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Entanglement entropies of an interval in the free Schrödinger field theory on the half line

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Abstract: We study the entanglement entropies of an interval adjacent to the boundary of the half line for the free fermionic spinless Schrödinger field theory at finite density and zero temperature, with either Neumann or Dirichlet boundary conditions. They are finite functions of the dimensionless parameter given by the product of the Fermi momentum and the length of the interval. The entanglement entropy displays an oscillatory behaviour, differently from the case of the interval on the whole line. This behaviour is related to the Friedel oscillations of the mean particle density on the half line at the entangling point. We find analytic expressions for the expansions of the entanglement entropies in the regimes of small and large values of the dimensionless parameter. They display a remarkable agreement with the curves obtained numerically. The analysis is extended to a family of free fermionic Lifshitz models labelled by their integer Lifshitz exponent, whose parity determines the properties of the entanglement entropies. The cumulants of the local charge operator and the Schatten norms of the underlying kernels are also explored.

Keywords: Boundary Quantum Field Theory, Field Theories in Lower Dimensions, Space-Time Symmetries

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1 Introduction

The bipartite entanglement corresponding to a spatial bipartition has been intensively investigated in the past three decades by employing methods of quantum field theories, quantum may-body systems and quantum gravity (see e.g. the reviews [1–7]).

Consider a quantum system in a state ρ and a bipartition of the space $A \cup B$ which provides a corresponding factorisation of the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. When ρ is pure, the bipartite entanglement is measured by the entanglement entropy S_A, which is defined as the von Neumann entropy of the reduced density matrix $\rho_A \equiv \mathrm{Tr}_{\mathcal{H}_B}(\rho)$, namely

$$S_A \equiv -\mathrm{Tr}(\rho_A \log \rho_A) = \lim_{\alpha \to 1} S_A^{(\alpha)} \tag{1.1}$$

(hereafter the notation $\mathrm{Tr}(\dots) \equiv \mathrm{Tr}_{\mathcal{H}_A}(\dots)$ is adopted). The entanglement entropy can be obtained also through the replica limit, i.e. the analytic continuation $\alpha \to 1$ of the Rényi entropies

$$S_A^{(\alpha)} \equiv \frac{1}{1-\alpha} \log[\mathrm{Tr}(\rho_A^\alpha)] \tag{1.2}$$

where $\alpha \neq 1$ is a real and positive parameter; hence we identify $S_A^{(1)} \equiv S_A$. The single copy entanglement [8–10] is obtained as the limit $\alpha \to +\infty$ of the Rényi entropies (1.2), where $\mathrm{Tr}(\rho_A^\alpha) = \sum_j \lambda_j^\alpha$ in terms of the eigenvalues $\lambda_j \in [0,1]$ of ρ_A; hence $S_A^{(\infty)} = -\log(\lambda_{\max})$, with $\lambda_{\max}$ being the largest eigenvalue of ρ_A. The entanglement entropies include the entanglement entropy S_A, the Rényi entropies $S_A^{(\alpha)}$ and the single copy entanglement $S_A^{(\infty)}$.

For relativistic quantum field theories in $d+1$ spacetime dimensions and in their ground state, the entanglement entropies of a region A are divergent quantities as the ultraviolet (UV) cutoff ϵ vanishes and the leading divergence $S_A^{(\alpha)} \propto \mathrm{Area}(\partial A)/\epsilon^{d-1} + \cdots$ provides the area law [11–14], where the dots denote subleading terms as $\epsilon \to 0$. An important exception to this behavior is observed for conformal field theories in $d = 1$, where for the entanglement entropies of an interval $A = [-R, R]$ on the line we have $S_A^{(\alpha)} = \frac{c}{6}(1 + \frac{1}{\alpha}) \log(2R/\epsilon) + \cdots$ as $\epsilon \to 0$, being c the central charge of the model [15–17]. In the presence of spatial boundaries, the entanglement entropies depend also on the boundary conditions (b.c.). For instance, in a $d = 1$ boundary conformal field theory on the half line $x \geqslant 0$ and in its ground state, for the entanglement entropies of an interval $A = [0, R]$ adjacent to the boundary it has been found [17] that $S_A^{(\alpha)} = \frac{c}{12}(1 + \frac{1}{\alpha}) \log(2R/\epsilon) + \cdots$ and that the subleading constant term contains the Affleck-Ludwig boundary entropy [18], which encodes the boundary conditions and provides a monotonic function along a boundary renormalization group flow [19–21].

The properties of the bipartite entanglement in a quantum field theory depend on the nature of the spacetime symmetry. In order to gain some new insights on this relation, it is worth investigating the bipartite entanglement in non-relativistic quantum field theories. Insightful non-relativistic models exhibit the Lifshitz invariance [22–26],where the time and space coordinates scale in a different way, characterised by the Lifshitz exponent $z > 0$ (relativistic field theories have $z = 1$). Various quantities in these models have been studied, also in higher dimensions, including the entanglement entropies [27–38]. A remarkable property of the entanglement entropies for the free fermions at finite density in generic dimension is the violation of the area law [28, 29, 39].

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We focus on the $d = 1$ free fermionic spinless Schrödinger field theory at zero temperature and finite density μ. This is a free non-relativistic quantum field theory with $z = 2$ which describes the dilute spinless Fermi gas in $d = 1$ [40, 41]. When this model is defined on the line, the entanglement entropies of an interval $[-R, R] \subset \mathbb{R}$ have been studied in [42], finding that they are finite functions of the dimensionless parameter $\eta \equiv R\, k_{\rm F} \geqslant 0$, where $k_{\rm F}$ is the Fermi momentum, and that the entanglement entropy S_A is a monotonically increasing function of η. The $\mu = 0$ case has been considered earlier in [43–45].

In this manuscript we investigate the above mentioned Schrödinger field theory on the half line $x \geqslant 0$ with scale invariant boundary conditions (that are of either Neumann or Dirichlet type) imposed at the origin $x = 0$. In these models the mean value of the particle density exhibits Friedel oscillations, which depend on the boundary conditions and decay with the distance from the boundary [46]. We study the entanglement entropies of the interval $A = [0, R]$ adjacent to the boundary of the half line. We find that also these entanglement entropies are finite functions of the dimensionless parameter $\eta \equiv R\, k_{\rm F} \geqslant 0$. In these models the entanglement entropy displays an oscillatory behaviour, differently from the entanglement entropy of the interval on the line considered in [42]. We remark that in our analyses the dispersion relation $\omega(k) \propto k^2$ is not approximated through a linear dispersion relation at the Fermi points (Tomonaga's approximation) [47, 48].

The finiteness of the entanglement entropies in these models on the half line is a consequence of the analogous property which holds for the entanglement entropies of an interval on the line [42]. The latter follows from the properties of the solution of the sine kernel spectral problem in the interval on the line, which has been found in a series of seminal papers by Slepian, Pollak and Landau [49–52] and it is written in terms of the prolate spheroidal wave functions (PSWF) of order zero (see also the overview [53] and the recent book [54]). Also the numerical evaluation of these functions has been carefully investigated (see [54] and references therein). The relevance of this spectral problem for the entanglement in free fermionic systems has been highlighted in [55].

The procedure described in [42] for the entanglement entropies of the interval on the line, which follows the one discussed in [56–59] for some lattice models, can be adapted to the entanglement entropies of an interval adjacent to the boundary of the half line in a straightforward way and this leads us to write analytic expressions for the expansions of the entanglement entropies in the regimes of small and large values of η. These results are based on the expansions of the Bessel kernel tau function reported in [60–63], specialised to two specific values of the parameter in the Bessel kernel. We remark that the complete expansions found in [61, 62] have been obtained by applying to the Painlevé III_1 equation the method (Kyiv formula) introduced in [64] for the Painlevé VI equation. Also some results [65–69] obtained in lattice models are relevant for our analyses.

The outline of this manuscript is as follows. In section 2 we briefly describe the free fermionic spinless Schrödinger field theory on the half line at finite density and finite temperature, focussing on the zero temperature limit and on the scale invariant boundary conditions. The entanglement entropies of the interval $A = [0, R]$ for this model, which are the main results of this manuscript, are discussed in section 3. In section 4 we extend the analysis to a hierarchy of Lifshitz fermion fields with integer Lifshitz exponents $z \geqslant 1$. The

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expansions of the entanglement entropies as $\eta \to 0$ and $\eta \to \infty$ are investigated in section 5 and section 6 respectively. In section 7 we explore the Schatten norms and the relation between S_A and the charge cumulants [70–72]. Some conclusions are drawn in section 8. The appendices A, B and C contain the derivations of some results reported in the main text and also further technical details.

2 Free Schrödinger field theory on the half line at finite density

The dynamics of the free fermionic Schrödinger field theory on the half line $x \geqslant 0$ is defined by the equation of motion

$$\left(\mathrm{i}\,\partial_t + \frac{1}{2m}\,\partial_x^2\right)\psi(t,x) = 0 \tag{2.1}$$

where $m > 0$ is the mass and ψ is a complex quantum field. This field satisfies the equal-time canonical anticommutation relations

$$\{\psi(t,x_1)\,,\,\psi^*(t,x_2)\} = \delta(x_1 - x_2) \qquad \{\psi(t,x_1)\,,\,\psi(t,x_2)\} = \{\psi^*(t,x_1)\,,\,\psi^*(t,x_2)\} = 0 \tag{2.2}$$

and the boundary condition

$$\lim_{x\to 0^+}(\,\partial_x - \vartheta\,)\psi(t,x) = 0 \tag{2.3}$$

where the parameter ϑ has dimension of mass and parametrizes all self-adjoint extensions of the Hamiltonian $-\frac{1}{2m}\,\partial_x^2$ on the half line [73].

The solution of the boundary value problem defined by (2.1)–(2.3) for $\vartheta \geqslant 0$ reads

$$\psi(t,x) = \int_0^\infty \mathrm{e}^{-\mathrm{i}\omega(k)t}\left(\mathrm{e}^{\mathrm{i}kx} + \frac{k+\mathrm{i}\vartheta}{k-\mathrm{i}\vartheta}\,\mathrm{e}^{-\mathrm{i}kx}\right)a(k)\,\frac{\mathrm{d}k}{2\pi} \qquad \omega(k) = \frac{k^2}{2m} \tag{2.4}$$

where the oscillators $\{a(k)\,:\,k \geqslant 0\}$ and their Hermitian conjugates $\{a^*(k)\,:\,k \geqslant 0\}$ generate a standard canonical anticommutation relation algebra $\mathcal{A}$

$$\{a(k)\,,\,a^*(p)\} = 2\pi\,\delta(k-p) \qquad \{a(k)\,,\,a(p)\} = \{a^*(k)\,,\,a^*(p)\} = 0\,. \tag{2.5}$$

The phase factor $\frac{k+\mathrm{i}\vartheta}{k-\mathrm{i}\vartheta}$ in the integrand of (2.4) describes the reflection from the boundary at $x = 0$. For $\vartheta < 0$, in addition to the scattering states there exists a bound state $\mathrm{e}^{\vartheta x}$ with energy $\omega_{\mathrm{b}}(\vartheta) = \vartheta^2/(2m)$. Since in this paper we focus on the scale invariant points $\vartheta = 0$ and $\vartheta = \infty$, for details in treating this bound state we refer to [74].

In order to implement the finite density condition, we adopt the Gibbs representation of the algebra $\mathcal{A}$. In this representation, the basic two-point correlators are [75]

$$\langle a^*(p)\,a(k)\rangle_{\beta,\mu} = \frac{1}{1+\mathrm{e}^{\beta[\omega(k)-\mu]}}\,2\pi\,\delta(k-p) \tag{2.6}$$

$$\langle a(p)\,a^*(k)\rangle_{\beta,\mu} = \frac{\mathrm{e}^{\beta[\omega(k)-\mu]}}{1+\mathrm{e}^{\beta[\omega(k)-\mu]}}\,2\pi\,\delta(k-p) \tag{2.7}$$

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where $\beta > 0$ is the inverse temperature and μ is the chemical potential in the Fermi distribution in (2.6). Combining (2.4) with (2.6) and (2.7), one obtains the following two-point functions

$$\langle\psi^*(t_1,x_1)\psi(t_2,x_2)\rangle_{\beta,\mu}=\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega(k)(t_{12}-\mathrm{i}\varepsilon)}}{1+\mathrm{e}^{\beta[\omega(k)-\mu]}}\left(\mathrm{e}^{-\mathrm{i}kx_{12}}+\mathrm{e}^{-\mathrm{i}k\widetilde{x}_{12}}+\frac{2\mathrm{i}\vartheta}{k-\mathrm{i}\vartheta}\mathrm{e}^{-\mathrm{i}k\widetilde{x}_{12}}\right)\frac{\mathrm{d}k}{2\pi} \tag{2.8}$$

$$\langle\psi(t_1,x_1)\psi^*(t_2,x_2)\rangle_{\beta,\mu}=\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}\omega(k)(t_{12}-\mathrm{i}\varepsilon)}\mathrm{e}^{\beta[\omega(k)-\mu]}}{1+\mathrm{e}^{\beta[\omega(k)-\mu]}}\left(\mathrm{e}^{\mathrm{i}kx_{12}}+\mathrm{e}^{\mathrm{i}k\widetilde{x}_{12}}-\frac{2\mathrm{i}\vartheta}{k+\mathrm{i}\vartheta}\mathrm{e}^{\mathrm{i}k\widetilde{x}_{12}}\right)\frac{\mathrm{d}k}{2\pi} \tag{2.9}$$

where

$$t_{12}\equiv t_1-t_2 \qquad x_{12}\equiv x_1-x_2 \qquad \widetilde{x}_{12}\equiv x_1+x_2\,. \tag{2.10}$$

We remark that, for the free models that we are considering, the two-point functions (2.8) and (2.9) fully characterize the model at quantum level. From these functions, the Hilbert space $\mathcal{H}$, the action of the quantum field on $\mathcal{H}$ and the symmetry content (a unitary group representation of the spacetime symmetries on $\mathcal{H}$) can be reconstructed [76]. Notice that space translations are broken because of the presence of a boundary, while dilatations are broken at finite density $\mu > 0$.

As mentioned above, in this manuscript we study the limiting regimes where $\vartheta = 0$ and $\vartheta \to \infty$, which are scale invariant and define respectively the Neumann (+) and Dirichlet (−) boundary conditions, i.e.

$$\lim_{x\to 0^+}\partial_x\psi_+(t,x)=0 \qquad \lim_{x\to 0^+}\psi_-(t,x)=0\,. \tag{2.11}$$

From (2.4) one gets

$$\psi_\pm(t,x)=\int_0^\infty \mathrm{e}^{-\mathrm{i}\omega(k)t}\left(\mathrm{e}^{\mathrm{i}kx}\pm\mathrm{e}^{-\mathrm{i}kx}\right)a(k)\,\frac{\mathrm{d}k}{2\pi}\,. \tag{2.12}$$

Taking the limits $\vartheta \to 0$ and $\vartheta \to \infty$ in (2.8) and (2.9), one finds

$$\langle\psi_\pm^*(t_1,x_1)\,\psi_\pm(t_2,x_2)\rangle_{\beta,\mu}=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega(k)(t_{12}-\mathrm{i}\varepsilon)}\,\frac{\left(\mathrm{e}^{-\mathrm{i}kx_{12}}\pm\mathrm{e}^{-\mathrm{i}k\widetilde{x}_{12}}\right)}{1+\mathrm{e}^{\beta[\omega(k)-\mu]}}\,\frac{\mathrm{d}k}{2\pi} \tag{2.13}$$

$$\langle\psi_\pm(t_1,x_1)\,\psi_\pm^*(t_2,x_2)\rangle_{\beta,\mu}=\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega(k)(t_{12}-\mathrm{i}\varepsilon)}\,\frac{\left(\mathrm{e}^{\mathrm{i}kx_{12}}\pm\mathrm{e}^{\mathrm{i}k\widetilde{x}_{12}}\right)\mathrm{e}^{\beta[\omega(k)-\mu]}}{1+\mathrm{e}^{\beta[\omega(k)-\mu]}}\,\frac{\mathrm{d}k}{2\pi}\,. \tag{2.14}$$

At equal times $t_1 = t_2 \equiv t$ and in the zero temperature limit $\beta \to \infty$, the integration over k in (2.13) and (2.14) can be easily performed and gives

$$\langle\psi_\pm^*(t,x_1)\,\psi_\pm(t,x_2)\rangle_{\infty,\mu}=\frac{\sin(k_{\mathrm{F}}x_{12})}{\pi\,x_{12}}\pm\frac{\sin(k_{\mathrm{F}}\widetilde{x}_{12})}{\pi\,\widetilde{x}_{12}} \tag{2.15}$$

$$\langle\psi_\pm(t,x_1)\,\psi_\pm^*(t,x_2)\rangle_{\infty,\mu}=\delta(x_{12})-\left[\frac{\sin(k_{\mathrm{F}}x_{12})}{\pi\,x_{12}}\pm\frac{\sin(k_{\mathrm{F}}\widetilde{x}_{12})}{\pi\,\widetilde{x}_{12}}\right] \tag{2.16}$$

where k_{F} is the Fermi momentum

$$k_{\mathrm{F}}\equiv\sqrt{2m\mu}\,. \tag{2.17}$$

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In this regime, the correlators (2.8) and (2.9) can be expressed as

$$\langle \psi^*(t,x_1)\,\psi(t,x_2)\rangle_{\infty,\mu} = \langle \psi^*_+(t,x_1)\,\psi_+(t,x_2)\rangle_{\infty,\mu} + 2\mathrm{i}\,\vartheta \int_{-k_\mathrm{F}}^{k_\mathrm{F}} \frac{\mathrm{e}^{-\mathrm{i}k\widetilde{x}_{12}}}{k-\mathrm{i}\vartheta}\,\frac{\mathrm{d}k}{2\pi} \tag{2.18}$$

$$\langle \psi(t,x_1)\,\psi^*(t,x_2)\rangle_{\infty,\mu} = \langle \psi_+(t,x_1)\,\psi^*_+(t,x_2)\rangle_{\infty,\mu} - 2\mathrm{i}\,\vartheta \int_{-k_\mathrm{F}}^{k_\mathrm{F}} \frac{\mathrm{e}^{\mathrm{i}k\widetilde{x}_{12}}}{k+\mathrm{i}\vartheta}\,\frac{\mathrm{d}k}{2\pi}\,. \tag{2.19}$$

The expression (2.15) allows us to evaluate the mean value of the particle densities for the two b.c.'s that we are considering. The result is

$$\langle \varrho_\pm(t,x)\rangle_{\infty,\mu} \equiv \langle \psi^*_\pm(t,x)\,\psi_\pm(t,x)\rangle_{\infty,\mu} = \frac{k_\mathrm{F}}{\pi} \pm \frac{\sin(2k_\mathrm{F}x)}{2\pi\,x} = \langle \varrho(t,x)\rangle_{\infty,\mu} \pm \frac{\sin(2k_\mathrm{F}x)}{2\pi\,x} \tag{2.20}$$

where $\langle \varrho(t,x)\rangle_{\infty,\mu} = k_\mathrm{F}/\pi$ is the mean value of the particle density on the whole line, which is independent of the position. Thus, (2.20) shows the Friedel-type oscillations [46] around the particle density on the line, whose amplitude decays with the distance from the boundary. The densities vanish for $\mu = 0$, as expected. We find it worth considering the following normalised densities

$$\frac{\langle \varrho_\pm(t,x)\rangle_{\infty,\mu}}{k_\mathrm{F}} = \frac{\langle \varrho(t,x)\rangle_{\infty,\mu}}{k_\mathrm{F}} \pm \frac{\sin(2\chi)}{2\pi\,\chi} \qquad \chi \equiv k_\mathrm{F}x \tag{2.21}$$

which are functions of the dimensionless parameter χ.

In this paper we study the entanglement entropies of the bipartition $[0,R]\cup[R,\infty)$ of the half line for the system described above. This bipartition of the half line naturally leads us to consider the normalised densities (2.21) evaluated at the entangling point $x = R$, which read

$$\frac{\langle \varrho_\pm(t,R)\rangle_{\infty,\mu}}{k_\mathrm{F}} = \frac{\langle \varrho(t,R)\rangle_{\infty,\mu}}{k_\mathrm{F}} \pm \frac{\sin(2\eta)}{2\pi\,\eta} \qquad \eta \equiv k_\mathrm{F}R\,. \tag{2.22}$$

Another natural quantity to introduce is the mean particle number $N_{A,\pm}$ in the interval $A = [0,R]$, namely

$$N_{A,\pm} \equiv \int_0^R \langle \varrho_\pm(t,x)\rangle_{\infty,\mu}\,\mathrm{d}x \;=\; \frac{\eta}{\pi} \pm \frac{\mathrm{Si}(2\eta)}{2\pi} \tag{2.23}$$

where $\mathrm{Si}(z) \equiv \int_0^z \frac{\sin(t)}{t}\,\mathrm{d}t$ is the sine integral function. The dimensionless parameter η plays a fundamental role throughout our analysis. In the regime of large η, for (2.23) we have[1]

$$N_{A,\pm} = \frac{\eta}{\pi} \pm \left(\frac{1}{4} - \frac{\cos(2\eta)}{4\pi\eta}\right) + O(1/\eta^2) \qquad \eta \to \infty\,. \tag{2.25}$$

Since for the Schrödinger problem on the whole line [42] the mean particle number in the interval $[-R,R]$ is $N_{2A\subset\mathbb{R}} = 2\eta/\pi$, one can rewrite (2.23) as

$$N_{A,\pm} - \frac{1}{2}\,N_{2A\subset\mathbb{R}} \;=\; \pm\,\frac{\mathrm{Si}(2\eta)}{2\pi}\,. \tag{2.26}$$

The expressions in (2.21), (2.26) and (2.25) provide the red and blue curves in figure 4.

[1] The expansion of $\mathrm{Si}(z)$ as $z \to \infty$ reads [77]

$$\mathrm{Si}(z) = \frac{\pi}{2} - \frac{\cos(z)}{z}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{z^{2n}} - \frac{\sin(z)}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!}{z^{2n}} \tag{2.24}$$

where the two series are asymptotic.

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3 Entanglement entropies

The main quantities investigated in this manuscript are the entanglement entropies (see (1.1) and (1.2)) for the free Schrödinger field at finite density and zero temperature on the half line $x \geqslant 0$ when the spatial bipartition is given by the interval $A = [0, R]$ and its complement. The Gaussian nature of the state in this free fermionic model allows to compute these entanglement entropies through the spectra associated to the spectral problems described in section 3.1. The entanglement entropies are then evaluated in section 3.2.

3.1 Spectral problems

Since we are dealing with a free fermionic model, the entanglement entropies can be evaluated from the two-point functions on the half line for either Neumann (+) or Dirichlet (−) b.c., namely (see (2.15))

$$K_{\pm}(k_{\text{F}}; x, y) \equiv \frac{\sin[k_{\text{F}}(x-y)]}{\pi(x-y)} \pm \frac{\sin[k_{\text{F}}(x+y)]}{\pi(x+y)} \,. \tag{3.1}$$

These kernels satisfy

$$\int_0^\infty K_{\pm}(k_{\text{F}}; x, z)\, K_{\pm}(k_{\text{F}}; z, y)\, \mathrm{d}z = K_{\pm}(k_{\text{F}}; x, y) \qquad x, y \geqslant 0 \tag{3.2}$$

and therefore define projection operators on the half line. This property implies that the finite density states, which generate the correlation functions (2.15) and (2.16), are pure states [78].

