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Abstract

In this paper we initiate the study of broadcast dimension, a variant of metric dimension. Let G be a graph with vertex set $V(G)$, and let $d(u, w)$ denote the length of a $u - w$ geodesic in G. For $k \geq 1$, let $d_k(x, y) = \min\{d(x, y), k+1\}$. A function $f : V(G) \to \mathbb{Z}^+ \cup \{0\}$ is called a resolving broadcast of G if, for any distinct $x, y \in V(G)$, there exists a vertex $z \in V(G)$ such that $f(z) = i > 0$ and $d_i(x, z) \neq d_i(y, z)$. The broadcast dimension, $\text{bdim}(G)$, of G is the minimum of $bc_f(G) = \sum_{v \in V(G)} f(v)$ over all resolving broadcasts of G, where $bc_f(G)$ can be viewed as the total cost of the transmitters (of various strength) used in resolving the entire network described by the graph G. Note that $\text{bdim}(G)$ reduces to $\text{adim}(G)$ (the adjacency dimension of G, introduced by Jannesari and Omoomi in 2012) if the codomain of resolving broadcasts is restricted to $\{0, 1\}$. We determine its value for cycles, paths, and other families of graphs. We prove that $\text{bdim}(G) = \Omega(\log n)$ for all graphs G of order n, and that the result is sharp up to a constant factor. We show that $\frac{\text{adim}(G)}{\text{bdim}(G)}$ and $\frac{\text{bdim}(G)}{\dim(G)}$ can both be arbitrarily large, where $\dim(G)$ denotes the metric dimension of G. We also examine the effect of vertex deletion on both the adjacency dimension and the broadcast dimension of graphs.

1 Introduction

Let G be a finite, simple, and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The *distance* between two vertices $x, y \in V(G)$, denoted by $d(x, y)$, is the length of a shortest path between x and y in G; if x and y belong to different components of G, we define $d(x, y) = \infty$. Metric dimension, introduced by Slater [33] and by Harary and Melter [20], is a graph parameter that has been studied extensively.

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A vertex $z \in V(G)$ *resolves* a pair of vertices $x, y \in V(G)$ if $d(x, z) \neq d(y, z)$. A set $S \subseteq V(G)$ is a *resolving set* of G if, for any distinct $x, y \in V(G)$, there exists $z \in S$ such that $d(x, z) \neq d(y, z)$. The *metric dimension* of G, denoted by $\dim(G)$, is the minimum cardinality over all resolving sets of G. Khuller et al. [28] considered robot navigation as one of the applications of metric dimension, where a robot that moves from node to node knows its distances to landmarks that are placed on the vertices of the resolving set. Metric dimension minimizes the size of a set of landmarks, so that the robot is able to determine its location in the graph by computing its distances to the landmarks, no matter where it is located. A number of variants of metric dimension have also been investigated; see for example [12, 17, 27, 36, 38, 39].

For $x \in V(G)$ and $S \subseteq V(G)$, let $d(x, S) = \min\{d(x, y) : y \in S\}$. Meir and Moon [29] introduced distance-k domination. For any positive integer k, a set $D \subseteq V(G)$ is called a *distance-k dominating set* of G if, for each $u \in V(G) - D$, $d(u, D) \leq k$. The *distance-k domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality over all distance-k dominating sets of G; the distance-1 domination number is the well-known domination number. Erwin [9, 10] introduced the concept of broadcast domination, where cities with broadcast stations have transmission power that enable them to broadcast messages to cities at distances greater than one, depending on the transmission power of broadcast stations. More explicitly, following [9, 10], a function $f : V(G) \to \{0, 1, 2, \ldots, \text{diam}(G)\}$ is called a *dominating broadcast* of G if, for each vertex $x \in V(G)$, there exists a vertex $y \in V(G)$ such that $f(y) > 0$ and $d(x, y) \leq f(y)$. The *broadcast (domination) number*, $\gamma_b(G)$, of G is the minimum of $D_f(G) := \sum_{v \in V(G)} f(v)$ over all dominating broadcasts f of G; here, $D_f(G)$ can be viewed as the total cost of the transmitters used to achieve full coverage of a network of cities described via the graph G being considered. Note that $\gamma_b(G)$ reduces to $k \cdot \gamma_k(G)$ if the codomain of dominating broadcasts is restricted to $\{0, k\}$.

Jannesari and Omoomi [25] introduced the *adjacency dimension* of G, denoted by $\text{adim}(G)$, as a tool to study the metric dimension of the lexicographic product graphs; they defined the adjacency distance between two vertices $x, y \in V(G)$ to be $0, 1, 2$, respectively, if $d(x, y) = 0$, $d(x, y) = 1$, and $d(x, y) \geq 2$. Adjacency resolving sets and adjacency dimension are defined analogously in [25]. Assuming that a landmark that can detect long distance can be costly, the authors of [25] considered a robot that detects its position only from landmarks adjacent to it; this can be viewed as combining the concept of a resolving set and a dominating set. More generally, we can apply the concept of a distance-k dominating set to a resolving set. If a robot can detect up to distance $k > 0$ from each landmark, the minimum number of such landmarks to determine the robot's position on the graph is called the distance-k dimension of G, denoted by $\dim_k(G)$; note that $\dim_1(G) = \text{adim}(G)$. For articles on the distance-k dimension of graphs, see [1, 11, 13, 15, 19, 35].

Now, we apply the concept of a dominating broadcast to a resolving set. For any nonnegative integer k and for $x, y \in V(G)$, let $d_k(x, y) = \min\{d(x, y), k + 1\}$. Let $f : V(G) \to \mathbb{Z}^+ \cup \{0\}$ be a function. We define $\text{supp}_G(f) = \{v \in V(G) : f(v) > 0\}$. We say that f is a *resolving broadcast* of G if, for any distinct $x, y \in V(G)$, there exists a vertex $z \in \mathrm{supp}_G(f)$ such that $d_{f(z)}(x, z) \neq d_{f(z)}(y, z)$. The *broadcast dimension* of G, denoted by $\mathrm{bdim}(G)$, is the minimum of $bc_f(G) = \sum_{v \in V(G)} f(v)$ over all resolving broadcasts f of G, where $bc_f(G)$ can be viewed as the total cost of the transmitters (of various strength) used in resolving the entire network described via the graph G being considered. Like metric dimension and adjacency dimension, broadcast dimension also has a natural application to robot navigation. Again the robot uses its distances to the landmarks to determine its location, but in the case of broadcast dimension, the landmarks may have different strengths. If the robot is too far from a landmark, then the robot gets no signal from that landmark. If we assume that the cost of the landmarks increases linearly with their strength, then broadcast dimension minimizes the total cost of a set of landmarks (of possibly different strengths) that the robot can use to determine its location in the graph, no matter where it is located. Whereas the existing variants of metric dimension require the cost to be the same for all of the landmarks, this new variant offers an improvement by allowing the landmark costs to vary, which allows landmark configurations that are less expensive overall.

Note that, if the codomain of resolving broadcasts is restricted to $\{0, k\}$, where k is a positive integer, then $\mathrm{bdim}(G)$ reduces to $k \cdot \dim_k(G)$. For an ordered set $S = \{u_1, u_2, \ldots, u_k\} \subseteq V(G)$ of distinct vertices, the metric code, the adjacency code, and the broadcast code, respectively, of $v \in V(G)$ with respect to S are the k-vectors $r_S(v) = (d(v, u_1), d(v, u_2), \ldots, d(v, u_k))$, $a_S(v) = (d_1(v, u_1), d_1(v, u_2), \ldots, d_1(v, u_k))$, and $b_S(v) = (d_{i_1}(v, u_1), d_{i_2}(v, u_2), \ldots, d_{i_k}(v, u_k))$, where $f(u_j) = i_j > 0$ for a resolving broadcast f being considered.

Suppose $f(x)$ and $g(x)$ are two functions defined on some subset of real numbers. We write $f(x) = O(g(x))$ if there exist positive constants N and C such that $|f(x)| \leq C|g(x)|$ for all $x > N$, $f(x) = \Omega(g(x))$ if $g(x) = O(f(x))$, and $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$.

