Chromatic polynomials

Zeros, algorithms and computational complexity

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Chromatic polynomials:

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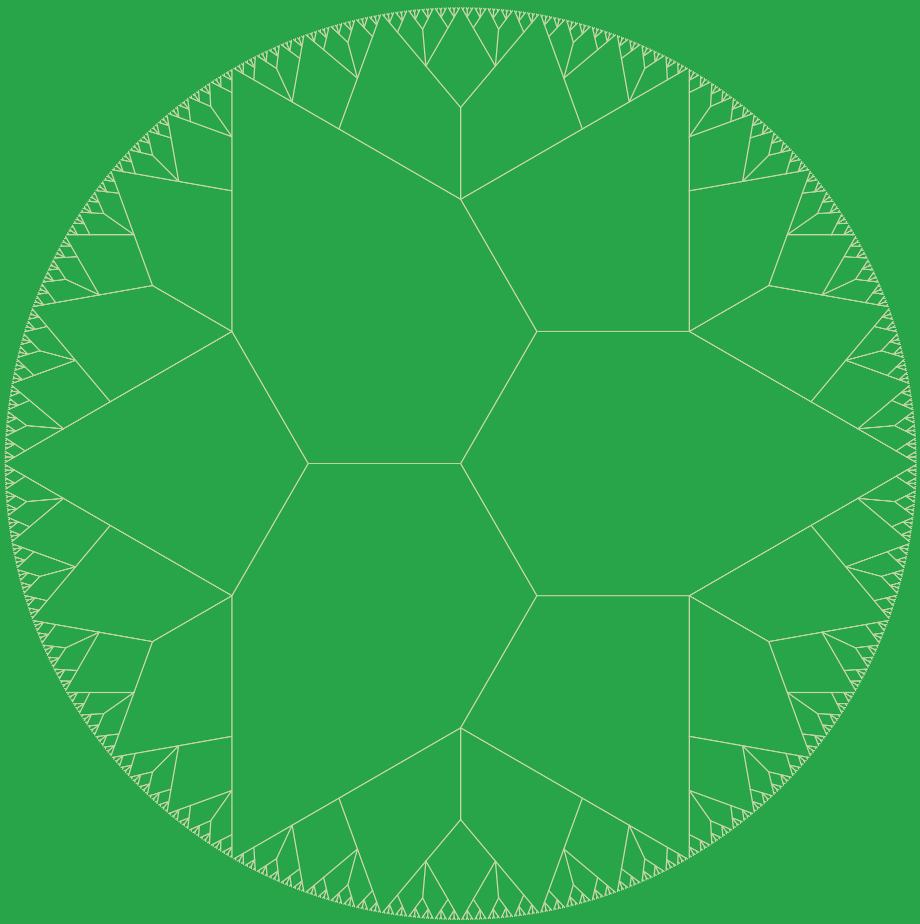

Jeroen Huijben

Chromatic polynomials: zeros, algorithms and computational complexity

Jeroen Huijben

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Chromatic polynomials: zeros, algorithms and computational complexity

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CHAPTER 1

Introduction

The protagonist of this thesis is the chromatic polynomial. This graph-theoretic object was introduced by Birkhoff in 1912 [12], and has been studied extensively since then. The polynomial was generalised by Tutte to a 2-variable polynomial [70], now known as the Tutte polynomial, which is essentially equivalent to the partition function of the Potts model from statistical physics [27]. We will first introduce these objects and then state the main questions that we consider in this thesis.

1.1 Colourings

Let $G = (V, E)$ be a graph and q a positive integer. A *proper q-colouring of G* is an assignment $\varphi : V \to [q] \coloneqq \{1, \ldots, q\}$ (we think of the numbers $1, \ldots, q$ as the colours) such that any two adjacent vertices receive a different colour. For every graph G we can now define a function $Z(G; \bullet)$ such that for a positive integer q, the number of proper q-colourings of G equals $Z(G; q)$. Perhaps surprisingly, this function is a polynomial in q and therefore it is called the *chromatic polynomial*.

In Chapters 3 and 4 we consider a generalization of the chromatic polynomial. This generalization has several equivalent forms, which we discuss in Section 1.3. One of these equivalent forms is the partition function of the *Potts model*; this is most directly a generalization of the chromatic polynomial, and will often be the most intuitive form to think about. In the Potts model we consider colourings $\varphi : V \to [q]$, but now they are not required to be proper. Actually, the proper colourings receive a weight of 1, whereas non-proper colourings receive a weight based on the number of *monochromatic edges*, edges between vertices with the same colour. The partition function depends on an additional variable y, and is defined for positive integers q as

$$Z(G; q, y) \coloneqq \sum_{\varphi : V \to [q]} y^{\#\text{monochromatic edges in } \varphi}.$$

Clearly $Z(G;q)$ equals the special case $Z(G;q,0)$. It is well-known that $Z(G;q,y)$ can be written in the following form

$$Z(G;q,y) = \sum_{A \subseteq E} q^{k(A)}(y-1)^{|A|},$$

where $k(A)$ counts the number of connected components of the subgraph (V, A). This form is also called the partition function of the *random cluster model*. (We give a proof of this equality in Proposition 1.1.) Because this summation is also defined when q is not a positive integer, we take this as the definition of $Z(G;q,y)$ for those values.