It is straightforward to observe that the sine kernel, which provides the two-point function of the same model on the line, is related to the kernels (3.1) as follows

$$K_{\text{sine}}(k_{\text{F}}; x, y) \equiv \frac{\sin[k_{\text{F}}(x-y)]}{\pi(x-y)} = \frac{K_+(k_{\text{F}}; x, y) + K_-(k_{\text{F}}; x, y)}{2} \,. \tag{3.3}$$

Considering the kernels (3.1) reduced to $A = [0, R] \subset \mathbb{R}^+$, after rescaling R, the corresponding spectral problems read

$$\int_0^1 K_{\pm}(\eta; x, y)\, f_n^{\pm}(\eta; y)\, \mathrm{d}y = \gamma_n^{\pm}\, f_n^{\pm}(\eta; x) \qquad x \in [0, 1] \tag{3.4}$$

where $\gamma_n^{\pm} = \gamma_n^{\pm}(\eta)$ are functions of η. In order to solve (3.4), first we consider the auxiliary spectral problem associated to the sine kernel, i.e.

$$\int_{-1}^1 K_{\text{sine}}(\eta; x, y)\, f_n(\eta; y)\, \mathrm{d}y = \gamma_n\, f_n(\eta; x) \qquad x \in [-1, 1] \tag{3.5}$$

whose eigenvalues and eigenfuctions can be expressed in terms of the prolate spheroidal wave functions (PSWF) [79–81]. The eigenvalues in (3.5) can be written in terms of the radial PSWF of zero order $\mathcal{R}_{0n}$ [49, 54, 82]

$$\gamma_n = \frac{2\eta}{\pi}\, \mathcal{R}_{0n}(\eta, 1)^2 \qquad n \in \mathbb{N}_0 \tag{3.6}$$

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while the corresponding eigenfunctions are expressed through the angular PSWF of zero order $\mathcal{S}_{0n}$ as follows

$$f_n(\eta;x) \,=\, \sqrt{n+\frac{1}{2}}\,\mathcal{S}_{0n}(\eta,x) \tag{3.7}$$

which also satisfy

$$f_n(\eta;-x) = (-1)^n f_n(\eta;x)\,. \tag{3.8}$$

The spectral problems (3.4) can be related to the sine kernel spectral problem (3.5) by first rewriting the latter one in the form

$$\int_{-1}^{0} \frac{\sin[\eta(x-y)]}{\pi(x-y)}\, f_n(\eta;y)\,\mathrm{d}y \,+\, \int_{0}^{1} \frac{\sin[\eta(x-y)]}{\pi(x-y)}\, f_n(\eta;y)\,\mathrm{d}y \,=\, \gamma_n\, f_n(\eta;x) \qquad x\in[-1,1]\,. \tag{3.9}$$

Changing the variable $y \mapsto -y$ in the first integral of (3.9) and using the parity condition (3.8), one obtains

$$(-1)^n \int_{0}^{1} \frac{\sin[\eta(x+y)]}{\pi(x+y)}\, f_n(\eta;y)\,\mathrm{d}y \,+\, \int_{0}^{1} \frac{\sin[\eta(x-y)]}{\pi(x-y)}\, f_n(\eta;y)\,\mathrm{d}y \,=\, \gamma_n\, f_n(\eta;x)\,. \tag{3.10}$$

Finally, by comparing (3.4) and (3.10), for Neumann b.c. we have

$$\gamma_n^+ = \gamma_{2n} \qquad f_n^+(\eta;x) = f_{2n}(\eta;x) \qquad n\in\mathbb{N}_0 \qquad x\in[0,1] \tag{3.11}$$

while for Dirichlet b.c. one gets

$$\gamma_n^- = \gamma_{2n+1} \qquad f_n^-(\eta;x) = f_{2n+1}(\eta;x) \qquad n\in\mathbb{N}_0 \qquad x\in[0,1]\,. \tag{3.12}$$

The spectrum $\{\gamma_n\}$ in (3.5) has been extensively discussed in the literature [54]. In particular, $\gamma_n \in (0,1)$ for any $n\in\mathbb{N}$ and any $\eta>0$. For a fixed value of η, these eigenvalues are non-degenerate and decrease with n. Furthermore, $\gamma_n \to 0$ as $n\to\infty$ in a super-exponential way. The critical index

$$n_0 \equiv \left\lfloor \frac{2\eta}{\pi} \right\rfloor \in \mathbb{N}_0 \tag{3.13}$$

can be identified where $\gamma_{n_0} \simeq 1/2$. This critical index allows to partition the spectrum in three different sets where γ_n behave in a characteristic way [83, 84]. For numerical purposes, we have used that $\gamma_n \approx 0$ when $n \geqslant 2(n_0+2)$.

3.2 Entanglement entropies

The eigenvalues of the spectral problems (3.4) provide the entanglement entropies of an interval $A=[0,R]\subset\mathbb{R}^+$. For Neumann and Dirichlet b.c., they are given respectively by

$$S_{A,+}^{(\alpha)} = \sum_{n=0}^{\infty} s_\alpha(\gamma_n^+) \qquad\qquad S_{A,-}^{(\alpha)} = \sum_{n=0}^{\infty} s_\alpha(\gamma_n^-) \tag{3.14}$$

where

$$s_\alpha(x) \equiv \frac{1}{1-\alpha}\,\log\!\left[x^\alpha + (1-x)^\alpha\right]. \tag{3.15}$$

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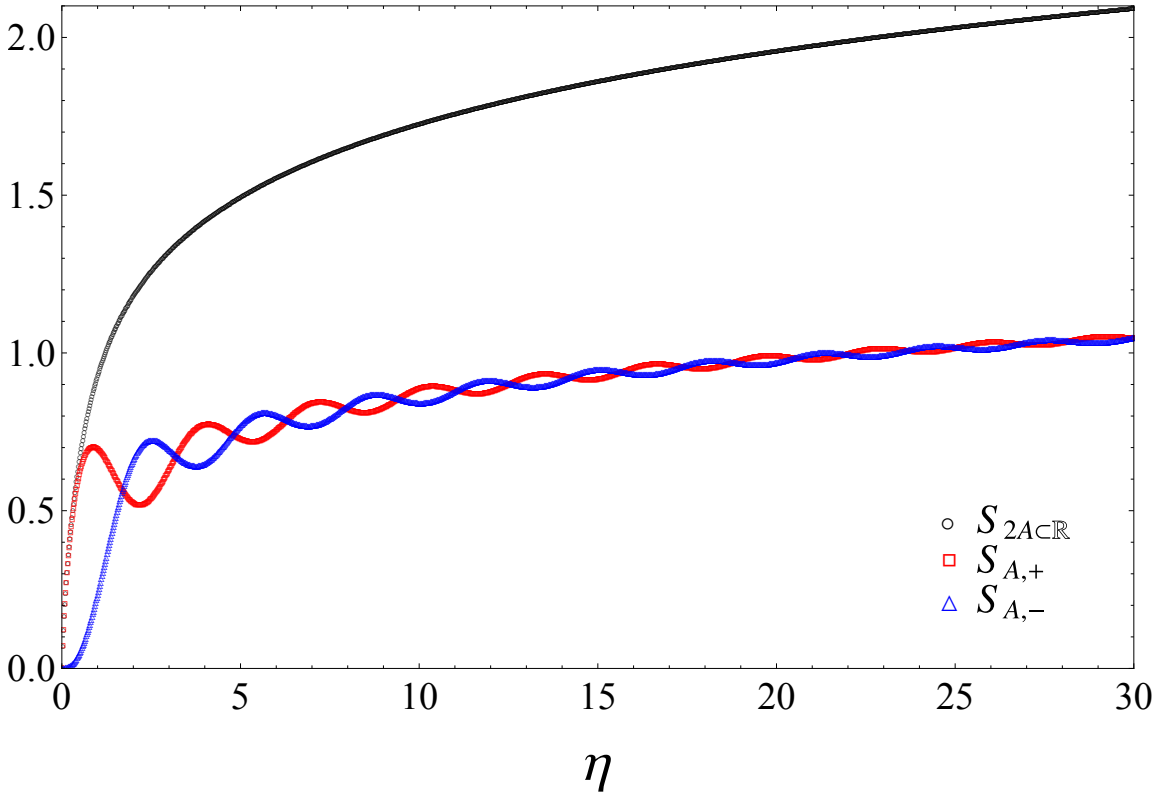

Figure 1. Entanglement entropy of an interval $A = [0, R]$ adjacent to the boundary of the half-line, for Neumann b.c. (red curve $S_{A,+}$) and Dirichlet b.c. (blue curve $S_{A,-}$), obtained numerically from (3.14). The black curve corresponds to the entanglement entropy $S_{2A\subset\mathbb{R}}$ of an interval of length $2R$ on the line. The relation (3.18) occurs among these quantities.

In the limits $\alpha \to 1$ and $\alpha \to \infty$, this function becomes respectively

$$s(x) \equiv -x\log(x) - (1-x)\log(1-x) \qquad s_\infty(x) \equiv \begin{cases} -\log(1-x) & x \in [0, 1/2] \\ -\log(x) & x \in (1/2, 1] \end{cases} \tag{3.16}$$

that are employed in (3.14) to evaluate the entanglement entropy $S_{A,\pm}$ and the single copy entanglement $S^{(\infty)}_{A,\pm}$ as follows

$$\lim_{\alpha\to 1} S^{(\alpha)}_{A,\pm} = S_{A,\pm} \qquad \lim_{\alpha\to\infty} S^{(\alpha)}_{A,\pm} = S^{(\infty)}_{A,\pm}\,. \tag{3.17}$$

Summing up the two expressions for the entanglement entropies in (3.14) corresponding to the two different boundary conditions, one obtains

$$S^{(\alpha)}_{2A\subset\mathbb{R}} = S^{(\alpha)}_{A,+} + S^{(\alpha)}_{A,-} \tag{3.18}$$

where $S^{(\alpha)}_{2A\subset\mathbb{R}}$ are the entanglement entropies of an interval of length $2R$ on the line for the Schrödinger field theory at zero temperature and finite density, which have been studied in [42]. Since $S^{(\alpha)}_{A,\pm}$ are positive functions of η and $S^{(\alpha)}_{2A\subset\mathbb{R}}$ is finite for any given η (the proof has been reported in section 4 of [42]), also $S^{(\alpha)}_{A,\pm}$ are finite functions of η.

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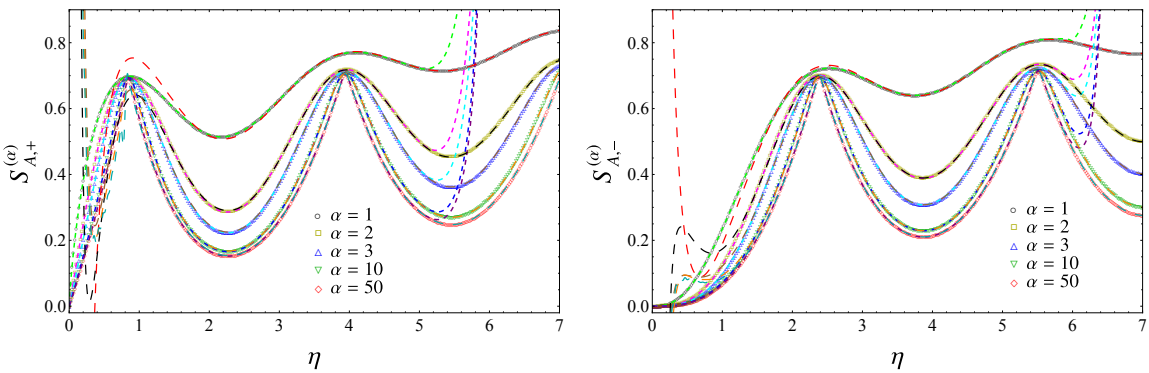

Figure 2. Entanglement entropies for either Neumann (left panel) or Dirichlet (right panel) boundary conditions. The data points have been obtained numerically from (3.14), while the dashed lines correspond to the small (see section 5) and large (see section 6) interval expansion.

In figure 1 we show $S_{A,\pm}$, evaluated numerically from (3.14), and compare them with the entanglement entropy $S_{2A\subset\mathbb{R}}$ of an interval of length $2R$ on the line. These three quantities are related through (3.18). The numerical analysis has been performed as explained in [42], by employing an optimised Fortran code provided to us by Vladimir Rokhlin. In particular, the infinite sums (3.14) have been truncated to $n \leqslant 2(n_0+2)$, where n_0 is the critical index (3.13). We checked numerically that the entanglement entropies do not change significantly by including more terms. This truncation criterion, which is the one adopted in [42], has been applied to evaluate numerically all the quantities in this manuscript that involve a sum over the spectra (3.11) and (3.12).

The main feature to highlight in figure 1 is the fact that $S_{A,\pm}$ are not monotonic functions. Instead, $S_{2A\subset\mathbb{R}}$ is a monotonic function as proved in [42]. The proof of this feature of $S_{2A\subset\mathbb{R}}$ exploits the invariance under translations, which does not hold for the model on the half line.

In figure 2 and figure 3 we also show the entanglement entropies $S^{(\alpha)}_{A,\pm}$ for different values of α, obtained numerically from (3.14).

Since $\gamma_n^{\pm} \in (0,1)$ for any $n \in \mathbb{N}$ and $\eta > 0$ in (3.14), we can adapt to our case the procedure employed in [56, 57] to evaluate the entanglement entropies in some spin chains, as done in [42] for the entanglement entropies of an interval on the line for the Schrödinger model at finite density and zero temperature. This allows to write (3.14) as the following contour integral in the complex plane

$$S^{(\alpha)}_{A,\pm} = \lim_{\epsilon,\delta\to 0} \frac{1}{2\pi \mathrm{i}} \oint_{\mathfrak{C}} s_\alpha(z)\, \partial_z \log(\tau_\pm)\, \mathrm{d}z \tag{3.19}$$

where $s_\alpha(z)$ is the holomorphic function obtained from (3.15). The closed path $\mathfrak{C}$ encircles the interval $[0,1] \subset \mathbb{R}$ and is parameterised by the infinitesimal parameters ϵ and δ through its decomposition $\mathfrak{C} = \mathfrak{C}_0 \cup \mathfrak{C}_- \cup \mathfrak{C}_1 \cup \mathfrak{C}_+$, where $\mathfrak{C}_0$ and $\mathfrak{C}_1$ are two arcs of radius $\epsilon/2$ centered in 0 and 1 respectively, while $\mathfrak{C}_\pm$ are the segments belonging to the horizontal lines $x \pm \mathrm{i}\delta$ with $x \in \mathbb{R}$ and intersecting $\mathfrak{C}_0$ and $\mathfrak{C}_1$ (see e.g. figure 1 of [56], where a similar path is

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shown); hence $\epsilon \to 0$ implies $\delta \to 0$. The functions $\tau_\pm$ in the integrand of (3.19) are the tau functions associated to the kernels (3.1)

$$\tau_\pm \equiv \det(I - z^{-1} K_\pm) = \prod_{n=0}^{\infty} (1 - z^{-1} \gamma_n^\pm) \tag{3.20}$$

i.e. the Fredholm determinants of the corresponding kernels, where I denotes the identity operator, $z \in \mathbb{C}$ and $\gamma_n^\pm$ are the eigenvalues of $K_\pm$, which are obtained from the eigenvalues of the sine kernel K_{sine} (see (3.11) and (3.12)). From this relation, it is straightforward to observe that the tau function $\tau_{\text{sine}} \equiv \det(I - z^{-1} K_{\text{sine}})$ associated to the sine kernel can be written in terms of the tau functions in (3.20) as follows[2]

$$\tau_{\text{sine}} = \tau_+ \tau_- \tag{3.21}$$

(see also Proposition 1 in [85] with $a_{\text{there}} = 0$).

The relation (3.21), combined with (3.19), provides (3.18) in a straightforward way. This observation can be extended to a class of quantities having the form $G_A = \sum_{n \geqslant 0} g(\gamma_n)$, where $g(0) = 0$. Indeed, by writing these quantities like in (3.19) and exploiting (3.21), one finds the relation $G_{2A \subset \mathbb{R}} = G_{A,+} + G_{A,-}$, where $G_{2A \subset \mathbb{R}}$ corresponds to G_A for the interval $[-R, R]$ on the line, and $G_{A,\pm}$ to G_A for the interval $[0, R]$ adjacent to the boundary of the half line where either Neumann $(+)$ or Dirichlet $(-)$ are imposed at the origin. Also the Schatten norms (7.3) (see section 7) belong to this class of quantities and (7.6) gives the above mentioned relation for them.

Fredholm determinants of integrable kernels occur in many interesting problems in physics and mathematics. In particular, the Fredholm determinants (3.20) are related to some probability distribution of the level spacings for random matrices [86] and to the inverse scattering problem [87].

In our analyses we exploit in a crucial way the fact that the Fredholm determinants (3.20) are related to the solutions of a particular Painlevé III differential equation [60, 88, 89]. In particular, the kernels (3.1) can be obtained as special cases of a Bessel kernel (see (A.5)) and the relation between the corresponding spectral problems is discussed in the appendix A. This allows to write the tau functions $\tau_\pm$ in (3.19) as special cases of the tau function of this Bessel kernel (see (A.11)). The auxiliary sigma function associated to this Bessel kernel tau function satisfies a particular Painlevé III_1 differential equation (see (A.8) and (A.9)). Combining these observations, the relation between the Painlevé III_1 and III_1' (see (A.16)) and the small η expansion of $\tau_\pm$ given in [60] (see appendix B.2), we obtain

$$\tau_\pm(\eta) = \frac{2^{1/8}}{\pi^{\pm 1/2}\, \mathrm{e}^{\eta^2/8}\, \eta^{1/8}}\, \tau_{\text{III}'}(\eta^2/4)\big|_{\theta_* = \theta_\star = \pm 1/4} \tag{3.22}$$

where the tau function $\tau_{\text{III}'}(t)$ in [61] is employed. The explicit expressions of the tau functions occurring in the r.h.s. of (3.22) are discussed below (see (5.9) and (6.1)).

Analytic expressions for the expansions of $\tau_\pm$ as $\eta \to 0$ and $\eta \to \infty$ have been obtained in [61, 62]. In section 5.2 and section 6, we have employed them into (3.19) to get analytic

[2]The relation (3.21) implies $\sigma_{\text{sine}} = \sigma_+ + \sigma_-$ for the corresponding auxiliary functions (see [61] and also appendix A).

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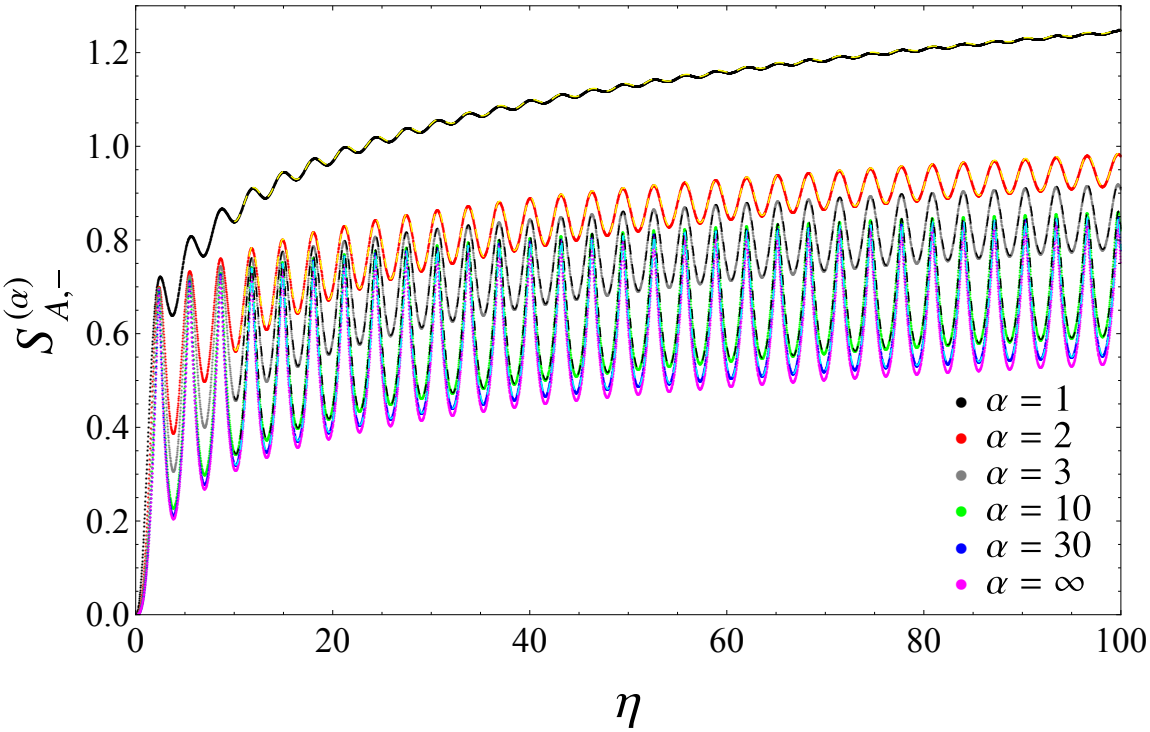

Figure 3. Entanglement entropies $S^{(\alpha)}_{A,-}$ for Dirichlet boundary conditions. The dashed lines (shown only for $\eta \geqslant 10$) correspond to the large η expansion discussed in section 6.

results for the corresponding expansions of $S^{(\alpha)}_{A,\pm}$. In figure 2 and figure 3, the dashed curves have been found from these analytic expansions, while the curves identified by the empty markers have been obtained numerically through (3.14), in the same way described for figure 1. In particular, we have used (5.12) for small η and the expressions discussed in section 6 (see e.g. (6.7), (6.9) and (6.11)) for large η. We emphasise that $S^{(\alpha)}_{A,\pm}$ are oscillating functions of η for any value of $\alpha > 0$. A remarkable agreement between the numerical results and the analytic expressions for the small and large η expansions is observed. Furthermore, in figure 2 an intermediate regime of η can be identified where the curves corresponding to the small and large η expansions overlap. The size of this crossover regime depends both on the boundary condition and on the value of α.

While in figure 2 the entanglement entropies $S^{(\alpha)}_{A,\pm}$ are shown only for $\eta \in [0,7]$, a larger domain has been considered in figure 3, where only $S^{(\alpha)}_{A,-}$ are reported because the curves for $S^{(\alpha)}_{A,+}$ are qualitatively very similar. In figure 3 one observes the logarithmic growth of the entanglement entropies (in particular, from (6.7) and (6.9) we have that $S^{(\alpha)}_{A,\pm} = \frac{1}{12}(1+\frac{1}{\alpha})\log(\eta)+O(1)$ as $\eta \to \infty$) and also their oscillatory behaviour. For a given value of α, the amplitude of the oscillations vanishes as $\eta \to \infty$. Instead, this amplitude increases with α for a given value of η.

We find it worth introducing the following combinations of entanglement entropies

$$B^{(\alpha)}_{A,\pm} \equiv S^{(\alpha)}_{A,\pm} - \frac{1}{2}\, S^{(\alpha)}_{2A\subset\mathbb{R}} = \pm\, \frac{S^{(\alpha)}_{A,+} - S^{(\alpha)}_{A,-}}{2} \tag{3.23}$$

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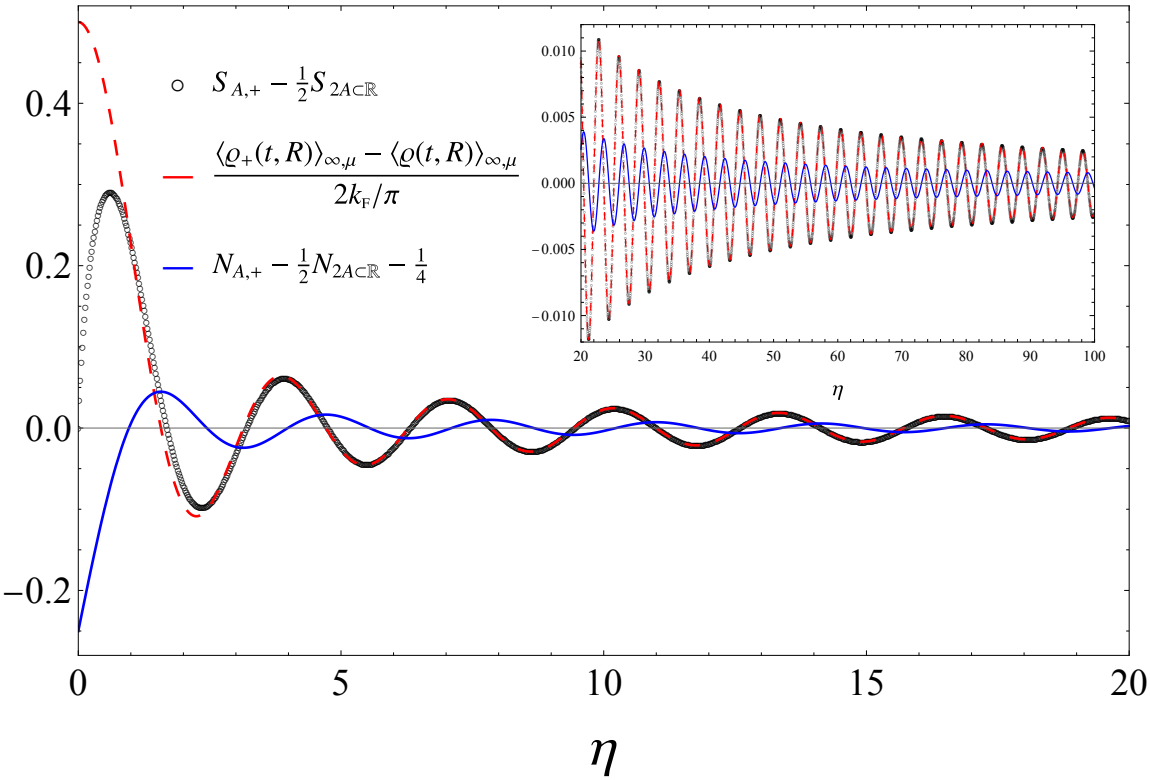

Figure 4. Entanglement entropy, normalised density at the entangling point (2.22) and mean particle number in the interval (2.23) in the case of Neumann b.c., where half of the corresponding quantity on the line has been subtracted (see (3.23) for $\alpha = 1$, (2.26) and (2.25)) to highlight the oscillatory behaviour.

where (3.18) has been employed. For boundary conformal field theories in $d = 1$, the r.h.s. of (3.23) can be defined for any conformally invariant boundary condition: the resulting combination, which depends on the boundary condition imposed at the origin of the half line, is UV finite [17]. In the Schrödinger models that we are considering, both $S^{(\alpha)}_{A,\pm}$ and $S^{(\alpha)}_{2A\subset\mathbb{R}}$ in the r.h.s. of (3.23) are finite functions of η; hence this property holds for any linear combination of these two quantities. In the closing paragraph of section 6 an interesting feature of the special combination (3.23) is highlighted.