In this paper, we initiate the study of broadcast dimension. In Section 2, we discuss some general results on the metric dimension, the adjacency dimension, and the broadcast dimension of graphs. For example, it is easy to see that for any graph G, $\dim(G) \leq \mathrm{bdim}(G) \leq \mathrm{adim}(G)$. We also find the broadcast dimension of paths and cycles. In Section 3, we prove that $\mathrm{bdim}(G) = \Omega(\log n)$ for all graphs G of order n, and that the result is sharp up to a constant factor. We also characterize the family of graphs of adjacency dimension k for each k. In Section 4, we characterize all graphs G for which $\mathrm{bdim}(G)$ equals 1, 2, and $|V(G)| - 1$. It is noteworthy that $\mathrm{bdim}(G) = 2$ ($\mathrm{adim}(G) = 2$, respectively) implies that G is planar, whereas an example of non-planar graph G with $\dim(G) = 2$ was given in [28]. In Section 5, we provide graphs G such that both $\mathrm{adim}(G) - \mathrm{bdim}(G)$ and $\mathrm{bdim}(G) - \dim(G)$ can be arbitrarily large. We also show that, for two connected graphs G and H with $H \subset G$, $\dim(H) - \dim(G)$ ($\mathrm{bdim}(H) - \mathrm{bdim}(G)$ and $\mathrm{adim}(H) - \mathrm{adim}(G)$, respectively) can be arbitrarily large. In addition, we find all trees T such that $\mathrm{bdim}(T) = \dim(T)$. In Section 6, we examine the effect of vertex deletion on adjacency dimension and broadcast dimension. We also investigate the effect of edge deletion on adjacency dimension. In Section 7, we conclude with some open problems.

We conclude the introduction with some terminology and notation that we will use throughout the paper. The *diameter*, $\mathrm{diam}(G)$, of G is $\max\{d(x,y) : x, y \in V(G)\}$. The *open neighborhood* of a vertex $v \in V(G)$ is $N(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex u in G, denoted by $\deg(u)$, is $|N(u)|$. An *end vertex* is a vertex of degree one, and a *major vertex* is a vertex of degree at least three. The *join* of any two graphs H_1 and H_2, denoted by $H_1 + H_2$, is the graph obtained from the disjoint union of H_1 and H_2 by joining every vertex of H_1 with every vertex of H_2. Let P_n, C_n, and K_n respectively denote the path, cycle, and complete graph of order n, and let $K_{m,n}$ denote the complete bipartite graph with parts of size m and n. We denote by $\mathbf{1}_\alpha$ and $\mathbf{2}_\alpha$, respectively, the α-vector with 1 on each entry and the α-vector with 2 on each entry.

2 General results

In this section, we discuss some general results on the metric dimension, the adjacency dimension, and the broadcast dimension of graphs. We also determine the broadcast dimension of paths and cycles. For distinct $u, w \in V(G)$, if $N(u) - \{w\} = N(w) - \{u\}$, then u and w are called *twin vertices* of G.

Observation 2.1. *Let u and w be twin vertices of a graph G. Then,*

(a) [23] *for any resolving set S of G, $S \cap \{u, w\} \neq \emptyset$;*

(b) [25] *for any adjacency resolving set A of G, $A \cap \{u, w\} \neq \emptyset$;*

(c) for any resolving broadcast f of G, either $f(u) > 0$ or $f(w) > 0$.

Proposition 2.2. [25]

(a) If G is a connected graph, then $\mathrm{adim}(G) \geq \dim(G)$.

(b) If G is a connected graph with $\mathrm{diam}(G) = 2$, *then* $\mathrm{adim}(G) = \dim(G)$. *Moreover, there exists a graph G such that* $\mathrm{adim}(G) = \dim(G)$ *and* $\mathrm{diam}(G) > 2$.

(c) For every graph G, $\mathrm{adim}(G) = \mathrm{adim}(\overline{G})$, *where $\overline{G}$ denotes the complement of G.*

Observation 2.3.

(a) For any graph G of order $n \geq 2$,

$$1 \leq \dim(G) \leq \mathrm{bdim}(G) \leq \mathrm{adim}(G) \leq n - 1.$$

(b) For any graph G with $\mathrm{diam}(G) \in \{1, 2\}$, $\dim(G) = \mathrm{bdim}(G) = \mathrm{adim}(G)$.

Next, we consider graphs G with $\mathrm{diam}(G) \leq 2$. For any graphs H_1 and H_2, $\mathrm{diam}(H_1 + H_2) \leq 2$; thus, $\dim(H_1 + H_2) = \mathrm{bdim}(H_1 + H_2) = \mathrm{adim}(H_1 + H_2)$ by Observation 2.3(b).

Theorem 2.4. [3, 32] *For* $n \geq 3$*,*

$$\dim(C_n + K_1) = \begin{cases} 3 & \text{if } n \in \{3, 6\}, \\ \lfloor \frac{2n+2}{5} \rfloor & \text{otherwise.} \end{cases}$$

Theorem 2.5. [4] *For* $n \geq 1$*,*

$$\dim(P_n + K_1) = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n \in \{2, 3\}, \\ 3 & \text{if } n = 6, \\ \lfloor \frac{2n+2}{5} \rfloor & \text{otherwise.} \end{cases}$$

Proposition 2.2(b), along with Theorems 2.4 and 2.5, implies the following proposition.

Proposition 2.6. [25] *For* $n \geq 7$*, if* $G \in \{P_n, C_n\}$*, then* $\text{adim}(G + K_1) = \lfloor \frac{2n+2}{5} \rfloor$.

As an immediate consequence of Observation 2.3(b) and Theorems 2.4 and 2.5, we have the following corollary.

Corollary 2.7. *For* $n \geq 3$*, let* $G \in \{P_n, C_n\}$*. Then,*

$$\text{bdim}(G + K_1) = \begin{cases} 2 & \text{if } n = 3 \text{ and } G = P_3, \\ 3 & \text{if } n = 3 \text{ and } G = C_3, \\ 3 & \text{if } n = 6, \\ \lfloor \frac{2n+2}{5} \rfloor & \text{otherwise.} \end{cases}$$

The metric dimension and the adjacency dimension of any complete k-partite graph were determined in [31] and [25], respectively.

Theorem 2.8. [25, 31] *For* $k \geq 2$*, let* $G = K_{a_1, a_2, \ldots, a_k}$ *be any complete* k*-partite graph of order* $n = \sum_{i=1}^{k} a_i$*. Let* s *be the number of partite sets of* G *consisting of exactly one element. Then,*

$$\dim(G) = \text{adim}(G) = \begin{cases} n - k & \text{if } s = 0, \\ n + s - k - 1 & \text{if } s \neq 0. \end{cases}$$

As an immediate consequence of Observation 2.3(b) and Theorem 2.8, we have the following corollary.

Corollary 2.9. *For* $k \geq 2$*, let* $G = K_{a_1, a_2, \ldots, a_k}$ *be any complete* k*-partite graph of order* $n = \sum_{i=1}^{k} a_i$*. Let* s *be the number of partite sets of* G *consisting of exactly one element. Then,*

$$\text{bdim}(G) = \begin{cases} n - k & \text{if } s = 0, \\ n + s - k - 1 & \text{if } s \neq 0. \end{cases}$$

Now, we recall the metric dimension of the Petersen graph.

Theorem 2.10. [26] *For the Petersen graph $\mathcal{P}$,* $\dim(\mathcal{P}) = 3$.

Since $\operatorname{diam}(\mathcal{P}) = 2$, Observation 2.3(b) and Theorem 2.10 imply the following corollary.

Corollary 2.11. *For the Petersen graph $\mathcal{P}$,* $\operatorname{bdim}(\mathcal{P}) = \operatorname{adim}(\mathcal{P}) = 3$.

Next, we consider paths and cycles.

Proposition 2.12. [25] *For $n \geq 4$,* $\operatorname{adim}(P_n) = \operatorname{adim}(C_n) = \lfloor \frac{2n+2}{5} \rfloor$.

Theorem 2.13. *For $n \geq 4$,* $\operatorname{bdim}(P_n) = \operatorname{bdim}(C_n) = \lfloor \frac{2n+2}{5} \rfloor$

Proof. Let G be P_n or C_n, with vertices $v_0, \ldots, v_{n-1}$ in order, where $n \geq 4$. By Observation 2.3(a) and Proposition 2.12, $\operatorname{bdim}(G) \leq \lfloor \frac{2n+2}{5} \rfloor$ for $n \geq 4$. Thus, it suffices to prove that $\sum_{v \in V(G)} f(v)$ is minimized when $f(v) \leq 1$ for all $v \in V(G)$.

Suppose that f is a resolving broadcast that achieves $\operatorname{bdim}(G)$. If $f(v) \leq 1$ for all $v \in V(G)$, then we are done. Otherwise, we recursively modify f to obtain a new resolving broadcast f' for which $\sum_{v \in V(G)} f'(v) \leq \sum_{v \in V(G)} f(v)$ and $f'(v) \leq 1$ for all $v \in V(G)$.