1.2 Motivation, background and contents

The two main motivating questions for this thesis are the following.

> *Where are the zeros of $Z(G;q)$?*
>
> *For which values of q and y is it easy to compute $Z(G;q,y)$?*

These questions are stated very broadly, and the answer will at least depend on the graphs G that we are considering. In the next sections we will therefore describe these questions, and the contents of this thesis, more precisely.

Before diving in, we need to make one important remark. Even though those two questions seem very different, they turn out to be closely related! And this holds not only for the Potts model, but for other graph polynomials as well. One direction of this relation has been established for a large class of polynomials, based on the interpolation method [4, 60]. This method shows that it is easy to approximate a graph polynomial on certain open subsets of the complex plane that are zero-free. In the other direction it is known for some graph polynomials, such as the independence polynomial [10, 23, 11, 31] and the partition function of the Ising model [21, 63, 18], that in the vicinity of zeros it is computationally hard to approximate the graph polynomial. Combining the results of Chapters 2 and 3, we prove a similar relation for the chromatic polynomial.

1.2.1 Chromatic zeros

As mentioned, one main interest of this thesis is the zeros of the chromatic polynomial, which we call *chromatic zeros*. This section will first cover some of the known results about them, most of this can be found in [45].

The chromatic polynomial was initially introduced as a tool that might help proving the Four Colour Theorem. In fact, it was conjectured that $Z(G;q) > 0$

for any planar graph G and any *real* value $q \geq 4$, or in other words that the interval $[4, \infty)$ is free of chromatic zeros of planar graphs. There is a relatively simple induction proof to show that the interval $[5, \infty)$ is indeed zero-free, and the Four Colour Theorem has been proved in a completely different way [1, 2]. But still the conjecture that the interval $(4, 5)$ is zero-free, is widely open! The picture on the rest of the real line is almost completely resolved. The intervals $(-\infty, 0), (0, 1)$ and $(1, \frac{32}{27}]$ are zero-free for all graphs. On the interval $(\frac{32}{27}, 4)$ the chromatic zeros of planar graphs are dense, with the exception of one small interval around $\frac{5+\sqrt{5}}{2}$,[1] but it is conjectured that they are dense on this interval as well.

We do not need to stop at plugging in real numbers, and can even plug in complex numbers. Even though there are some real zero-free intervals, it is known that the chromatic zeros of all graphs are dense in the entire complex plane [66]. Again, this picture changes for smaller graph classes. If the degree of every vertex is bounded by Δ, it is known that the absolute values of the chromatic zeros are bounded by 6.91Δ [28]. This zero-free region 'around ∞', combined with the interpolation method [60] mentioned before, yields a fast algorithm to approximate $Z(G; q)$ for any q such that $|q| > 6.91\Delta$. The fact that this algorithm also works for real q, provides some extra motivation for looking at the complex zeros.

The bound of 6.91Δ is probably far from sharp. It is conjectured that the optimal bound (as $\Delta \to \infty$) is $\approx 1.6\Delta$ and that it is reached for the complete bipartite graph $K_{\Delta,\Delta}$. A related conjecture is that the real parts of the chromatic zeros are bounded by Δ [67]. If we lift the degree constraint on one vertex, the chromatic zeros are still bounded, this time the best known bound is $\approx 7.96\Delta + 1$ [65].[2] For this family the same bound on the real part of the zeros was conjectured by Sokal [67]. In the same paper it was already remarked that this is not true for $\Delta = 3$, but in this thesis we go further and prove the following.

Theorem (Theorem 2.4)**.** *For every large enough Δ, there exists a graph G where all vertices but one have degree at most Δ, and which has a chromatic zero with real part bigger than Δ.*

The examples to prove this theorem (and disprove Sokal's conjecture) are all *series-parallel graphs*. This is a subclass of the planar graphs and has been studied extensively. For example, [66] proves as a main result that the chromatic zeros of a family of series-parallel graphs are dense in the region $\{q \in \mathbb{C} \mid |1 - q| > 1\}$. The main goal of Chapter 2 is to study the chromatic zeros of the entire family of series-parallel graphs. To do this, we turn the question into a sort of dynamical system and analyse its behaviour. Critical in this analysis is the use of Montel's

[1] This is related to a result of Tutte, saying that $Z(G; \frac{5+\sqrt{5}}{2}) > 0$ for all planar graphs G.
[2] We cannot lift the degree constraint at two vertices, as shown by [66].

theorem. A weaker version of this theorem was already used in [66], but required a family of graphs with a closed form description of their chromatic polynomial. The way we use Montel is closer to that in [23], and allows us to look at the entire family of series-parallel graphs.

