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Wai Chee Shiua
Technology Research Institute

Gee Choon Lau
Universiti Teknologi MARA

Ho Kuen Ng
San Jose State University, ho-kuen.ng@sjsu.edu

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Edge-Coloring Vertex-Weighting of Graphs

Wai-Chee Shiu[a], Gee-Choon Lau[b*], Ho-Kuen Ng[c]

[a]College of Global Talents, Bejing Institute of Technology,
Zhuhai, China. &
Department of Mathematics, The Chinese University of Hong Kong,
Shatin, Hong Kong.
[b]Faculty of Computer & Mathematical Sciences,
Universiti Teknologi MARA Johor
85000 Segamat, Malaysia.
[c]Department of Mathematics, San José State University,
San José CA 95192 USA.

E-mail: wcshiu@associate.hkbu.edu.hk
E-mail: geeclau@yahoo.com
E-mail: ho-kuen.ng@sjsu.edu

Abstract. Let $G = (V(G), E(G))$ be a simple, finite and undirected graph of order n. A k-vertex weighting of a graph G is a mapping $w : V(G) \to \{1, \ldots, k\}$. A k-vertex weighting induces an edge labeling $f_w : E(G) \to \mathbb{N}$ such that $f_w(uv) = w(u) + w(v)$. Such a labeling is called an *edge-coloring k-vertex weighting* if $f_w(e) \neq f_w(e')$ for any two adjacent edges e and e'. Denote by $\mu'(G)$ the minimum k for G to admit an edge-coloring k-vertex weighting. In this paper, we determine $\mu'(G)$ for some classes of graphs.

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*Corresponding Author

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1. Introduction

Let $G = (V(G), E(G))$ (or simply $G = (V, E)$ for short, if there is no ambiguity) be a simple, finite and undirected graph of order $|V| = v$ and size $|E| = e$. All notation not defined in this paper can be found in [1].

The first paper on graph labeling was introduced by Rosa in 1967. Since then, there have been more than 1500 research papers on graph labelings being published (see the dynamic survey by Gallian [4]).

In [7], the concept of vertex-coloring k-edge weighting was introduced.

Definition 1.1. A mapping $w : E(G) \to \{1, \dots, k\}$ induces a vertex labeling $f_w : V(G) \to \mathbb{N}$ such that $f_w(v)$ is the sum of the weighting of the edges incident to v. Such a labeling is called a *vertex-coloring k-edge weighting* if $f_w(u) \neq f_w(v)$ for any edge uv.

Denote by $\mu(G)$ the minimum k such that G has a vertex-coloring k-edge weighting. Clearly, such a graph G does not have a K_2 as a component. We say a graph is non-trivial if it does not contain a K_2 as a component. It is conjectured in [7] that $\mu(G) \leq 3$ for all non-trivial graph G.

Several results on vertex-coloring k-edge weighting graphs can be found in [2, 3, 5, 6, 9]. In this paper, we introduce a dual version of vertex-coloring k-edge weighting which is defined as follow.

Definition 1.2. A mapping $w : V(G) \to \{1, \dots, k\}$ induces an edge labeling $f_w : E(G) \to \mathbb{N}$ such that $f_w(uv) = w(u) + w(v)$. Such a labeling is called an *edge-coloring k-vertex weighting* if $f_w(e) \neq f_w(e')$ for any two adjacent edges e and e'.

Denote by $\mu'(G)$ the minimum k for G to admit an edge-coloring k-vertex weighting.

The following facts follow directly from the definition.

Fact 1. $\mu'(G) = 1$ if and only if every component of G is a K_2.

Fact 2. Suppose w is an edge-coloring k-vertex weighting of G. If u and v have a common neighbor in G, then $w(u) \neq w(v)$. This is also a sufficient condition for an edge-coloring vertex weighting.

By the definition of edge-coloring k-vertex weighting, it induces an edge-coloring for the concerning graph. Precisely, it induces a k-edge-coloring. Following is the detail.

Fact 3. Let $\chi'(G)$ be the chromatic index of G. Then $\mu'(G) \geq \chi'(G)$. Hence $\mu'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of G.

Proof. Suppose w is an edge-coloring k-vertex weighting of G. Let A be the multiplication table of $\mathbb{Z}_k$. One may choose any symmetric Latin square of order k. Define $f : E(G) \to \mathbb{Z}_k$ by $f(uv) = (A)_{w(u),w(v)}$, that is the $(w(u), w(v))$-entry of A. Clearly, f is a proper k-edge-coloring. □

2. $\mu'(G)$ for Some Classes of Graphs

Proposition 2.1. *For $n \geq 2$, $\mu'(P_n) = \Delta(P_n)$.*

Proof. Let $P_n = v_1 v_2 \cdots v_n$. For $n = 2$, it follows from Fact 1. For $n \geq 3$, $\Delta(P_n) = 2$. It follows from the fact that the following mapping w is an edge-coloring 2-vertex weighting for $n \geq 3$: $w(v_i) = 1$ for $i \equiv 1, 2 \pmod 4$, and $w(v_i) = 2$ for $i \equiv 0, 3 \pmod 4$. □

Proposition 2.2. *For $n \geq 3$, $\mu'(C_n) = 2$ if $n \equiv 0 \pmod 4$ and $\mu'(C_n) = 3$ otherwise.*

Proof. Let $C_n = u_1 u_2 \cdots u_n u_1$. It follows from Fact 3 that $\mu'(C_n) \geq 2$. From Fact 2, it is clear that $\mu'(C_3) = 3$. We now assume $n \geq 4$.