Start by defining f_0 such that $f_0(v) = f(v)$ for all $v \in V(G)$. Given f_i, let v_j be any vertex in $V(G)$ such that $f_i(v_j) > 1$. If $f_i(v_j) = 2$, then we define $f_{i+1}(v_{(j-1) \bmod n}) = f_{i+1}(v_{(j+1) \bmod n}) = 1$ and $f_{i+1}(v_j) = 0$, unless v_j is an end vertex of P_n. If $G = P_n$ and $f_i(v_0) = 2$, then we define $f_{i+1}(v_0) = 1$ and $f_{i+1}(v_1) = 1$. If $G = P_n$ and $f_i(v_{n-1}) = 2$, then we define $f_{i+1}(v_{n-1}) = 1$ and $f_{i+1}(v_{n-2}) = 1$.

Otherwise if $f_i(v_j) = x > 2$, then we define $f_{i+1}(v_j) = x - 2$ and

$$f_{i+1}(v_{(j-x+1) \bmod n}) = f_{i+1}(v_{(j+x-1) \bmod n}) = 1.$$

If any vertices are assigned multiple values for f_{i+1}, only the maximum value is used. If any vertex v is assigned no values for f_{i+1}, then $f_{i+1}(v) = f_i(v)$. By construction, if f_i is a resolving broadcast, then so is f_{i+1}.

The process will end in finitely many steps, so suppose that k is an integer such that $f_k(v) \leq 1$ for all $v \in V(G)$. Then, we let $f' = f_k$, and $\sum_{v \in V(G)} f'(v) \leq \sum_{v \in V(G)} f(v)$ by construction. Since $f = f_0$ is a resolving broadcast, f_i is a resolving broadcast for each $0 \leq i \leq k$ by induction, and so in particular $f' = f_k$ is a resolving broadcast. Thus, $\operatorname{bdim}(P_n) = \operatorname{adim}(P_n) = \operatorname{bdim}(C_n) = \operatorname{adim}(C_n) = \lfloor \frac{2n+2}{5} \rfloor$. □

3 Extremal bounds and characterization

In this section, we prove that $\operatorname{bdim}(G) = \Omega(\log n)$ for all graphs G of order n, and that the result is sharp up to a constant factor. We also obtain bounds for the clique number and maximum degree of graphs with adjacency dimension k or broadcast dimension k. Furthermore, we characterize the family of graphs of adjacency dimension k. First, we recall some known bounds for the metric dimension of graphs.

Hernando et al. [23] proved the next theorem, which improves on an earlier result of [6].

Theorem 3.1. [23] *Let G be a connected graph of order n, diameter d, and* $\dim(G) = k$. *Then*

$$n \leq \left(\left\lfloor \frac{2d}{3} \right\rfloor + 1 \right)^k + k \sum_{i=1}^{\lceil \frac{d}{3} \rceil} (2i-1)^{k-1}.$$

As a corollary of Observation 2.3(a) and Theorem 3.1, we obtain bounds on the maximum order of any graph G with $\text{diam}(G) = d$ and $\text{bdim}(G) = k$.

Corollary 3.2. *For any graph G with* $\text{diam}(G) = d$ *and* $\text{bdim}(G) = k$,

$$|V(G)| \leq \left(\left\lfloor \frac{2d}{3} \right\rfloor + 1 \right)^k + k \sum_{i=1}^{\lceil \frac{d}{3} \rceil} (2i-1)^{k-1}.$$

Proof. If G has $\text{bdim}(G) = k$, then $\dim(G) \leq k$ by Observation 2.3(a). So, the desired result follows from Theorem 3.1. □

We also obtain bounds on the maximum order of any subgraph of G with $\text{diam}(G) = d$ and $\text{bdim}(G) = k$. Based on a result from [17] and using the same reasoning that led to the previous corollary, we obtain the next corollary.

Corollary 3.3. *For any graph G with* $\text{bdim}(G) = k$ *and any subgraph H of G with* $\text{diam}(H) = d$, $|V(H)| \leq (d+1)^k$.

Remark 3.4. *By Observation 2.3(a), Corollaries 3.2 and 3.3 hold when* $\text{bdim}(G) = k$ *is replaced by* $\text{adim}(G) = k$.

The next result shows that Corollary 3.2 is sharp for $d = 2$. This result uses a family of graphs from [39, 17] with diameter two. Hence, all dimensions considered in this paper coincide.

Theorem 3.5. *There exist graphs G of order n with* $\text{bdim}(G) = O(\log n)$.

Proof. We construct a graph G of order $n = k + 2^k$ by starting with k vertices $v_1, \ldots, v_k$ in a clique, and adding 2^k new vertices $\{u_b\}_{b \in \{0,1\}^k}$ also in a clique labeled with binary strings of length k such that u_b has an edge with v_j if and only if the j^{th} digit of b is 1.

Define the resolving broadcast f such that $f(v_i) = 1$ for all $1 \leq i \leq k$ and $f(u_b) = 0$ for all $b \in \{0,1\}^k$. Since $n = k + 2^k$ and $\sum_{v \in V(G)} f(v) = k$, we have $\text{bdim}(G) = O(\log n)$. For any n not of the form $k + 2^k$, we can define n' to be the least number greater than n that is of the form $k + 2^k$, construct G' with n' vertices as described, and delete any number of vertices u_b from G' until the remaining graph G has n vertices. □

Based on the proof of Theorem 3.5, we have the following corollary.

Corollary 3.6. *There exist graphs G of order n with* $\text{adim}(G) = O(\log n)$.

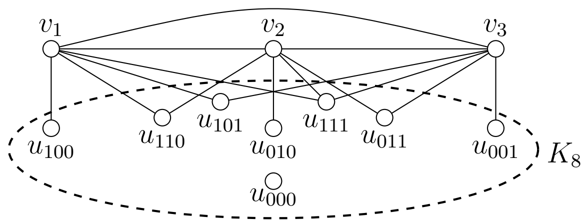

Figure 1: A graph G of order n satisfying $\text{bdim}(G) = O(\log n)$; here $k = 3$ for G described in the proof of Theorem 3.5.

The construction in Theorem 3.5 can also be used to recursively characterize the graphs G with $\text{adim}(G) = k$. Given any graph G_1 on k vertices $v_1, \ldots, v_k$ and G_2 on 2^k vertices $\{u_b\}_{b \in \{0,1\}^k}$, define the graph $B(G_1, G_2)$ to be obtained by connecting v_i and u_b if and only if the i^{th} digit of b is 1. Moreover, define $\mathcal{B}(G_1, G_2)$ to be the family of induced subgraphs of $B(G_1, G_2)$ that contain every vertex in G_1. Finally, define $\mathcal{H}_0 = \emptyset$ and for each $k > 0$ define $\mathcal{H}_k$ to be the family of graphs obtained from taking the union of $\mathcal{B}(G_1, G_2)$ over all graphs G_1 with j vertices $v_1, \ldots, v_j$ and G_2 with 2^j vertices $\{u_b\}_{b \in \{0,1\}^j}$, for each $1 \leq j \leq k$.

Theorem 3.7. *For each $k \geq 1$, the set of graphs G with $\text{adim}(G) = k$ is $\mathcal{H}_k - \mathcal{H}_{k-1}$ up to isomorphism.*

Proof. It suffices to show that the set of graphs G with $\text{adim}(G) \leq k$ is $\mathcal{H}_k$. By construction, every graph in $\mathcal{H}_k$ has $\text{adim}(G) \leq k$, since the vertices $v_1, \ldots, v_j$ are an adjacency resolving set. Thus, it suffices to show that every graph G with $\text{adim}(G) \leq k$ is in $\mathcal{H}_k$. Fix an arbitrary graph G with $\text{adim}(G) \leq k$. Let $X = \{x_1, \ldots, x_j\}$ be an adjacency resolving set for G with $j \leq k$. Let G_1 be the induced subgraph of G restricted to X, and let G_2 be the induced subgraph of G restricted to $\overline{X}$. Label the vertex v of G_2 as u_b with a binary string b so that the i^{th} digit of b is 1 if and only if there is an edge between v and x_i. Note that every vertex gets a unique label, or else X would not be an adjacency resolving set. Let G_2' be any graph on 2^j vertices $\{u_b\}_{b \in \{0,1\}^j}$ such that $G_2'|_{V(G_2)} = G_2$. Then, G is an induced subgraph of $B(G_1, G_2')$ that contains every vertex in G_1, so G is in $\mathcal{H}_k$. $\square$

As a corollary, we obtain an upper bound on the maximum order of any graph of adjacency dimension k. The graph in Theorem 3.5 shows that the bound is sharp. The following result (with a different terminology) is contained in the proof of Theorem 45 of [14], and was also proved independently in [24].