Using this we manage to do the following two things.

Theorem (Theorem 2.3). *Chromatic zeros of series-parallel graphs are dense in the halfplane* $\{q \in \mathbb{C} \mid \Re(q) > \frac{3}{2}\}$.

Theorem (Theorem 2.1). *There exists an open set* $U \subseteq \mathbb{C}$ *around* $(0, \frac{32}{27})$ *such that* $U \setminus \{1\}$ *is free of chromatic zeros of series-parallel graphs.*

Recall that in [66] it was already proved that series-parallel chromatic zeros are dense in $\{q \mid |1-q| > 1\}$, and with Theorem 2.3 we extend the region where zeros are dense. We can push the results even further with some computer calculations. We also use a computer calculation to create a picture of the open set in Theorem 2.1. Both additional results are summarized in Figure 2.1. These two results do not yet form a complete picture; we expect the 'undecided region' to be mostly zero-free, but do not yet have the tools to prove this.

1.2.2 Computational complexity

The other main motivation for this thesis is understanding the computational complexity of (approximately) computing $Z(G; q, y)$.

In the previous section we already mentioned the graph class of bounded degree graphs, for which there is a fast algorithm (running in polynomial time in the size of G) to approximate the chromatic polynomial $Z(G; q)$ as long as q is large enough. The requirement that q is large enough is essential here, and it might well be that for other values of q there cannot exist such an algorithm. To make this question more precise, we have to consider complexity classes of computational problems. We will give an informal introduction to some of these classes, formal definitions can for example be found in [3].

Most well-known is the class NP consisting of certain decision problems. All of these problems ask a question of the form 'Does there exist some object satisfying these restrictions?', and the possible answers are 'Yes' and 'No'. Of course, the answer should only be 'Yes' if there indeed exists an example satisfying the restrictions. Further it should be possible to check quickly whether an example indeed satisfies the restrictions, meaning that this can be done by a computer, with a running time that is polynomial in the size of the restrictions and the example.

The class NP has a subclass of problems called P which consist of problems that can actually be solved in polynomial time. Although it is widely believed that this is a strict subclass, proving this is one of the greatest open problems in theoretical computer science, and is one of the seven Millennium Prize Problems [22].

We can get an example of an NP-problem for every positive integer q, by asking the question 'Does the graph G admit a proper q-colouring?'. The object we are looking for is the colouring, and the graph G presents the restrictions. Indeed this problem is in NP, because it is possible to check whether any colouring is proper by checking all the edges of G one by one, which is doable in polynomial time. When q equals 1 or 2 this problem is even contained in the class P: for $q = 1$ one needs to check whether the graph contains any edges, and for $q = 2$ whether the graph is bipartite. For $q \geq 3$ it is not known whether the problems are in the class P, but these problems have been proven to be NP-*complete* [52]. Meaning that if any of these problems would be in P, it must be that P = NP. If we assume that P $\neq$ NP, as is widely believed, there cannot exist a polynomial time algorithm for an NP-complete problem, and therefore we think of these problems as being hard.

Next we look at the class #P. The problems in the class #P are related to problems in NP, but they ask the question 'How many objects are there, satisfying these restrictions?'. Clearly this question is at least as hard as the corresponding decision question, but it is believed that in general the problems in #P are harder than those in NP.

Continuing our example, we ask the question 'How many proper q-colourings does the graph G admit?', or equivalently we ask for the number $Z(G; q)$ for some fixed q. This problem is in #P, because we already saw that the related decision question is in NP. For $q = 1, 2$ these counting problems can still be solved in polynomial time, while for $q \geq 3$ the problems are #P-complete [47].

Because #P-complete problems are considered to be very hard, it is natural to explore related questions which might be easier. In this thesis we mainly consider the following two relaxations: restricting the class of input graphs, and asking for an approximation instead of the exact number. One of the main problems we are considering is the following, for a fixed complex number q.

Name:	q-Planar-Abs-Chromatic
Input:	A planar graph G.
Output:	A rational number r such that $e^{-\frac{1}{4}} \leq \frac{r}{\lvert Z(G;q)\rvert} \leq e^{\frac{1}{4}}$ if $Z(G; q) \neq 0$.

(The precise numbers $e^{-\frac{1}{4}}$ and $e^{\frac{1}{4}}$ are irrelevant, the computational complexity is the same for other choices.) In Chapter 3 we investigate when this problem is #P-*hard*. We cannot call this problem #P-complete, because it's not contained in #P. But the notion #P-hard has a similar meaning: if we can solve this problem in polynomial time, we can solve all #P-problems in polynomial time. We obtain the following results.

Theorem (Theorem 3.1). *For each non-real algebraic number $q \in \mathbb{C}$ such that $|1-q| > 1$ or $\Re(q) > 3/2$, the problem q-*PLANAR-ABS-CHROMATIC *is #P-hard.*

An immediate consequence of this result is that approximating the absolute value $|Z(G;q)|$ where G is allowed to be any graph, is a #P-hard problem for every non-real algebraic number q.

