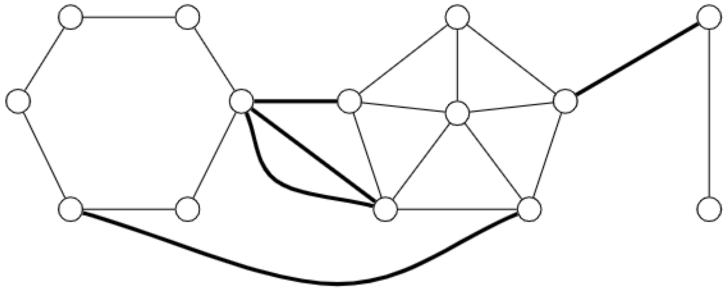

Figure 7: A weak join of C_6, the wheel graph W_6 and P_2.

$f_0(\underline{x}) \in \mathbb{Z}_3[x_1, x_2, \ldots, x_{15}]$, where

$$\begin{aligned} f_0(\underline{x}) = {} & [1-(0-(x_1+x_7+x_8+x_{14}+x_{15}))^2]\cdot[1-(0-(x_1+x_2+x_{11}+x_{12}))^2]\cdot \\ & [1-(0-(x_2+x_3+x_8+x_9))^2]\cdot[1-(0-(x_3+x_4+x_{12}+x_{13}))^2]\cdot \\ & [1-(0-(x_4+x_5+x_9+x_{10}+x_{15}))^2]\cdot[1-(0-(x_5+x_6+x_{13}+x_{14}))^2]\cdot \\ & [1-(0-(x_6+x_7+x_{10}+x_{11}))^2]. \end{aligned}$$

Using Mathematica 12, we see that $\deg(f_0(\underline{x})) = 14$ and that $f_0(\underline{x})$ contains the monomial term $-6400x_1x_2\cdots x_{11}x_{12}x_{14}x_{15} \equiv 2x_1x_2\cdots x_{11}x_{12}x_{14}x_{15} \pmod 3$. Let $S_i = \{1,2\}$, for $i = 1, 2, \ldots, 12, 14, 15$ and $S_{13} = \{1\}$. So by Theorem 3.1, we have that $f_0(\underline{x}') \neq 0$, for some $\underline{x}' \in S_1 \times S_2 \times \cdots \times S_{15}$. Thus, $f_0(\underline{x}') = 1$ and G_3 is a Hartke $\mathbb{Z}_3$-magic graph with magic value 0.

Now, let G_4 (graph G1121 from [28]) be the bottom half of the graph in Figure 8. Then, $f_0(\underline{y}) \in \mathbb{Z}_3[y_1, y_2, \ldots, y_{14}]$, where

$$\begin{aligned} f_0(\underline{y}) = {} & [1-(0-(y_1+y_5+y_6+y_9+y_{10}+y_{14}))^2]\cdot[1-(0-(y_1+y_2+y_{11}))^2]\cdot \\ & [1-(0-(y_2+y_3+y_7+y_8+y_{10}))^2]\cdot[1-(0-(y_3+y_4+y_{13}+y_{14}))^2]\cdot \\ & [1-(0-(y_4+y_5))^2]\cdot[1-(0-(y_6+y_7+y_{11}+y_{12}))^2]\cdot \\ & [1-(0-(y_8+y_9+y_{12}+y_{13}))^2]. \end{aligned}$$

Using Mathematica 12, we see that $\deg(f_0(\underline{y})) = 14$ and that $f_0(\underline{y})$ contains the monomial term $-4096y_1y_2\cdots y_{13}y_{14} \equiv 2y_1y_2\cdots y_{13}y_{14} \pmod 3$. Let $S_i = \{1,2\}$, for $i = 1, 2, \ldots, 14$. So by Theorem 3.1, we have that $f_0(\underline{y}') \neq 0$, for some $\underline{y}' \in S_1 \times S_2 \times \cdots \times S_{14}$. Thus, $f_0(\underline{y}') = 1$ and G_4 is a Hartke $\mathbb{Z}_3$-magic graph with magic value 0.

The graph G in Figure 8 is a weak join of G_3 and G_4, where z_1, z_2 and z_3 are the additional simple and multiedges used to create the weak join. Let $f_0(\underline{r}) \in \mathbb{Z}_3[r_1, r_2, \ldots, r_{32}]$, where

$$r_i = \begin{cases} x_i & \text{if } 1 \leq i \leq 15; \\ y_{i-15} & \text{if } 16 \leq i \leq 29; \\ z_{i-29} & \text{if } 30 \leq i \leq 32, \end{cases}$$

be a Hartke polynomial of G. Using Mathematica 12, we see that $\deg(f_0(\underline{r})) = 28$ and that $f_0(\underline{r})$ contains the monomial term

$$\begin{aligned}(-6400r_1r_2\cdots r_{11}r_{12}r_{14}r_{15})&\cdot(-4096r_{16}r_{17}\cdots r_{28}r_{29})\\ &\equiv(2r_1r_2\cdots r_{11}r_{12}r_{14}r_{15})\cdot(2r_{16}r_{17}\cdots r_{28}r_{29}) \pmod 3\\ &\equiv r_1r_2\cdots r_{11}r_{12}r_{14}r_{15}r_{16}r_{17}\cdots r_{28}r_{29} \pmod 3.\end{aligned}$$

Let $S_i=\{1,2\}$, for $i=1,2,\dots,12,14,15,16,17\dots28,29$ and $S_{13}=S_{30}=S_{31}=S_{32}=\{1\}$. So by Theorem 3.1, we have that $f_0(\underline{r}')\neq 0$, for some $\underline{r}'\in S_1\times S_2\times\cdots\times S_{32}$. Thus, $f_0(\underline{r}')=1$ and G is a Hartke $\mathbb{Z}_3$-magic graph with magic value 0.