Corollary 3.8. *The maximum order of any graph of adjacency dimension k is $k+2^k$.*

We also obtain a sharp upper bound on the maximum degree of any graph of adjacency dimension k.

Corollary 3.9. *The maximum possible degree of any vertex in any graph of adjacency dimension k is $k + 2^k - 1$.*

Proof. The upper bound is immediate from Corollary 3.8, while the upper bound is achieved by the vertex u_{1^k} in $B(K_k, K_{2^k})$. ☐

In addition, we obtain a sharp upper bound on the clique number of graphs of adjacency dimension k and graphs of broadcast dimension k.

Corollary 3.10. *The maximum possible clique number of any graph of adjacency dimension k is 2^k. Similarly, the maximum possible clique number of any graph of broadcast dimension k is 2^k.*

Proof. The upper bound follows from Corollary 3.3. The bound is achieved by the graph $G = B(K_k, K_{2^k})$, which has $\mathrm{adim}(G) = \mathrm{bdim}(G) = k$. ☐

The next result is sharp up to a constant factor, as shown by paths, cycles, and grid graphs.

Proposition 3.11. *For graphs G of diameter d,* $\mathrm{adim}(G) \geq \mathrm{bdim}(G) \geq \frac{d}{3}$.

Proof. Suppose that G is a graph of diameter d, and let f be a resolving broadcast that achieves $\mathrm{bdim}(G)$. Since G has diameter d, there exist $d+1$ vertices which form a geodesic path in G. For each vertex x on the geodesic path, there must exist some vertex $v \in \mathrm{supp}_G(f)$ with $d(v,x) \leq f(v)$, or else f would not be a resolving broadcast of G. Moreover there are at most $2f(v)+1$ vertices y on the geodesic path with $d(v,y) \leq f(v)$, or else the path would not be geodesic. Thus, $|\mathrm{supp}_G(f)| + 2\sum_{v \in V(G)} f(v) \geq d+1$, which implies that $\mathrm{bdim}(G) \geq \frac{d}{3}$. ☐

Thus, we have a sharp bound on $\mathrm{bdim}(G)$ up to a constant factor for any graph G with $\dim(G) = O(1)$, where the upper bound follows from the definition of $\mathrm{bdim}(G)$.

Theorem 3.12. *For every graph G of diameter d,* $\frac{d}{3} \leq \mathrm{bdim}(G) \leq \dim(G)(d-1)$.

Corollary 3.13. *If G has diameter d and* $\dim(G) = O(1)$, *then* $\mathrm{bdim}(G) = \Theta(d)$. *If G has diameter $d = O(1)$, then* $\mathrm{bdim}(G) = \Theta(\dim(G))$.

The next result is sharp up to a constant factor by Theorem 3.5.

Theorem 3.14. *For all graphs G of order n,* $\mathrm{adim}(G) \geq \mathrm{bdim}(G) = \Omega(\log n)$.

Proof. Suppose that f is a resolving broadcast that achieves $\mathrm{bdim}(G)$, and let $y = |\mathrm{supp}_G(f)|$. The $\Omega(\log n)$ bound holds if $y > \ln(\frac{n}{2})$, so we suppose that $y \leq \ln(\frac{n}{2})$. There are a total of y vertices v for which $f(v) > 0$, so there are y vertices that have a 0 in their broadcast code with respect to f. All vertices must have a unique broadcast code with respect to f, and there are $f(v)+1$ *nonzero* choices for the v-coordinate of the broadcast code with respect to f, so the number of vertices in G with no 0 in their broadcast code with respect to f is at most $\prod_{v \in \mathrm{supp}_G(f)}(f(v)+1)$. Thus, we must have $y + \prod_{v \in \mathrm{supp}_G(f)}(f(v)+1) \geq n$, since G has order n.

If we rewrite the last inequality in an equivalent form as $(\prod_{v \in \mathrm{supp}_G(f)}(f(v)+1))^{1/y} \geq (n-y)^{1/y}$ and note that $\frac{\sum_{v \in \mathrm{supp}_G(f)}(f(v)+1)}{y} \geq (\prod_{v \in \mathrm{supp}_G(f)}(f(v)+1))^{1/y}$ by the arithmetic mean - geometric mean inequality, we obtain that $y+\sum_{v\in V(G)} f(v) = \sum_{v\in \mathrm{supp}_G(f)}(f(v)+1) \geq y(n-y)^{1/y}$, or equivalently $\sum_{v\in V(G)} f(v) \geq y(n-y)^{1/y} - y$. Since $y(n-y)^{1/y} \geq y(\frac{n}{2})^{1/y} \geq ye$ for n sufficiently large, we have $\sum_{v\in V(G)} f(v) \geq y(\frac{n}{2})^{1/y} - y \geq \frac{e-1}{e} y(\frac{n}{2})^{1/y}$.

Define $g(y) = \ln(\frac{e-1}{e} y(\frac{n}{2})^{1/y})$, so $g'(y) = \frac{1}{y} - \frac{\ln(\frac{n}{2})}{y^2}$, which has one root at $y = \ln(\frac{n}{2})$. This is a minimum of g, since $g''(\ln(\frac{n}{2})) > 0$. Since $\ln(x)$ is an increasing function, $\frac{e-1}{e} y(\frac{n}{2})^{1/y}$ is also minimized at $y = \ln(\frac{n}{2})$, where it has value $(e-1)\ln(\frac{n}{2})$. Thus, $\sum_{v\in V(G)} f(v) \geq (e-1)\ln(\frac{n}{2})$ in this case. □

4 Graphs G having $\mathrm{bdim}(G)$ equal to 1, 2, and $|V(G)|-1$

Next, we characterize graphs G having $\mathrm{bdim}(G)$ equal to 1, 2, and $|V(G)|-1$. We begin with the following known results on metric dimension and adjacency dimension.

Theorem 4.1. [6] *Let G be a connected graph of order n. Then,*

(a) $\dim(G) = 1$ *if and only if* $G = P_n$*;*

(b) for $n \geq 4$, $\dim(G) = n-2$ *if and only if* $G = K_{s,t}$ $(s,t \geq 1)$, $G = K_s + \overline{K}_t$ $(s \geq 1, t \geq 2)$, *or* $G = K_s + (K_1 \cup K_t)$ $(s,t \geq 1)$*;*

(c) $\dim(G) = n-1$ *if and only if* $G = K_n$.

Theorem 4.2. [25] *Let G be a graph of order n. Then,*

(a) $\mathrm{adim}(G) = 1$ *if and only if* $G \in \{P_1, P_2, P_3, \overline{P}_2, \overline{P}_3\}$*;*

(b) $\mathrm{adim}(G) = n-1$ *if and only if* $G \in \{K_n, \overline{K}_n\}$.

Note that, if f is a resolving broadcast of G with $f(v) = 2$ and $f(w) = 0$ for each $w \in V(G) - \{v\}$, then v is an end vertex of P_4 or v is an end vertex of $P_3 \cup P_1$, and $\mathrm{adim}(P_4) = \mathrm{adim}(P_3 \cup P_1) = 2$ as shown in Theorem 3.7. Also, note that $\mathrm{adim}(G) = 2$ implies $\mathrm{bdim}(G) = 2$. So, Observation 2.3(a), Theorems 3.7, 4.1 and 4.2 imply the following proposition.

Proposition 4.3. *Let G be a graph of order n. Then,*

(a) $\mathrm{bdim}(G) = 1$ *if and only if* $G \in \{P_1, P_2, P_3, \overline{P}_2, \overline{P}_3\}$*;*

(b) $\mathrm{bdim}(G) = 2$ *if and only if* $G \in \mathcal{H}_2 - \mathcal{H}_1$ *as described in Theorem 3.7 (see Figure 2);*

(c) $\mathrm{bdim}(G) = n-1$ *if and only if* $G \in \{K_n, \overline{K}_n\}$.

The graphs G with $\mathrm{adim}(G) = 2$ were determined in [24], as well as the graphs G with $\mathrm{adim}(G) = n-2$. The next question about graphs with high broadcast dimension is open.

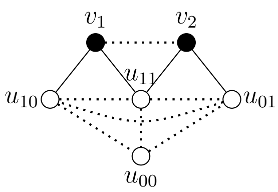

Figure 2: Graphs G satisfying $\mathrm{bdim}(G) = 2$ are the graphs satisfying $\mathrm{adim}(G) = 2$, described in Theorem 3.7. Black vertices must be present, a solid edge must be present whenever the two vertices incident to the solid edge are in the graph, but a dotted edge is not necessarily present.