The values of q in this Theorem are restricted to the region $\{q \mid |1-q| > 1 \text{ or } \Re(q) > \frac{3}{2}\}$, and recall from the previous section that the chromatic zeros of series-parallel graphs are dense in this region. This is no coincidence, because both results follow from the same condition. We can even push the relation a bit further.

Theorem (Corollary 3.13(c)). *If $q \in \mathbb{C} \setminus \mathbb{R}$ is the chromatic zero of a planar graph, the problem q-*PLANAR-ABS-CHROMATIC *is #P-hard.*

The methods we used for the chromatic polynomial extend easily to the random cluster partition function $Z(G;q,y)$. In this case we prove for example that approximating its absolute value is #P-hard when q, y are both algebraic numbers, at least one of them non-real and $|y| > 1$. When q is a positive integer, this is a complex version of the ferromagnetic Potts model.

A novel part of our method is the reduction strategy: assuming a polynomial time algorithm to approximate $|Z(G;q,y)|$ for planar graphs G, we construct a polynomial time algorithm to compute $Z(G;q,y)$ *exactly* for planar graphs G, at the same values of q and y. Other inapproximability results would often reduce from a problem at different parameters where the graph polynomial is zero-free. This raises more technical issues, which our methods circumvent.

Our work still leaves open many interesting questions. For example, there are open neighbourhoods of the intervals $(0,1)$ and $(1, \frac{32}{27})$ where we cannot find chromatic zeros of planar graphs. Meaning that for values of q in these opens we cannot prove that q-PLANAR-ABS-CHROMATIC is #P-hard. It could very well be possible that chromatic zeros do exist there, but we just have not found them yet. On the other hand it might be that there are actually zero-free regions.

For bounded degree graphs we mentioned that a zero-free region was the key in actually getting a polynomial time algorithm to approximate the chromatic polynomial, using the interpolation method. However, we do not see yet how to apply this method to possible zero-free regions around $(0,1)$ and $(1, \frac{32}{27})$. So even if we could find a zero-free region, it is still a big question what happens with the computational complexity.

1.2.3 Markov chain sampling

The previous section focused on inapproximability results. We also mentioned the interpolation method, which can be used to actually construct polynomial time approximation algorithms. In this section we will look at another way of approximating $Z(G;q,y)$ by using Markov chains.

We first introduce the Potts model as a probability distribution. For this we require that q is a positive integer and y is non-negative. Given a graph $G=(V,E)$, we assign to every colouring $\varphi : V \to [q]$ the probability

$$\tfrac{1}{Z(G;q,y)} \cdot y^{\#\text{monochromatic edges in } \varphi}.$$

Because $Z(G;q,y)$ is difficult to compute exactly (in fact, #P-hard for most values of q, y [47]), we cannot easily compute these probabilities. Therefore we set another goal, namely finding a way of (approximately) sampling from this distribution, without knowing the exact probabilities. If this succeeds, we will use this sampling method to approximate $Z(G;q,y)$.

The idea for this is to choose an event and approximate its probability by drawing samples from (an approximation to) the model. For example, if we choose any edge e of the graph G, the probability that its endpoints receive the same colour is $\frac{yZ(G/e;q,y)}{Z(G;q,y)}$. If we compute a similar ratio for the graph G/e, and graphs with more edges contracted, we can determine $Z(G;q,y)$ by a telescoping product.

A common way to approximate the Potts model is to construct a Markov chain with a unique stationary distribution that equals the Potts model. If we simulate enough steps of this Markov chain, we hope to end up with a distribution that is close enough. Perhaps the simplest Markov chains for this is the *Glauber dynamics*. In every step of the Glauber dynamics, if we are in the state φ, we choose uniformly at random a vertex v of the graph, and 'forget' its colour. That is, we condition the Potts model on the colouring $\varphi|_{V\setminus\{v\}}$. The distribution of the colour on the vertex v is now easily computed, it only depends on the colours on the neighbours of v. To finish the step, we choose a new colour for v according to this conditional distribution.

The Glauber dynamics indeed has the Potts model as its stationary distribution, and will always converge towards it. So the main question is the speed of this convergence. It turns out that for bounded degree graphs, with q large enough and y close to 1, the convergence is fast enough that we can find a polynomial-time approximation algorithm for the partition function that succeeds with probability 3/4 [13].[3] Note that this algorithm uses randomness, and thus has a probability of failing.

Heuristically we can understand why y should not be very large.[4] In that case, the colourings with many monochromatic edges are preferred, and those with

[3] The exact probability is not important. Running this algorithm multiple times and taking the median, it is possible to achieve arbitrarily high success probabilities.