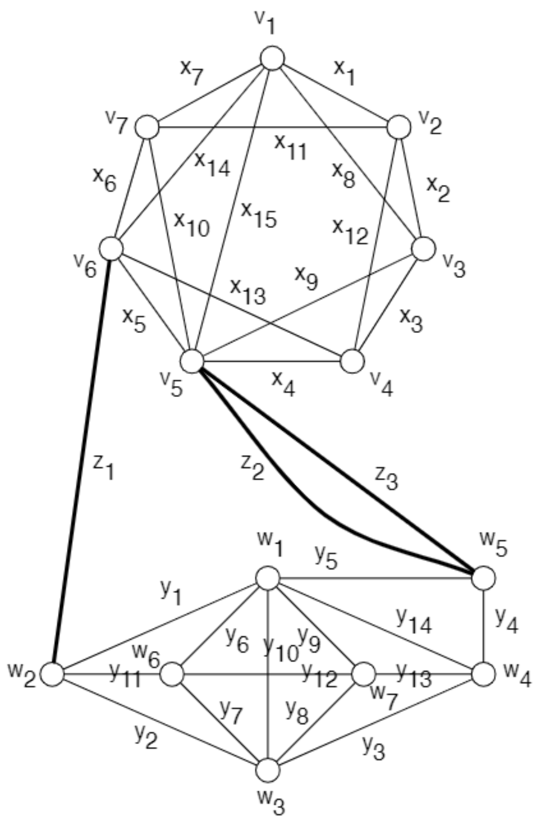

Figure 8: A weak join of Hartke $\mathbb{Z}_3$-magic graphs G_3 and G_4 is a Hartke $\mathbb{Z}_3$-magic graph.

In [22], the $\mathbb{Z}_k$-magic property was analyzed for various classical graph products. There, it was shown that if G and H are connected $\mathbb{Z}_k$-magic graphs, then the Cartesian and lexicographic products of G and H are $\mathbb{Z}_k$-magic, for $k\in\{2,3,4,\dots\}$. However, if instead we strengthen the restriction on G and weaken the restriction on H, then we obtain additional results. To this end, recall the following definitions [5].

Definition 5.17. Let G and H be connected graphs. Then, the *Cartesian* product $G\square H$ is a graph which has vertex set $V(G\square H)=\{(g,h) : g\in V(G) \text{ and } h\in V(H)\}$ and edge set $E(G\Box H)$, where two vertices (g,h) and (g',h') are adjacent if ($g=g'$ and h adj h') or ($h=h'$ and g adj g').

Definition 5.18. Let G and H be connected graphs. Then, the *lexicographic* product $G\circ H$ is a graph which has vertex set $V(G\circ H)=\{(g,h): g\in V(G) \text{ and } h\in V(H)\}$ and edge set $E(G\circ H)$, where two vertices (g,h) and (g',h') are adjacent if ($g=g'$ and h adj h') or (g adj g').

Definition 5.19. Let G and H be connected graphs. Then, the *strong* product $G\boxtimes H$ is a graph which has vertex set $V(G\boxtimes H)=\{(g,h): g\in V(G) \text{ and } h\in V(H)\}$ and edge set $E(G\boxtimes H)$, where two vertices (g,h) and (g',h') are adjacent if ($g=g'$ and h adj h') or ($h=h'$ and g adj g') or (h adj h' and g adj g').

Of these three graph products, only the lexicographic product is not commutative.

Example 5.20. Figure 9 illustrates $P_2\Box P_3$, $P_2\circ P_3$, $P_3\circ P_2$ and $P_2\boxtimes P_3$.

Corollary 5.21. *Let $p\geq 3$ be prime. Suppose that G is a Hartke $\mathbb{Z}_p$-magic graph and H is a graph. Then, the Cartesian product $G\Box H$ is a Hartke $\mathbb{Z}_p$-magic graph.*

Proof. Let T be a spanning tree of H. $G\Box T$ is obtained by replacing each vertex of T with a copy of G and replacing each edge of T with edges connecting the corresponding vertices of copies of G. Since G is a Hartke $\mathbb{Z}_p$-magic graph, $G\Box T$ is a weak join of Hartke $\mathbb{Z}_p$-magic graphs. By Theorem 5.15, $G\Box T$ is a Hartke $\mathbb{Z}_p$-magic graph. Finally, for each of the edges in H which are not in T, add edges connecting the corresponding vertices of copies of G in $G\Box T$ to obtain $G\Box H$. Since $G\Box T$ is a Hartke $\mathbb{Z}_p$-magic graph, we see that $G\Box H$ is a Hartke $\mathbb{Z}_p$-magic graph, by Theorem 5.3. □