Case 1. $n \equiv 0 \pmod 4$. Define $w(u_i) = 1$ for $i \equiv 1, 2 \pmod 4$, and $w(u_i) = 2$ for $i \equiv 0, 3 \pmod 4$. Clearly, w is an edge-coloring 2-vertex weighting.

Case 2. $n = 4k + r$, $1 \leq r \leq 3$. Assign u_i as in Case 1 for $i \leq 4k$. If $r = 1, 2$, then assign the remaining vertices by 3. If $r = 3$, then assign u_{n-2} and u_{n-1} by 3 and u_n by 2. Clearly, the mapping is an edge-coloring 3-vertex weighting.

Suppose $\mu'(C_n) = 2$. Without loss of generality, we assume $w(u_1) = 1$. By Fact 2, we must have $w(u_3) = 2$. Hence, $w(u_2) = 1$ or $w(u_2) = 2$. If the former holds, then we must assign u_i as in Case 1 for $i \leq 4k$. Now, the remaining vertices must be assigned with 1 or 2, contradicting Fact 2. If the later holds, then we must assign u_i by 1 for $i \equiv 0, 1 \pmod 4$, and by 2 for $i \equiv 2, 3 \pmod 4$. Again, assigning the remaining vertices by 1 or 2 contradicts Fact 2.

□

Proposition 2.3. *For $n \geq 3$, $\mu'(K_n) = n$.*

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. By Fact 3, $\mu'(K_n) \geq n - 1$. If the equality holds, then at least two vertices will be assigned the same integer in $\{1, 2, \ldots, n-1\}$. This contradicts Fact 2 since every two vertices of K_n have $n - 2$ common neighbors. Hence, $\mu'(K_n) \geq n$. The following mapping w show that the equality holds: $w(v_i) = i$ for $1 \leq i \leq n$. □

Proposition 2.4. *For $1 \leq m \leq n$, $\mu'(K_{m,n}) = \Delta(K_{m,n}) = n$.*

Proof. Let the vertices in the two partite sets be $u_i, 1 \leq i \leq m$ and $v_i, 1 \leq i \leq n$. By Fact 3, $\mu'(K_{m,n}) \geq n$. The following mapping w shows that the equality holds: $w(u_i) = w(v_i) = i$. □

Let W_n be the wheel graph of order $n + 1$ with a central vertex u of degree n.

Theorem 2.5. *For $n \geq 4$, $\mu'(W_n) = n$. Moreover, $\mu'(W_3) = 4$.*

Proof. Note that $W_3 \cong K_4$. By Proposition 2.3, $\mu'(W_3) = 4$. Suppose $n \geq 4$. By Fact 3, $\mu'(W_n) \geq n$. The following mapping w shows that the equality holds: $w(u) = 1$, assign the remaining vertices by 1 to n consecutively. □

Definition 2.6. Let $G_n, n \geq 2$, denote the *gear graph* obtained from a wheel graph of order $2n+1$ by deleting n spokes where no two of the spokes are consecutive.

Theorem 2.7. *For $n \geq 4$, $\mu'(G_n) = n$. Moreover, $\mu'(G_2) = 3$ and $\mu'(G_3) = 4$.*

Proof. Let u be the central vertex of degree n and the induced cycle C_{2n} be $v_1v_2\cdots v_{2n}$ such that v_i is of degree 3 for odd i. Suppose $n = 2$. By Fact 3, $\mu'(G_2) \geq 3$. Assign u by 3 and v_1 to v_4 by 1,1,2,2. We have $\mu'(G_2) = 3$. Note that $\Delta(G_2) = \mu'(G_2)$.

Suppose $n = 3$. By Fact 3, $\mu'(G_3) \geq 3$. Suppose the equality holds. Since u is adjacent to vertices v_{2i-1}, we assign v_{2i-1} by i for $1 \leq i \leq 3$. Without loss of generality, assume u is assigned 1. By Fact 2, none of $v_{2i}, i = 1, 2, 3$, is assigned by 1. Hence, at least two of them must get the same label. This contradicts Fact 2. Hence, $\mu'(G_3) \geq 4$. Now, assign v_1 to v_6 by $2, 2, 3, 3, 4, 4$ consecutively. We have $\mu'(G_3) = 4$.

Now assume $n \geq 4$ so that $\Delta(G_n) = n$. By Fact 3, $\mu'(G_n) \geq n$. The following mapping w shows that equality holds: $w(u) = 1$, $w(v_{2i-1}) = i$ for $1 \leq i \leq n$, $w(v_{2i}) = i+1$ for $1 \leq i \leq n-1$, and $w(v_{2n}) = n-1$. □

Note that $\mu'(G_3) = \Delta(G_3) + 1$. This shows that not all bipartite graphs G have $\mu'(G) = \Delta(G)$.