Question 4.4. *Which graphs G of order n satisfy* $\mathrm{bdim}(G) = n-2$*?*

A graph is *planar* if it can be drawn in a plane without edge crossing. For any graphs H and G, H is called a *minor* of G if H can be obtained from G by vertex deletion, edge deletion, or edge contraction.

Theorem 4.5. [37] *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.*

Remark 4.6. *It was shown in [28] that there exists a non-planar graph G with* $\dim(G) = 2$*. However,* $\mathrm{adim}(G) = 2$ *(*$\mathrm{bdim}(G) = 2$*, respectively) implies G is planar (see Figure 2). Also, note that, for each $k \geq 3$, there exists a non-planar graph G satisfying* $\mathrm{bdim}(G) = k$ *and* $\mathrm{adim}(G) = k$*, respectively. For example, the graph G of order $n = k + 2^k$ with* $\mathrm{bdim}(G) = k$ *(*$\mathrm{adim}(G) = k$*, respectively) described in the proof of Theorem 3.5 contains K_{2^k} as a subgraph. Since K_{2^k}, for $k \geq 3$, contains K_5 as a minor, G is not planar by Theorem 4.5.*

5 Comparing $\dim(G)$, $\mathrm{adim}(G)$, and $\mathrm{bdim}(G)$

Next, we provide a connected graph G such that both $\mathrm{adim}(G) - \mathrm{bdim}(G)$ and $\mathrm{bdim}(G) - \dim(G)$ can be arbitrarily large. In fact, we obtain the stronger result that $\frac{\mathrm{adim}(G)}{\mathrm{bdim}(G)}$ and $\frac{\mathrm{bdim}(G)}{\dim(G)}$ can be arbitrarily large. We first recall some results on grid graphs.

Proposition 5.1. [5] *For the grid graph $G = P_m \times P_n$ $(m, n \geq 2)$,* $\dim(G) = 2$.

Proposition 5.2. [28] *For the d-dimensional grid graph $G = \prod_{i=1}^{d} P_{n_i}$, where $d \geq 2$ and $n_i \geq 2$ for each $i \in \{1, \ldots, d\}$,* $\dim(G) \leq d$.

With Theorem 3.12, propositions 5.1 and 5.2 immediately imply the following corollary.

Corollary 5.3. *If G is the grid graph $P_{m+1} \times P_{n+1}$ $(m, n \geq 1)$, then* $\mathrm{bdim}(G) = \Theta(m+n)$*. More generally, if G is the d-dimensional grid graph $\prod_{i=1}^{d} P_{n_i+1}$ with $n_i \geq 1$ for each $i = 1, \ldots, d$, then* $\mathrm{bdim}(G) = \Theta(\sum_{i=1}^{d} n_i)$*, where the constant in the upper bound depends on $d \geq 2$.*

Proof. The lower bound follows since Theorem 3.12 implies that $\mathrm{bdim}(G) \geq \frac{\sum_{i=1}^{d} n_i}{3}$. For the upper bound, it is easy to see that we can use the same configuration as the one used in [28] which showed that $\dim(H) \leq d$ for every d-dimensional grid graph H. Specifically in the case $d = 2$, we can define a resolving broadcast f of G such that $f(u) = f(v) = m + n$ for two non-opposite corner vertices u, v of the grid G and $f(w) = 0$ for all other vertices $w \in V(G)$. Since the diameter of G is $m + n$, the metric code of each vertex $w \in V(G)$ with respect to $\{u, v\}$ coincides with the broadcast code of w with respect to f. □

Theorem 5.4. *For $k \geq 2$, let G be the d-dimensional grid graph $\prod_{i=1}^{d} P_k$. Then,* $\mathrm{bdim}(G) = \Theta(k)$, *and* $\mathrm{adim}(G) = \Theta(k^d)$, *where the constants in the bounds depend on d. So, $\frac{\mathrm{adim}(G)}{\mathrm{bdim}(G)}$ and $\frac{\mathrm{bdim}(G)}{\dim(G)}$ can be arbitrarily large.*

Proof. Note that $\dim(G) \leq d$ by Proposition 5.2 and $\mathrm{bdim}(G) = \Theta(k)$ by Corollary 5.3. To see that $\mathrm{adim}(G) = \Theta(k^d)$, first note that $\mathrm{adim}(G) = O(k^d)$ since $|V(G)| = O(k^d)$. Moreover, any adjacency resolving set of G must contain at least one vertex from every $\prod_{i=1}^{d} P_3$ subgraph of G except for at most one, so $\mathrm{adim}(G) = \Omega(k^d)$. □

In the next result, we show that the multiplicative gap between $\mathrm{bdim}(G)$ and $\mathrm{adim}(G)$ in Theorem 5.4 is tight up to a constant factor. To state this result, we define $\Delta'(G)$ to be the maximum value of t for which there exists a positive integer j and a vertex $v \in V(G)$ such that there exist at least t distinct vertices $u_1, \ldots, u_t \in V(G)$ with $d_G(u_i, v) = j$ for each $i = 1, \ldots, t$. Note that when $G = P_k \times P_k$, we have $\Delta'(G) = \Theta(k)$, so $\frac{\mathrm{adim}(G)}{\mathrm{bdim}(G)} = \Theta(k) = \Theta(\Delta'(G))$.

Proposition 5.5. *For all graphs G, $\frac{\mathrm{adim}(G)}{\mathrm{bdim}(G)} = O(\Delta'(G))$.*

Proof. Given a resolving broadcast f of G with $\sum_{v \in V(G)} f(v) = \mathrm{bdim}(G)$, we show how to convert f into an adjacency resolving set for G which uses at most $(\Delta'(G) + 1)\mathrm{bdim}(G)$ vertices. Let v be a vertex $v \in V(G)$ with $f(v) > 0$. If $f(v) = 1$, then we put the vertex v into the adjacency resolving set for G. If $f(v) > 1$, then for each k, $0 < k \leq f(v)$, we list the vertices $u_1, \ldots, u_t$ with $d_G(u_i, v) = k$, and we add each vertex $u_1, \ldots, u_t$ into the adjacency resolving set for G, as well as the vertex v. Thus, we add at most $(\Delta'(G) + 1)$ vertices to the adjacency resolving set for each vertex $v \in V(G)$ with $f(v) > 0$ and each positive integer k with $k \leq f(v)$. This implies that $\mathrm{adim}(G) \leq (\Delta'(G) + 1)\mathrm{bdim}(G)$. □

The proof of the last proposition also implies the following proposition.

Proposition 5.6. *For all graphs G of order n,* $\mathrm{bdim}(G)\Delta'(G) = \Omega(n)$.

It was shown in [8] that metric dimension is not a monotone parameter on subgraph inclusion; see [8] for an example satisfying $H \subset G$ and $\dim(H) > \dim(G)$. Next, we show for graphs H and G with $H \subset G$ that $\dim(H) - \dim(G)$, $\mathrm{bdim}(H) - \mathrm{bdim}(G)$, and $\mathrm{adim}(H) - \mathrm{adim}(G)$ can be arbitrarily large. In fact, we obtain the stronger result that $\frac{\dim(H)}{\dim(G)}$, $\frac{\mathrm{bdim}(H)}{\mathrm{bdim}(G)}$, and $\frac{\mathrm{adim}(H)}{\mathrm{adim}(G)}$ can be arbitrarily large.

Theorem 5.7. *There exist connected graphs H and G such that $H \subset G$ and $\frac{\dim(H)}{\dim(G)}$, $\frac{\text{bdim}(H)}{\text{bdim}(G)}$, and $\frac{\text{adim}(H)}{\text{adim}(G)}$ can be arbitrarily large.*

Proof. For $k \geq 3$, let $H = K_{\frac{k(k+1)}{2}}$; let $V(H)$ be partitioned into $V_1, V_2, \ldots, V_k$ such that $V_i = \{w_{i,1}, w_{i,2}, \ldots, w_{i,i}\}$ with $|V_i| = i$, where $i \in \{1, 2, \ldots, k\}$. Let G be the graph obtained from H and k isolated vertices $u_1, u_2, \ldots, u_k$ as follows: u_1 is adjacent to each vertex of $V_1 \cup (\cup_{j=2}^{k}\{w_{j,1}\})$, u_2 is adjacent to each vertex of $V_2 \cup (\cup_{j=3}^{k}\{w_{j,2}\})$, u_3 is adjacent to each vertex of $V_3 \cup (\cup_{j=4}^{k}\{w_{j,3}\})$, and so on, i.e., for each $i \in \{1, 2, \ldots, k\}$, u_i is adjacent to each vertex of $V_i \cup (\cup_{j=i+1}^{k}\{w_{j,i}\})$ (see the graph G in Figure 3 when $k = 4$). Since $\text{diam}(H) = 1$ and $\text{diam}(G) = 2$, $\dim(H) = \text{bdim}(H) = \text{adim}(H)$ and $\dim(G) = \text{bdim}(G) = \text{adim}(G)$ by Observation 2.3(b). Note that $H \subset G$ and $\dim(H) = \frac{k(k+1)}{2} - 1$ by Theorem 4.1(c). Since $\{u_1, u_2, \ldots, u_k\}$ forms a resolving set of G, $\dim(G) \leq k$. So, $\frac{\dim(H)}{\dim(G)} = \frac{\text{bdim}(H)}{\text{bdim}(G)} = \frac{\text{adim}(H)}{\text{adim}(G)} \geq \frac{k^2+k-2}{2k} \to \infty$ as $k \to \infty$. $\square$