[4] In [13, 15] they even show slow convergence for large y on some class of graphs.

only a single colour have the highest weight. If the Glauber dynamics converges towards the Potts model quickly, it should at least be possible to transition between these single-colour states. The problem is that the intermediate states have at least both colours, so much fewer monochromatic edges and hence a much smaller weight. If y is very close to 0, similar situations can occur. For example, when $q = 2$ and the graph is bipartite, there exist two proper colourings, and it is difficult to transition between those two.

With this limitation on the Glauber dynamics for large y, it is interesting to find alternative ways of sampling the Potts model for such y. If G happens to be a planar graph, there is a way to do this. Take x such that $q = (x-1)(y-1)$, then $Z(G; q, y)$ and $Z(G^*; q, x)$ differ only by a trivial prefactor, where G^* is the planar dual of G. If y is very large, this means that x is close to 1, so we can estimate $Z(G^*; q, x)$ by using the Glauber dynamics.

In Chapter 4 we extend this idea to non-planar graphs. For any graph G, the partition function $Z(G; q, y)$ is, up to a prefactor, equal to another partition function with x as parameter. This new partition function has *flows* as its states. We define the notion of a flow and this partition function in section 1.3.

In the results for y close to 1, the graphs needed to have bounded degree. Similarly in our results for large y, we have to put restrictions on the graphs, but now on the cycles in the graphs. The exact requirements are somewhat technical, and will be explained in Chapter 4, with the main result being Theorem 4.2. One example of graphs to which our results apply, are 'simply connected' subgraphs of the lattice $\mathbb{Z}^d$.

1.3 Potts model preliminaries

In this section we collect several equivalent forms of the partition function of the Potts model, and give the proofs that they are equivalent. All these results, and in fact much more, can also be found in [27].

Given a graph G and an edge e, we can construct new graphs by either deleting e or contracting e. The *deletion* of e is denoted by $G \setminus e$ and arises simply from deleting e from the edge set. The *contraction* of e is denoted by G/e, for this we first delete the edge e, and next identify its endpoints into a single vertex. Note that this can introduce multiple edges between the same pair of vertices. The chromatic polynomial, as well as the Potts model and random cluster partition function, satisfy the so-called *deletion-contraction* relation:

$$Z(G; q, y) = Z(G \setminus e; q, y) + (y-1)Z(G/e; q, y).$$

This relation is the basis for the most well-known equivalent form of the random cluster partition function, the *Tutte polynomial* $T(G; x, y)$. This is the universal polynomial that satisfies the deletion-contraction relation $T(G) = T(G \setminus e) + T(G/e)$ for any edge e which is not a loop or a bridge. (A bridge is an edge e such that the deletion $G \setminus e$ has more connected components than G.) The base case is $T(G; x, y) = x^i y^j$ if the only edges of G are i bridges and j loops.

The last equivalent form we introduce is the *flow partition function.* To define a *flow* (also called a circulation), we first need to choose an orientation of all the edges. A flow is an assignment $f : E \to \mathbb{Z}/q\mathbb{Z}$ such that for every vertex the sum of $f(e)$ over all incoming edges e, is the same as the sum over all outgoing edges e. The partition function is now defined as follows:

$$Z_{\text{flow}}(G; q, x) = \sum_{\substack{f:E\to\mathbb{Z}/q,\\ \text{flow}}} x^{\#\text{zero-valued edges in } f}.$$

Note that if we would reverse the orientation of one edge, we can retain the flow condition by multiplying the value on that edge with -1. Since the weight given to a flow in the partition function depends on the number of non-zero values and reversing edges does not change this weight, the chosen orientation is irrelevant for the value of the flow partition function. For this reason we will usually not specify an orientation of the input graph G, and allow ourselves in arguments to choose an arbitrary orientation.

Counting the number of flows, which equals $Z_{\text{flow}}(G; q, 1)$, is actually easy. In every component of the graph we choose a spanning tree, and on the remaining edges we can pick any value. Then there exists a unique way to complete this into a flow by choosing values on the edges of the spanning trees. This comes up to a total of $q^{k(E)-|V|+|E|}$ flows.

To prove the relation between all these different forms, we will use Z_{Potts} and Z_{RC} for respectively the Potts model and the random cluster partition function. The exact relation is captured in the following Proposition.

Proposition 1.1. *Assuming that $q = (x-1)(y-1)$, we have the following equalities:*

$$\begin{aligned} Z_{\text{Potts}}(G; q, y) = Z_{\text{RC}}(G; q, y) &= \tfrac{q^{|V|}}{(x-1)^{|E|}} \cdot Z_{\text{flow}}(G; q, x) \\ &= (y-1)^{|V|}(x-1)^{k(E)} \cdot T(G; x, y). \end{aligned}$$