Problem 2.1. *Find necessary and/or sufficient condition for a bipartite graph G to have $\mu'(G) = \Delta(G)$.*

Remark 2.8. It would be more natural to ask first whether the exact 2-distance chromatic number of bipartite graphs is computable in polynomial time. The following argument shows that the problem is NP-complete, which means that "algorithmically simple" conditions that are both necessary and sufficient very likely do not exist.

Let H be a 3-regular graph and let G be the graph obtained from H by replacing every edge of G by a path of length 2 (that is, G is the 1-subdivision of H). Clearly, G is a bipartite graph, with one part, A, formed by the vertices of G and the other part, B, formed by the new degree-2 vertices. The exact 2-distance chromatic number of G is now equal to the maximum of $\chi(H)$ and $\chi'(H)$ (the chromatic number and the chromatic index of H). Indeed, the maximum number of colors needed to color the vertices in A is equal to $\chi(H)$, and the maximum number of colors needed to color the vertices in B is equal to $\chi'(H)$.

By Brooks theorem, $\chi(H) = 3$ whenever H is not isomorphic to K_4. By Vizing's theorem, $\chi'(H)$ is equal to 3 or 4, so whenever H is not isomorphic to K_4, then the exact 2-distance chromatic number of G is equal to $\chi'(H)$. Finally, computing the chromatic index of a 3-regular graph is an NP-complete problem [8].

Theorem 2.9. *For a grid $P_m \times P_n, m, n \geq 3$, we have $\mu'(P_m \times P_n) = 4$.*

Proof. By Fact 3, we have $\mu'(P_m \times P_n) \geq 4$. We now show that equality holds.

View the grid as a collection of horizontal paths, with the paths stacked one above another. Label the vertices as follows, path by path, from top to bottom.

1 2 2 1 1 2 2 1 ...
3 4 4 3 3 4 4 3 ...
2 1 1 2 2 1 1 2 ...
4 3 3 4 4 3 3 4 ...

Continue with the same pattern. It can be verified that all adjacent edges will get distinct weights. □

From Fact 1, we know all 1-regular graphs G have $\mu'(G) = 1$. From Proposition 2.2, we know that all 2-regular graphs G have $\mu'(G) = 2$ if and only if each component of G is an n-cycle with $n \equiv 0 \pmod 4$. From Proposition 2.3, we know that for each $r \geq 2$, there exists an r-regular graph G such that $\mu'(G) = r + 1$. In next theorem, we show that there are also 3-regular graphs G with $\mu'(G) = 3$ or 4.

Theorem 2.10. *For $n \geq 3$, we have*

$$\mu'(P_2 \times C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3; \\ 4 & \text{if } n \not\equiv 0 \pmod 3. \end{cases}$$

Proof. Let $G = P_2 \times C_n$ with the 2 induced C_n given by $C_1 = u_1u_2 \cdots u_nu_1$ and $C_2 = v_1v_2 \cdots v_nv_1$, where u_iv_i are edges, $1 \leq i \leq n$. By Fact 3, $\mu'(G) \geq 3$.

(1) $n \equiv 0 \pmod 3$. Assign the vertices u_i and v_i by 1,2,3 periodically for $1 \leq i \leq n$. It is clear that the labeling is an edge-coloring 3-vertex weighting.

(2) Suppose $n \equiv 1 \pmod 6$. Assume $n = 7$. Suppose $\mu'(P_2 \times C_7) = 3$. By Fact 2, and without loss of generality, assume that u_2, u_7, v_1 are assigned by $1, 2, 3$ respectively. It follows that we must assign v_3, u_4, v_5, u_6, v_7 by $2, 3, 1, 2, 3$ respectively. This implies that v_6, u_5, v_4, u_3, v_2 must be assigned by $1, 3, 2, 1, 3$ respectively. We get a contradiction since v_2 and v_7 are assigned by 3. Hence, $\mu'(P_2 \times C_7) \geq 4$. For $n \geq 13$, we can show similarly that $\mu'(G) \geq 4$.

Suppose $n \equiv 4 \pmod 6$. Assume $n = 4$. Suppose $\mu'(P_2 \times C_4) = 3$. By Fact 2, and without loss of generality, assume that u_2, u_4, v_1 are assigned $1, 2, 3$ respectively. It follows that v_3 cannot be assigned since it has a common neighbor with u_2, u_4, v_1, respectively. Hence, $\mu'(P_2 \times C_4) \geq 4$. For $n \geq 10$, we can show similarly that $\mu'(G) \geq 4$.

To show that equality holds for $n \equiv 1 \pmod 3$, we assign u_i and v_i by $1, 2, 3$ periodically for $1 \le i \le n-1$, and assign u_n, v_n by 4.

(3) Suppose $n \equiv 5 \pmod 6$. Assume $n = 5$. Suppose $\mu'(P_2 \times C_5) = 3$. By Fact 2, and without loss of generality, assume that u_2, u_5, v_1 are assigned by $1, 2, 3$ respectively. It follows that we must assign v_3, u_4, v_5 by $2, 3, 1$ respectively. This implies that v_4, u_3, v_2 must be assigned by $1, 3, 2$ respectively. Now u_1 cannot be assigned. Hence, $\mu'(P_2 \times C_5) \ge 4$. Clearly $\mu'(P_2 \times C_5) = 4$ by the same assignment as above and assign u_1 by 4. For $n \ge 11$, we can show similarly that $\mu'(G) \ge 4$. To show that equality holds, we can assign u_1 by 4 and u_2 to u_n by $1, 3, 3, 2, 2, 1$ periodically. We then assign v_1 to v_n by $3, 2, 2, 1, 1, 3$ periodically.