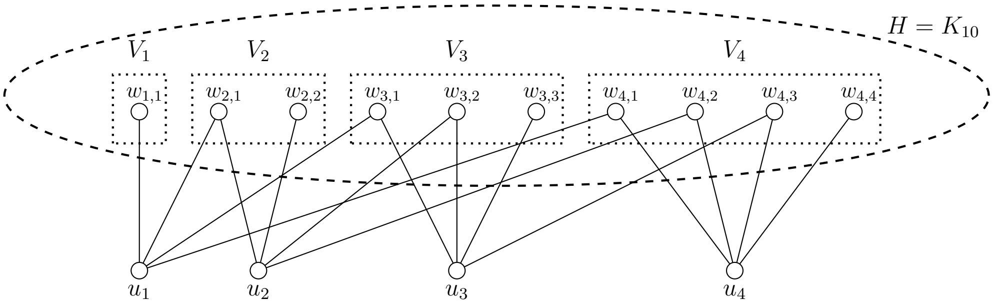

Figure 3: A graph G such that $H \subset G$ and $\frac{\dim(H)}{\dim(G)} = \frac{\text{bdim}(H)}{\text{bdim}(G)} = \frac{\text{adim}(H)}{\text{adim}(G)}$ can be arbitrarily large; here, $k = 4$ and $H = K_{10}$ for the example described in Theorem 5.7.

Next we find all trees T for which $\dim(T) = \text{bdim}(T)$. First we recall some terminology. Fix a tree T. An end vertex ℓ is called a *terminal vertex* of a major vertex v if $d(\ell, v) < d(\ell, w)$ for every other major vertex w in T. The *terminal degree*, $ter(v)$, of a major vertex v is the number of terminal vertices of v in T, and an *exterior major vertex* is a major vertex that has positive terminal degree. We denote by $ex(T)$ the number of exterior major vertices of T, and $\sigma(T)$ the number of end vertices of T.

Theorem 5.8. [6, 28, 30] *For a tree T that is not a path,* $\dim(T) = \sigma(T) - ex(T)$.

Theorem 5.9. [30] *Let T be a tree with $ex(T) = k \geq 1$, and let $v_1, v_2, \ldots, v_k$ be the exterior major vertices of T. For each i $(1 \leq i \leq k)$, let $\ell_{i,1}, \ell_{i,2}, \ldots, \ell_{i,\sigma_i}$ be the terminal vertices of v_i with $ter(v_i) = \sigma_i \geq 1$, and let $P_{i,j}$ be the $v_i - \ell_{i,j}$ path, where $1 \leq j \leq \sigma_i$. Let $W \subseteq V(T)$. Then, W is a minimum resolving set of T if and only if W contains exactly one vertex from each of the paths $P_{i,j} - v_i$ $(1 \leq j \leq \sigma_i$ and $1 \leq i \leq k)$ with exactly one exception for each i with $1 \leq i \leq k$ and W contains no other vertices of T.*

Proposition 5.10. *Let T be a non-trivial tree. Then,* $\dim(T) = \mathrm{bdim}(T)$ *if and only if $T \in \{P_2, P_3\}$ or T is a tree obtained from the star $K_{1,x}$ ($x \geq 3$) by subdividing at most $x-1$ edges exactly once.*

Proof. ($\Leftarrow$) First, note that $\mathrm{bdim}(T) = 1 = \dim(T)$ for $T \in \{P_2, P_3\}$ by Theorem 4.1 and Proposition 4.3. Second, let T be a tree obtained from the star $K_{1,x}$ ($x \geq 3$) by subdividing at most $x-1$ edges exactly once. Let w be the major vertex of T, and let $\ell_1, \ell_2, \ldots, \ell_x$ be the terminal vertices of w in T such that $d(w, \ell_1) \geq d(w, \ell_2) \geq \ldots \geq d(w, \ell_x)$; then $d(w, \ell_x) = 1$. If $f : V(T) \to \mathbb{Z}^+ \cup \{0\}$ is a function defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in N(w) - \{\ell_x\} \\ 0 & \text{otherwise,} \end{cases}$$

then f is a resolving broadcast of T, and thus $\mathrm{bdim}(T) \leq x - 1 = \dim(T)$ by Theorem 5.8. By Observation 2.3(a), $\mathrm{bdim}(T) = \dim(T)$.

($\Rightarrow$) Let $\dim(T) = \mathrm{bdim}(T)$. Let $f : V(T) \to \mathbb{Z}^+ \cup \{0\}$ be a resolving broadcast of T with $bc_f(T) = \dim(T)$, and let $R = \mathrm{supp}_T(f)$. First, let $ex(T) = 0$, i.e., T is a path; then $bc_f(T) = 1$ by Theorem 4.1(a). So, $b_R(u) \in \{0, 1, 2\}$ for each $u \in V(T)$. Thus, $T \in \{P_2, P_3\}$.

Second, let $ex(T) = 1$ and suppose that v is the exterior major vertex of T. Note that R has nonempty intersection with all but one of the paths $P^1, \ldots, P^x$ hanging from v; otherwise, there are two neighbors of v with the same broadcast code. For each i, $1 \leq i \leq x$, let ℓ_i be the terminal vertex of v belonging to P^i and let s_i be the vertex of P^i that is adjacent to v. Since $\dim(T) = \mathrm{bdim}(T)$, R must contain exactly one vertex u_i of each path P^i that has nonempty intersection with R, and $f(u_i) = 1$ for each such path. Without loss of generality, suppose that P^x has no vertex in R. Then, the order of P^x is 1, since otherwise the vertices of P^x all have broadcast code $\mathbf{2}_{x-1}$. Furthermore, the order of P^i for each $i \neq x$ is at most 2. Otherwise, if $u_i = s_i$, then ℓ_i and ℓ_x have the same broadcast code $\mathbf{2}_{x-1}$. If $u_i = \ell_i$, then s_i and ℓ_x have the same broadcast code $\mathbf{2}_{x-1}$. Finally if $u_i \neq s_i$ and $u_i \neq \ell_i$, then the neighbors of u_i have the same broadcast code.

Next, let $ex(T) \geq 2$. As in the preceding case, R contains exactly one vertex of all but one of the paths hanging from each exterior major vertex $v_1, v_2, \ldots, v_{ex(T)}$, and since $\dim(T) = \mathrm{bdim}(T)$, R must contain exactly one vertex of each one of these paths with cost equal to one. However, the vertices adjacent to v_1 and v_2 that belong to the paths which have no intersection with R must have the same broadcast code $\mathbf{2}_{|R|}$. Thus, in this case we cannot have $\dim(T) = \mathrm{bdim}(T)$. $\square$

Proposition 5.10 implies the following corollary.

Corollary 5.11. *For any non-trivial tree T,* $\dim(T) = \mathrm{adim}(T)$ *if and only if $T \in \{P_2, P_3\}$ or T is a tree obtained from the star $K_{1,x}$ ($x \geq 3$) by subdividing at most $x-1$ edges exactly once.*

6 The effect of vertex or edge deletion on the adjacency dimension and the broadcast dimension of graphs

Throughout this section, let v and e, respectively, denote a vertex and an edge of a connected graph G such that both $G - v$ and $G - e$ are connected graphs. First, we consider the effect of vertex deletion on adjacency dimension and broadcast dimension. It is known that both $\dim(G) - \dim(G - v)$ and $\dim(G - v) - \dim(G)$ can be arbitrarily large; see [3] and [7], respectively. We show that $\frac{\text{bdim}(G)}{\text{bdim}(G-v)}$ can be arbitrarily large, whereas $\text{adim}(G) \leq \text{adim}(G - v) + 1$. We also show that both $\text{bdim}(G - v) - \text{bdim}(G)$ and $\text{adim}(G - v) - \text{adim}(G)$ can be arbitrarily large.

We recall the following useful result.