Proof. The way we will prove this, is to relate all functions to Z_{RC}. For the relation between Z_{RC} and T we simply have to check the base cases and the deletion-contraction relation, which we leave as an exercise. Next we relate Z_{RC} and Z_{Potts}. Here we introduce the colourings by interpreting $q^{k(A)}$ as colouring every component of (V, A) with one of q colours:

$$\begin{aligned}
Z_{\mathrm{RC}}(G;q,y) &= \sum_{A\subseteq E} \sum_{\substack{\varphi:V\to[q],\\ \text{locally constant on } (V,A)}} (y-1)^{|A|} \\
&= \sum_{\varphi:V\to[q]} \sum_{\substack{A\subseteq E,\\ \text{monochromatic for } \varphi}} (y-1)^{|A|} \\
&= \sum_{\varphi:V\to[q]} y^{\#\text{monochromatic edges in } \varphi} \\
&= Z_{\mathrm{Potts}}(G;q,y).
\end{aligned}$$

The relation between Z_{RC} and Z_{flow} is proven in a similar way, but now we interpret $q^{k(A)-|V|+|A|}$ as the number of flows on (V, A):

$$\begin{aligned}
\tfrac{(x-1)^{|E|}}{q^{|V|}} \cdot Z_{\mathrm{RC}}(G;q,y) &= \sum_{A\subseteq E} q^{k(A)-|V|+|A|}(x-1)^{|E|-|A|} \\
&= \sum_{A\subseteq E} \sum_{\substack{f:A\to\mathbb{Z}/q,\\ \text{flow}}} (x-1)^{|E\setminus A|} \\
&= \sum_{\substack{f:E\to\mathbb{Z}/q,\\ \text{flow}}} \sum_{\substack{B\subseteq E,\\ f \text{ is zero on } B}} (x-1)^{|B|} \\
&= \sum_{\substack{f:E\to\mathbb{Z}/q,\\ \text{flow}}} x^{\#\text{zero-valued edges in } f} \\
&= Z_{\mathrm{flow}}(G;q,x).
\end{aligned}$$

□

1.4 Organisation of the thesis

Each of the three following chapters is based on a single paper. As such, they can be read independently. Here follows a short technical summary of every chapter, together with a reference to the paper on which the chapter is based.

Chapter 2 In this chapter we consider the zeros of the chromatic polynomial of series-parallel graphs. Complementing a result of Sokal, showing density outside the disk $|q-1| \leq 1$, we show density of these zeros in the half plane $\Re(q) > 3/2$ and we show there exists an open region U containing the interval $(0, 32/27)$ such that $U \setminus \{1\}$ does not contain zeros of the chromatic polynomial of series-parallel graphs.

We also disprove a conjecture of Sokal by showing that for each large enough integer Δ there exists a series-parallel graph for which all vertices but one have degree at most Δ and whose chromatic polynomial has a zero with real part exceeding Δ.

This chapter is based on [8], two extra sections have been added answering questions posed in the paper.

Chapter 3 We show that for any non-real algebraic number q such that $|q - 1| > 1$ or $\Re(q) > \frac{3}{2}$ it is #P-hard to compute a multiplicative (resp. additive) approximation to the absolute value (resp. argument) of the chromatic polynomial evaluated at q on planar graphs. This implies #P-hardness for all non-real algebraic q on the family of all graphs. We moreover prove several hardness results for q such that $|q - 1| \leq 1$.

Our hardness results are obtained by showing that a polynomial time algorithm for *approximately* computing the chromatic polynomial of a planar graph at non-real algebraic q (satisfying some properties) leads to a polynomial time algorithm for *exactly* computing it, which is known to be hard by a result of Vertigan. Many of our results extend in fact to the more general partition function of the random cluster model.

This chapter is based on [7].

Chapter 4 In this chapter we consider the algorithmic problem of sampling from the Potts model and computing its partition function at low temperatures. Instead of directly working with colourings, we consider the equivalent problem of sampling flows. We show, using path coupling, that a simple and natural Markov chain on the set of flows is rapidly mixing. As a result we find a δ-approximate sampling algorithm for the Potts model at low enough temperatures, whose running time is bounded by $O(m^2 \log(m\delta^{-1}))$ for graphs G with m edges.

This chapter is based on [43].

To each of these papers, all authors contributed equally.

Chapter 2

Chromatic zeros of series-parallel graphs

2.1 Introduction

Recall that the chromatic polynomial of a graph $G = (V, E)$ is defined as

$$Z(G; q) := \sum_{A \subseteq E} (-1)^{|A|} q^{k(A)},$$

where $k(A)$ denotes the number of components of the graph (V, A). We call a number $q \in \mathbb{C}$ a *chromatic zero* if there exists a graph G such that $Z(G; q) = 0$.

About twenty years ago Sokal [66] proved that the set of chromatic zeros of all graphs is dense in the entire complex plane. In fact, he only used a very small family of graphs to obtain density. In particular, he showed that the chromatic zeros of all generalized theta graphs (parallel compositions of equal length paths) are dense outside the disk $\overline{B}(1, 1)$. (We denote for $c \in \mathbb{C}$ and $r > 0$ by $\overline{B}(c, r)$ the closed disk centered at c of radius r.) Extending this family of graphs by taking the disjoint union of each generalized theta graph with an edge and connecting the endpoints of this edge to all other vertices, he then obtained density in the entire complex plane.