Suppose $n \equiv 2 \pmod 6$. Assume $n = 8$. Suppose $\mu'(P_2 \times C_8) = 3$. By Fact 2, and without loss of generality, assume that u_2, u_8, v_1 are assigned by $1, 2, 3$ respectively. It follows that we must assign v_3, u_4, v_5 by $2, 3, 1$ respectively. Then u_6 cannot be assigned. Hence, $\mu'(P_2 \times C_8) \ge 4$. For $n \ge 14$, we can show similarly that $\mu'(G) \ge 4$. Now we suppose to give a vertex-coloring 4-edge weighting for this case. $n \equiv 0 \pmod 4$. Assign u_1 to u_n by $1, 2, 3, 4$ periodically. Then assign the label to v_i by the same label of u_i. Suppose $n \equiv 2 \pmod 4$. Assign u_1 to u_{n-2} by $1, 2, 3, 4$ periodically. Then assign v_1 to v_{n-1} by $3, 4, 1, 2$ periodically. Finally, assign u_{n-1}, u_n, v_{n-1} and v_n by 4, 1, 2, and 3, respectively.

□

Definition 2.11. Let $DS(m, n)$ denote the *double star graph* obtained from $K_{1,m}$ and $K_{1,n}$ by adding an edge joining the two vertices of the two bipartite graphs with maximum degree.

Theorem 2.12. *For $1 \le m \le n$, $\mu'(DS(m, n)) = n + 1$.*

Proof. Let the vertex sets of $K_{1,m}$ and $K_{1,n}$ be $\{u\} \cup \{u_i \mid 1 \le i \le m\}$ and $\{v\} \cup \{v_i \mid 1 \le i \le n\}$, where u and v are the central vertices, respectively.

By Fact 3, we have $\mu'(DS(m, n)) \ge n + 1$. The following mapping w shows that the equality holds: $w(u) = n + 1, w(u_i) = i + 1, w(v) = 1, w(v_i) = i$. □

Theorem 2.13. *All trees T have $\mu'(T) = \Delta(T)$.*

Proof. By Fact 3, $\mu'(T) \ge \Delta(T)$. It suffices to show that the equality holds. Let u be a vertex of T and call it the root. Label u with 1. Now u has $d \le \Delta(T)$ children, say $u_1, u_2, \ldots, u_d$. Label these children using distinct integers in $\{1, 2, \ldots, d\}$. Suppose $deg(u_i) = d_i \ge 2$. Observe that each of u_i $(1 \le i \le d)$ has its parent u and at most $d_i - 1$ children as adjacent vertices. Since u_i already has a label, we label the children of u_i using distinct integers in $\{1, 2, \ldots, d_i\}$ that are not labels of the parent of u_i. This process can be repeated until all the vertices with a common neighbor are labeled with distinct integers. Since T is a tree, we cannot go back to a previously labeled vertices. This guarantees that all integers in $\{1, 2, \dots, \Delta(T)\}$ are being used and that all adjacent edges have distinct labels. □

Definition 2.14. A *tadpole graph* $T_{n,l}$ is a simple graph obtained from an n-cycle by attaching a path of length l, where $n \geq 3$ and $l \geq 1$. Let the n-cycle be $u_1 u_2 \cdots u_{n-1} u_n$ and the attached path be $u_n v_1 \cdots v_l$.

Theorem 2.15. *For $n \geq 3$ and $l \geq 1$, $\mu'(T_{n,l}) = 3$.*

Proof. By Fact 3, we know $\mu'(T_{n,l}) \geq 3$. We now show that equality holds. The weights of the vertices belong to the n-cycle are assigned as in Proposition 2.2 such that

(i) for $n \equiv 0 \pmod 4$, $w(u_{n-1}) = w(u_n) = 2$. Assign v_1 by 3 and the remaining vertices by 1,1,2,2 periodically;
(ii) for $n \equiv 1 \pmod 4$, $w(u_n) = 3$. Assign v_1 by 3 and the remaining vertices by 1,1,2,2 periodically;
(iii) for $n \equiv 2 \pmod 4$, $w(u_{n-1}) = w(u_n) = 3$. Assign the vertices v_1 to v_l by 2,2,1,1 periodically;
(iv) for $n \equiv 3 \pmod 4$, $w(u_{n-2}) = w(u_{n-1}) = 3$, $w(u_n) = 2$. Assign the vertices v_1 to v_l by 1,1,2,2 periodically.

□

Definition 2.16. A *lollipop graph* $L_{n,l}$ is a simple graph obtained from a complete graph K_n attaching a path of length l, where $n \geq 3$ and $l \geq 1$. Let vertices of K_n-cycle be $u_1, \dots, u_n$ and the attached path be $u_1 v_1 \cdots v_l$.