Since $S_{2A\subset\mathbb{R}}$ is a monotonic function [42] while $S_{A,\pm}$ display an oscillatory behaviour (see figure 1), the combinations (3.23) in the special case of $\alpha = 1$ are oscillating functions of η for both Neumann and Dirichlet boundary conditions. In figure 4 we focus on Neumann b.c. as prototypical case, and compare the combination (3.23) for $\alpha = 1$ with the density at the entangling point (2.22), which exhibits the Friedel oscillations, and the mean particle number in the interval (2.23). The oscillatory behaviours are highlighted when half of the corresponding quantities on the line are subtracted (see (3.23) for $\alpha = 1$ and (2.26)). The density at the entangling point (2.22) can be rewritten as follows

$$\frac{1}{2k_F/\pi}\left[\langle \varrho_\pm(t,R)\rangle_{\infty,\mu} - \frac{1}{2}\big(\langle \varrho(t,R)\rangle_{\infty,\mu} + \langle \varrho(t,-R)\rangle_{\infty,\mu}\big)\right] = \pm\frac{\sin(2\eta)}{4\,\eta} \tag{3.24}$$

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where the r.h.s. provides the Friedel oscillations and the quantity within the square brackets has a form similar to the one of the other quantities displayed in figure 4. The functions in (3.24) oscillate around zero with decreasing amplitudes and their zeros correspond to $2\eta = n\pi$, where $n \in \mathbb{N}$ and $\eta > 0$. These zeros correspond to the values of η where the critical index (3.13) has jump discontinuities. The parity of the critical index is a π-periodic function of η taking values $+1$ for $\eta \in [k\pi, (k+1/2)\pi)$ and -1 for $\eta \in [(k+1/2)\pi, (k+1)\pi)$, where $k \in \mathbb{N}_0$. When n_0 is even, the eigenvalue γ_{n_0} contributes to $S_{A,+}$; hence one expects $S_{A,+} \gtrsim S_{A,-}$ (i.e. $B_{A,+} \gtrsim 0$), as observed for this curve in figure 4 for $\eta \gtrsim 5$. Analogously, $B_{A,+} \lesssim 0$ when n_0 is odd.

In figure 4, a remarkable agreement between $B_{A,+}$ and (3.24) is observed for $\eta > 4$. This can be explained by anticipating some results discussed in section 6 about the expansion of $S_{A,\pm}$ at large η (see (6.9), (6.12) and (2.22)), which allow to write

$$S_{A,\pm} = \frac{1}{6}\log(4\eta) + \frac{E_1}{2} + \frac{\langle \varrho_\pm(t,R)\rangle_{\infty,\mu} - \langle \varrho(t,R)\rangle_{\infty,\mu}}{2k_{\mathrm{F}}/\pi} + O(1/\eta^2) \tag{3.25}$$

where E_1 is the constant (6.10) in the special case of $\alpha = 1$. The expansion (3.25) tells us that the Friedel oscillations occurring in the normalised density at the entangling point provide the first subleading correction of the entanglement entropy in the regime of large η, which vanish as $\eta \to \infty$. As for the mean particle number in the interval (see the blue solid curve in figure 4, obtained from (2.26) and (2.25)), its oscillations have the same frequency and they are shifted by $\pi/4$.

The entanglement entropies of the interval adjacent to the boundary of the half line display oscillations (see e.g. figure 1, figure 2 and figure 3); hence it is worth asking whether a monotonically increasing function of η can be constructed. Since $S^{(\alpha)}_{A,\pm} \geqslant 0$, it is natural to consider

$$\mathcal{S}^{(\alpha)}_{A,\pm} \equiv \int_0^\eta S^{(\alpha)}_{A,\pm}(\xi)\,\mathrm{d}\xi\,. \tag{3.26}$$

Let us investigate the class of functions of η whose generic element is $G(\eta) = \sum_{n\geqslant 0} g(\gamma_n)$, where $g(x) \to 0$ as $x \to 0$ in a proper way to guarantee the convergence of the series that defines G. Since the spectrum of the sine kernel satisfies the following property (see eq. (3.51) in [54])

$$\gamma'_n = \frac{2}{\eta}\,\gamma_n\, f_n(\eta;1)^2 \tag{3.27}$$

we have that $G'(\eta) = \sum_{n\geqslant 0} g'(\gamma_n)\,\gamma'_n$ with $\gamma'_n \geqslant 0$; hence the condition $g'(x) \geqslant 0$ for $x \in (0,1)$ implies $G'(\eta) \geqslant 0$. The expressions in (3.26) correspond to the particular choice given by $g'(x) = s_\alpha(x)$ and to the restriction to the eigenvalues of the sine kernel spectral problem labelled by either even or odd values of n.

4 Integer Lifshitz exponents

In this section we study a hierarchy of two component Lifshitz fermion fields $\psi(t,x)$ whose time evolution on the half line $x \geqslant 0$ is given by

$$\left[\mathrm{i}\sigma_0\,\partial_t - \frac{1}{(2m)^{z-1}}\left(-\mathrm{i}\sigma_3\,\partial_x\right)^z\right]\psi(t,x) = 0 \qquad\qquad z \in \mathbb{N} \tag{4.1}$$

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where

$$\psi(t,x) = \begin{pmatrix} \psi_1(t,x) \\ \psi_2(t,x) \end{pmatrix} \qquad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.2}$$

We assume in addition that $\psi_i(t,x)$ satisfy the equal-time anti-commutation relations (2.2) and introduce the generalised Fermi momentum

$$k_{\mathrm{F},z} \equiv (2m)^{1-1/z} \mu^{1/z} \,. \tag{4.3}$$

Notice that $k_{\mathrm{F},1} = \mu$, $k_{\mathrm{F},2} = k_{\mathrm{F}}$, and $k_{\mathrm{F},z} \to 2m$ for $z \to \infty$.

From (4.1) it follows that for $z = 1$ the fields $\psi_i(t,x)$ are the left and right moving components of a massless Dirac fermion $\psi(t,x)$ on the half line. For $z = 2$ one has instead two independent Schrödinger fields (2.1). It is useful to consider first these two cases because the ones corresponding to $z = 2n-1$ and $z = 2n$ for any $n \in \mathbb{N}$ can be studied as direct extensions of the models having $z = 1$ and $z = 2$ respectively.

The Dirac case $z = 1$ has been considered in detail in [90]. It has two types of boundary conditions at $x = 0$ that ensure energy conservation. The boundary condition

$$\psi_1(t,0) = \mathrm{e}^{\mathrm{i}\alpha_{\mathrm{v}}}\, \psi_2(t,0) \qquad \alpha_{\mathrm{v}} \in [0,2\pi) \qquad t \in \mathbb{R} \tag{4.4}$$

preserves the electric charge but not the helicity, while the opposite holds for the boundary condition

$$\psi_1(t,0) = \mathrm{e}^{-\mathrm{i}\alpha_{\mathrm{a}}}\, \psi_2^*(t,0) \qquad \alpha_{\mathrm{a}} \in [0,2\pi) \qquad t \in \mathbb{R}\,. \tag{4.5}$$

The boundary conditions (4.4) and (4.5) define respectively the vector and axial phases of the massless Dirac fermion on the half line. Notice that both (4.4) and (4.5) provide a scale invariant coupling of components with different chirality at $x = 0$. For this reason the kernels of the spectral problem for the entanglement entropies in the two phases have off-diagonal elements. In fact, by imposing (4.4) or (4.5) for any $z = 2n-1$, one finds

$$\boldsymbol{K}_{2n-1}(x,y;\alpha) = \begin{pmatrix} K(k_{\mathrm{F},2n-1}; x-y) & \mathrm{e}^{\mathrm{i}\alpha}\, K(k_{\mathrm{F},2n-1}; x+y) \\ \mathrm{e}^{-\mathrm{i}\alpha}\, K(k_{\mathrm{F},2n-1}; -x-y) & K(k_{\mathrm{F},2n-1}; -x+y) \end{pmatrix} \qquad \alpha \in \{\alpha_{\mathrm{v}}\,, \alpha_{\mathrm{a}}\} \tag{4.6}$$

where

$$K(\mu;\zeta) \equiv \frac{\mathrm{e}^{-\mathrm{i}\zeta\mu}}{2\pi\mathrm{i}\,(\zeta - \mathrm{i}\varepsilon)} \qquad \varepsilon \to 0^+ \tag{4.7}$$

hence $\boldsymbol{K}_{2n-1}$ can be written in terms of the correlators as discussed in [90].

For $z = 2n$ one finds instead two fully decoupled Schrödinger fields and either the Neumann or the Dirichlet boundary conditions (see (2.11)) can be imposed for each of them (the case $z = 2$ has been discussed in section 2). Accordingly,

$$\boldsymbol{K}_{2n}(x,y;\kappa_1,\kappa_2) = \begin{pmatrix} K_{\kappa_1}(k_{\mathrm{F},2n}; x,y) & 0 \\ 0 & K_{\kappa_2}(k_{\mathrm{F},2n}; x,y) \end{pmatrix} \qquad \kappa_1\,, \kappa_2 \in \{+\,,-\} \tag{4.8}$$

where $K_\pm$ are given by (3.1).

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When $z = 2n-1$ the entanglement entropies for the interval $A = [0, R]$ can be expressed in terms of the eigenvalues ζ_s of the spectral problem

$$\int_0^R \boldsymbol{K}_{2n-1}(x, y; \alpha)\, \Phi(y, s)\, \mathrm{d}y \;=\; \zeta_s\, \Phi(x, s) \qquad \Phi(x, s) \equiv \begin{pmatrix} \Phi_1(x, s) \\ \Phi_2(x, s) \end{pmatrix} \qquad s \in \mathbb{R}\,. \tag{4.9}$$

The solution of (4.9) is given by

$$\zeta_s = \frac{1 - \tanh(\pi s)}{2} \tag{4.10}$$

and

$$\Phi_1(x, s) = \mathrm{e}^{\mathrm{i}k_{\mathrm{F},2n-1}x}\phi_s(x) \qquad \Phi_2(x, s) = \mathrm{e}^{-\mathrm{i}k_{\mathrm{F},2n-1}x}\mathrm{e}^{-\mathrm{i}\alpha}\phi_s(-x) \tag{4.11}$$

for $x \in [0, R]$, and ζ_s and $\phi_s(x)$ satisfy the simpler and well known [91–93] spectral problem

$$\int_{-R}^{R} \frac{1}{2\pi\mathrm{i}\,(x - y - \mathrm{i}\varepsilon)}\, \phi_s(y)\, \mathrm{d}y \;=\; \zeta_s\, \phi_s(x) \qquad x \in [-R, R]\,. \tag{4.12}$$

We remark that the dependence on $k_{\mathrm{F},2n-1}$ and α is carried by the eigenfunctions, while the eigenvalues ζ_s are independent of these parameters. The explicit form [91–93] of $\phi_s(x)$ is not needed because the entanglement entropies are fully expressed in terms of the eigenvalues. Thus, all the Lifshitz fermions with odd z have the entanglement entropies of the relativistic massless Dirac fermion, i.e.

$$S_A^{(\alpha)} \;=\; \frac{1}{12}\left(1 + \frac{1}{\alpha}\right)\log(2R/\epsilon) + O(1) \qquad z = 2n - 1\,. \tag{4.13}$$

It is worth mentioning that the independence of the spectrum on $k_{\mathrm{F},2n-1}$ leads to a well known logarithmic ultraviolet divergency, which induces the presence of the UV cutoff ϵ in (4.13).

When $z = 2n$ we have two independent Schrödinger fields and each of them satisfy either the Neumann or the Dirichlet boundary condition. Therefore

$$S_{A;\kappa_1,\kappa_2}^{(\alpha)} \;=\; S_{A,\kappa_1}^{(\alpha)} + S_{A,\kappa_2}^{(\alpha)} \qquad \kappa_1\,, \kappa_2 \in \{+\,, -\} \qquad z = 2n \tag{4.14}$$

where $S_{A,\kappa}^{(\alpha)}$ is given by (3.14) with the substitution $k_{\mathrm{F}} \mapsto k_{\mathrm{F},2n}$.

The mean particle density (2.20) of the Schrödinger fermion exhibits Friedel oscillations. It turns out that such oscillations are absent for the massless Dirac fermion $z = 1$ on the half line. In fact, in this case, for both the vector and axial phases, one finds [90]

$$\langle \varrho(t, x)\rangle_{\infty,\mu} \equiv \langle :\psi_1^*\, \psi_1 : (t, x)\rangle_{\infty,\mu} + \langle :\psi_2^*\, \psi_2 : (t, x)\rangle_{\infty,\mu} = \frac{|\mu|}{\pi} \tag{4.15}$$

where $: \cdots :$ denotes the normal product. The mixed correlation functions in the vector and axial phases are given by

$$\langle :\psi_1^*\, \psi_2 : (t, x)\rangle_{\infty,\mu} = \mathrm{e}^{-\mathrm{i}\alpha_{\mathrm{v}}}\, \frac{\mathrm{e}^{\mathrm{i}\mu x}}{2\pi x}\, \sin(\mu x) \qquad \langle :\psi_1\, \psi_2 : (t, x)\rangle_{\infty,\mu} = \mathrm{e}^{-\mathrm{i}\alpha_{\mathrm{a}}}\, \frac{\mathrm{e}^{\mathrm{i}\mu x}}{2\pi x}\, \sin(\mu x) \tag{4.16}$$

respectively, which display an oscillatory behaviour.

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5 Small η expansion

In this section we investigate the expansion of the entanglement entropies as $\eta \to 0$ by employing two different methods. The first approach (section 5.1) is based on the expansion of the PSWF, while the second one (section 5.2) exploits the expansion of the tau functions (3.22).

5.1 PSWF approach

The asymptotic behaviour of the PSWF leads to the following small η expansion for the eigenvalues [54, 94]

$$\gamma_n \,=\, \tilde{g}_n\, \eta^{2n+1}\, [1 + \tilde{a}_n\, \eta^2 + O(\eta^4)] \tag{5.1}$$

where

$$\tilde{g}_n = \frac{2}{\pi} \left(\frac{2^{2n} (n!)^3}{(2n)!\,(2n+1)!} \right)^2 \qquad\qquad \tilde{a}_n = -\frac{2n+1}{(2n-1)^2\,(2n+3)^2}\,. \tag{5.2}$$

By using (5.1) combined with either (3.11) for Neumann b.c. or (3.12) for Dirichlet b.c. into (3.14), we obtain the expansions of the entanglement entropies reported below.

As for the entanglement entropy, for Neumann b.c. we find

$$\begin{aligned} S_{A,+} \,=\, & -\frac{2}{\pi}\,\eta \log(\eta) + \frac{2}{\pi}\,[1 - \log(2/\pi)]\,\eta - \frac{2}{\pi^2}\,\eta^2 + \frac{2}{9\pi}\,\eta^3 \log(\eta) \\ & + \frac{2}{3\pi}\left(\frac{1}{3}\log(2/\pi) - \frac{2}{\pi^2}\right)\eta^3 + \frac{4}{3\pi^2}\left(\frac{1}{3} - \frac{1}{\pi^2}\right)\eta^4 + O(\eta^5 \log(\eta)) \end{aligned} \tag{5.3}$$

which comes only from γ_0 because $s(\gamma_{2n}) = O(\eta^5 \log(\eta))$ when $n \geqslant 1$. Instead, for Dirichlet b.c. we obtain

$$\begin{aligned} S_{A,-} \,=\, & -\frac{2}{3\pi}\,\eta^3 \log(\eta) + \frac{2}{9\pi}\,[1 - \log(2/(9\pi))]\,\eta^3 + \frac{2}{25\pi}\,\eta^5 \log(\eta) \\ & + \frac{2}{75\pi}\log(2/(9\pi))\,\eta^5 - \frac{2}{81\pi^2}\,\eta^6 + O(\eta^7 \log(\eta)) \end{aligned} \tag{5.4}$$

where only γ_1 has been employed because $s(\gamma_{2n+1}) = O(\eta^7 \log(\eta))$ when $n \geqslant 1$. Comparing the leading terms of (5.3) and (5.4), we have that $S_{A,+} > S_{A,-}$ when $\eta \to 0$, which can be observed from figure 1 and from the top panels of figure 5, where the solid blue lines correspond to the expressions in (5.3) and (5.4).

The analysis performed for the entanglement entropy can be adapted to find the expansion of the Rényi entropies (3.14) with $\alpha \neq 1$ as $\eta \to 0$. We focus on the cases where $\alpha > 1$ and finite. For Neumann b.c. we find

$$\begin{aligned} S^{(\alpha)}_{A,+} \,=\, \frac{\alpha}{\alpha-1} \Bigg\{ & \left[1 + \frac{\eta}{\pi} + \left(\frac{4}{3\pi^2} - \frac{1}{3}\right)\eta^2 + \frac{2}{\pi}\left(\frac{1}{\pi^2} - \frac{1}{\pi}\right)\eta^3 \right] \frac{2\eta}{\pi} \\ & - \left[\frac{1}{\alpha} + \frac{2\eta}{\pi} + \left(\frac{2(\alpha+1)}{\pi^2} - \frac{1}{9}\right)\eta^2 + \frac{2(\alpha+1)}{9\pi}\left(\frac{6(\alpha+2)}{\pi^2} - 1\right)\eta^3 \right] \left(\frac{2\eta}{\pi}\right)^{\alpha} \\ & + \left[\frac{1}{2\alpha} + \frac{2\eta}{\pi} + \left(\frac{2(2\alpha+1)}{\pi^2} - \frac{1}{9}\right)\eta^2 \right] \left(\frac{2\eta}{\pi}\right)^{2\alpha} \\ & - \left(\frac{1}{3\alpha} + \frac{2\eta}{\pi}\right)\left(\frac{2\eta}{\pi}\right)^{3\alpha} + \frac{1}{4\alpha}\left(\frac{2\eta}{\pi}\right)^{4\alpha} \Bigg\} + O(\eta^5) \end{aligned} \tag{5.5}$$

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while for Dirichlet b.c. we obtain

$$S_{A,-}^{(\alpha)} = \frac{\alpha}{\alpha-1}\left\{\left(1-\frac{9}{75}\,\eta^2+\frac{1}{9\pi}\,\eta^3\right)\frac{2\eta^3}{9\pi}\right. \tag{5.6}$$
$$\left. -\left(\frac{1}{\alpha}-\frac{3}{25}\,\eta^2+\frac{2}{9\pi}\,\eta^3\right)\left(\frac{2\eta^3}{9\pi}\right)^{\alpha}+\frac{1}{2\alpha}\left(\frac{2\eta^3}{9\pi}\right)^{2\alpha}\right\}+O(\eta^7)\,.$$

The derivation of the expansions (5.5) and (5.6) is reported in appendix B.1. Notice that the relevance of the various terms in these expansions as $\eta \to 0$ depends on α.

The expansion of the single copy entanglement as $\eta \to 0$ can be studied in a similar way, by employing the function $s_\infty(x)$ in (3.16). When Neumann b.c. are imposed, we find

$$S_{A,+}^{(\infty)} = \frac{2}{\pi}\,\eta+\frac{2}{\pi^2}\,\eta^2+\frac{2}{3\pi}\left(\frac{4}{\pi^2}-\frac{1}{3}\right)\eta^3+\frac{4}{\pi^2}\left(\frac{1}{\pi^2}-\frac{1}{9}\right)\eta^4+O(\eta^5) \tag{5.7}$$

while for Dirichlet b.c. the expansion of the single copy entanglement reads

$$S_{A,-}^{(\infty)} = \frac{2}{9\pi}\,\eta^3-\frac{2}{75\pi}\,\eta^5+\frac{2}{81\pi^2}\,\eta^6+O(\eta^7)\,. \tag{5.8}$$

The relation (3.18) can be employed to check the above expansions. Indeed, by summing up either (5.3) and (5.4), or (5.5) and (5.6), or (5.7) and (5.8), we recover the expansions found in section 7.2 of [42].

In figure 5 we compare the small η expansions reported above (blue solid lines) with the corresponding exact curves, obtained numerically through (3.14) (black empty circles). The cases $\alpha = 1$, $\alpha = 3$ and $\alpha \to \infty$ are considered, for both boundary conditions. Notice that these small η expansions do not capture the first local maximum of the corresponding curves.

5.2 Tau function approach

The expansion of the entanglement entropies as $\eta \to 0$ can be found by plugging into (3.19) the expansion of $\tau_\pm$ in this regime. In the appendix B.2 the latter expansion is obtained as a special case of the expansion given in the Conjecture 4 of [61]. The result reads

$$\tau_\pm = \frac{1}{\pi^{\pm 1/2}\,\mathrm{e}^{\eta^2/4}}\sum_{n=0}^{\infty} C_{\mathrm{III}'}\left(\pm\frac{1}{4},\pm\frac{1}{4},\frac{1}{4}\mp n\right)\frac{(\eta/2)^{n(2n\mp 1)}}{z^n}\,\mathcal{B}_\pm(n;\eta^2/4) \tag{5.9}$$

where the coefficients are written in terms of the Barnes G-function $G(z)$ as follows

$$C_{\mathrm{III}'}(\theta,\theta,\hat\sigma) \equiv \frac{\left[G(1+\theta+\hat\sigma)\,G(1+\theta-\hat\sigma)\right]^2}{G(1+2\hat\sigma)\,G(1-2\hat\sigma)}\,. \tag{5.10}$$

The functions $\mathcal{B}_\pm(n;t)$ are defined in (B.16) and the first terms of their Taylor expansions for small t are reported in (B.17) and (B.18). Thus, the expressions in (5.9) are double expansions; both in positive powers of η and in negative powers of z.

Approximate analytic expressions for the entanglement entropies in the regime of small η can be found by adapting the analysis performed in [42] for the interval on the line to the cases we are considering on the half line.

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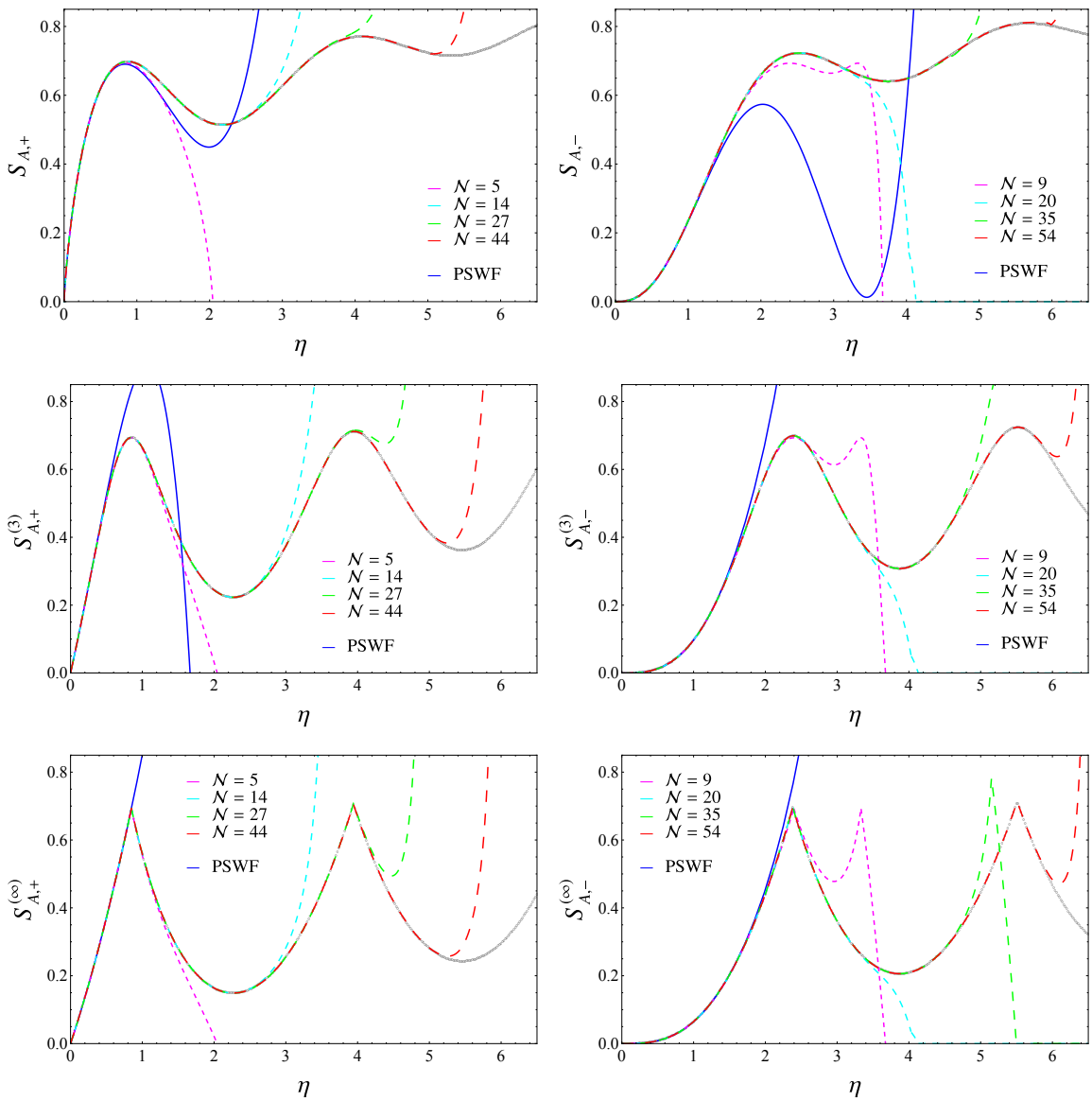

Figure 5. Entanglement entropies $S_A^{(\alpha)}$ for $\alpha = 1$ (top panels), $\alpha = 3$ (middle panels) and $\alpha \to \infty$ (bottom panels) in the small η regime. The blue solid curves correspond to the expansions obtained through the PSWF (see section 5.1), while the coloured dashed curves have been found through the expansion (5.9) of the tau function (see section 5.2 and (5.12)).