Corollary 5.22. *Let $p\geq 3$ be prime. Suppose that G is a Hartke $\mathbb{Z}_p$-magic graph and H is a graph. Then, $G\circ H$, $H\circ G$ and $G\boxtimes H$ are Hartke $\mathbb{Z}_p$-magic graphs.*

Proof. First, note that $V(G\Box H)=V(G\circ H)=V(H\circ G)=V(G\boxtimes H)$. We also see that the edge sets of $G\circ H$, $H\circ G$, and $G\boxtimes H$ contain the edge set of $G\Box H$. Since $G\Box H$ is Hartke $\mathbb{Z}_p$-magic by Corollary 5.21, these other products are Hartke $\mathbb{Z}_p$-magic, by Theorem 5.3. □

6 Further directions and some open questions

Throughout this paper, we used the Combinatorial Nullstellensatz in the construction of Hartke $\mathbb{Z}_p$-magic graphs, for prime $p\geq 3$. Graphs of this type were found to have an edge-stability property. This was used to further construct non-trivial Hartke $\mathbb{Z}_p$-magic graphs.

It is natural for the reader to wonder if connected simple Hartke $\mathbb{Z}_p$-magic graphs exist, for all orders $n\geq 6$ and prime $p\geq 3$. The authors of this paper believe that this is true.

Conjecture 6.1. *Let $p\geq 3$ be prime. Then, there exists a connected simple Hartke $\mathbb{Z}_p$-magic graph G, for all $|V(G)|\geq 6$.*

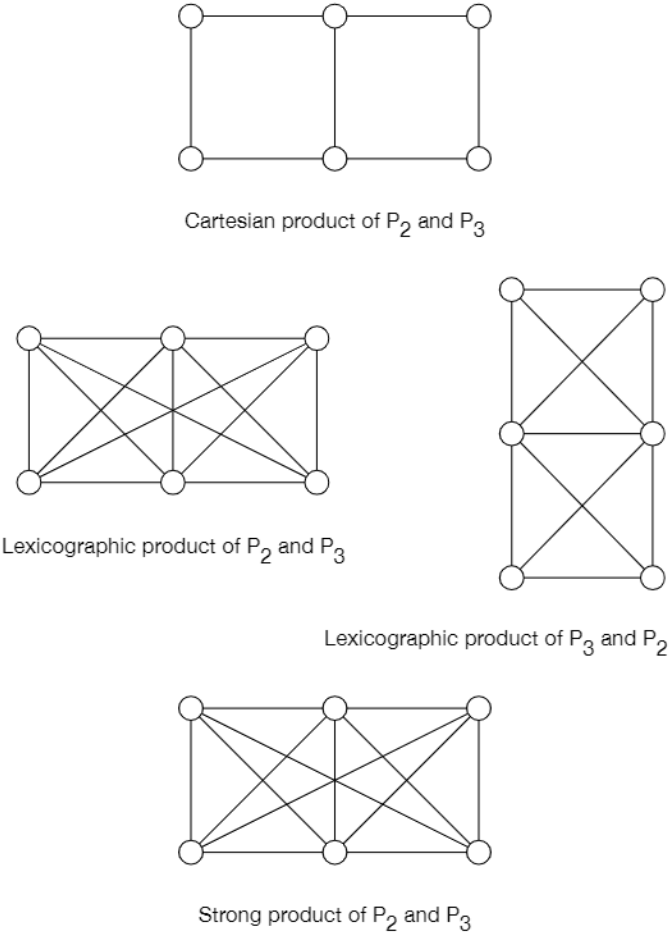

Figure 9: $P_2 \square P_3$, $P_2 \circ P_3$, $P_3 \circ P_2$ and $P_2 \boxtimes P_3$.

The Combinatorial Nullstellensatz can be generalized in different ways. Theorem 3.1 is true over integral domains. The Generalized Combinatorial Nullstellensatz [35] sharpens Theorem 3.1; instead of analyzing a monomial with degree $= \deg(f)$, it suffices to consider a monomial that does not divide any other monomial term in f. In [26], Michalek remarks that the Combinatorial Nullstellensatz is true over any commutative ring R with unity, as long as $a - b$ is not a zero divisor in R, for any $a, b \in S_i$ $(i = 1, 2, \ldots, m)$. Can any of these generalizations of the Combinatorial Nullstellensatz help us in analyzing the $\mathbb{Z}_p$-magic graph labeling problem (prime $p \geq 3$)?

Here are some other questions one might consider.

1. Are there natural classes of Hartke $\mathbb{Z}_p$-magic graphs?
2. Let $p \geq 3$ be prime and G be a connected simple graph satisfying $|E(G)| \geq \frac{p-1}{p-2} \cdot |V(G)|$. What is the probability that G is a Hartke $\mathbb{Z}_p$-magic graph?

3. Are there other types of graph labeling problems where the Combinatorial Nullstellensatz or its various generalizations can be used?

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