Proposition 6.1. [12] *Let H be a graph of order $n \geq 2$. Then,* $\text{adim}(K_1 + H) \geq \text{adim}(H)$.

Remark 6.2. *The value of $\frac{\text{bdim}(G)}{\text{bdim}(G-v)}$ can be arbitrarily large, as G varies.*

Proof. Let $G = (P_k \times P_k) + K_1$, and let v be the vertex in the K_1. Then, $\text{bdim}(G - v) = \Theta(k)$ by Theorem 5.4, but $\text{bdim}(G) = \text{adim}(G) \geq \text{adim}(G - v) = \Theta(k^2)$ by Proposition 6.1 and Theorem 5.4. □

Proposition 6.3. *For any graph G,* $\text{adim}(G) \leq \text{adim}(G - v) + 1$, *where the bound is sharp.*

Proof. Let S be a minimum adjacency resolving set of $G - v$. Note that, for any vertex x in $G - v$, $a_S(x)$ in $G - v$ remains the same in G. So, $S \cup \{v\}$ forms an adjacency resolving set of G, and hence $\text{adim}(G) \leq |S| + 1 = \text{adim}(G - v) + 1$. For the sharpness of the bound, let $G = K_n$ for $n \geq 3$; then $\text{adim}(G) = n - 1$ and $\text{adim}(G - v) = n - 2$, for any $v \in V(G)$, by Theorem 4.2(b). □

Remark 6.4. *The value of* $\text{bdim}(G-v)-\text{bdim}(G)$ *and* $\text{adim}(G-v)-\text{adim}(G)$ *can be arbitrarily large, as G varies.*

Proof. Let G be the graph in Figure 4, where $k \geq 2$. Note that $\text{diam}(G) = \text{diam}(G-v) = 2$; thus, $\dim(G) = \text{bdim}(G) = \text{adim}(G)$ and $\dim(G - v) = \text{bdim}(G - v) = \text{adim}(G - v)$ by Observation 2.3(b).

First, we show that $\dim(G) = k + 1$. Let S be any minimum resolving set of G. Note that, for each $i \in \{1, 2, \ldots, k\}$, x_i and z_i are twin vertices of G; thus $|S \cap \{x_i, z_i\}| \geq 1$ by Observation 2.1(a). Without loss of generality, let $S' = \cup_{i=1}^{k}\{x_i\} \subseteq S$. Since $r_{S'}(y_i) = r_{S'}(z_i)$ for each $i \in \{1, 2, \ldots, k\}$, $|S| \geq k + 1$; thus $\dim(G) \geq k + 1$. On the other hand, $S' \cup \{v\}$ forms a resolving set of G, and thus $\dim(G) \leq k + 1$. So, $\dim(G) = k + 1$.

Second, we show that $\dim(G - v) = 2k$. Let R be any minimum resolving set of $G - v$. Note that, for each $i \in \{1, 2, \ldots, k\}$, any two vertices in $\{x_i, y_i, z_i\}$ are twin vertices of $G-v$. By Observation 2.1(a), $|R\cap\{x_i, y_i, z_i\}| \geq 2$ for each $i \in \{1, 2, \ldots, k\}$;

thus $|R| \geq 2k$. Since $\cup_{i=1}^{k}\{x_i, y_i\}$ forms a resolving set of $G - v$, $\dim(G - v) \leq 2k$. Thus, $\dim(G - v) = 2k$.

Therefore, $\dim(G - v) - \dim(G) = \mathrm{bdim}(G - v) - \mathrm{bdim}(G) = \mathrm{adim}(G - v) - \mathrm{adim}(G) = 2k - (k + 1) = k - 1 \to \infty$ as $k \to \infty$. □

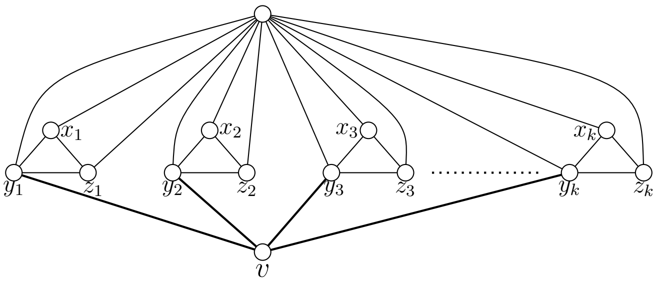

Figure 4: A graph G such that $\dim(G - v) - \dim(G) = \mathrm{bdim}(G - v) - \mathrm{bdim}(G) = \mathrm{adim}(G - v) - \mathrm{adim}(G)$ can be arbitrarily large, where $k \geq 2$.

Next, we consider the effect of edge deletion. We recall the following result on the effect of edge deletion on metric dimension.

Theorem 6.5. [7]

(a) For any graph G and any edge $e \in E(G)$, $\dim(G - e) \leq \dim(G) + 2$.

(b) The value of $\dim(G) - \dim(G - e)$ can be arbitrarily large (see Figure 5).

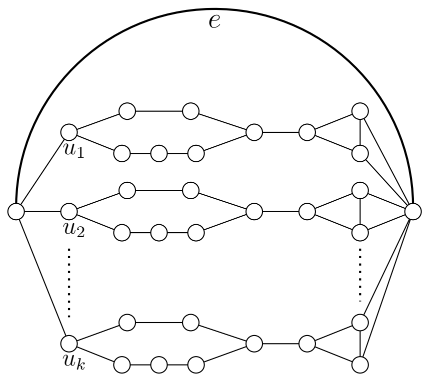

Figure 5: A graph G such that $\dim(G) - \dim(G - e)$ can be arbitrarily large, where $k \geq 2$.

Now, we consider the effect of edge deletion on adjacency dimension. We begin with the following lemma, which is used in proving Theorem 6.7.

Lemma 6.6. *For any graph G, let $e = xy \in E(G)$.*

(a) If S is an adjacency resolving set of G, then $S \cup \{x, y\}$ is an adjacency resolving set of $G - e$.

(b) If R is an adjacency resolving set of $G-e$, then $R \cup \{x, y\}$ is an adjacency resolving set of G.

Proof. Let $e = xy \in E(G)$.

(a) Since S is an adjacency resolving set of G, $S' = S \cup \{x, y\}$ is also an adjacency resolving set of G. Since the adjacency code of each vertex, excluding x and y, with respect to S' in G remains the same in $G-e$, S' is an adjacency resolving set of $G-e$.

(b) Since R is an adjacency resolving set of $G-e$, $R' = R \cup \{x, y\}$ is an adjacency resolving set of $G-e$. Since the adjacency code of each vertex, excluding x and y, with respect to R' in $G-e$ remains the same in G, R' is an adjacency resolving set of G. □

Theorem 6.7. *For every graph G and every edge $e \in E(G)$,* $\text{adim}(G)-1 \leq \text{adim}(G-e) \leq \text{adim}(G)+1$.

Proof. We denote by $d_{H,1}(x, y)$ the adjacency distance between two vertices x and y in a graph H.

First, we show that $\text{adim}(G-e) \leq \text{adim}(G)+1$. Let S be a minimum adjacency resolving set of G, and let $e \in E(G)$. Let $x, y \in V(G-e) - S = V(G) - S$ such that $z \in S$ with $d_{G,1}(x, z) \neq d_{G,1}(y, z)$. Without loss of generality, let $d_{G,1}(x, z) = 1$ and $d_{G,1}(y, z) = 2$; then $xz \in E(G)$. If $d_{G-e,1}(x, z) = d_{G-e,1}(y, z)$, then $e = xz$. Since $z \in S$, $S \cup \{x\}$ forms an adjacency resolving set of $G-e$ by Lemma 6.6(a). Thus $\text{adim}(G-e) \leq |S|+1 = \text{adim}(G)+1$.

Second, we show that $\text{adim}(G)-1 \leq \text{adim}(G-e)$. Let R be any minimum adjacency resolving set of $G-e$, and let $e = uv \in E(G)$. If $|R \cap \{u, v\}| = 0$, then each entry of $a_R(u)$ and $a_R(v)$ is 1 or 2; thus, the adjacency code of each vertex with respect to R in $G-e$ remains the same in G, and hence R is an adjacency resolving set of G. If $|R \cap \{u, v\}| = 1$, say $u \in R$ and $v \notin R$ without loss of generality, then $R \cup \{v\}$ forms an adjacency resolving set of G by Lemma 6.6(b). If $|R \cap \{u, v\}| = 2$ (i.e., $u, v \in R$), then R is an adjacency resolving set of G by Lemma 6.6(b). Therefore, $\text{adim}(G) \leq |R|+1 = \text{adim}(G-e)+1$. □

The bounds in Theorem 6.7 are sharp, as shown in the next proposition.