Theorem 2.17. *For $n \geq 3$ and $l \geq 1$, $\mu'(L_{n,l}) = n$.*

Proof. By Fact 3, we know $\mu'(L_{n,l}) \geq n$. The following mapping w shows that the equality holds: $w(u_i) = i$, $1 \leq i \leq n$, which is same as in Proposition 2.3. Assign the vertices v_1 to v_l by 1,2,2,1 periodically. □

Definition 2.18. Given $t \geq 2$ paths, $P_{n_j} = v_{j,1} \cdots v_{j,n_j}$, of order $n_j \geq 2$, $(1 \leq j \leq t)$. A *spider graph* $SP(n_1, \dots, n_t)$ is the one-point union of the t paths at vertex $v_{j,1}$.

Theorem 2.19. *For $t \geq 2$, $\mu'(SP(n_1, \dots, n_t)) = t$.*

Proof. Let the merged vertex be denoted by $v_{1,1}$. Assign $v_{j,2}$ to v_{j,n_j} by $j, j, 1, 1$ periodically if $2 \leq j \leq t$. Assign $v_{1,1}$ to v_{1,n_1} by $1, 1, 2, 2$ periodically. Then we have an edge-coloring t-vertex weighting. Hence by Fact 3, we know $\mu'(SP(n_1, \dots, n_t)) = t$. □

Definition 2.20. For $t \geq 2$, a *one point union* of t cycles is a graph obtained from t cycles, say C_{n_i} for $n_i \geq 3, 1 \leq i \leq t$, by identifying one vertex from each cycle. We denote such a graph by $U(n_1, \dots, n_t)$. Without loss of generality, we always assume that $3 \leq n_1 \leq \cdots \leq n_t$.

Theorem 2.21. *For $t \geq 2$, $\mu'(U(n_1, \dots, n_t)) = 2t + 1$ if $n_j = 3$ for $1 \leq j \leq t$ and $\mu'(U(n_1, \dots, n_t)) = 2t$ otherwise.*

Proof. Let the t cycles be $C_{n_j} = v_{j,1} \cdots v_{j,n_j} v_{j,1}$. We merge $v_{1,1}$, $v_{2,1}$, $v_{3,1}$, ..., $v_{t,1}$ into one vertex, say $v_{1,1}$ again.

Suppose $n_t = 3$. By Fact 2, all the vertices must get different labels. Hence, $\mu'(U(n_1, \dots, n_t)) = 2t + 1$. This can be attained by defining $w(v_{1,1}) = 1$, $w(v_{j,2}) = 2j$ and $w(v_{j,3}) = 2j + 1$ for $1 \leq j \leq t$.

Suppose $n_t \geq 4$. Consider the star induced by the set $X = \{v_{1,1}\} \cup \{v_{j,2}, v_{j,n_j} \mid 1 \leq j \leq t\}$. Assign $v_{1,1}$ and $v_{t,2}$ by 1, v_{t,n_t} by $2t$, $v_{j,2}$ by $2j$ for $1 \leq j \leq t-1$ and v_{j,n_j} by $2j + 1$ for $1 \leq j \leq t-1$. The subgraph $U(n_1, \dots, n_t) - X$ is a disjoint union of some paths. Assign the path $v_{1,3}v_{1,4} \cdots v_{1,n_1-1}$ by 4,4,2,2 periodically and each of other path by 2,2,3,3 periodically. □

Definition 2.22. A *cycle with a long chord* (or *theta graph*) is a graph obtained from a cycle C_m, $m \geq 4$, by adding a chord of length l where $l \geq 1$. Namely, let $C_m = u_0 u_1 \cdots u_{m-1} u_0$. Without loss of generality, we may assume the long chord joins u_0 with u_a, where $2 \leq a \leq m-2$. That is, $u_0 u_m u_{m+1} \cdots u_{m+l-2} u_a$ is the chord. We denote this graph by $C_m(a; l)$. Note that, by symmetry we may assume that $2 \leq a \leq \lfloor m/2 \rfloor$; when $l = 1$, the chord is $u_0 u_a$.

Theorem 2.23. *For $a \geq 2$, $l \geq 1$, $\mu'(C_m(a; l)) = 3$ except a graph isomorphic to $C_6(2; 2)$, $C_6(3; 2)$ or $C_6(3; 3)$. Moreover $\mu'(C_6(2; 2) = \mu'(C_6(3; 2)) = \mu'(C_6(3; 3)) = 4$.*

Proof. The theta graph contains three cycles which are isomorphic to C_m, C_{a+l} and C_{m-a+l}. Hence at least one cycle is even. By Fact 3, $\mu'(C_m(a, l)) \geq 3$.