Given a positive integer $\mathcal{N} \geqslant 1$, let us truncate (5.9) by discarding the terms of order $o(\eta^{\mathcal{N}})$. This condition provides also a truncation of the series in n occurring in (5.9) to $n \leqslant N_\pm$, where $N_\pm = N_\pm(\mathcal{N})$. The functional form of $N_\pm(\mathcal{N})$ depends on the boundary condition. Since $o(1/z^{N_\pm})$ terms in (5.9) have been neglected, denoting by $\tilde{\tau}_{\mathcal{N},N_\pm}$ the resulting finite sum, we have that $\tilde{\tau}_{\mathcal{N},N_\pm} = z^{-N_\pm} P_{N_\pm,\mathcal{N}}(z)$, where $P_{N_\pm,\mathcal{N}}(z)$ is a polynomial of degree $N_\pm$ whose coefficients are polynomials in η of different degrees that are smaller

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than or equal to $\mathcal{N}$. Thus

$$\partial_z \log(\tilde{\tau}_{\mathcal{N},N_\pm}) = \sum_{i=1}^{N_\pm} \frac{1}{z-z_i} - \frac{N_\pm}{z} \tag{5.11}$$

where $z_i \in \mathcal{P}_{N_\pm,\mathcal{N}}$ are the zeros of $P_{N_\pm,\mathcal{N}}(z)$, that are highly non trivial functions of η. According to the Abel-Ruffini theorem, the roots of a polynomial of degree five or higher cannot be written through radicals. This fundamental algebraic obstruction tells us that analytic results can be found only for $N_\pm \leqslant 4$.

Plugging the finite sum (5.11) into (3.19) and exploiting that $s_\alpha(0) = 0$, one obtains

$$\widetilde{S}^{(\alpha)}_{A;\mathcal{N}} \equiv \sum_j s_\alpha(\tilde{z}_j) \qquad \tilde{z}_j \in \mathcal{P}_{N_\pm,\mathcal{N}} \cap [0,1] \tag{5.12}$$

where only the zeros of $P_{N_\pm,\mathcal{N}}(z)$ belonging to $[0,1]$ contribute. The finite sums (5.12) approximate the entanglement entropies in the small η regime. The analytic expressions for (5.12) have been obtained by combining the procedure discussed in the appendix E.2 of [42] with the results in the appendix B.2. The results are quite lengthy and not very instructive; hence they have not been reported here.

Some curves obtained from (5.12) are shown in figure 5 and figure 2, for either Neumann (left panels) or Dirichlet (right panels) boundary conditions, and they correspond to coloured dashed lines. The exact curves obtained numerically are indicated through empty markers. In figure 5 we have considered $N_+, N_- \in \{1,2,3,4\}$ and the largest values of $\mathcal{N}$ providing them. In particular, we have that $N_+(5) = 1$, $N_+(14) = 2$, $N_+(27) = 3$, and $N_+(44) = 4$ for Neumann boundary condition and $N_-(9) = 1$, $N_-(20) = 2$, $N_-(35) = 3$, and $N_-(54) = 4$ for Dirichlet boundary condition. In figure 2 we have reported the curves corresponding to $N_+(44) = 4$ and $N_-(54) = 4$, for various values of α. It is evident in figure 5 that the agreement between the approximations obtained from (5.12) and the corresponding exact curves improve as $N_\pm$ increases. The best approximations for small η obtained from (5.12) (see figure 2 and the red curves in figure 5) nicely reproduce the first two oscillations of the entanglement entropies.

6 Large η expansion

In this section we study the expansions of the entanglement entropies as $\eta \to \infty$, which are obtained by plugging the expansions of the tau functions $\tau_\pm$ in this regime into (3.19).

The expansion of the Painlevé III'_1 tau function given in eq. (A.30) of [62], properly specialised to the cases we are considering, provides the large η expansions of the tau functions $\tau_\pm$ in (3.22). This analysis is described in the appendix C.1 and the results read

$$\tau_\pm = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pm \mathrm{i}\frac{\pi}{2}(\nu+n)} \frac{\mathrm{e}^{\mathrm{i}2\eta(n+\nu)}}{(4\eta)^{(\nu+n)^2}} G(1+\nu+n)\, G(1-\nu-n) \sum_{p=0}^{\infty} \frac{\mathcal{D}_p(\nu+n)}{(2\eta)^p} \tag{6.1}$$

where

$$\nu = \frac{1}{2\pi\mathrm{i}} \log(1-1/z) \tag{6.2}$$

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and

$$\mathcal{D}_0(\nu) = 1 \qquad \mathcal{D}_1(\nu) = -\,\mathrm{i}\,\nu^3 \qquad \mathcal{D}_2(\nu) = -\frac{1}{4}\,\nu^2(2\nu^4 + 5\nu^2 + 1)\,. \tag{6.3}$$

We find it worth remarking that the expansions (6.1) is an asymptotic series in $1/\eta$ [62]; hence also the corresponding expansions of the entanglement entropies derived from them are asymptotic. The expansion (6.1) is not valid for $z \in [0, 1]$, where a different expansion of $\tau_\pm$ is expected (see e.g. [95] for $z = 1$).

In order to study the expansion of the entanglement entropies (3.19) as $\eta \to \infty$, we find it convenient to write (6.1) as follows

$$\tau_\pm \;=\; \tilde{\tau}_{\pm,\infty}\,\mathcal{T}_{\pm,\infty} \tag{6.4}$$

where $\tilde{\tau}_{\pm,\infty}$ is the term corresponding to $n = 0$ and $p = 0$ in (6.1), namely

$$\tilde{\tau}_{\pm,\infty} \;=\; \frac{\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}\nu}\,\mathrm{e}^{\mathrm{i}2\nu\eta}\,G(1+\nu)\,G(1-\nu)}{(4\eta)^{\nu^2}} \tag{6.5}$$

while $\mathcal{T}_{\pm,\infty}$ reads

$$\mathcal{T}_{\pm,\infty} = \sum_{n\in\mathbb{Z}} (\pm\mathrm{i})^n\,\frac{\mathrm{e}^{\mathrm{i}2n\eta}}{(4\eta)^{n(n+2\nu)}}\,\frac{G(1+\nu+n)\,G(1-\nu-n)}{G(1+\nu)\,G(1-\nu)}\sum_{p=0}^{\infty}\frac{\mathcal{D}_p(\nu+n)}{(2\eta)^p}\,. \tag{6.6}$$

The term (6.5) corresponds to eq. (1.35) of [63] specialised to our cases (see appendix C.1).

Plugging (6.4) into (3.19), the following decomposition for $S^{(\alpha)}_{A,\pm}$ is obtained

$$S^{(\alpha)}_{A,\pm} = S^{(\alpha)}_{A,\pm,\infty} + \widetilde{S}^{(\alpha)}_{A,\pm,\infty} \tag{6.7}$$

where $S^{(\alpha)}_{A,\pm,\infty}$, which originates from $\log(\tilde{\tau}_{\pm,\infty})$, provides the leading contributions to the entanglement entropies for large η; while $\widetilde{S}^{(\alpha)}_{A,\pm,\infty}$ comes from $\log(\mathcal{T}_{\pm,\infty})$ and gives the subleading corrections that vanish when $\eta \to \infty$.

The leading terms occurring in $S^{(\alpha)}_{A,\pm,\infty}$ can be found by first taking the logarithm of (6.5), i.e.

$$\log(\tilde{\tau}_{\pm,\infty}) = \mathrm{i}\,2\,\nu\,\eta - \nu^2\log(4\eta) \pm \mathrm{i}\,\frac{\pi}{2}\,\nu + \log\big[G(1-\nu)\,G(\nu+1)\big] \tag{6.8}$$

and then plugging the resulting expression into (3.19). While the integrals corresponding to the linear terms in ν in the r.h.s. of (6.8) vanish, the remaining terms provide non vanishing contributions to the entanglement entropies and the result reads

$$S^{(\alpha)}_{A,\pm,\infty} = \frac{1}{12}\left(1+\frac{1}{\alpha}\right)\log(4\eta) + \frac{E_\alpha}{2} \tag{6.9}$$

where the constant term E_α is

$$E_\alpha \equiv \left(1+\frac{1}{\alpha}\right)\int_0^\infty\left(\frac{\alpha\,\mathrm{csch}(t)}{\alpha^2-1}\,(\mathrm{csch}(t/\alpha) - \alpha\,\mathrm{csch}(t)) - \frac{\mathrm{e}^{-2t}}{6}\right)\frac{\mathrm{d}t}{t}\,. \tag{6.10}$$

We remark that (6.9) is independent of the boundary condition. Furthermore, (6.9) is equal to half of the corresponding terms in the large η expansion of $S^{(\alpha)}_{2A\subset\mathbb{R}}$ (see eq. (8.16)

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of [42]), which has been previously found in the lattice [56, 57, 59] by using the Fisher-Hartwig conjecture, and also in the continuum by employing a result of Slepian [94], as shown in [55, 96] (see also [34] for a rigorous derivation of the logarithm term). The fact that the leading terms in the large η expansions of $S^{(\alpha)}_{A,\pm}$ are half of the corresponding ones for $S^{(\alpha)}_{2A\subset\mathbb{R}}$ can be observed in figure 1.

The term $\widetilde{S}^{(\alpha)}_{A,\pm,\infty}$ in (6.7) is obtained from (6.6) and contains all the subleading contributions in the large η expansion of $S^{(\alpha)}_{A,\pm}$, which vanish for $\eta\to\infty$. It can be written as follows

$$\widetilde{S}^{(\alpha)}_{A,\pm,\infty} = \sum_{N=0}^{\infty} \frac{\widetilde{S}^{(\alpha)}_{A,\pm,\infty,N}}{(4\eta)^N} \,. \tag{6.11}$$

In the appendix C.3 the derivation of the coefficients $\widetilde{S}^{(\alpha)}_{A,\pm,\infty,N}$ for $N\in\{0,1,2\}$ is described. These are non trivial functions of η that vanish when $\eta\to\infty$ and their expressions are reported in the following.

When $N=0$, for the entanglement entropy and the Rényi entropies with $\alpha\neq 1$ we find respectively (see the appendix C.3.3)

$$\widetilde{S}_{A,\pm,\infty,0} \;=\; \pm\sin(2\eta)\sum_{k=1}^{\infty}(-1)^{k+1}\,\frac{(2k-1)\,[(k-1)!]^2}{(4\eta)^{2k-1}} \;=\; \pm\,\frac{\sin(2\eta)}{4\eta}\left(1-\frac{3}{16\eta^2}\right)+O(1/\eta^5) \tag{6.12}$$

and

$$\widetilde{S}^{(\alpha)}_{A,\pm,\infty,0} \;=\; \frac{2}{\alpha-1}\sum_{j=1}^{\infty}\frac{1}{j}\,\cos\!\left[\left(2\eta-\frac{\pi}{2}\right)j\right]\sum_{k=1}^{\infty}\left(\frac{\pm\,\Omega(\hat{y}_k/\alpha)}{(4\eta)^{(2k-1)/\alpha}}\right)^{j} \tag{6.13}$$

where $\hat{y}_k\equiv \mathrm{i}(k-1/2)$ and $\Omega(\hat{y}_k/\alpha)\equiv\frac{\Gamma(1/2-1/(2\alpha)+k/\alpha)}{\Gamma(1/2+1/(2\alpha)-k/\alpha)}$ (see (C.32) and (C.53) respectively).

The coefficient $\widetilde{S}^{(\alpha)}_{A,\pm,\infty,1}$ is derived in the appendix C.3.4. For $\alpha=1$ and $\alpha\neq 1$ we obtain respectively

$$\widetilde{S}_{A,\pm,\infty,1} \;=\; \pm\cos(2\eta)\sum_{k=1}^{\infty}(-1)^{k+1}\,\frac{(2k-1)\,[(k-1)!]^2\,\mathcal{P}_1(\hat{y}_k)}{(4\eta)^{2k-1}} \;=\; \mp\,\frac{\cos(2\eta)}{2\eta}+O(1/\eta^3) \tag{6.14}$$

with $\mathcal{P}_1(\hat{y}_k)=-6k^2+6k-2$ coming from (C.59), and

$$\widetilde{S}^{(\alpha)}_{A,\pm,\infty,1} \;=\; -\frac{2}{\alpha-1}\sum_{j=1}^{\infty}\sin\!\left[\left(2\eta-\frac{\pi}{2}\right)j\right]\sum_{k=1}^{\infty}\mathcal{P}_1(y_k/\alpha)\left(\frac{\pm\,\Omega(\hat{y}_k/\alpha)}{(4\eta)^{(2k-1)/\alpha}}\right)^{j} \tag{6.15}$$

where $\mathcal{P}_1(y_k/\alpha)=-\frac{3}{2}\big[(2k-1)/\alpha\big]^2-\frac{1}{2}$.

Finally, for $\widetilde{S}^{(\alpha)}_{A,\pm,\infty,1}$ with either $\alpha=1$ or $\alpha\neq 1$ we find (see the appendix C.3.5)

$$\begin{aligned}\widetilde{S}_{A,\pm,\infty,2} &= -\frac{1}{6}\pm\sin(2\eta)\sum_{k=1}^{\infty}(-1)^{k+1}\,\frac{(2k-1)\,[(k-1)!]^2\big[\mathcal{P}_{2,c}(\hat{y}_k)-\mathcal{P}_{2,a}(\hat{y}_k)\big]}{(4\eta)^{2k-1}}\\ &= -\frac{1}{6}\mp\frac{7\sin(2\eta)}{4\eta}+O(1/\eta^3)\end{aligned} \tag{6.16}$$

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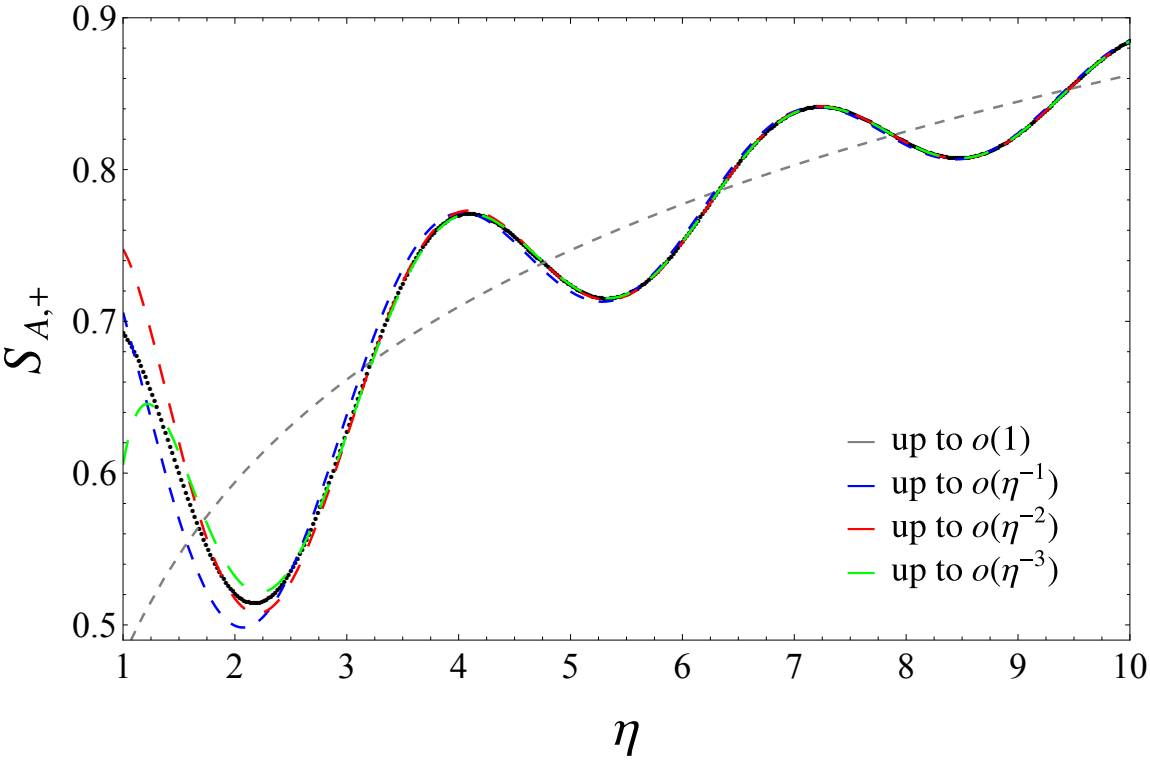

Figure 6. Entanglement entropy for Neumann b.c. in a range where the large η expansion (see (6.7), (6.9), and (6.11)) deviates from the curve obtained numerically (black points).

where $\mathcal{P}_{2,c}(\hat{y}_k) - \mathcal{P}_{2,a}(\hat{y}_k) = -18k^4 + 16k^3 - 8k + 3$ is obtained from (C.74) and (C.76), and

$$\widetilde{S}^{(\alpha)}_{A,\pm,\infty,2} = \frac{(\alpha+1)(3\alpha^2-7)}{48\alpha^3} - \frac{2}{\alpha-1}\sum_{j=1}^{\infty}\cos\left[\left(2\eta-\frac{\pi}{2}\right)j\right]\sum_{k=1}^{\infty}\widetilde{\mathcal{P}}_2(j;y_k/\alpha)\left(\frac{\pm\Omega(\hat{y}_k/\alpha)}{(4\eta)^{(2k-1)/\alpha}}\right)^j \tag{6.17}$$

with $\widetilde{\mathcal{P}}_2(j;y)$ defined as (from (C.76), (C.77) and (C.78))

$$\widetilde{\mathcal{P}}_2(j;y) \equiv -\mathcal{P}_{2,c}(y) + \mathcal{P}_{2,d}(y) + (1-j)\,\mathcal{P}_{2,e}(y) = 18j\,y^4 - 3j\,y^2 + \frac{j}{8} + 20\mathrm{i}\,y^3 - 5\mathrm{i}\,y\,. \tag{6.18}$$

In the appendix C.3.6 some consistency checks for the analytic expressions of $\widetilde{S}^{(\alpha)}_{A,\pm,\infty,N}$ reported above have been discussed. In particular, we have considered the limit $\alpha \to 1$, the relation (3.18) and the double scaling limit of the lattice results obtained in [69].

In figure 6, figure 7 and figure 8, the curves for the entanglement entropies found numerically are compared with the curves corresponding to the analytic expressions valid at larg η obtained from (6.7), (6.9) and (6.11), where $\widetilde{S}_{A,\pm,\infty,N}$ and $\widetilde{S}^{(\alpha)}_{A,\pm,\infty,N}$ are respectively (6.12) and (6.13) for $N = 0$, (6.14) and (6.15) for $N = 1$, and (6.16) and (6.17) for $N = 2$. We show these results only for Neumann b.c. because the ones for Dirichlet b.c. are qualitatively very similar.

In figure 6 the entanglement entropy $S_{A,+}$ is considered: the exact curve is labelled by the black circles, the dashed grey line correspond only to the leading terms (6.9) and the dashed coloured lines are obtained by including also the three subleading terms having

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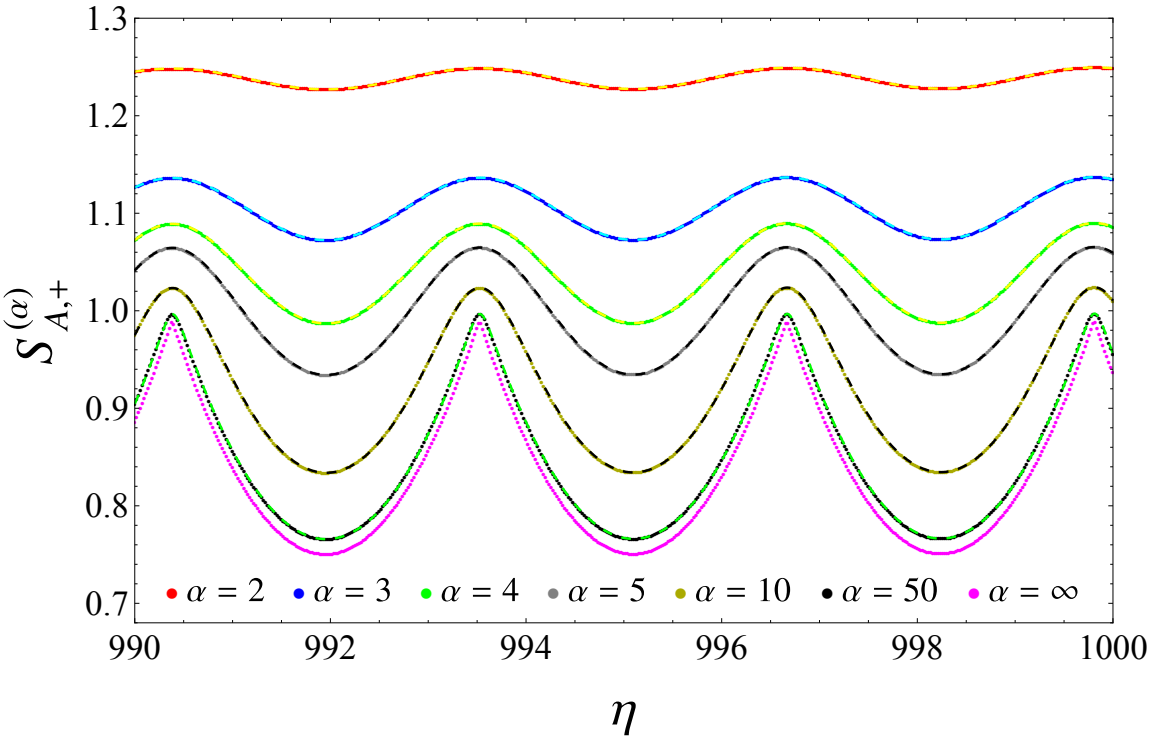

Figure 7. Oscillatory behaviour of the entanglement entropies with $\alpha > 1$ for Neumann b.c. in the large η regime. The dashed curves are obtained from (6.7), (6.9) and (6.11) with $N \leqslant 2$.

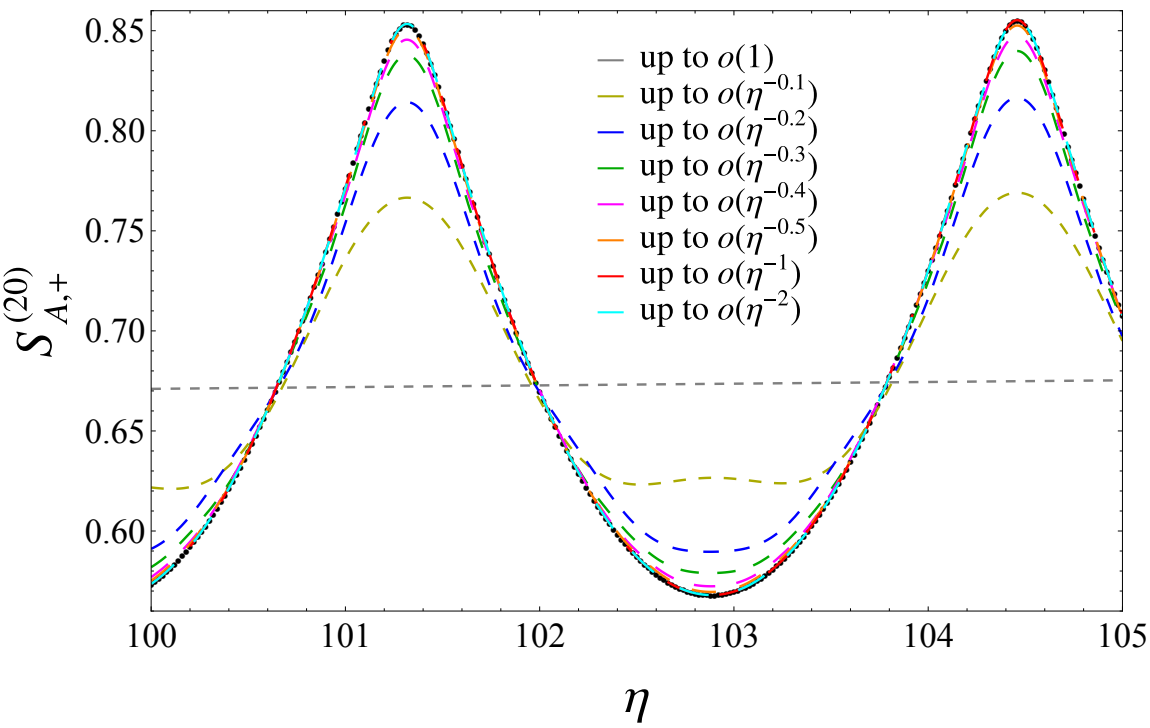

Figure 8. The Rényi entropy of order $\alpha = 20$ for Neumann b.c. in the regime of large η.

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$N \in \{0,1,2\}$ in (6.11). The analytic approximation that takes into account also the subleading terms nicely reproduce the exact curve for $\eta \gtrsim 6.5$. For smaller values of η, the subleading term corresponding to $N = 0$ is not enough to capture the exact curve. When $\eta < 2.5$ all the approximate curves obtained from an analytic expression that we have considered deviate from the one of $S_{A,+}$.

In figure 7 we focus on the range $\eta \in [990, 1000]$, where η takes large values and the amplitude of the oscillations is small. The curves for the Rényi entropies found numerically (coloured circles) are nicely captured by the corresponding approximate analytic expressions (dashed lines) found by including all the subleading terms that we have evaluated (i.e. the ones having $N \in \{0,1,2\}$ in (6.11)), truncated at order $O(1/\eta^3)$.

In figure 8 we consider $S^{(20)}_{A,+}$ in a regime where η is large enough and show how the agreement between the exact curve (black circles) and the ones obtained from the analytic expressions of the large η expansion truncated at some order improves as the number of subleading terms in this truncated sum increases, i.e. when higher orders in $1/\eta$ are included. The dashed grey line corresponds to the leading terms (6.9).

We find it worth remarking that the oscillating terms in $\widetilde{S}_{A,+,\infty,N}$ and $\widetilde{S}_{A,-,\infty,N}$ have opposite signs (see (6.12), (6.14) and (6.16)), while the non oscillating ones (only the constant in (6.16) for the terms we are considering) are equal to half of the corresponding ones in the large η expansion of $S^{(\alpha)}_{2A\subset\mathbb{R}}$ [42]. This implies that the oscillating terms cancel in the r.h.s. of (3.18) (see also (C.86)) and therefore $S_{2A\subset\mathbb{R}}$ does not oscillate, as found in [42]. Notice that, instead, this exact cancellation of the oscillating terms in the r.h.s. of (3.18) does not occur when $\alpha \neq 1$.