Proposition 6.8.
(a) If G is the complete graph of order at least three, then $\text{adim}(G-e) = \text{adim}(G)-1$ *for every edge $e \in E(G)$.*

(b) If G is the graph in Figure 6, then $\text{adim}(G-e) = \text{adim}(G)+1$.

Proof. (a) Let $G = K_n$ for $n \geq 3$. Then, for every edge $e \in E(G)$, $\text{adim}(G) = n-1$ and $\text{adim}(G-e) = n-2$; thus, $\text{adim}(G-e) = \text{adim}(G)-1$.

(b) Let G be the graph in Figure 6. Let $N(u_1) - \{u_2\} = \cup_{i=1}^{a}\{x_i\}$, $N(u_2) - \{u_1, u_3\} = \cup_{i=1}^{b}\{y_i\}$, and $N(u_3) - \{u_2\} = \cup_{i=1}^{c}\{z_i\}$, where $a, c \geq 3$ and $b \geq 2$. We show that $\text{adim}(G-e) = \text{adim}(G)+1$.

First, we show that $\mathrm{adim}(G-e)=a+b+c-1$. Let S be a minimum adjacency resolving set of $G-e$. Since any two vertices in $\cup_{i=1}^{a}\{x_i\}$, $\cup_{i=1}^{b}\{y_i\}$, and $\cup_{i=1}^{c}\{z_i\}$, respectively, are twin vertices in $G-e$, by Observation 2.1(b), we have $|S\cap(\cup_{i=1}^{a}\{x_i\})|\ge a-1$, $|S\cap(\cup_{i=1}^{b}\{y_i\})|\ge b-1$ and $|S\cap(\cup_{i=1}^{c}\{z_i\})|\ge c-1$. Let $S'=(\cup_{i=2}^{a}\{x_i\})\cup(\cup_{i=2}^{b}\{y_i\})\cup(\cup_{i=2}^{c}\{z_i\})\subseteq S$. Note that (i) the adjacency code $a_{S'}(u_1)$ of u_1 with respect to S' is the $(a+b+c-3)$-vector with 1 on the first $(a-1)$ entries and 2 on the rest of the entries; (ii) $a_{S'}(u_2)$ is the $(a+b+c-3)$-vector with 1 on the ath through $(a+b-2)$th entries and 2 on the rest of the entries; (iii) $a_{S'}(u_3)$ is the $(a+b+c-3)$-vector with 2 on the first $(a+b-2)$ entries and 1 on the rest of the entries; (iv) $a_{S'}(x_1)=a_{S'}(y_1)=a_{S'}(z_1)=\mathbf{2}_{a+b+c-3}$. Since S' fails to be an adjacency resolving set of $G-e$ and, for any $w\in V(G-e)-S'$, $S'\cup\{w\}$ fails to be an adjacency resolving set of $G-e$, $\mathrm{adim}(G-e)\ge a+b+c-1$. On the other hand, $S'\cup\{x_1,y_1\}$ forms an adjacency resolving set of $G-e$, and hence $\mathrm{adim}(G-e)\le a+b+c-1$. Thus, $\mathrm{adim}(G-e)=a+b+c-1$.

Next, let $e=x_1z_1$. We show that $\mathrm{adim}(G)=a+b+c-2$. By Theorem 6.7, $\mathrm{adim}(G)\ge a+b+c-2$. Since $R=(\cup_{i=1}^{a}\{x_i\})\cup(\cup_{i=2}^{b}\{y_i\})\cup(\cup_{i=2}^{c}\{z_i\})$ forms an adjacency resolving set of G with $|R|=a+b+c-2$, $\mathrm{adim}(G)\le a+b+c-2$. Thus, $\mathrm{adim}(G)=a+b+c-2$. □

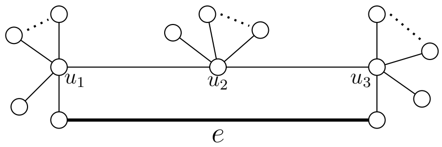

Figure 6: A graph G with $\mathrm{adim}(G-e)=\mathrm{adim}(G)+1$.

Question 6.9. *Is* $\mathrm{bdim}(G-e)\le\mathrm{bdim}(G)+d_{G-e}(u,v)-1$ *for every graph* G*, where* $e=uv\in E(G)$*?*

Question 6.10. *Is there a family of graphs* G *such that* $\mathrm{bdim}(G)-\mathrm{bdim}(G-e)$ *can be arbitrarily large?*

7 Open Problems

Below are some open problems about broadcast dimension that are only partially answered by the results in this paper.

Question 7.1. *Which graphs* G *satisfy* $\dim(G)=\mathrm{bdim}(G)$*?*

Question 7.2. *Which graphs* G *satisfy* $\mathrm{bdim}(G)=\mathrm{adim}(G)$*?*

Proposition 5.10, Corollary 5.11, and the results in Section 4 make some progress toward answering Questions 7.1 and 7.2.

Question 7.3. *Is there a family of graphs* G_k *with* $\mathrm{bdim}(G)=k$ *for which* $\mathrm{adim}(G)=2^{\Omega(k)}$*?*

Theorem 5.4 shows that for each $d \geq 1$ there is a family of graphs G_k with $\mathrm{bdim}(G) = k$ for which $\mathrm{adim}(G) = \Omega(k^d)$.

Question 7.4. *What are the values of* $\mathrm{bdim}(T)$ *and* $\mathrm{adim}(T)$ *for every tree* T*?*

Proposition 5.10 and Corollary 5.11 make progress on Question 7.4.

It is known that determining the domination number of a general graph is an NP-hard problem (see [16]), as are some of its variants [22, 34]. It is also known that determining the metric dimension (adjacency dimension, respectively) of a graph is an NP-hard problem; see [16] ([14], respectively). However, Heggernes and Lokshtanov [21] found a polynomial-time algorithm for computing the broadcast domination number $\gamma_b(G)$. This leads to the next question.

Question 7.5. *Is there a polynomial-time algorithm to determine the value of* $\mathrm{bdim}(G)$ *for every graph* G*?*

Another natural algorithmic problem is to list all minimum resolving broadcasts of a given graph. In the worst-case, any algorithm to solve this problem must take $2^{\Omega(n)}$ time for a graph of order n. We find an algorithm that takes $2^{O(n)}$ time to list all minimum resolving broadcasts of any given graph of order n.

Theorem 7.6. *There is an algorithm that takes* $2^{O(n)}$ *time to list all minimum resolving broadcasts of any given graph of order* n*. Any algorithm for listing all minimum resolving broadcasts of a given graph of order* n *must take* $2^{\Omega(n)}$ *time in the worst-case.*

Proof. For the worst-case, note that the graph H_k on $2k$ vertices consisting of k copies of K_2 has $2^{\Omega(k)}$ minimum resolving broadcasts, and so does the graph H'_k on $2k + 1$ vertices consisting of k copies of K_2 and an isolated vertex, so any algorithm for listing all minimum resolving broadcasts of a given graph of order n must take $2^{\Omega(n)}$ on the families H_k and H'_k.

For an algorithm to list all minimum resolving broadcasts of any given graph G of order n, we let $v_1, \ldots, v_n$ be the vertices of G. Let $s = 0$ and perform the following steps:

1. Increment s. Let $S = \emptyset$.
2. For each nonnegative integer solution $(x_1, \ldots, x_n)$ to the equation $x_1 + \cdots + x_n = s$, determine if the function f defined by setting $f(v_i) = x_i$ is a resolving broadcast for G. If f is a resolving broadcast, add f to S. If S is nonempty after checking every solution $(x_1, \ldots, x_n)$, return S and halt. Otherwise go back to step 1.

There are $\binom{s+n-1}{n-1}$ nonnegative integer solutions $(x_1, \ldots, x_n)$ to the equation $x_1 + \cdots + x_n = s$, so the algorithm only checks at most $\binom{2n-2}{n-1} = 2^{O(n)}$ solutions for each value of s. For each solution $(x_1, \ldots, x_n)$, it takes polynomial time in n to determine whether the solution corresponds to a resolving broadcast for G. Thus, the algorithm has $2^{O(n)}$ running time for graphs G of order n. □

Finally, we note that in the definition of broadcast dimension, we used a cost function that increased linearly with the strengths of the landmarks. It would be interesting to investigate variants of broadcast dimension with other cost functions.

Note: The present paper is based on [18], which was posted on arXiv on May 15, 2020. The article [2] was brought to our attention by one of the four referees. Theorem 6.7 and the sharpness of the bounds were also obtained in [2]. We also note that Corollary 3.6 was shown in [2].

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