(1) Suppose there is a $4k$-cycle. Then the graph is isomorphic to $C_{4k}(a; l)$ for some $l \geq 1$ and $2 \leq a \leq 2k$. The weights of the vertices belong to the $4k$-cycle is assigned as in Proposition 2.2 starting at u_0. Let such labeling be w. Hence $w(u_0) = 1$ and $w(u_a) = 1$ or 2.
   - (a) $l \equiv 0 \pmod 4$. Reassign u_1 by 3 and assign or reassign the path $u_0 u_{4k} \cdots u_{4k+l-2}$ by 1,1,3,3 periodically.
   - (b) $l \equiv 1 \pmod 4$. If $l = 1$, then reassign u_a by 3. If $l \geq 5$, then assign or reassign the path $u_1 u_0 u_{4k} \cdots u_{4k+l-2}$ by 3,3,1,1 periodically
   - (c) $l \equiv 2 \pmod 4$. Assign or reassign the path $u_0 u_{4k} \cdots u_{4k+l-2}$ by 3,3,1,1 periodically.
   - (d) $l \equiv 3 \pmod 4$. Assign the path $u_{4k} \cdots u_{4k+l-2}$ by 3,3,1,1 periodically.

   Hence, $\mu'(C_{4k}(a; l)) = 3$.

(2) Suppose there is a $2k$-cycle for some odd $k \geq 3$. Then the graph is isomorphic to $C_{2k}(a; l)$ for some $l \geq 1$ and $2 \leq a \leq k$. The weights of the vertices belong to the $2k$-cycle is assigned as in Proposition 2.2 starting at

u_1. Let such labeling be w. Hence $w(u_0) = 3 = w(u_{2k-1})$, $w(u_{2k-2}) = 2$ and $w(u_a) = 1$ or 2.

(a) $l \equiv 0 \pmod 4$. Suppose $w(u_a) = 2$. Let $w(u_{2k}) = 2$ and assign the path $u_{2k+1} \cdots u_{2k+l-2}$ by 1,3,3,1 periodically. Suppose $w(u_a) = 1$. Assign the path $u_{2k} \cdots u_{2k+l-2}$ by 2,2,3,3 periodically.

(b) $l \equiv 1 \pmod 4$. Suppose $l \geq 5$. Then assign the path $u_{2k} \cdots u_{2k+l-2}$ by 2,2,3,3 periodically.
Suppose $l = 1$. If $w(u_a) = 2$, then nothing needs to do. Suppose $w(u_a) = 1$. If $a \equiv 2 \pmod 4$, then $w(u_{a-1}) = 1$. For $k = 3$, reassign the vertices u_0 to u_5 by 1,2,3,3,2,1 respectively. For $k \geq 5$, remove the chord u_0u_a and add the chord $u_{2k-1}u_{a-1}$. The resulting graph is isomorphic to the original and the weights assignment is proper. If $a \equiv 1 \pmod 4$, then $w(u_{a-1}) = 2$. Reassign the vertices u_{2k-3}, u_{2k-2}, u_{2k-1}, u_0, u_1 by 3,2,2,3,3, respectively.

(c) $l \equiv 2 \pmod 4$. Suppose $l \geq 6$. Let $w(u_{2k}) = 2$ and assign the path $u_{2k+1} \cdots u_{2k+l-2}$ by 1,1,3,3 periodically.
Suppose $l = 2$. Assume $k \geq 5$. If $w(u_a) = 1$, then reassign or assign the vertices u_{2k-3}, u_{2k-2}, u_{2k-1}, u_0, u_{2k} by 3,3,2,2,3, respectively. If $w(u_a) = 2$, then $a \equiv 3, 0 \pmod 4$. Since $k \geq 5$ and $a \neq 5$, $a < 2k-5$. Reassign or assign the vertices u_{2k-3}, u_{2k-2}, u_{2k-1}, u_0, u_{2k}, u_a by 3,3,2,2,3,3, respectively. For $k = 3$, we will deal with $C_6(2;2)$ and $C_6(3;2)$ later.

(d) $l \equiv 3 \pmod 4$. Suppose $l \geq 7$. Let $w(u_{2k}) = w(u_{2k+1}) = 2$ and assign the path $u_{2k+2} \cdots u_{2k+l-2}$ by 1,1,3,3 periodically.
Suppose $l = 3$. When $k \geq 5$, $a < 2k-4$. Reassign or assign the vertices u_{2k-3}, u_{2k-2}, u_{2k-1}, u_0, u_{2k}, u_{2k+1} by 3,3,2,2,3,3, respectively. When $k = 3$, we only need to deal with $C_6(2;3)$. Figure 1(a) shows the proper assignment of $C_6(2;3)$.

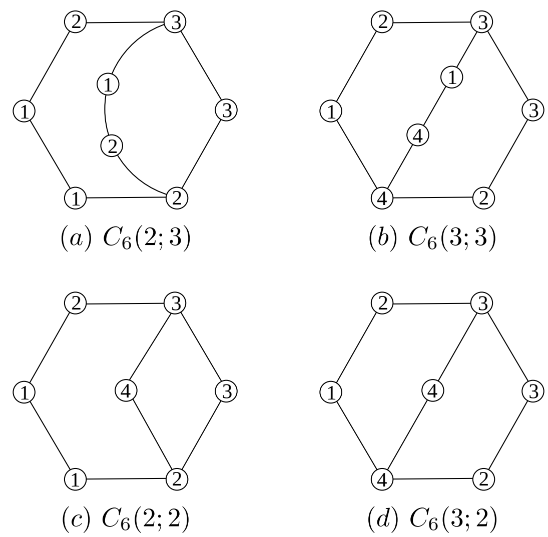


FIGURE 1. Proper edge-colorings vertex weighting of graphs.