As far as we know it is still open whether the chromatic zeros of all planar graphs or even series-parallel graphs are dense in the complex plane. Motivated by this question and Sokal's result we investigate in the present chapter what happens inside the disk $\overline{B}(1, 1)$ for the family of series-parallel graphs. See Section 2.2 for a formal definition of series-parallel graphs. Our first result implies that the chromatic zeros of series-parallel are *not* dense in the complex plane.

Theorem 2.1. *There exists an open set U containing the open interval $(0, 32/27)$ such that $Z(G; q) \neq 0$ for any $q \in U \setminus \{1\}$ and for all series-parallel graphs G.*

We note that the interval $(0, 32/27)$ is tight, as shown in [44, 69]. In fact, Jackson [44] even showed that there are no chromatic zeros in the interval $(1, 32/27)$. Unfortunately, we were not able to say anything about larger families of graphs.

In fact, Theorem 2.1 does not hold for planar graphs, because there are planar graphs whose chromatic roots approach 1, see Section 2.6.

In terms of chromatic zeros of series-parallel graphs inside the disk $\overline{B}(1,1)$ we have found an explicit condition, Theorem 2.16 below, that allows us to locate many zeros inside this disk. Concretely, we have the following results.

Theorem 2.2. *Let $q > 32/27$. Then there exists q' arbitrarily close to q and a series-parallel graph G such that $Z(G; q') = 0$.*

This result may be seen as a a variation on Thomassen's result [69] saying that real chromatic zeros (of not necessarily series-parallel graphs) are dense in $(32/27, \infty)$.

Another result giving many zeros inside $\overline{B}(1,1)$ is the following.

Theorem 2.3. *The set of chromatic zeros of all series-parallel graphs is dense in the set $\{q \mid \Re(q) > 3/2\}$.*

After inspecting our proof of Theorem 2.3 (given in Section 2.4) it is clear that one can obtain several strengthenings of this result. Figure 2.1 shows a computer generated picture displaying where chromatic zeros of series-parallel graphs can be found as well as the zero-free region from Theorem 2.1.

We next restrict our attention to a subclass of series-parallel graphs. A *leaf joined tree* is a graph $\widehat{T}$ obtained from a rooted tree (T, v) by identifying all its leaves except possibly v into a single vertex. A while ago Sokal conjectured [67, Conjecture 9.5'] that for each integer $\Delta \geq 3$ the chromatic zeros of all graphs all of whose vertices have degree at most Δ except possibly one vertex are contained in the half plane $\{q \mid \Re(q) \leq \Delta\}$. This conjecture was disproved by Royle for $\Delta = 3$, as Sokal mentions in footnote 31 in [67]. Here we show that this is no coincidence, as we disprove this conjecture for all Δ large enough.

Theorem 2.4. *There exists $\Delta_0 > 0$ such that for all integers $\Delta \geq \Delta_0$ there exists a leaf joined tree $\widehat{T}$ obtained from a tree T of maximum degree Δ such that $\widehat{T}$ has a chromatic zero q with $\Re(q) > \Delta$.*

The proof of this theorem, together with some explicit calculations, also allows us to find such chromatic zeros for $4 \leq \Delta \leq 45$. Table 2.1 in Section 2.6 records values of q, which are accumulation points of chromatic zeros of leaf joined trees, corresponding with the given Δ.

2.1.1 Approach

Very roughly the main tool behind the proofs of our results is to write the chromatic polynomial $Z(G; q)$ as the sum of two other polynomials $Z_1(G; q) + Z_2(G; q)$ which can be iteratively computed for all series-parallel graphs, see Section 2.2 for

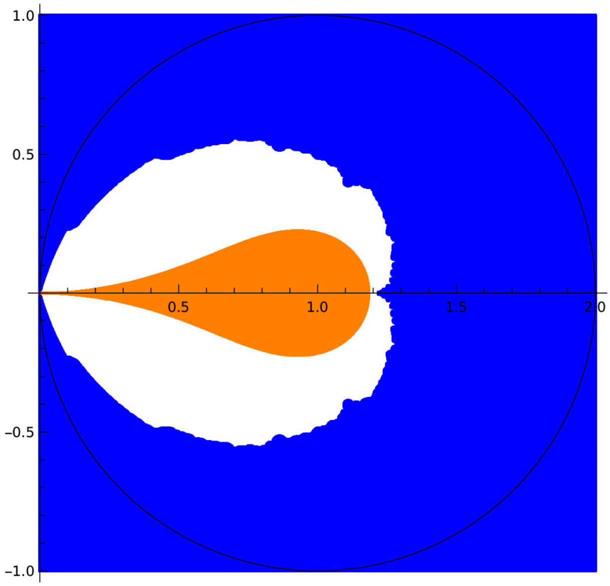

Figure 2.1: A pixel-picture of chromatic zeros and zero-free regions for series-parallel graphs, with a resolution of 1001 × 1001 pixels. Every orange pixel represent a provably zero-free value of q, while every blue pixel represents a value of q in the closure of the set of all chromatic zeros of series-parallel graphs. The region depicted in the picture ranges from $-i$ to $2+i$.

the precise definitions. We also define the rational function $R(G;q) := \frac{Z_1(G;q)}{Z_2(G;q)}$ and clearly $R(G;q) = -1$ implies $Z(G;0) = 0$. A certain converse also holds under some additional conditions.