By applying the above observations to the combination (3.23), we find that the non oscillating terms simplify, for any $\alpha > 0$. This cancellation is due to the relative factor of $-1/2$ between $S^{(\alpha)}_{A,\pm}$ and $S^{(\alpha)}_{2A\subset\mathbb{R}}$ in (3.23).

7 Cumulants expansion

In this section we discuss the relation, found in [70–72], between entanglement entropy and the charge cumulants for the models that we are considering.

The cumulants of the time independent local charge operator $Q_{A,\pm} = \int_A \varrho_\pm(t=0,x)\,\mathrm{d}x$ are $\mathcal{C}^{(k)}_A \equiv \big[\partial^k_{\mathrm{i}\zeta} \log(\langle \mathrm{e}^{\mathrm{i}\zeta Q_{A,\pm}}\rangle)\big]\big|_{\zeta=0}$, where $k \geqslant 1$. Their generating function can be expressed in terms of the tau function (3.20) as follows [42, 70, 72, 97]

$$\log\big[\langle \mathrm{e}^{\mathrm{i}\zeta Q_{A,\pm}}\rangle\big] = \mathrm{Tr}\big[\log\big(I + (\mathrm{e}^{\mathrm{i}\zeta}-1)\,K_\pm\big)\big] = \mathrm{Tr}\big[\log\big(I - z^{-1}\,K_\pm\big)\big] = \log(\tau_\pm) \tag{7.1}$$

where I is the identity operator, $K_\pm$ are the kernels (3.1) and $\zeta = 2\pi\nu$, with $\nu = \nu(z)$ being defined in (6.2).

From (7.1), one finds that the first cumulants are given by

$$\mathcal{C}^{(1)}_{A,\pm} = \mathrm{Tr}(K_\pm) \qquad \mathcal{C}^{(2)}_{A,\pm} = \mathrm{Tr}(K_\pm - K_\pm^2) \qquad \mathcal{C}^{(3)}_{A,\pm} = \mathrm{Tr}(K_\pm - 3K_\pm^2 + 2K_\pm^3) \tag{7.2}$$

where the Schatten p-norm of the kernels (3.1) is defined as

$$\mathrm{Tr}(K_\pm^p) = \sum_{n=0}^{\infty} (\gamma_n^\pm)^p \qquad p \geqslant 1\,. \tag{7.3}$$

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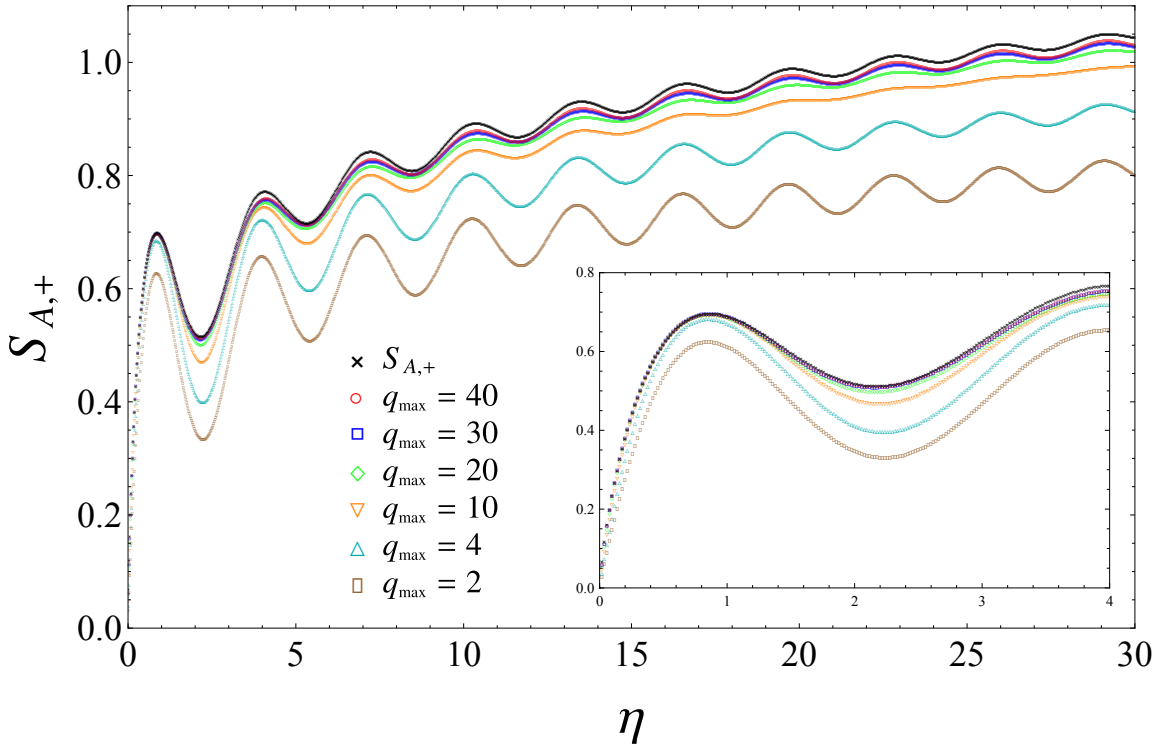

Figure 9. Cumulant expansion of the entanglement entropy for Neumann b.c. (see (7.4)). The inset zooms in on a range of small values of η, where a complete overlap is observed when $q_{\max}$ is large enough.

The n-th cumulant $\mathcal{C}^{(n)}_{A,\pm}$ is a finite linear combinations of the Schatten norms (7.3) with $p \in \mathbb{N}$ and $p = 1, \dots, n$; hence we can evaluate numerically the cumulants by computing the Schatten norms from (7.3), as done for the entanglement entropies.

In [70–72] a remarkable relation between the entanglement entropies and the charge cumulants has been studied (the final form has been reported in [71]). A similar relation has been found also for the Rényi entropies with integer index [72]. Focussing only on the entanglement entropy for simplicity, for the models we are considering this relation reads

$$S_{A,\pm} = \lim_{q\to\infty} \sum_{n=1}^{q+1} a_n(q)\, \mathcal{C}^{(n)}_{A,\pm} \qquad a_n(q) \equiv \begin{cases} 0 & \text{odd } n \\ 2 \displaystyle\sum_{k=n-1}^{q} \frac{S_1(k,n-1)}{k!\,k} & \text{even } n \end{cases} \tag{7.4}$$

where $S_1(k,n)$ are the unsigned Stirling numbers of the first kind. We remark that the coefficients in (7.4) are independent of η; hence the dependence on η of $S_{A,\pm}$ is encoded only in the cumulants.

In figure 9 we show some results about the cumulant expansion (7.4) in the case of Neumann b.c. (the curves for $S_{A,-}$ are very similar). The black crosses correspond to the curve obtained numerically, while the curves identified by the empty markers are given by the finite sums obtained by restricting (7.4) to $n \leqslant q_{\max} + 1$, for different values of $q_{\max}$. The approximation of the exact curve improves as $q_{\max}$ increases. Notice that a complete agreement is obtained for small values of η, as highlighted in the inset of figure 9.

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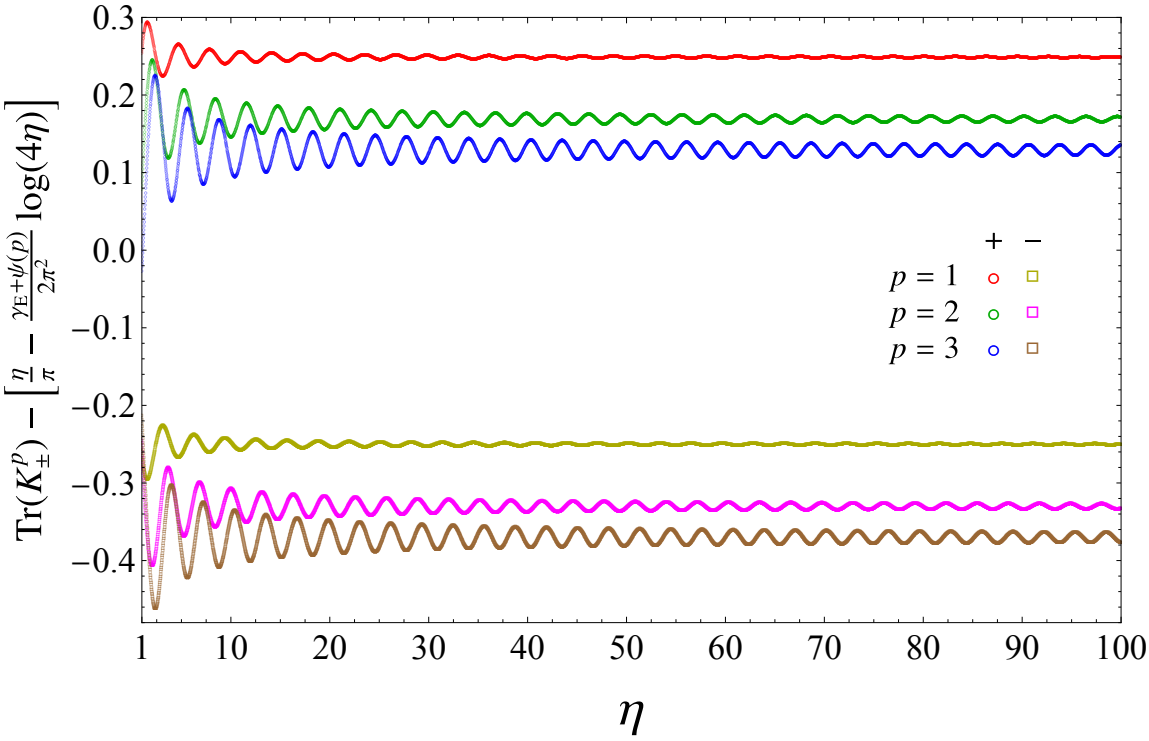

Figure 10. Schatten norms (7.3) for some values of p, when either Neumann (+) or Dirichlet (−) b.c. are imposed, in the large η regime (see (7.8)).

In [70] the cumulant expansion $S_{A,\pm} = \lim_{q\to\infty} \sum_{n=1}^{q} \tilde{a}_n \, \mathcal{C}_{A,\pm}^{(n)}$ has been first proposed, where $\tilde{a}_n = \lim_{q\to\infty} a_n(q) = 2\zeta(n)$ and $\zeta(z)$ is the Riemann zeta function. This expansion is divergent [71]. Indeed, testing numerically this expansion, we found that its deviation from the entanglement entropy increases with q.

The Schatten norms, and consequently also the cumulants, can be expressed in terms of the tau function (3.20) by adapting to (7.3) the procedure leading to the contour integral (3.19) for the entanglement entropies. The result reads

$$\mathrm{Tr}(K_\pm^p) \;=\; \lim_{\epsilon,\delta\to 0} \frac{1}{2\pi \mathrm{i}} \oint_{\mathfrak{C}} z^p \, \partial_z \log(\tau_\pm) \, \mathrm{d}z \qquad\qquad p \in \mathbb{N} \,. \tag{7.5}$$

This expression allows us to write the expansions of the Schatten norms in the regimes of small and large η by adapting to these quantities the analyses discussed in section 5 and section 6 for the entanglement entropies, as done in [42] for the interval on the line.

From (3.21) and (7.5), it is straightforward to observe that

$$\mathrm{Tr}(K_{\mathrm{sine}}^p) \;=\; \mathrm{Tr}(K_+^p) + \mathrm{Tr}(K_-^p) \tag{7.6}$$

where $\mathrm{Tr}(K_{\mathrm{sine}}^p)$ are the Schatten norms for the interval $[-R, R]$ on the line considered in [42]. The case $p = 1$ of (7.6) is interesting because the l.h.s. is known analytically. In particular, by employing eq. (3.55) in [54], we have

$$\frac{2\eta}{\pi} \;=\; \mathrm{Tr}(K_+) + \mathrm{Tr}(K_-) \,. \tag{7.7}$$

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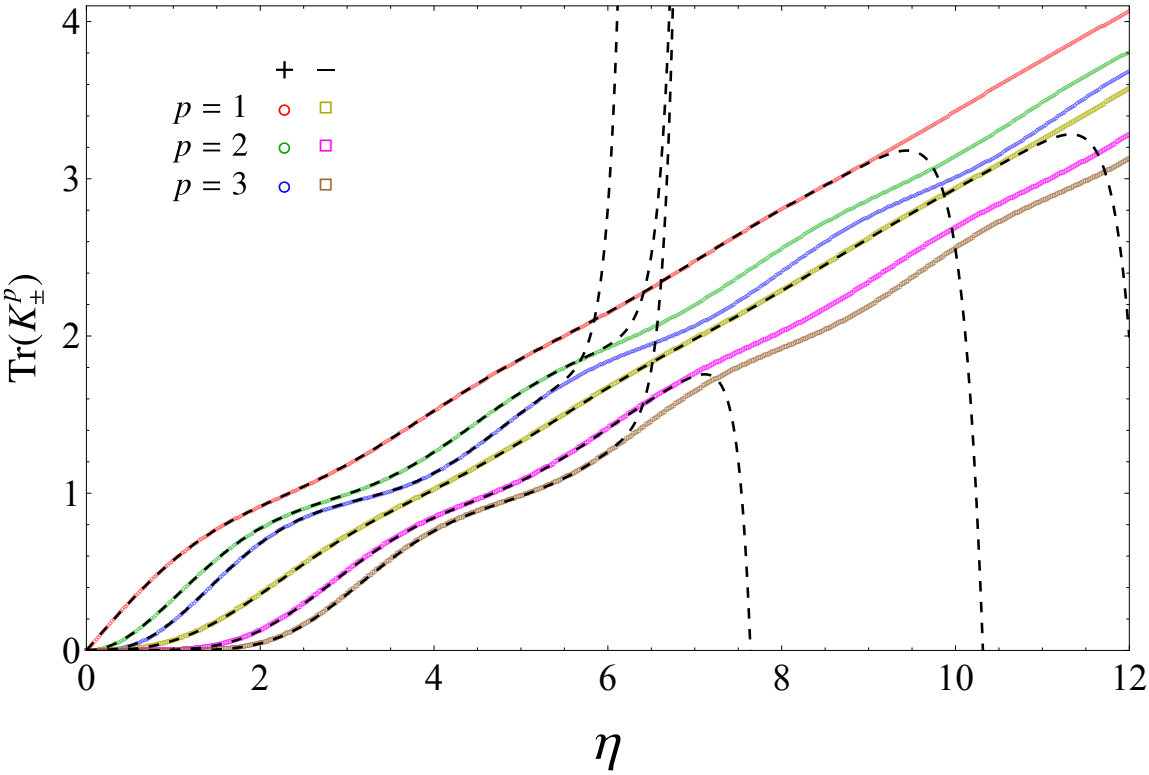

Figure 11. Schatten norms (7.3) for some values of p in the small η regime, when either Neumann (+) or Dirichlet (−) b.c. are imposed. The dashed black curves have been obtained through (7.9) in the best approximation where analytic expressions can be found (see section 5.2).

In the regime of large η, by using (C.5) into (7.5) we find that

$$\mathrm{Tr}(K_{\pm}^{p}) \;=\; \frac{\eta}{\pi} - \frac{\gamma_{\mathrm{E}} + \psi(p)}{2\pi^2}\,\log(4\eta) \pm \frac{1}{4} + C_0(p) + o(1) \tag{7.8}$$

where $\gamma_{\mathrm{E}} \simeq 0.577$ is the Euler-Mascheroni constant, $\psi(x)$ is the digamma function (notice that $\psi(1) = -\gamma_{\mathrm{E}}$) and $C_0(p)$ is independent of the boundary conditions because it comes from the term $\log[G(1+\nu)\,G(1-\nu)]$ in (C.5). The linear and the logarithmic terms in the expansion (7.8) are independent of the boundary conditions and they are equal to 1/2 of the corresponding terms in the expansion of the Schatten norms of the two-point function for the interval on the line (see eq. (9.17) of [42]). The dependence on the boundary conditions occurs in the constant term of (7.8).

The proper combination of the expansions (7.8) provide the corresponding expansions of the cumulants (see (7.2)). In particular, one finds that the linear terms of (7.8) simplify in the combinations (7.2) and therefore $\mathcal{C}_{A,\pm}^{(n)} = \beta_{\pm}^{(n)} \log(4\eta) + O(1)$ as $\eta \to \infty$. By employing this observation in (7.4), we conclude that all the cumulants contribute to the leading logarithmic term of $S_{A,\pm}$ in (6.9).

In figure 10 we show the numerical results of the Schatten norms for different orders p, evaluated numerically through (7.3) in the range $\eta \in [1, 100]$. Subtracting the leading divergent terms for large values of η, the resulting expression tends to a constant as $\eta \to \infty$

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which depend both on p and on the boundary condition. The dependence on the boundary conditions of the $O(1)$ term in (7.8) occurs only through the term $\pm 1/4$; indeed, by using the curves reported in figure 10, we have checked that the curves for $\mathrm{Tr}(K_\pm^p) - \left[\frac{\eta}{\pi} - \frac{\gamma_{\mathrm{E}}+\psi(p)}{2\pi^2}\log(4\eta) \pm \frac{1}{4}\right]$ associated to different boundary conditions oscillate around the same constant value, which corresponds to $C_0(p)$.

In the regime of small η, the analysis discussed in section 5.2 for the entanglement entropies can be applied to the Schatten norms in a straightforward way. The approximate analytic expressions for $\mathrm{Tr}(K_\pm^p)$ as $\eta \to 0$ read

$$\sum_j \tilde{z}_j^p \qquad\qquad \tilde{z}_j \in \mathcal{P}_{N_\pm,\mathcal{N}} \cap [0,1] \tag{7.9}$$

where $\mathcal{P}_{N_\pm,\mathcal{N}}$ and $\tilde{z}_j$ have been introduced in section 5.2. In figure 11 we compare the numerical results for $\mathrm{Tr}(K_\pm^p)$ when $\eta \in [0,12]$ with the best analytic expressions (dashed black curves) obtained from (7.9) as discussed in section 5.2 for the entanglement entropies. The agreement is remarkable in a range of η whose width decreases with p.

8 Conclusions

We studied the entanglement entropies $S_{A,\pm}^{(\alpha)}$ of an interval adjacent to the boundary of the half line for the free fermionic spinless Schrödinger field theory at finite density μ and zero temperature, along the lines of the analysis made in [42] of these quantities for an interval on the line. We have considered the models characterised by scale invariant boundary conditions (2.11) at the origin of the half line, which are of either Neumann (+) or Dirichlet (−) type.

The spectral problems (3.4) can be solved (see section 3.1) through the sine kernel spectral problem in the interval on the line, whose solution has been found by Slepian, Pollak and Landau in the seminal papers [49–53]. The corresponding eigenvalues $\gamma_n^\pm$, which are functions of the dimensionless parameter η introduced in (2.22), can be written in terms of the PSWF, as shown in (3.11), (3.12) and (3.6). The relation (3.18) and the results of [42] allow to prove that $S_{A,\pm}^{(\alpha)}$ are finite functions of η (see e.g. in figure 1, figure 2 and figure 3). We remark that $S_{A,\pm}$ display an oscillatory behaviour, differently from $S_{2A\subset\mathbb{R}}$, as shown in figure 1.

The numerical evaluation of $S_{A,\pm}^{(\alpha)}$ has been performed as described in [42]. In this manuscript we have obtained analytic expressions for the expansions of $S_{A,\pm}$, both in the small η regime (see (5.12) and figure 5) and in the large η regime (see (6.7), (6.9), (6.11) with the coefficients corresponding to $N \in \{0,1,2\}$ written explicitly in section 6, and figure 3). These analytic results are based on the expansions of the Fredholm determinants $\tau_\pm$ in (3.20), that are given by (5.9) and (6.1). Since $\tau_\pm$ are special cases of the Bessel kernel tau function [60, 88, 89] (see (A.11)), these expansions have been obtained from the expansions found in [61] and [62] for the Painlevé III_1' tau functions as $\eta \to 0$ and $\eta \to \infty$ respectively (see the appendices B.2 and C.1). Some terms of the small η expansions of $S_{A,\pm}^{(\alpha)}$ have been obtained also through the properties of the PSWF (see section 5.1).

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The large η expansion of $S_{A,\pm}$ can be also written as in (3.25) and this form highlights the fact that the first correction vanishing as $\eta \to \infty$ can be expressed through the Friedel oscillations occurring in the normalised density at the entangling point (2.22) (see figure 4).

In section 4 we have shown that the results obtained for the Schrödinger model, whose Lifshitz exponent is $z = 2$, can be employed to obtain the corresponding ones for a hierarchy of two component Lifshitz fermion fields on the half line having even z. In the same section, we have considered also a hierarchy of Lifshitz fermion fields on the half line with odd values of z, whose first model ($z = 1$) corresponds to the massless Dirac fermion on the half line [90].

Finally, we have discussed the expansion (7.4), found in [70–72], of S_A in terms of charge cumulants (see figure 9), and the Schatten norms (7.3) by adapting the procedure described for the entanglement entropies (see figure 10 and figure 11).

Various directions can be explored to extend the results discussed in this manuscript. Considering free non-relativistic models, it would be interesting to study the entanglement entropies when the system is in a generic Gibbs state, where both the density and the temperature are non vanishing [98–100]. Furthermore, it would be very insightful to investigate the same problems also for non-relativistic bosonic fields. Besides a physical boundary, a point-like defect provides another way to break the invariance under translations that would be interesting to explore for non-relativistic field theories [101–107]. It is worth considering also the entanglement entropies of more complicated spatial bipartitions, given e.g. by two disjoint intervals on the line [92, 108–114] or by a single interval not adjacent to the boundary of the half line [69, 90]. Interesting models where it is important to understand the properties of the bipartite entanglement include the relativistic massive models and their non-relativistic limit [115–117]. The most important generalisations to study are the non-relativistic interacting models, like e.g. the $d = 1$ spinfull fermionic field with a quartic interaction [40, 118, 119]. Another important direction to pursue involves the analysis of the entanglement entropies in gravitational theories with non-relativistic symmetries [120–123].

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A A Bessel kernel and a Painlevé III

Let us consider the Bessel kernel [60, 89, 124]

$$K_{\rm B}(a;x,y) \equiv \frac{\sqrt{y}\, J_a(\sqrt{x}\,)\, J_a'(\sqrt{y}\,) - \sqrt{x}\, J_a(\sqrt{y}\,)\, J_a'(\sqrt{x}\,)}{2\,(x-y)} \tag{A.1}$$

$$= \frac{\sqrt{x}\, J_{a+1}(\sqrt{x}\,)\, J_a(\sqrt{y}\,) - \sqrt{y}\, J_{a+1}(\sqrt{y}\,)\, J_a(\sqrt{x}\,)}{2\,(x-y)} \tag{A.2}$$

$$= \frac{1}{4}\int_0^1 J_a(\sqrt{xs}\,)\, J_a(\sqrt{ys}\,)\, {\rm d}s \qquad\qquad a > -1 \tag{A.3}$$

where $x > 0$ and $y > 0$ with $x \neq y$, $J_a'(\xi) \equiv \partial_\xi J_a(\xi)$ and some identities for Bessel functions have been employed.[3]

The kernels (3.1) can be expressed in terms of the Bessel kernel (A.1) as follows

$$K_\pm(\eta;x,y) \;=\; 2\,\eta^2\sqrt{x\,y}\; K_{\rm B}\Big(\mp\frac{1}{2}\,;(\eta\,x)^2,(\eta\,y)^2\Big)\,. \tag{A.5}$$

This identity leads us to write the spectral problems (3.4) as

$$2\,\eta^2\int_0^1 {\rm d}y\,\sqrt{x\,y}\; K_{\rm B}\Big(\mp\frac{1}{2}\,;(\eta\,x)^2,(\eta\,y)^2\Big)\, f_n^\pm(\eta;y) \;=\; \gamma_n^\pm\, f_n^\pm(\eta;x)\,. \tag{A.6}$$

In terms of $\tilde{x} \equiv (\eta\,x)^2$ and of the integration variable $\tilde{y} \equiv (\eta\,y)^2$, this becomes

$$\int_0^{\eta^2} {\rm d}\tilde{y}\; K_{\rm B}\Big(\mp\frac{1}{2}\,;\tilde{x},\tilde{y}\Big)\,\frac{f_n^\pm(\eta;\sqrt{\tilde{y}}/\eta\,)}{\sqrt[4]{\tilde{y}}} \;=\; \gamma_n^\pm\,\frac{f_n^\pm(\eta;\sqrt{\tilde{x}}/\eta\,)}{\sqrt[4]{\tilde{x}}}\,. \tag{A.7}$$

Comparing this expression with (3.4), one realises that the spectral problem associated to $K_{\rm B}(\mp\frac{1}{2}\,;x,y)$ in the interval $[0,\eta^2] \in \mathbb{R}^+$ and the one associated to $K_\pm(\eta;x,y)$ in the interval $[0,1] \in \mathbb{R}^+$ discussed in section 3 have the same spectrum; hence they share the same tau function.