For $C_6(3;3)$, suppose there is an edge-coloring 3-vertex weighting w of it. By symmetry and without loss of generality, we may assume $w(u_0) = 3$, $w(u_1) = 1$, $w(u_5) = 3$ and $w(u_6)$=2. By Fact 2, u_2, u_4, u_7 cannot be assigned by 3. Hence at least two of them are assigned the same weight. This contradicts Fact 2. So $\mu'(C_6(3;3) \geq 4$. Figure 1(b) shows that $\mu'(C_6(3;3)) = 4$.

For $C_6(2;2)$, suppose there is an edge-coloring 3-vertex weighting w of it. By Fact 2, $w(u_1)$ and $w(u_6)$ are distinct. Without loss of generality, we may assume $w(u_1) = 1$ and $w(u_6) = 2$. By Fact 2 again, we have $w(u_3) = 3 = w(u_5)$, a contradiction. So $\mu'(C_6(2;2)) \geq 4$. Figure 1(c) shows that $\mu'(C_6(2;2)) = 4$.

For $C_6(3;2)$, suppose there is an edge-coloring 3-vertex weighting w of it. Without loss of generality, we may assume $w(u_0) = 3$. Suppose $w(u_6) = 3$. Then $w(u_1)$, $w(u_3)$ and $w(u_5)$ are distinct and cannot be 3, a contradiction. So $w(u_6) \neq 3$. Without loss of generality, we may assume $w(u_6) = 1$. By symmetry, we may assume $w(u_1) = 3$. Hence, $w(u_5) = 2$ and $w(u_3) = 1$. So, $w(u_2)$ and $w(u_4)$ must be distinct and cannot be 1 or 3, a contradiction. Therefore, $\mu'(C_6(3;2)) \geq 4$. Figure 1(d) shows that $\mu'(C_6(3;2)) = 4$. □

Definition 2.24. A *long dumbbell graph* is a graph obtained from two cycles C_a and C_b, by joining a path P_{k+1} of length k for $a, b \geq 3$ and $k \geq 1$. Without loss of generality, we may assume $a \geq b$ and

$$C_a = u_1 \cdots u_a u_1, \quad P_{k+1} = u_1 w_1 \cdots w_{k-1} v_1 \quad \text{and } C_b = v_1 \cdots v_b v_1.$$

This graph is denoted by $D(a, b; k)$. When $k = 1$, $P_2 = u_1 v_1$ and $D(a, b; k)$ is called a *dumbbell graph*.

Theorem 2.25. *For $a \geq b \geq 3$ and $k \geq 1$, $\mu'(D(a,b;k)) = 3$ except that $\mu'(D(3,3;2)) = 4$.*

Proof. Suppose $a = b = 3$. We define $w(u_i) = i$, $1 \leq i \leq 3$. Assume $k \neq 2$. Assign the path P_{k+1} by 1,1,3,3 periodically. If $w(w_{k-1}) \neq w(v_1)$, reassign w_{k-1} and v_1 by 2. It is now easy to assign weights to the two remaining vertices of C_b to get $\mu'(D(3,3;k)) = 3$. Now assume $\mu'(D(3,3;2)) = 3$. By Fact 2, $w(u_1), w(u_2), w(u_3)$ (respectively, $w(u_2), w(u_3), w(w_1)$) are distinct. Without loss of generality, we define $w(u_i) = i$, $1 \leq i \leq 3$. Hence, we must define $w(w_1) = 1$. By symmetry, we must have $w(v_1) = 1$, contradicting Fact 2. Now, define $w(v_i) = i + 1$. We have $\mu'(D(3,3;2)) = 4$.

Suppose $a \geq 4$. The weights of the vertices belong to the a-cycle are assigned as in Proposition 2.2 starting at u_a, then u_1 and so on. Thus $w(u_a) = w(u_1) = 1$ and $w(u_2) = 2$. Assign the path $w_1 \cdots w_{k-1}v_1$ by 3,3,2,2 periodically for $k \geq 2$; let $w(v_1) = 3$ for $k = 1$. For this partial assignment, we will deal with the following cases:

(1) $k \equiv 0 \pmod 4$, i.e., $w(w_{k-1}) = 2$, $w(v_1) = 2$. If $b \equiv 0 \pmod 4$, then redefine $w(v_1) = 1$ and weights of the vertices belong to the b-cycle are assigned as in Proposition 2.2 starting at v_1 by using weights 1 and 3. If $b \equiv 1, 2 \pmod 4$, then redefine $w(v_1) = 1$ and weights of the vertices belong to the b-cycle are assigned as in Proposition 2.2 starting at v_1. If $b \equiv 3 \pmod 4$ and $b \geq 7$, then weights of the vertices belong to the b-cycle are assigned as in Proposition 2.2 starting at v_2 and ending at v_1, which has been assigned by 2. If $b = 3$, then define $w(v_2) = 1$ and $w(v_3) = 3$.