The Fredholm determinant $\tau_{\rm B}(t) \equiv \det(I - z^{-1}K_{\rm B})$ associated to the integral operator $K_{\rm B}$ acting on the interval $[0,t]$, whose kernel is (A.1), can be studied by introducing the auxiliary function

$$\sigma_{\rm B}(t) \equiv -\,t\,\partial_t \log\big[\tau_{\rm B}(t)\big]\,. \tag{A.8}$$

This function satisfies the following Painlevé III equation [60, 88, 89]

$$\big(\,t\,\sigma_{\rm B}''\big)^2 + \big(4\,\sigma_{\rm B}' - 1\big)\,\big(\sigma_{\rm B} - t\,\sigma_{\rm B}'\big)\,\sigma_{\rm B}' - a^2\big(\sigma_{\rm B}'\big)^2 \;=\; 0 \tag{A.9}$$

with the boundary condition

$$\sigma_{\rm B}(t) \;=\; \frac{t^{1+a}}{z\,2^{2(1+a)}\,\Gamma(1+a)\,\Gamma(2+a)} + \ldots \qquad\qquad t \to 0^+ \tag{A.10}$$

[3]The expressions in (A.2) and (A.3) have been obtained by using respectively $\xi J_a'(\xi) = aJ_a(\xi) - \xi J_{a+1}(\xi)$ and

$$\int z\,J_a(\tilde{x}z)J_a(\tilde{y}z)\,{\rm d}z \;=\; z\,\frac{\tilde{x}J_{a+1}(\tilde{x}z)J_a(\tilde{y}z) - \tilde{y}J_a(\tilde{x}z)J_{a+1}(\tilde{y}z)}{\tilde{x}^2 - \tilde{y}^2}\,. \tag{A.4}$$

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where the dots correspond to subleading terms. Notice that, while the differential equation (A.9) is not affected by the sign of a, its boundary condition (A.10) depends on it.

Combining the observations collected above, we have that

$$\tau_{\pm}(\eta) \;=\; \tau_{\rm B}(\eta^2)\big|_{a=\mp 1/2}\,. \tag{A.11}$$

We remark that two different versions of the Painlevé III_1 differential equation have been introduced in the literature, which are usually denoted by Painlevé III_1 and Painlevé III_1' [125]. In the σ-form of Jimbo, Miwa and Okamoto, the Painlevé III_1 reads

$$\left(t\,\sigma_{\rm III}''\right)^2 + \left(4\,\sigma_{\rm III}' - 1\right)\left(\sigma_{\rm III} - t\,\sigma_{\rm III}'\right)\sigma_{\rm III}' - 4\,\theta_*\theta_\star\left(\sigma_{\rm III}'\right)^2 - \frac{(\theta_*-\theta_\star)^2}{4} \;=\; 0 \tag{A.12}$$

while the Painlevé III_1' is

$$\left(t\,\sigma_{\rm III'}''\right)^2 - \left[4\left(\sigma_{\rm III'}'\right)^2 - 1\right]\left(\sigma_{\rm III'} - t\,\sigma_{\rm III'}'\right) + 4\,\theta_*\theta_\star\,\sigma_{\rm III'}' - \left(\theta_*^2+\theta_\star^2\right) \;=\; 0 \tag{A.13}$$

where the notation of [61] has been adopted. These differential equations are invariant under $(\theta_*,\theta_\star)\to(-\theta_*,-\theta_\star)$. The solutions of (A.12) and (A.13) are related as follows (see e.g. remark 2 in [126])

$$\sigma_{\rm III}(t) = -\,\sigma_{\rm III'}(t/4) + \frac{t}{8} + \theta_*\theta_\star\,. \tag{A.14}$$

The tau functions associated to the solutions of (A.12) and (A.13) are defined respectively by

$$\sigma_{\rm III}(t) \equiv -\,t\,\partial_t\log\big[\tau_{\rm III}(t)\big] \qquad\qquad \sigma_{\rm III'}(t) \equiv\; t\,\partial_t\log\big[\tau_{\rm III'}(t)\big]\,. \tag{A.15}$$

From these definitions and the relation (A.14), one finds that

$$\tau_{\rm III}(t) \;\propto\; \frac{\tau_{\rm III'}(t/4)}{\mathrm{e}^{t/8}\,t^{\theta_*\theta_\star}}\,. \tag{A.16}$$

As for the Bessel kernel (A.1), the differential equation (A.9) satisfied by its auxiliary function (A.8) corresponds to (A.12) in the special case given by $\theta_*=\theta_\star=\pm a/2$. In our analysis we set $\theta_*=\theta_\star=-a/2$.

The relations (A.11) and (A.16) provide (3.22) up to a proportionality constant whose derivation is reported in the appendix B.2 (see (B.15)).

B On the small η expansion

In this appendix we describe the technical details underlying some results concerning the expansions of the entanglement entropies as $\eta\to 0$ reported in section 5.

B.1 PSWF approach

In the following we discuss the derivation of the expansions of $S_{A,\pm}^{(\alpha)}$ given in (5.5) and (5.6).

For finite $\alpha\neq 1$, the expression (3.15) can be written as $s_\alpha(x)=s_{\alpha,1}(x)+s_{\alpha,2}(x)$, with

$$s_{\alpha,1}(x) \equiv \frac{\alpha}{1-\alpha}\log(1-x) = \frac{\alpha}{\alpha-1}\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+O(x^5)\right] \tag{B.1}$$

$$s_{\alpha,2}(x) \equiv \frac{1}{1-\alpha}\,\log\left[1+\chi_\alpha(x)\right] = \frac{1}{1-\alpha}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\,\chi_{j\alpha}(x) \tag{B.2}$$

where we have introduced $\chi_\beta(x)\equiv[x/(1-x)]^\beta$.

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The contribution to (5.5) and (5.6) coming from $s_{\alpha,1}(x)$ is given by

$$s_{\alpha,1}(\gamma_0) = \frac{\alpha}{\alpha-1}\left[\tilde{g}_0\eta + \frac{\tilde{g}_0^2}{2}\eta^2 + \left(\frac{\tilde{g}_0^2}{3}+\tilde{a}_0\right)\tilde{g}_0\eta^3 + \left(\frac{\tilde{g}_0^2}{4}+\tilde{a}_0\right)\tilde{g}_0^2\eta^4\right] + O(\eta^5) \tag{B.3}$$

$$s_{\alpha,1}(\gamma_1) = \frac{\alpha}{\alpha-1}\left[\tilde{g}_1\eta^3 + \tilde{g}_1\tilde{a}_1\eta^5 + \frac{\tilde{g}_1^2}{2}\eta^6\right] + O(\eta^7) \tag{B.4}$$

in terms of (5.2); while $s_{\alpha,1}(\gamma_{2n}) = O(\eta^5)$ and $s_{\alpha,1}(\gamma_{2n+1}) = O(\eta^7)$ when $n \geqslant 1$. As for the contribution to (5.5) and (5.6) originating from $s_{\alpha,2}(x)$, all the eigenvalues must be considered; indeed for generic α we have

$$\begin{aligned}\chi_\beta(\gamma_0) = \tilde{g}_0^\beta\eta^\beta\Big\{1 + \beta\tilde{g}_0\eta + \left(\beta\tilde{a}_0 + \frac{\beta(\beta+1)}{2}\tilde{g}_0^2\right)\eta^2 \\ + \frac{\beta(\beta+1)}{6}\tilde{g}_0\left(6\tilde{a}_0 + (\beta+2)\tilde{g}_0^2\right)\eta^3\Big\} + O(\eta^{4+\beta})\end{aligned} \tag{B.5}$$

$$\chi_\beta(\gamma_{2n}) = \tilde{g}_{2n}^\beta\eta^{\beta(4n+1)}\Big\{1+\beta\tilde{a}_{2n}\eta^2\Big\} + O(\eta^{4+\beta(4n+1)}) \qquad n \geqslant 1 \tag{B.6}$$

and

$$\chi_\beta(\gamma_1) = \tilde{g}_1^\beta\eta^{3\beta}\Big\{1+\beta\tilde{a}_1\eta^2+\beta\tilde{g}_1\eta^3\Big\} + O(\eta^{4+3\beta}) \tag{B.7}$$

$$\chi_\beta(\gamma_{2n+1}) = \tilde{g}_{2n+1}^\beta\eta^{\beta(4n+3)}\Big\{1+\beta\tilde{a}_{2n+1}\eta^2\Big\} + O(\eta^{4+\beta(4n+3)}) \qquad n \geqslant 1\,. \tag{B.8}$$

Combining the above results, we arrive to

$$\begin{aligned}S_{A,+}^{(\alpha)} = \frac{\alpha}{\alpha-1}\Bigg\{&\left[\tilde{g}_0\eta+\frac{\tilde{g}_0^2}{2}\eta^2+\left(\frac{\tilde{g}_0^3}{3}+\tilde{g}_0\tilde{a}_0\right)\eta^3+\left(\frac{\tilde{g}_0^4}{4}+\tilde{g}_0^2\tilde{a}_0\right)\eta^4\right] \\ &+\sum_{j=1}^{\infty}(-1)^j\left[\frac{1}{j\alpha}+\tilde{g}_0\eta+\left(\tilde{a}_0+(j\alpha+1)\frac{\tilde{g}_0^2}{2}\right)\eta^2+(j\alpha+1)\tilde{g}_0\left(\tilde{a}_0+(j\alpha+2)\frac{\tilde{g}_0^2}{6}\right)\eta^3\right]\tilde{g}_0^{j\alpha}\eta^{j\alpha} \\ &+\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(-1)^j\left(\frac{1}{j\alpha}+\tilde{a}_{2n}\eta^2\right)\tilde{g}_{2n}^{j\alpha}\eta^{j\alpha(4n+1)}\Bigg\}+O(\eta^{\min\{5,4+\alpha\}})\end{aligned} \tag{B.9}$$

and

$$\begin{aligned}S_{A,-}^{(\alpha)} = \frac{\alpha}{\alpha-1}\Bigg\{&\left(\tilde{g}_1\eta^3+\tilde{g}_1\tilde{a}_1\eta^5+\frac{\tilde{g}_1^2}{2}\eta^6\right)+\sum_{j=1}^{\infty}(-1)^j\tilde{g}_1^{j\alpha}\eta^{3j\alpha}\left(\frac{1}{j\alpha}+\tilde{a}_1\eta^2+\tilde{g}_1\eta^3\right) \\ &+\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(-1)^j\tilde{g}_{2n+1}^{j\alpha}\eta^{j\alpha(4n+3)}\left(\frac{1}{j\alpha}+\tilde{a}_{2n+1}\eta^2\right)\Bigg\}+O(\eta^{\min\{7,4+3\alpha\}})\,.\end{aligned} \tag{B.10}$$

We remark that, despite (B.9) and (B.10) contain an infinite number of terms, only a finite number of them are $O(\eta^{\min\{5,4+\alpha\}})$ or $O(\eta^{\min\{7,4+3\alpha\}})$ once α has been fixed. In particular, when $\alpha > 1$, from (B.9) and (B.10) we obtain the expansions (5.5) and (5.6) by discarding the terms of order $O(\eta^5)$ and $O(\eta^7)$ respectively.

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B.2 Tau function approach

In this appendix we derive (5.9) as special cases of the expansion of the Painlevé III_1' tau function found in [61].

The Conjecture 4 of [61], combined with (A.11) and (A.16), provides the following ansatz

$$\tau_\pm = \frac{\mathcal{C}_{0,\pm}}{\mathrm{e}^{\eta^2/8}\,\eta^{1/8}} \sum_{n\in\mathbb{Z}} C_{\mathrm{III}'}\left(\pm\frac{1}{4},\pm\frac{1}{4},\hat\sigma_\pm+n\right) s^n_{\mathrm{III}',\pm}\left(\frac{\eta}{2}\right)^{2(\hat\sigma_\pm+n)^2} \mathcal{B}_{\mathrm{III}'}\left(\pm\frac{1}{4},\pm\frac{1}{4},\hat\sigma_\pm+n;\,\eta^2/4\right) \tag{B.11}$$

where the explicit expressions of $C_{\mathrm{III}'}(\theta_*,\theta_\star,\hat\sigma)$ and $\mathcal{B}_{\mathrm{III}'}(\theta_*,\theta_\star,\hat\sigma;t)$ are reported in [61].[4] In our cases we need the special case given by $C_{\mathrm{III}'}(\theta,\theta,\hat\sigma)$ (see (5.10)).

The parameters $\mathcal{C}_{0,\pm}$, $\hat\sigma_\pm$ and $s_{\mathrm{III}',\pm}$ in (B.11) are fixed by imposing the proper behaviour as $\eta\to 0$. This behaviour is obtained by employing the small η expansion of the eigenvalues (see (5.1) and (5.2)) into the definition (3.20) of the tau functions $\tau_\pm$, finding

$$\tau_+ = 1-\tilde g_0\,\frac{\eta}{z}-\tilde g_0\,\tilde a_0\,\frac{\eta^3}{z}+O(\eta^5) = 1-\frac{2}{\pi}\,\frac{\eta}{z}+\frac{2}{9\pi}\,\frac{\eta^3}{z}+O(\eta^5) \tag{B.12}$$

$$\tau_- = 1-\tilde g_1\,\frac{\eta^3}{z}-\tilde g_1\,\tilde a_1\,\frac{\eta^5}{z}+O(\eta^7) = 1-\frac{2}{9\pi}\,\frac{\eta^3}{z}+\frac{2}{75\pi}\,\frac{\eta^5}{z}+O(\eta^7) \tag{B.13}$$

which agree with the expansions obtained from eq. (1.22) of [60].

In order to get consistency between (B.11) and the expansions in (B.12) and (B.13), first we observe that both $\tau_\pm$ and $\mathcal{B}_{\mathrm{III}'}(\pm\frac14,\pm\frac14,\hat\sigma_\pm+n;\eta^2/4)$ do not diverge as $\eta\to 0$. This tells us that the factor $1/\eta^{1/8}$ multiplying the series in the r.h.s. of (B.11) must simplify with the factor $(\eta/2)^{2\hat\sigma_\pm^2+4\hat\sigma_\pm n+2n^2}$ in the summand of the series in (B.11); hence $\hat\sigma_\pm=1/4$. For this specific value of $\hat\sigma_\pm$, we have that

$$C_{\mathrm{III}'}\left(+\frac14,+\frac14,\frac14+n\right)\Big|_{n>0} = 0 \qquad\qquad C_{\mathrm{III}'}\left(-\frac14,-\frac14,\frac14+n\right)\Big|_{n<0} = 0 \tag{B.14}$$

which simplify (B.11) in a significant way. For $n=0$ we have $C_{\mathrm{III}'}(\pm\frac14,\pm\frac14,\frac14)=\pi^{\pm1/2}$.

Then, by considering the terms corresponding to $n\in\{-1,0,1\}$ in (B.11), which involve also $\mathcal{B}_{\mathrm{III}'}(\pm\frac14,\pm\frac14,\frac12+n;\,\eta^2/4)$ for these values of n, agreement with (B.12) and (B.13) is obtained when

$$\mathcal{C}_{0,\pm} = \frac{2^{1/8}}{\pi^{\pm1/2}} \qquad\qquad s_{\mathrm{III}',\pm} = z^{\pm1}\,. \tag{B.15}$$

Combining the above results, we find that the expansion (B.11) simplifies to (5.9), with

$$\mathcal{B}_\pm(n;t) = \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}^{\mathrm{III}'}_{\lambda,\mu}\left(\pm\frac14,\pm\frac14,\frac14\mp n\right) t^{|\lambda|+|\mu|} \tag{B.16}$$

where $\mathbb{Y}$ is the set of all Young diagrams (see section 3.1 of [61] for a more detailed explanation) and the coefficients $\mathcal{B}^{\mathrm{III}'}_{\lambda,\mu}(\theta_*,\theta_\star,\sigma)$ are given by eq. (4.19) of [61]. From (B.16), we

[4] The parameter $\hat\sigma$ corresponds to the one denoted by σ in [61].

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obtain the following expansions of $\mathcal{B}_+(n;t)$ for some values of n

$$\begin{aligned}
\mathcal{B}_+(0;t) &= 1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+\frac{t^6}{720}+\frac{t^7}{5040}+\frac{t^8}{40320}+\frac{t^9}{362880}+\dots \\
\mathcal{B}_+(1;t) &= 1+\frac{5}{9}t+\frac{121}{450}t^2+\frac{733}{7350}t^3+\frac{187483}{7144200}t^4+\frac{4481899}{864448200}t^5+\frac{241147213}{292183491600}t^6+\dots \\
\mathcal{B}_+(2;t) &= 1+\frac{25}{49}t+\frac{1003}{7350}t^2+\frac{67979}{2668050}t^3+\frac{338600363}{88376488200}t^4+\frac{74803}{146076840}t^5+\dots \\
\mathcal{B}_+(3;t) &= 1+\frac{61}{121}t+\frac{5289}{40898}t^2+\frac{1216997}{54108054}t^3+\frac{22574889473}{7568418161304}t^4+\dots \\
\mathcal{B}_+(4;t) &= 1+\frac{113}{225}t+\frac{2797321}{21978450}t^2+\frac{6943216561}{320013558150}t^3+\frac{34818356971}{12442127140872}t^4+\dots
\end{aligned} \tag{B.17}$$

where the dots denote subleading terms; while for the expansions of $\mathcal{B}_-(n;t)$ corresponding to the same values of n we find

$$\begin{aligned}
\mathcal{B}_-(0;t) &= 1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+\frac{t^6}{720}+\frac{t^7}{5040}+\frac{t^8}{40320}+\frac{t^9}{362880}+\dots \\
\mathcal{B}_-(1;t) &= 1+\frac{13}{25}t+\frac{369}{2450}t^2+\frac{6887}{198450}t^3+\frac{680459}{96049800}t^4+\frac{20582899}{16232416200}t^5+\dots \\
\mathcal{B}_-(2;t) &= 1+\frac{41}{81}t+\frac{126409}{960498}t^2+\frac{1266497}{54108054}t^3+\frac{4321645}{1348539192}t^4+\frac{45832437061}{126661543608600}t^5+\dots \\
\mathcal{B}_-(3;t) &= 1+\frac{85}{169}t+\frac{130929}{1022450}t^2+\frac{526486511}{23934532050}t^3+\frac{16747684673123}{5840887463353800}t^4+\dots \\
\mathcal{B}_-(4;t) &= 1+\frac{145}{289}t+\frac{38897}{306850}t^2+\frac{2786241737}{129592267350}t^3+\frac{8223548620739}{2986329051875400}t^4+\dots
\end{aligned} \tag{B.18}$$

and $\mathcal{B}_\pm(n;t) = 1+\dots$ when $n \geqslant 5$. We have access to more terms in these expansions, but they have not been written here to avoid lengthy expressions. In the case of $n=0$, these expansions lead us to conjecture that $\mathcal{B}_\pm(0;t) = \mathrm{e}^t$.

The best results reported in section 5.2 for the small η expansions of the entanglement entropies involve a polynomial of fourth degree in z. This is achieved by employing the expansion of $\mathcal{B}_+(n;t)$ and $\mathcal{B}_-(n;t)$ up to $O(t^p)$, where the pairs (n,p) are given respectively by

$$(n,p) \in \{(0,23),(1,22),(2,20),(3,15),(4,9),(5,0)\} \tag{B.19}$$

$$(n,p) \in \{(0,28),(1,26),(2,23),(3,17),(4,10),(5,0)\} \tag{B.20}$$

and whose first terms are reported in (B.17) and (B.18).

C On the large η expansion

In this appendix we discuss the derivation of some results o the expansion of the entanglement entropies as $\eta \to \infty$ employed in section 6.

C.1 Tau functions

Consider the expansion of the Painlevé III_1' tau function given in eq. (A.30) of [62]. In the cases we are exploring $\theta_* = \theta_\star = \pm 1/4$ (see the appendix A); hence this expansion becomes

$$\tau_{\mathrm{III}'}(t) = s^{-\frac{1}{24}} \sum_{n\in\mathbb{Z}} \mathsf{P}^n \mathscr{C}(\hat{\nu}+n,s) \sum_{p=0}^{\infty} \frac{\mathcal{D}_p(\hat{\nu}+n)}{s^p} \qquad t=\frac{s^2}{16} \tag{C.1}$$

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where

$$\mathscr{C}(\hat{\nu},s) \equiv \mathrm{e}^{\frac{s^2}{32}+\mathrm{i}\hat{\nu}s+\mathrm{i}\frac{\pi}{2}\hat{\nu}^2}\,\frac{G(1+\hat{\nu})^2}{(2\pi)^{\hat{\nu}}\,2^{\hat{\nu}^2}\,s^{\hat{\nu}^2-1/6}} \tag{C.2}$$

and $\mathcal{D}_p(\hat{\nu})$ for $p\in\{0,1,2\}$ have been reported in (6.3). The function (C.1) is parameterised by P and $\hat{\nu}$.

Combining the expansion (C.1) with (A.11) and (A.16), we find that the tau functions occurring in (3.19) can be written as

$$\tau_\pm = \mathsf{N}_\pm\,\frac{\tau_{\mathrm{III}'}(\eta^2/4)}{\mathrm{e}^{\eta^2/8}\,\eta^{1/8}} \tag{C.3}$$

$$= 2^{\frac{1}{8}}\,\mathsf{N}_\pm \sum_{n\in\mathbb{Z}} \mathsf{P}_\pm^n\,\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\nu_\pm+n)^2+\mathrm{i}2\eta(\nu_\pm+n)}\,\frac{G(1+\nu_\pm+n)^2}{(2\pi)^{(\nu_\pm+n)}\,(4\eta)^{(\nu_\pm+n)^2}}\sum_{p=0}^{\infty}\frac{\mathcal{D}_p(\nu_\pm+n)}{(2\eta)^p} \tag{C.4}$$

where the parameters $\mathsf{N}_\pm$, $\nu_\pm$ and $\mathsf{P}_\pm$ (which depend on z but are independent of η) occur in the first terms of the large η expansion of $\tau_\pm$.

The leading terms in the expansion of $\log\tau_\pm$ as $\eta\to\infty$ have been reported in eq. (1.35) of [63] and, in our notation,[5] they read (see also (6.8))

$$\log(\tau_\pm) \;=\; \mathrm{i}\,2\nu\,\eta-\nu^2\log(4\eta)\pm\mathrm{i}\,\frac{\pi}{2}\,\nu+\log\big[G(1+\nu)\,G(1-\nu)\big]+O(1/\eta) \tag{C.5}$$

where ν is given by (6.2). Comparing (C.5) with the term having $n=p=0$ in (C.4), we find

$$\mathsf{N}_\pm = \frac{(2\pi)^\nu}{2^{1/8}}\,\frac{G(1-\nu)}{G(1+\nu)}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\pm\nu-\nu^2)} \qquad\qquad \nu_\pm=\nu \tag{C.6}$$

which are independent of η, as expected. The constant $\mathsf{P}_\pm$ cannot be obtained from the terms reported in (C.5) because it occurs in the subleading contributions of (C.4) having $n\neq 0$.

We find the parameter P_- in terms of z by imposing that τ_- in (C.4) agrees with the proper limit of the corresponding lattice result obtained for the XX model in the semi-infinite chain with open boundary conditions [65–69]. In particular, from eq. (39) of [69] and by employing the notation adopted there, we have the following lattice result[6]

$$\begin{aligned}\frac{D_\ell(\lambda)}{(\lambda+1)^\ell} &= \mathrm{e}^{-2\mathrm{i}\beta k_{\mathrm{F,there}}\,\ell}\sum_{n\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\beta+n)}\big[\,4(\ell+1/2)\,\big|\sin(k_{\mathrm{F,there}})\big|\,\big]^{-(n+\beta)^2}\\ &\quad\times\mathrm{e}^{-\mathrm{i}\beta k_{\mathrm{F,there}}}\,\mathrm{e}^{-2\mathrm{i}k_{\mathrm{F,there}}n(\ell+1/2)}\,G(1+n+\beta)\,G(1-n-\beta)\end{aligned} \tag{C.7}$$

where ℓ is the number of consecutive sites of the block located at the beginning of the semi-infinite chain and $\beta\equiv\frac{1}{2\pi\mathrm{i}}\log(\frac{\lambda+1}{\lambda-1})$. The parameter λ in (C.7) and z are related by $\lambda=2z-1$; hence $\beta=-\nu$, with ν being defined in (6.2).

In the double scaling limit given by $\ell\to+\infty$ and $k_{\mathrm{F,there}}\to 0^+$ with $\ell\,k_{\mathrm{F,there}}=\eta$ kept fixed (which implies $L_{k,\mathrm{there}}=4(\ell+1/2)\,|\sin(k_{\mathrm{F,there}})|\to 4\eta$), the expression (C.7) becomes

$$\frac{D_\ell(\lambda)}{(\lambda+1)^\ell}\;\longrightarrow\;\sum_{n\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(\nu+n)}\,\mathrm{e}^{\mathrm{i}2\eta(\nu+n)}\,\frac{G(1+\nu+n)\,G(1-\nu-n)}{(4\eta)^{(\nu+n)^2}}\,. \tag{C.8}$$

[5] Comparing with the notation in eq. (1.35) of [63], we have $t_{\mathrm{there}}=\eta^2$, $v_{\mathrm{there}}=-2\pi\mathrm{i}\,\nu$ and $\alpha_{\mathrm{there}}=\mp 1/2$.

[6] The expression (C.7) has been obtained by simply removing a factor $\exp(-2\mathrm{i}\beta k_{\mathrm{F,there}}\,\ell)$ in eq. (39) of [69]. The proper limits of (C.7) agree with eq. (36) of [69] and with the expansion (C.5).

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By imposing that this expansion coincides with the series in n given by the truncation of τ_- in (C.4) obtained by considering only the term $p=0$ in the second series, we find[7]

$$\mathsf{P}_- \;=\; \mathrm{i}\,\frac{\mathrm{e}^{-\mathrm{i}2\pi\nu}}{z}\,. \tag{C.10}$$

Finally, plugging the expressions for ν_-, N_- and P_- (see (6.2), (C.6) and (C.10)) into (C.4), the expression for τ_- reported in (6.1) is obtained.

The ansatz for τ_+ can be found by using the relation (3.21), with [61]

$$\tau_{\text{sine}} = \sum_{n\,\in\,\mathbb{Z}} \mathrm{e}^{\mathrm{i}4\eta(\nu+n)} \left[\frac{G(1+\nu+n)\,G(1-\nu-n)}{(4\eta)^{(\nu+n)^2}}\right]^2 \sum_{p=0}^{\infty} \frac{\mathcal{D}_p^{\text{sine}}(\nu+n)}{(4\eta\,\mathrm{i})^p} \tag{C.11}$$

where ν and $\mathcal{D}_k^{\text{sine}}(\nu)$ are given respectively in (6.2) and in eqs. (8.3)-(8.5) of [42].[8] In particular, by using (6.1), (C.11) and (C.4) for τ_-, τ_{sine} and τ_+ respectively into (3.21), we obtain a relation that allows to determines the parameter P_+ as function of z. Denoting by n_+, n_- and n_{sine} and by p_+, p_- and p_{sine} the labels n and p respectively in the corresponding expressions (C.4), (6.1) and (C.11), from the term having $n_+ = n_- = n_{\text{sine}} \geqslant 1$ and $p_+ = p_- = p_{\text{sine}} = 0$ we arrive to

$$\mathsf{P}_+ \;=\; -\,\mathrm{i}\,\frac{\mathrm{e}^{-\mathrm{i}2\pi\nu}}{z}\,. \tag{C.12}$$

Finally, the expression for τ_+ in (6.1) is obtained by plugging (C.6) and (C.12) into (C.4).

Let us remark that it would be worth providing an alternative derivation of the parameters $\mathsf{N}_\pm$, $\nu_\pm$ and $\mathsf{P}_\pm$ in (C.4) through the connection formula for the Painlevé III_1, as done in [42] for the interval on the line, where the connection formula for the Painlevé V given in [127] has been employed.

C.2 A consistency check

In the final part of the appendix C.1, the ansatz for τ_+ in (6.1) has been obtained by requiring the validity of (3.21), but only few terms of the resulting series have been employed to fix the parameters occurring in (C.4) for τ_+. Hence, the relation (3.21) can be used as consistency check of the expressions for τ_+ and τ_{sine} given in (6.1) and (C.11) respectively, where free parameters do not occur. This analysis is performed by reorganising in the powers of η the expansions involved in (3.21).

As for τ_{sine} (see (C.11)), let us first change (n,k) into (n,j), where $j = 2n^2+k$; hence for each $n\in\mathbb{Z}$ we have $j\in\{2n^2, 2n^2+1, \cdots\}$. This leads to write (C.11) as follows

$$\tau_{\text{sine}} \;=\; \sum_{n\,\in\,\mathbb{Z}} \sum_{j=2n^2}^{\infty} \mathrm{e}^{\mathrm{i}4\eta(\nu+n)}\, \frac{G(1+\nu+n)^2\,G(1-\nu-n)^2}{\mathrm{i}^{j-2n^2}\,(4\eta)^{2\nu(\nu+2n)+j}}\, \mathcal{D}_{j-2n^2}^{\text{sine}}(\nu+n)\,. \tag{C.13}$$

[7] In this calculation we have used the identities

$$\frac{G(1-\nu)\,G(1+\nu+n)}{G(\nu+1)} = \mathrm{i}^{(n-1)n}\big[\Gamma(\nu+1)\Gamma(-\nu)\big]^n G(1-n-\nu) \qquad \Gamma(\nu+1)\,\Gamma(-\nu) = -\frac{\pi}{\sin(\pi\nu)} \tag{C.9}$$

and that $1/\sin(\pi\nu) = -2\mathrm{i}\,z\,\mathrm{e}^{\mathrm{i}\pi\nu}$ (from (6.2)).

[8] The expression (C.11) coincides with eq. (8.1) of [42] after some manipulations.

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The condition $j \geqslant 2n^2$ for any $n \in \mathbb{Z}$ is equivalent to $-n_{\rm s}(j) \leqslant n \leqslant n_{\rm s}(j)$ with $j \in \mathbb{N}_0$, where we have introduced $n_{\rm s}(j) \equiv \lfloor \sqrt{j/2} \rfloor$. Thus, $\tau_{\rm sine}$ can be written as

$$\tau_{\rm sine} = \sum_{j=0}^{\infty} \frac{1}{(4\eta)^j} \sum_{n=-n_{\rm s}(j)}^{n_{\rm s}(j)} (-1)^{n^2} {\rm e}^{i4\eta(\nu+n)} \, \frac{G(1+\nu+n)^2 \, G(1-\nu-n)^2}{{\rm i}^j (4\eta)^{2\nu(\nu+2n)}} \, \mathcal{D}^{\rm sine}_{j-2n^2}(\nu+n) \,. \tag{C.14}$$

Performing the same manipulations for $\tau_\pm$ in (6.1) as well, one obtains the following expansions

$$\tau_\pm = \sum_{j=0}^{\infty} \frac{1}{(4\eta)^j} \sum_{n=-n_{\rm h}(j)}^{n_{\rm h}(j)} {\rm e}^{\pm {\rm i}\frac{\pi}{2}(\nu+n)} \, {\rm e}^{{\rm i}2\eta(n+\nu)} \, \frac{G(1+\nu+n) \, G(1-\nu-n)}{(4\eta)^{\nu(\nu+2n)}} \, 2^{j-n^2} \mathcal{D}_{j-n^2}(\nu+n) \tag{C.15}$$

where $n_{\rm h}(j) \equiv \lfloor \sqrt{j} \rfloor$.

By writing the expressions (C.14) and (C.15) as $\tau_{\rm sine} \equiv \sum_{j=0}^{\infty} \tau_{{\rm sine},\, j}$ and $\tau_\pm \equiv \sum_{j=0}^{\infty} \tau_{\pm,\, j}$ respectively, we find that (3.21) is equivalent to

$$\tau_{{\rm sine},\, j} = \sum_{l=0}^{j} \tau_{+,\, l} \, \tau_{-,\, j-l} \qquad \forall \, j \in \mathbb{N}_0 \,. \tag{C.16}$$

We have checked the validity of this relation only for $j \in \{0,1,2\}$, finding agreement. In order to check (C.16) also for $j \geqslant 3$, one needs the explicit expressions for $\mathcal{D}_k(\nu)$ with $k \geqslant 3$.

C.3 Subleading terms of the entanglement entropies

In this subsection we discuss the evaluation of $\widetilde{S}^{(\alpha)}_{A,\kappa,\infty}$ in (6.7), where $\kappa \in \{+,-\}$, which contains the subleading terms of the entanglement entropies that vanish as $\eta \to \infty$. These terms are obtained from the expansion (6.6).

C.3.1 Expansion of the vanishing term

By using (3.19) and (6.4), for the term $\widetilde{S}^{(\alpha)}_{A,\kappa,\infty}$ in (6.7) we have

$$\widetilde{S}^{(\alpha)}_{A,\kappa,\infty} = \lim_{\epsilon,\delta\to 0} \frac{1}{2\pi i} \oint_{\mathfrak{C}} {\rm d}z \, s_\alpha(z) \, \partial_z \log\left(\mathcal{T}_{\kappa,\infty}\right) = -\lim_{\epsilon,\delta\to 0} \frac{1}{2\pi i} \oint_{\mathfrak{C}} {\rm d}z \, s'_\alpha(z) \, \log\left(\mathcal{T}_{\kappa,\infty}\right) \tag{C.17}$$

where an integration by parts has been performed and the closed path $\mathfrak{C}$ in the complex plane, which is parameterised by $\epsilon > 0$ and $\delta > 0$, has been described in the text below (3.19). The integrals along $\mathfrak{C}_0$ and $\mathfrak{C}_1$ in (C.17) vanish as $\epsilon \to 0$. In order to evaluated the remaining two terms, one needs the limit of $\mathcal{T}_{\kappa,\infty}$ as $z \to x \pm {\rm i}\,0^+$, with $x \in [0,1]$. The dependence on z in the expansion (6.6) occurs through $\nu = \nu(z)$ in (6.2), which gives

$$\lim_{\delta\to 0} \nu\big|_{z=x\pm {\rm i}\delta} = \lim_{\delta\to 0} \frac{1}{2\pi {\rm i}} \log\left[1-(x\pm {\rm i}\delta)^{-1}\right] = \pm\frac{1}{2} + \frac{1}{2\pi {\rm i}} \log(1/x-1) \,. \tag{C.18}$$

This suggests to adopt $y \equiv \frac{1}{2\pi}\log(1/x-1)$ as integration variable in the remaining two integrals; hence (C.17) becomes

$$\widetilde{S}^{(\alpha)}_{A,\kappa,\infty} = \frac{1}{2\pi {\rm i}} \int_{-\infty}^{+\infty} \hat{s}'_\alpha(y) \left[\log\Big(\mathcal{T}_{\kappa,\infty}\big|_{\nu=-{\rm i}y-\frac{1}{2}}\Big) - \log\Big(\mathcal{T}_{\kappa,\infty}\big|_{\nu=-{\rm i}y+\frac{1}{2}}\Big) \right] {\rm d}y \tag{C.19}$$

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where, for $\alpha = 1$ and $\alpha \in (0,1) \cup (1,\infty)$, we have respectively

$$\hat{s}_1'(y) = -\frac{\pi^2 y}{[\cosh(\pi y)]^2} \qquad \hat{s}_\alpha'(y) = \frac{\pi\alpha}{\alpha - 1}\left[\tanh(\pi y) - \tanh(\alpha\pi y)\right]. \tag{C.20}$$

The change of integration variable $y \to -y$ in the second term of (C.19) leads to

$$\widetilde{S}^{(\alpha)}_{A,\kappa,\infty} = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \mathrm{d}y\, \hat{s}_\alpha'(y) \left[\log\left(\mathcal{T}^-_{\kappa,\infty}\right) + \log\left(\mathcal{T}^+_{\kappa,\infty}\right)\right] \tag{C.21}$$

where

$$\mathcal{T}^\pm_{\kappa,\infty} \equiv \mathcal{T}_{\kappa,\infty}\big|_{\nu=\pm\tilde{\nu}} \qquad \tilde{\nu} \equiv \frac{1}{2} + \mathrm{i}\, y \tag{C.22}$$

which can be written explicitly by using (6.6) and the result reads

$$\mathcal{T}^\pm_{\kappa,\infty} = \sum_{n\in\mathbb{Z}} \sum_{k=0}^{\infty} \frac{(\mathrm{i}\kappa)^n \mathrm{e}^{\mathrm{i}2n\eta}\, 2^k}{(4\eta)^{\pm 2\mathrm{i}yn} (4\eta)^{n^2 \pm n + k}} \frac{G(1 \pm \tilde{\nu} + n)\, G(1 \mp \tilde{\nu} - n)}{G(1+\tilde{\nu})\, G(1-\tilde{\nu})}\, \mathcal{D}_k(\pm\tilde{\nu} + n). \tag{C.23}$$

Let us first perform the change of variable $n \to -n$ only for $\mathcal{T}^+_{\kappa,\infty}$. This allows to write (C.23) as

$$\mathcal{T}^\pm_{\kappa,\infty} = \sum_{n\in\mathbb{Z}} \sum_{k=0}^{\infty} \frac{(\mp\mathrm{i}\kappa)^n\, \mathrm{e}^{\mp\mathrm{i}2n\eta} (4\eta)^{2\mathrm{i}yn}}{(4\eta)^{n^2 - n + k}}\, 2^k\, \mathcal{G}_n(\tilde{\nu})\, \mathcal{D}_k(\pm\tilde{\nu} \mp n) \tag{C.24}$$

where

$$\mathcal{G}_n = \mathcal{G}_n(\tilde{\nu}) \equiv \frac{G(1+\tilde{\nu}-n)\, G(1-\tilde{\nu}+n)}{G(1+\tilde{\nu})\, G(1-\tilde{\nu})} \tag{C.25}$$

which can be written also as

$$\mathcal{G}_{n\leqslant -1}(\tilde{\nu}) = \prod_{j=1}^{-n} \frac{\Gamma(j+\tilde{\nu})}{\Gamma(1-j-\tilde{\nu})} \qquad \mathcal{G}_{n=0}(\tilde{\nu}) = 1 \qquad \mathcal{G}_{n\geqslant 1}(\tilde{\nu}) = \prod_{j=1}^{n} \frac{\Gamma(j-\tilde{\nu})}{\Gamma(1-j+\tilde{\nu})}. \tag{C.26}$$

The expression (C.24) suggests to introduce $j \equiv n^2 - n + k$ to replace the index k. Thus, for any $n \in \mathbb{Z}$, we have $j \geqslant n^2 - n$, which can be equivalently reformulated by introducing $\tilde{n}(j) \equiv \lfloor \sqrt{j + 1/4} - 1/2 \rfloor$ and considering the values of n such that $-\tilde{n}(j) \leqslant n \leqslant \tilde{n}(j) + 1$ for any $j \in \mathbb{N}_0$. For instance, we have $\tilde{n}(0) = \tilde{n}(1) = 0$, $\tilde{n}(2) = \tilde{n}(3) = \tilde{n}(4) = \tilde{n}(5) = 1$, etc.. These manipulations allow to write (C.24) as follows

$$\mathcal{T}^\pm_{\kappa,\infty} = \sum_{j=0}^{\infty} \frac{\mathcal{R}^\pm_{\kappa,j}}{(4\eta)^j} \qquad \mathcal{R}^\pm_{\kappa,j} \equiv \sum_{n=-\tilde{n}(j)}^{\tilde{n}(j)+1} (\mp\mathrm{i}\kappa)^n \mathrm{e}^{\mp\mathrm{i}2n\eta} (4\eta)^{2\mathrm{i}yn}\, 2^{j+n-n^2}\, \mathcal{G}_n(\tilde{\nu})\, \mathcal{D}_{j+n-n^2}(\pm\tilde{\nu} \mp n). \tag{C.27}$$

By introducing $\tilde{\mathcal{R}}^\pm_{\kappa,0}$ as follows

$$\mathcal{R}^\pm_{\kappa,0} = 1 + \tilde{\mathcal{R}}^\pm_{\kappa,0} \tag{C.28}$$

where

$$\tilde{\mathcal{R}}^\pm_{\kappa,0} \equiv \mp\mathrm{i}\kappa\, \mathrm{e}^{\mp\mathrm{i}2\eta}\, (4\eta)^{2\mathrm{i}y}\, \Omega(y) \qquad \Omega(y) \equiv \mathcal{G}_1(\tilde{\nu}) = \frac{\Gamma(\frac{1}{2} - \mathrm{i}y)}{\Gamma(\frac{1}{2} + \mathrm{i}y)} \tag{C.29}$$

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the expression of $\mathcal{R}^{\pm}_{\kappa,j}$ in (C.27) can be written as

$$\mathcal{R}^{\pm}_{\kappa,j} = \sum_{n=-\tilde{n}(j)}^{\tilde{n}(j)+1} (\tilde{\mathcal{R}}^{\pm}_{\kappa,0})^n \, \tilde{\mathcal{G}}_n(\tilde{\nu}) \, 2^{j+n-n^2} \mathcal{D}_{j+n-n^2}(\pm\tilde{\nu} \mp n) \qquad \tilde{\mathcal{G}}_n(\tilde{\nu}) \equiv \frac{\mathcal{G}_n(\tilde{\nu})}{\mathcal{G}_1(\tilde{\nu})^n} \,. \tag{C.30}$$

By exploiting the identity $\Gamma(x+1) = x\,\Gamma(x)$, one finds that

$$\tilde{\mathcal{G}}_n(\tilde{\nu}) = \begin{cases} \prod_{k=1}^{n-1} (y+\hat{y}_k)^{2(n-k)} & n \geqslant 2 \\ 1 & n \in \{0,1\} \\ \prod_{k=1}^{-n} (y-\hat{y}_k)^{2(-n+1-k)} & n \leqslant -1 \end{cases} \tag{C.31}$$

where

$$\hat{y}_k \equiv \mathrm{i}\left(k - \frac{1}{2}\right). \tag{C.32}$$

At this point, let us consider

$$\log(\mathcal{T}^{\pm}_{\kappa,\infty}) = \log(1+\tilde{\mathcal{R}}^{\pm}_{\kappa,0}) + \log\left(1 + \sum_{N=1}^{\infty} \frac{\mathcal{B}^{\pm}_{\kappa,N}}{(4\eta)^N}\right) = \sum_{N=0}^{\infty} \frac{\mathcal{Y}^{\pm}_{\kappa,N}}{(4\eta)^N} \tag{C.33}$$

where we have introduced

$$\mathcal{B}^{\pm}_{\kappa,N} \equiv \frac{\mathcal{R}^{\pm}_{\kappa,N}}{1+\tilde{\mathcal{R}}^{\pm}_{\kappa,0}} \tag{C.34}$$

and

$$\mathcal{Y}^{\pm}_{\kappa,0} \equiv \log(1+\tilde{\mathcal{R}}^{\pm}_{\kappa,0}) \qquad \mathcal{Y}^{\pm}_{\kappa,N\geqslant 1} \equiv \sum_{\Upsilon_N} (-1)^{\sum_{j=1}^{k} r_j + 1} \frac{(\sum_{j=1}^{k} r_j - 1)!}{\prod_{j=1}^{k} r_j!} \prod_{j=1}^{k} (\mathcal{B}^{\pm}_{\kappa,p_j})^{r_j} \tag{C.35}$$

being Υ_N defined as the set made by the integer decompositions of $N \in \mathbb{N}$, namely

$$\Upsilon_N \equiv \left\{ \Big((p_1,r_1),\ldots,(p_k,r_k)\Big) \in (\mathbb{N}^2)^k \ \text{ s.t. } \ p_1 > \cdots > p_k \ \text{ and } \ \sum_{j=1}^{k} p_j r_j = N \right\}. \tag{C.36}$$

For instance, for $N \in \{1,2,3,4\}$ we have

$$\Upsilon_1 = \Big\{((1,1))\Big\} \qquad \Upsilon_2 = \Big\{((2,1)),((1,2))\Big\} \tag{C.37}$$

$$\Upsilon_3 = \Big\{((3,1)),((2,1),(1,1)),((1,3))\Big\} \tag{C.38}$$

$$\Upsilon_4 = \Big\{((4,1)),((3,1),(1,1)),((2,2)),((2,1),(1,2)),((1,4))\Big\} \tag{C.39}$$

which respectively provide the following expression for $\mathcal{Y}^{\pm}_{\kappa,N}$ (from (C.35))

$$\mathcal{Y}^{\pm}_{\kappa,1} = \mathcal{B}^{\pm}_{\kappa,1} \qquad \mathcal{Y}^{\pm}_{\kappa,2} = \mathcal{B}^{\pm}_{\kappa,2} - \frac{1}{2}(\mathcal{B}^{\pm}_{\kappa,1})^2 \tag{C.40}$$

$$\mathcal{Y}^{\pm}_{\kappa,3} = \mathcal{B}^{\pm}_{\kappa,3} - \mathcal{B}^{\pm}_{\kappa,2}\mathcal{B}^{\pm}_{\kappa,1} + \frac{1}{3}(\mathcal{B}^{\pm}_{\kappa,1})^3 \tag{C.41}$$

$$\mathcal{Y}^{\pm}_{\kappa,4} = \mathcal{B}^{\pm}_{\kappa,4} - \mathcal{B}^{\pm}_{\kappa,3}\mathcal{B}^{\pm}_{\kappa,1} - \frac{1}{2}(\mathcal{B}^{\pm}_{\kappa,2})^2 + \mathcal{B}^{\pm}_{\kappa,2}(\mathcal{B}^{\pm}_{\kappa,1})^2 - \frac{1}{4}(\mathcal{B}^{\pm}_{\kappa,1})^4 \,. \tag{C.42}$$

In our analysis only $N=1$ and $N=2$ have been employed.

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Finally, by employing the expansions (C.33) into (C.21), the subleading terms of the entanglement entropies in (6.7) can be written in the form (6.11) with

$$\widetilde{S}^{(\alpha)}_{A,\kappa,\infty,N} = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \hat{s}'_\alpha(y) \Big(\mathcal{Y}^-_{\kappa,N} + \mathcal{Y}^+_{\kappa,N} \Big) \, \mathrm{d}y \tag{C.43}$$

where the function $\hat{s}'_\alpha(y)$ is given in (C.20).

C.3.2 Useful integrals

In the forthcoming analyses, we systematically encounter the integrals

$$\mathcal{I}^\pm_{\alpha,j,\kappa}[\mathcal{P}] \equiv \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}(1\mp 1)}}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \hat{s}'_\alpha(y) \Big[(\tilde{\mathcal{R}}^-_{\kappa,0})^j \pm (\tilde{\mathcal{R}}^+_{\kappa,0})^j \Big] \mathcal{P}(y) \, \mathrm{d}y \qquad j \in \mathbb{Z} \tag{C.44}$$

where $\mathcal{P}(y)$ is a polynomial and the factor $\mathrm{e}^{\mathrm{i}\frac{\pi}{4}(1\mp 1)}$ has been introduced for later convenience (in order to facilitate the construction of the trigonometric functions).

Since the integral (C.44) for $j = 0$ can be performed analytically, in the following we consider only the cases where $j \neq 0$. The integrand in (C.44) involves an integer power of $\tilde{\mathcal{R}}^\pm_{\kappa,0}$ defined in (C.29); hence the integral (C.44) can be evaluated by applying the residue theorem. It is convenient to choose a closed integration path that includes a half circumference at infinity lying either to the upper half plane or to the lower half plane, for $j > 0$ or $j < 0$ respectively.

As for the singularities of the integrand occurring in (C.44), the function $\Omega(y)$ has simple zeros in the upper half plane for $y = \hat{y}_k$ and simple poles in the lower half plane for $y = -\hat{y}_k$, where $k \in \mathbb{N}$. The opposite holds for $1/\Omega(y)$. Furthermore, for $k \in \mathbb{N}$ we have

$$\Omega(y) = \mathrm{i}(-1)^{k+1} \big[(k-1)!\big]^2 (y - \hat{y}_k) + O\big((y-\hat{y}_k)^2\big) \tag{C.45}$$

$$1/\Omega(y) = \mathrm{i}(-1)^{k} \big[(k-1)!\big]^2 (y + \hat{y}_k) + O\big((y+\hat{y}_k)^2\big) \, . \tag{C.46}$$

The function $\hat{s}'(y)$ has double poles for $y = \pm\, \hat{y}_k$ with $k \in \mathbb{N}$, and

$$\hat{s}'(y) = \frac{\hat{y}_k}{(y-\hat{y}_k)^2} + O\big((y-\hat{y}_k)^{-1}\big) \qquad \hat{s}'(y) = -\frac{\hat{y}_k}{(y+\hat{y}_k)^2} + O\big((y+\hat{y}_k)^{-1}\big) \, . \tag{C.47}$$

Instead, considering the function $\hat{s}'_\alpha(y)$ for finite $\alpha \neq 1$ and decomposing it as follows

$$\hat{s}'_\alpha(y) = \frac{\pi\alpha}{\alpha - 1} \tanh(\pi y) + \frac{\pi\alpha}{1-\alpha} \tanh(\alpha\pi y) \tag{C.48}$$

one observes that the first term has simple poles for $y = \pm\, \hat{y}_k$, with residues equal to $\alpha/(\alpha-1)$, while the second term has simple poles for $y = \pm\, \hat{y}_k/\alpha$, with $k \in \mathbb{N}$ and residues equal to $1/(1-\alpha)$.

The above observations lead us to evaluate the integral (C.44) by considering $\alpha = 1$ and finite $\alpha \neq 1$ separately.