(2) $k \equiv 3 \pmod 4$, i.e., $w(w_{k-1}) = 3$, $w(v_1) = 2$. If $b \geq 4$, then weights of the vertices belong to the b-cycle are assigned as in Proposition 2.2 starting at v_{b-1} and ending at v_{b-2}. So, $w(v_b) = 1$, $w(v_1) = 2$, $w(v_2) = 2$.

    Suppose $b = 3$. If $k = 3$, then redefine $w(w_2) = 2$. Clearly, it is easy to assign weights to C_b. If $k \geq 7$, then redefine $w(w_2) = 1$ and $w(v_1) = 1$. Clearly, it is easy to assign weights to C_b.

(3) $k \equiv 2 \pmod 4$, i.e., $w(w_{k-1}) = 3$, $w(v_1) = 3$. If $b = 3$, then it is easy to assign the weights to C_b. Suppose $b \geq 4$. Redefine $w(v_1) = 1$. Assign the vertices of C_b as in Proposition 2.2 starting at v_b and ending at v_{b-1}.

(4) $k \equiv 1 \pmod 4$. Suppose $k \geq 5$, i.e., $w(v_1) = 3$ and $w(w_{k-1}) = 2$. If $b = 3$, then redefine $w(w_{k-1}) = 1 = w(v_1)$. It is easy to have a proper assignment. Suppose $b \geq 4$. If $b \equiv 0 \pmod 4$, then weights of the vertices belong to the b-cycle are assigned as in Proposition 2.2 starting at v_1 by using weights 3 and 1. If $b \equiv 1, 2 \pmod 4$, then redefine $w(v_1) = 1$ and weights of the vertices belong to the b-cycle are assigned as in Proposition 2.2 starting at v_1. If $b \equiv 3 \pmod 4$, then weights of the vertices belong to the b-cycle are assigned as in Proposition 2.2 starting at v_1, but change the original weights 1,2,3 to 3,1,2 accordingly.

Suppose $k = 1$. If $b \equiv 0 \pmod 4$, assign the vertices of C_b as in Proposition 2.2 but change the original weights 1,2 to 3,2 accordingly. If $b \equiv 1, 2 \pmod 4$, assign the vertices of C_b as in Proposition 2.2 but change the original weights 1,2,3 to 3,1,2 accordingly. Now consider $b \equiv 3 \pmod 4$. If $b = 3$, assign v_2, v_3 by 2, 1 respectively. If $b \geq 7$, then weights of the vertices belong to the b-cycle are assigned as in Proposition 2.2 starting at v_1, but change the original weights 1,2,3 to 3,2,1 accordingly.

□

Example 2.26. The following figures are illustration of some cases in the proof of the above theorem when $k \equiv 1 \pmod 4$.

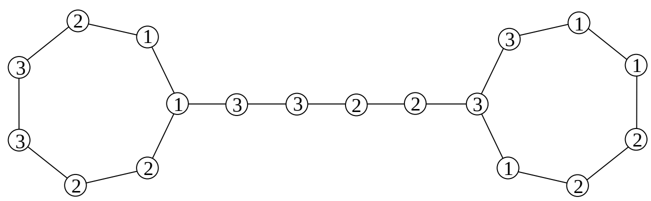


Figure 2. A proper edge-coloring 3-vertex weighting of $D(7, 7; 5)$

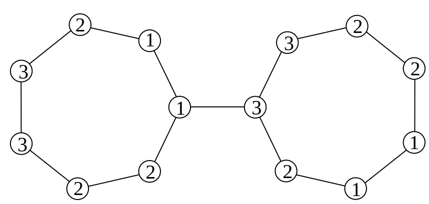
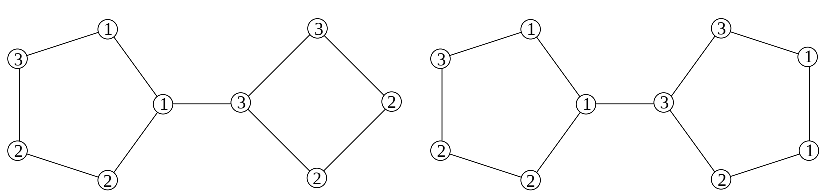


Figure 3. A proper edge-coloring 3-vertex weighting of $D(7, 7; 1)$



(a) $D(5, 4; 1)$ (b) $D(5, 5; 1)$

Figure 4. Proper edge-colorings 3-vertex weighting of some dumbbell graphs.

3. Remark

We may generalize the weights from positive integers to elements of an abelian. We will obtain the same results by the same arguments.

We may also generalize the definition of edge-coloring vertex weighting as follows:
Let S be a set of cardinal k. A k-vertex weighting of a graph G is a mapping $w : V(G) \to S$. This mapping is called *edge-coloring k-vertex weighting* if the weights of all neighbors of a vertex u are distinct, for each vertex u. Denote by $\mu'(G)$ the minimum k for G to admit an edge-coloring k-vertex weighting. Again, we will obtain the same results by the same arguments.

We end the paper with the following open problems.

Problem 3.1. *Characterize all 3-regular graphs G such that $\mu'(G) = 3$ or 4.*

Problem 3.2. *Find necessary and/or sufficient condition for graphs G such that $\mu'(G) = \Delta(G)$ or $\mu'(G) = \Delta(G) + 1$.*

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