When $\alpha = 1$, one finds $\mathcal{I}^\pm_{1,j,\kappa}[\mathcal{P}] = 0$ for $|j| \geqslant 2$ because the zeros of $\tilde{\mathcal{R}}^\pm_{\kappa,0}$ are simple while the poles of $\hat{s}'(y)$ are double. For $j \in \{1, -1\}$, by using (C.45), (C.46) and C.47), we

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get

$$\hat{s}'(y)\,\tilde{\mathcal{R}}^{\pm}_{\kappa,0}\,\mathcal{P}(y) = \mp\mathrm{i}\kappa\,(-1)^k(2k-1)\big[(k-1)!\big]^2\,\frac{\mathrm{e}^{\mp\mathrm{i}2\eta}\,\mathcal{P}(\hat{y}_k)}{2\,(4\eta)^{2k-1}}\,\frac{1}{y-\hat{y}_k} + O\big((y-\hat{y}_k)^0\big) \tag{C.49}$$

$$\hat{s}'(y)\,\frac{\mathcal{P}(y)}{\tilde{\mathcal{R}}^{\pm}_{\kappa,0}} = \pm\mathrm{i}\kappa(-1)^k(2k-1)\big[(k-1)!\big]^2\,\frac{\mathrm{e}^{\pm\mathrm{i}2\eta}\,\mathcal{P}(-\hat{y}_k)}{2\,(4\eta)^{2k-1}}\,\frac{1}{y+\hat{y}_k} + O\big((y+\hat{y}_k)^0\big) \tag{C.50}$$

where $k \in \mathbb{N}$. Thus, for the integral (C.44) in these cases we obtain (here $j \in \{-1\,,1\}$ and $\xi = \mathrm{sign}(j)$)

$$\mathcal{I}^{\pm}_{1,j,\kappa}[\mathcal{P}] \;=\; \kappa\sum_{k=1}^{\infty}(-1)^{k+1}(2k-1)\big[(k-1)!\big]^2\,\frac{\mathcal{P}(\xi\,\hat{y}_k)}{(4\eta)^{2k-1}}\cdot\begin{cases}\sin(2\eta) & \text{for } \mathcal{I}^{+}_{1,j,\kappa}\\ \cos(2\eta) & \text{for } \mathcal{I}^{-}_{1,j,\kappa}\,.\end{cases} \tag{C.51}$$

For finite $\alpha \neq 1$, the contribution of the first term in the r.h.s. of (C.48) vanish because its simple poles cancel with the simple zeros of $\tilde{\mathcal{R}}^{\pm}_{\kappa,0}$. As for the contribution of the second term in the r.h.s. of (C.48) to (C.44), we find

$$\begin{aligned}\mathcal{I}^{\pm}_{\alpha,j,\kappa}[\mathcal{P}] \;&=\; \frac{\xi\,\mathrm{e}^{\mathrm{i}\frac{\pi}{4}(1\mp1)}}{1-\alpha}\sum_{k=1}^{\infty}\mathcal{P}(\xi y_k/\alpha)\,\Big[(\tilde{\mathcal{R}}^{-}_{\kappa,0})^j\big|_{y=\xi y_k/\alpha} \pm (\tilde{\mathcal{R}}^{+}_{\kappa,0})^j\big|_{y=\xi y_k/\alpha}\Big] \\ &=\; \xi\,\frac{2\,(-\kappa)^j}{1-\alpha}\sum_{k=1}^{\infty}\mathcal{P}(\xi\,y_k/\alpha)\left(\frac{\Omega(\xi\,\hat{y}_k/\alpha)}{(4\eta)^{(2k-1)/\alpha}}\right)^j\cdot\begin{cases}\cos\big[(2\eta-\frac{\pi}{2})j\big] & \text{for } \mathcal{I}^{+}_{1,j,\kappa}\\ -\sin\big[(2\eta-\frac{\pi}{2})j\big] & \text{for } \mathcal{I}^{-}_{1,j,\kappa}\end{cases}\end{aligned} \tag{C.52}$$

where $\xi = \mathrm{sign}(j)$ again and

$$\Omega(\hat{y}_k/\alpha) = \frac{1}{\Omega(-\hat{y}_k/\alpha)} = \frac{\Gamma\Big(\frac{1}{2}-\frac{1}{2\alpha}+\frac{k}{\alpha}\Big)}{\Gamma\Big(\frac{1}{2}+\frac{1}{2\alpha}-\frac{k}{\alpha}\Big)}\,. \tag{C.53}$$

C.3.3 $N = 0$ term

The first term in (6.11) corresponds to $N = 0$ and its coefficient $\widetilde{S}^{(\alpha)}_{A,\pm,\infty,0}$ can be evaluated from (C.43) and (C.35) specialised to $N = 0$. First one expands (C.35) as follows

$$\mathcal{Y}^{\pm}_{\kappa,0} = \log(1+\tilde{\mathcal{R}}^{\pm}_{\kappa,0}) = \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\,(\tilde{\mathcal{R}}^{\pm}_{\kappa,0})^j \tag{C.54}$$

finding that (C.43) for $N = 0$ can be written as

$$\widetilde{S}^{(\alpha)}_{A,\kappa,\infty,0} \;=\; \frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\int_{-\infty}^{+\infty}\hat{s}'_{\alpha}(y)\,\Big[(\tilde{\mathcal{R}}^{-}_{\kappa,0})^j + (\tilde{\mathcal{R}}^{+}_{\kappa,0})^j\Big]\,\mathrm{d}y \tag{C.55}$$

which is a series whose coefficients are the integrals $\mathcal{I}^{+}_{\alpha,j}[\mathcal{P}]$ in (C.44) with $j \geqslant 1$ and $\mathcal{P}(y) = 1$ identically. When $\alpha = 1$, we can employ (C.51) specialised to this case (i.e. for $\mathcal{P}(y) = 1$, $\kappa = +1$ and $\xi = +1$), finding (6.12). For finite $\alpha \neq 1$, from (C.55) and (C.52) specialised to this case, we obtain (6.13).

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C.3.4 $N = 1$ term

As for the $N = 1$ term in (6.11), its coefficient $\widetilde{S}^{(\alpha)}_{A,\pm,\infty,1}$ can be found through (C.43) and (C.35) specialised to this case (the expression of $\mathcal{R}^{\pm}_{\kappa,1}$ can be obtained from (C.30)), that give

$$\mathcal{Y}^{\pm}_{\kappa,1} = \mathcal{B}^{\pm}_{\kappa,1} = 2\,\mathcal{D}_1(\pm\tilde{\nu}) + \frac{2\,\tilde{\mathcal{R}}^{\pm}_{\kappa,0}}{1+\tilde{\mathcal{R}}^{\pm}_{\kappa,0}}\left[\mathcal{D}_1(\pm\tilde{\nu}\mp 1) - \mathcal{D}_1(\pm\tilde{\nu})\right]. \tag{C.56}$$

From this expression and the expansion

$$\frac{1}{1+\tilde{\mathcal{R}}^{\pm}_{\kappa,0}} = \sum_{j=1}^{\infty}(-1)^{j+1}(\tilde{\mathcal{R}}^{\pm}_{\kappa,0})^{j-1} \tag{C.57}$$

we find that (C.43) in this case becomes

$$\begin{aligned}\widetilde{S}^{(\alpha)}_{A,\kappa,\infty,1} = \frac{1}{2\pi\mathrm{i}}\Bigg\{&2\int_{-\infty}^{+\infty}\hat{s}'_\alpha(y)\left[\mathcal{D}_1(\tilde{\nu})+\mathcal{D}_1(-\tilde{\nu})\right]\mathrm{d}y + \sum_{j=1}^{\infty}(-1)^{j+1}\int_{-\infty}^{+\infty}\hat{s}'_\alpha(y)\\ &\times 2\left[(\tilde{\mathcal{R}}^{-}_{\kappa,0})^{j}\left(\mathcal{D}_1(-\tilde{\nu}+1)-\mathcal{D}_1(-\tilde{\nu})\right)+(\tilde{\mathcal{R}}^{+}_{\kappa,0})^{j}\left(\mathcal{D}_1(\tilde{\nu}-1)-\mathcal{D}_1(\tilde{\nu})\right)\right]\mathrm{d}y\Bigg\}.\end{aligned} \tag{C.58}$$

By observing that $\mathcal{D}_1(\tilde{\nu}) + \mathcal{D}_1(-\tilde{\nu}) = 0$ and introducing

$$2\left[\mathcal{D}_1(\pm\tilde{\nu}\mp 1) - \mathcal{D}_1(\pm\tilde{\nu})\right] \equiv \mp\mathrm{i}\,\mathcal{P}_1(y) \qquad \mathcal{P}_1(y) \equiv 6y^2 - \frac{1}{2} \tag{C.59}$$

we find that (C.58) can be written as

$$\widetilde{S}^{(\alpha)}_{A,\kappa,\infty,1} = \frac{1}{2\pi}\sum_{j=1}^{\infty}(-1)^{j+1}\int_{-\infty}^{+\infty}\hat{s}'_\alpha(y)\left[(\tilde{\mathcal{R}}^{-}_{\kappa,0})^{j} - (\tilde{\mathcal{R}}^{+}_{\kappa,0})^{j}\right]\mathcal{P}_1(y)\,\mathrm{d}y \tag{C.60}$$

whose summand takes the form (C.44). Thus, when $\alpha = 1$, from (C.51) we obtain (6.14); while for finite $\alpha \neq 1$ we arrive to (6.15) by employing (C.52) in (C.60).

C.3.5 $N = 2$ term

The coefficient $\widetilde{S}^{(\alpha)}_{A,\pm,\infty,2}$ occurring in the term labelled by $N = 2$ in the r.h.s. of (6.11) is given by (C.43) and (C.35) specialised to $N = 2$.

In order to obtain $\mathcal{B}^{\pm}_{\kappa,2}$, first we construct $\mathcal{R}^{\pm}_{\kappa,2}$ from (C.30), finding

$$\mathcal{R}^{\pm}_{\kappa,2} = \frac{\tilde{\mathcal{G}}_{-1}(\tilde{\nu})}{\tilde{\mathcal{R}}^{\pm}_{\kappa,0}} + 4\,\mathcal{D}_2(\pm\tilde{\nu}) + 4\,\tilde{\mathcal{R}}^{\pm}_{\kappa,0}\,\mathcal{D}_2(\pm\tilde{\nu}\mp 1) + (\tilde{\mathcal{R}}^{\pm}_{\kappa,0})^2\,\tilde{\mathcal{G}}_2(\tilde{\nu}) \tag{C.61}$$

and then use (C.34), which leads to

$$\begin{aligned}\mathcal{B}^{\pm}_{\kappa,2} = &\frac{\tilde{\mathcal{G}}_{-1}(\tilde{\nu})}{\tilde{\mathcal{R}}^{\pm}_{\kappa,0}} + 4\,\mathcal{D}_2(\pm\tilde{\nu}) - \tilde{\mathcal{G}}_{-1}(\tilde{\nu})\\ &+\frac{\tilde{\mathcal{R}}^{\pm}_{\kappa,0}}{1+\tilde{\mathcal{R}}^{\pm}_{\kappa,0}}\left[\tilde{\mathcal{G}}_{-1}(\tilde{\nu}) - 4\,\mathcal{D}_2(\pm\tilde{\nu}) + 4\,\mathcal{D}_2(\pm\tilde{\nu}\mp 1)\right] + \frac{(\tilde{\mathcal{R}}^{\pm}_{\kappa,0})^2}{1+\tilde{\mathcal{R}}^{\pm}_{\kappa,0}}\,\tilde{\mathcal{G}}_2(\tilde{\nu}).\end{aligned} \tag{C.62}$$

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By using this expression and (C.56) into (C.35) for $N=2$, one obtains

$$\mathcal{Y}^{\pm}_{\kappa,2} = \mathcal{B}^{\pm}_{\kappa,2} - \frac{1}{2}\,(\mathcal{B}^{\pm}_{\kappa,1})^2 \;\equiv\; \mathcal{Y}^{\pm}_{\kappa,2,a} + \mathcal{Y}^{\pm}_{\kappa,2,b} + \mathcal{Y}^{\pm}_{\kappa,2,c} + \mathcal{Y}^{\pm}_{\kappa,2,d} + \mathcal{Y}^{\pm}_{\kappa,2,e} \tag{C.63}$$

where

$$\mathcal{Y}^{\pm}_{\kappa,2,a} \equiv \frac{\tilde{\mathcal{G}}_{-1}(\tilde{\nu})}{\tilde{\mathcal{R}}^{\pm}_{\kappa,0}} \qquad \mathcal{Y}^{\pm}_{\kappa,2,b} \equiv 4\mathcal{D}_2(\pm\tilde{\nu}) - 2\mathcal{D}_1(\pm\tilde{\nu})^2 - \tilde{\mathcal{G}}_{-1}(\tilde{\nu}) \tag{C.64}$$

$$\mathcal{Y}^{\pm}_{\kappa,2,c} \equiv \frac{\tilde{\mathcal{R}}^{\pm}_{\kappa,0}}{1+\tilde{\mathcal{R}}^{\pm}_{\kappa,0}}\Big\{\tilde{\mathcal{G}}_{-1}(\tilde{\nu}) - 4\mathcal{D}_2(\pm\tilde{\nu}) + 4\mathcal{D}_2(\pm\tilde{\nu}\mp 1) - 4\mathcal{D}_1(\pm\tilde{\nu})\big[\mathcal{D}_1(\pm\tilde{\nu}\mp 1) - \mathcal{D}_1(\pm\tilde{\nu})\big]\Big\} \tag{C.65}$$

$$\mathcal{Y}^{\pm}_{\kappa,2,d} \equiv \frac{(\tilde{\mathcal{R}}^{\pm}_{\kappa,0})^2}{1+\tilde{\mathcal{R}}^{\pm}_{\kappa,0}}\tilde{\mathcal{G}}_2(\tilde{\nu}) \qquad \mathcal{Y}^{\pm}_{\kappa,2,e} \equiv -2\left(\frac{\tilde{\mathcal{R}}^{\pm}_{\kappa,0}}{1+\tilde{\mathcal{R}}^{\pm}_{\kappa,0}}\right)^2\Big[\mathcal{D}_1(\pm\tilde{\nu}\mp 1) - \mathcal{D}_1(\pm\tilde{\nu})\Big]^2. \tag{C.66}$$

In (C.65) and (C.66), we can employ (C.57) and

$$\frac{1}{(1+\tilde{\mathcal{R}}^{\pm}_{\kappa,0})^2} = \sum_{j=2}^{\infty}(-1)^j(j-1)(\tilde{\mathcal{R}}^{\pm}_{\kappa,0})^{j-2}\,. \tag{C.67}$$

Plugging (C.63) into (C.43), we find that

$$\widetilde{S}^{(\alpha)}_{A,\kappa,\infty,2} = \mathcal{J}_{\alpha,a} + \mathcal{J}_{\alpha,b} + \mathcal{J}_{\alpha,c} + \mathcal{J}_{\alpha,d} + \mathcal{J}_{\alpha,e} \tag{C.68}$$

where the terms in the r.h.s. are the integrals provided by (C.64), (C.65) and (C.66), that are defined respectively by

$$\mathcal{J}_{\alpha,a} \equiv \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\hat{s}'_{\alpha}(y)\Big[(\tilde{\mathcal{R}}^{-}_{\kappa,0})^{-1} + (\tilde{\mathcal{R}}^{+}_{\kappa,0})^{-1}\Big]\mathcal{P}_{2,a}(y)\,\mathrm{d}y \tag{C.69}$$

$$\mathcal{J}_{\alpha,b} \equiv \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\hat{s}'_{\alpha}(y)\,\mathcal{P}_{2,b}(y)\,\mathrm{d}y \tag{C.70}$$

$$\mathcal{J}_{\alpha,c} \equiv \sum_{j=1}^{\infty}(-1)^{j+1}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\hat{s}'_{\alpha}(y)\Big[(\tilde{\mathcal{R}}^{-}_{\kappa,0})^{j} + (\tilde{\mathcal{R}}^{+}_{\kappa,0})^{j}\Big]\mathcal{P}_{2,c}(y)\,\mathrm{d}y \tag{C.71}$$

$$\mathcal{J}_{\alpha,d} \equiv \frac{1}{2\pi\mathrm{i}}\sum_{j=2}^{\infty}(-1)^{j}\int_{-\infty}^{+\infty}\hat{s}'_{\alpha}(y)\Big[(\tilde{\mathcal{R}}^{-}_{\kappa,0})^{j} + (\tilde{\mathcal{R}}^{+}_{\kappa,0})^{j}\Big]\mathcal{P}_{2,d}(y)\,\mathrm{d}y \tag{C.72}$$

$$\mathcal{J}_{\alpha,e} \equiv \frac{1}{2\pi\mathrm{i}}\sum_{j=2}^{\infty}(-1)^{j+1}(j-1)\int_{-\infty}^{+\infty}\hat{s}'_{\alpha}(y)\Big[(\tilde{\mathcal{R}}^{-}_{\kappa,0})^{j} + (\tilde{\mathcal{R}}^{+}_{\kappa,0})^{j}\Big]\mathcal{P}_{2,e}(y)\,\mathrm{d}y \tag{C.73}$$

in terms of the polynomials given respectively by

$$\mathcal{P}_{2,a}(y) \equiv \tilde{\mathcal{G}}_{-1}(\tilde{\nu}) = (y-\hat{y}_1)^2 = (y-\mathrm{i}/2)^2 \tag{C.74}$$

$$\begin{aligned}\mathcal{P}_{2,b}(y) &\equiv 4\,\mathcal{D}_2(-\tilde{\nu}) - 2\,\mathcal{D}_1(-\tilde{\nu})^2 + 4\,\mathcal{D}_2(\tilde{\nu}) - 2\,\mathcal{D}_1(\tilde{\nu})^2 - 2\,\tilde{\mathcal{G}}_{-1}(\tilde{\nu}) \\ &= -10\,y^4 + 20\mathrm{i}\,y^3 + 15\,y^2 - 5\mathrm{i}\,y - 5/8\end{aligned} \tag{C.75}$$

$$\begin{aligned}\mathcal{P}_{2,c}(y) &\equiv \tilde{\mathcal{G}}_{-1}(\tilde{\nu}) - 4\,\mathcal{D}_2(\pm\tilde{\nu}) + 4\,\mathcal{D}_2(\pm\tilde{\nu}\mp 1) - 4\,\mathcal{D}_1(\pm\tilde{\nu})\big[\mathcal{D}_1(\pm\tilde{\nu}\mp 1) - \mathcal{D}_1(\pm\tilde{\nu})\big] \\ &= -18\,y^4 - 20\mathrm{i}\,y^3 + 4\,y^2 + 6\mathrm{i}\,y - 3/8\end{aligned} \tag{C.76}$$

$$\mathcal{P}_{2,d}(y) \equiv \tilde{\mathcal{G}}_2(\tilde{\nu}) = (y+\hat{y}_1)^2 = (y+\mathrm{i}/2)^2 \tag{C.77}$$

$$\mathcal{P}_{2,e}(y) \equiv 2\big[\mathcal{D}_1(\pm\tilde{\nu}\mp 1) - \mathcal{D}_1(\pm\tilde{\nu})\big]^2 = -18\,y^4 + 3\,y^2 - 1/8\,. \tag{C.78}$$

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The integrals occurring in (C.69)–(C.73) have the form (C.44). Hence, when $\alpha = 1$ w can use (C.51) for these integrals, finding

$$\mathcal{J}_{1,a} = \kappa \, \sin(2\eta) \sum_{k=1}^{\infty} (-1)^k \, \frac{(2k-1)\big[(k-1)!\big]^2 \, \mathcal{P}_{2,a}(-\hat{y}_k)}{(4\eta)^{2k-1}} \qquad \mathcal{J}_{1,b} = -\frac{1}{6} \tag{C.79}$$

$$\mathcal{J}_{1,c} = -\kappa \, \sin(2\eta) \sum_{k=1}^{\infty} (-1)^k \, \frac{(2k-1)\big[(k-1)!\big]^2 \, \mathcal{P}_{2,c}(\hat{y}_k)}{(4\eta)^{2k-1}} \qquad \mathcal{J}_{1,d} = \mathcal{J}_{1,e} = 0 \tag{C.80}$$

where $\mathcal{P}_{2,a}(-\hat{y}_k) = (\hat{y}_k + \hat{y}_1)^2 = -k^2$ and $\mathcal{P}_{2,c}(\hat{y}_k) = -18k^4 + 16k^3 - k^2 - 8k + 3$. Then, the expression (6.16) is obtained by plugging (C.79) and (C.80) into (C.68) specialised to $\alpha = 1$.

For positive and finite $\alpha \neq 1$, by applying (C.52) to the integrals occurring in (C.69)–(C.73), we find

$$\mathcal{J}_{\alpha,a} = \frac{2\,\kappa}{1-\alpha} \, \sin(2\eta) \sum_{k=1}^{\infty} \mathcal{P}_{2,a}(-y_k/\alpha) \, \frac{\Omega(\hat{y}_k/\alpha)}{(4\eta)^{(2k-1)/\alpha}} \qquad \mathcal{J}_{\alpha,b} = \frac{(\alpha+1)(3\alpha^2-7)}{48\alpha^3} \tag{C.81}$$

$$\mathcal{J}_{\alpha,c} = -\frac{2}{1-\alpha} \sum_{j=1}^{\infty} \cos\left[\left(2\eta - \frac{\pi}{2}\right) j\right] \sum_{k=1}^{\infty} \mathcal{P}_{2,c}(y_k/\alpha) \left(\frac{\kappa\,\Omega(\hat{y}_k/\alpha)}{(4\eta)^{(2k-1)/\alpha}}\right)^j \tag{C.82}$$

$$\mathcal{J}_{\alpha,d} = \frac{2}{1-\alpha} \sum_{j=2}^{\infty} \cos\left[\left(2\eta - \frac{\pi}{2}\right) j\right] \sum_{k=1}^{\infty} \mathcal{P}_{2,d}(y_k/\alpha) \left(\frac{\kappa\,\Omega(\hat{y}_k/\alpha)}{(4\eta)^{(2k-1)/\alpha}}\right)^j \tag{C.83}$$

$$\mathcal{J}_{\alpha,e} = \frac{2}{1-\alpha} \sum_{j=2}^{\infty} (1-j) \cos\left[\left(2\eta - \frac{\pi}{2}\right) j\right] \sum_{k=1}^{\infty} \mathcal{P}_{2,d}(y_k/\alpha) \left(\frac{\kappa\,\Omega(\hat{y}_k/\alpha)}{(4\eta)^{(2k-1)/\alpha}}\right)^j \tag{C.84}$$

in terms of the polynomials in (C.74)–(C.78) evaluated at y_k/α. Finally, the expression (6.17) is obtained by combining (C.81)–(C.84) into (C.68).

C.3.6 Consistency checks

It is important to provide some consistency checks for the analytic expressions of the subleading terms obtained in this appendix and reported in section 6. In the following we consider the replica limit (3.17), the relation (3.18) and the double scaling limit of some lattice results.

As for the replica limit (3.17), in the expansion (6.11) it means that $\widetilde{S}^{(\alpha)}_{A,\pm,\infty,N} \to \widetilde{S}_{A,\pm,\infty,N}$ as $\alpha \to 1$ for any $N \in \mathbb{N}_0$. When $N = 0$, from (6.12), (6.13) and

$$\lim_{\alpha\to 1} \frac{\Omega(\hat{y}_k/\alpha)}{\alpha - 1} = (-1)^{k+1}(k-1/2)\big[(k-1)!\big]^2 \tag{C.85}$$

we conclude that only the term corresponding to $j = 1$ gives a non vanishing result in the sum over j occurring in (6.13); hence $\widetilde{S}^{(\alpha)}_{A,\pm,\infty,0} \to \widetilde{S}_{A,\pm,\infty,0}$ as $\alpha \to 1$. Similarly, from (6.14)–(6.17), we have checked that $\widetilde{S}^{(\alpha)}_{A,\pm,\infty,N} \to \widetilde{S}_{A,\pm,\infty,N}$ as $\alpha \to 1$ also for $N \in \{1, 2\}$.

Another consistency check is The validity of the relation (3.18) order by order in the large η expansion in another consistency check. From (6.9), it is straightforward to realise that this relation holds for the leading terms; hence (3.18) can be verified by checking that

$$\widetilde{S}^{(\alpha)}_{A,+,\infty,N} + \widetilde{S}^{(\alpha)}_{A,-,\infty,N} = 4^N \, \widetilde{S}^{(\alpha)}_{A,\infty,N} \qquad N \in \mathbb{N}_0 \tag{C.86}$$

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for the coefficients of the expansions (6.11), where $\widetilde{S}^{(\alpha)}_{A,\infty,N}$ for the interval $[-R,R]\subset\mathbb{R}$ on the line have been determined in [42] for $N\in\{0,1,2\}$.

When $N=0$, first we split (6.13) as follows

$$\widetilde{S}^{(\alpha)}_{A,\kappa,\infty,0} = \frac{2}{\alpha-1}\sum_{j=1}^{\infty}\frac{\cos\big[(2\eta-\frac{\pi}{2})2j\big]}{2j}\sum_{k=1}^{\infty}\left(\frac{\Omega(\hat{y}_k/\alpha)}{(4\eta)^{(2k-1)/\alpha}}\right)^{2j} + \frac{2\,\kappa}{\alpha-1}\sum_{j=1}^{\infty}\frac{\cos\big[(2\eta-\frac{\pi}{2})(2j-1)\big]}{2j-1}\sum_{k=1}^{\infty}\left(\frac{\Omega(\hat{y}_k/\alpha)}{(4\eta)^{(2k-1)/\alpha}}\right)^{2j-1} \tag{C.87}$$

because it straightforwardly leads to observe that the terms coming from the second line of (C.87) cancel in (C.86), while the remaining ones combine into $\widetilde{S}^{(\alpha)}_{A,\infty,0}$, as expected. The validity of (C.86) for $N=1$ has been checked by employing (6.15). Finally, we have checked (C.86) for $N=2$ by first observing that the terms corresponding to odd values of j cancel in the sums over j (see (6.17)) occurring in the l.h.s. of (C.86), and then that the remaining terms give

$$\widetilde{S}^{(\alpha)}_{A,+,\infty,2}+\widetilde{S}^{(\alpha)}_{A,-,\infty,2} = \frac{(\alpha+1)(3\alpha^2-7)}{24\alpha^3} + \frac{4}{1-\alpha}\sum_{j=1}^{\infty}(-1)^j\cos(4\eta j)\sum_{k=1}^{\infty}\widetilde{\mathcal{P}}_2(2j;\hat{y}_k/\alpha)\left(\frac{\Omega(\hat{y}_k/\alpha)}{(4\eta)^{(2k-1)/\alpha}}\right)^{2j} \tag{C.88}$$

which agrees with the result for $\widetilde{S}^{(\alpha)}_{A,\infty,2}$ found in [42].

It is important to verify that our results agree with the proper limit of the corresponding ones obtained on the lattice. In particular, taking the double scaling limit (defined in the text above (C.8)) of the expression in eq. (57) of [69], we find that $d_{\alpha,\text{there}}\to\widetilde{S}^{(\alpha)}_{A,-,\infty,0}$ (see (6.13)), as expected.

In the literature we have not found lattice results whose continuum limit provide the subleading terms corresponding to (6.15) and (6.17). These lattice results can be obtained by studying the subleading corrections to (C.7), as done e.g. in [59] for the block in the infinite XX chain.

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