To prove Theorem 2.1 we essentially show that these rational functions avoid the value -1. To prove presence of zeros we use that if the family rational functions $\{q \mapsto R(G;q)\}$ behaves chaotically (in some precise sense defined in Section 2.4) near some parameter q_0, then one can use the celebrated Montel theorem from complex analysis to conclude that there must be a nearby value q and a graph G for which $Z(G,q) = 0$.

Our approach to obtaining density of chromatic zeros is similar in spirit to Sokal's approach [66], but deviates from it in the use of Montel's theorem. Sokal uses Montel's 'small' theorem to prove the Beraha-Kahane-Weis theorem [9], which he is able to apply to the generalized theta graphs because their chromatic polynomials can be very explicitly described. It is not clear to what extent this applies to more complicated graphs. Our use of Montel's theorem is however directly inspired by [23], which in turn builds on [62, 10, 17]. Our approach in fact also allows us to give a relatively short alternative proof for density of chromatic zeros of generalized theta graphs outside the disk $\overline{B}(1,1)$, see Corollary 2.25.

Our proof of Theorem 2.4 makes use of an observation of Sokal and Royle in the appendix of the arXiv version of [64] (see `https://arxiv.org/abs/1307.1721`), saying that a particular recursion for ratios of leaf joined trees is up to a conjugation exactly the recursion for ratios of independence polynomial on trees. We make use of this observation to build on the framework of [23] allowing us to utilize some very recent work [6] giving an accurate description of the location of the zeros of the independence polynomial for the family of graphs with a given maximum degree.

Organization

The next section deals with formal definitions of series-parallel graphs and ratios. We also collect several basic properties there that are used in later sections. Section 2.3 is devoted to proving Theorem 2.1. In Section 2.4 we state a general theorem allowing us to derive various results on presence of chromatic zeros for series-parallel graphs. Finally in Section 2.5 we prove Theorem 2.4. We end the chapter with some questions in Section 2.6

2.2 Recursion for ratios of series-parallel graphs

We start with some standard definitions needed to introduce, and set up some terminology for series-parallel graphs. We follow Royle and Sokal [64] in their use of notation.

Let G_1 and G_2 be two graphs with designated start- and endpoints s_1, t_1, and s_2, t_2 respectively, referred to as *two-terminal graphs*. The *parallel composition* of G_1 and G_2 is the graph $G_1 \parallel G_2$ with designated start- and endpoints s, t obtained from the disjoint union of G_1 and G_2 by identifying s_1 and s_2 into a single vertex s and by identifying t_1 and t_2 into a single vertex t. The *series composition* of G_1 and G_2 is the graph $G_1 \bowtie G_2$ with designated start- and endpoints s, t obtained from the disjoint union of G_1 and G_2 by identifying t_1 and s_2 into a single vertex and by renaming s_1 to s and t_2 to t. Note that the order matters here. A two-terminal graph G is called *series-parallel* if it can be obtained from a single edge using series and parallel compositions. From now on we will implicitly assume the presence of the start- and endpoints when referring to a two-terminal graph G. We denote by $\mathcal{G}_{\mathrm{SP}}$ the collection of all series-parallel graphs and by $\mathcal{G}^*_{\mathrm{SP}}$ the collection of all series-parallel graphs G such that the vertices s and t are not connected by an edge.

Recall that for a positive integer q and a graph $G = (V, E)$ we have

$$Z(G; q) = \sum_{\varphi: V \to \{1,\dots,q\}} \prod_{uv \in E} (1 - \delta_{\varphi(u),\varphi(v)}),$$

where $\delta_{i,j}$ denotes the Kronecker delta. For a positive integer q and a two-terminal graph G, we can thus write[1],

$$Z(G; q) = Z^{\text{same}}(G; q) + Z^{\text{dif}}(G; q), \tag{2.1}$$

where $Z^{\text{same}}(G; q)$ collects those contribution where s, t receive the same color and where $Z^{\text{dif}}(G; q)$ collects those contribution where s, t receive the distinct colors. Since $Z^{\text{dif}}(G; q)$ is equal to $Z(G \parallel K_2; q)$, where K_2 denotes an edge, both these terms are polynomials in q. Therefore (2.1) also holds for any $q \in \mathbb{C}$.

We next collect some basic properties of Z, Z^{same} and Z^{dif} under series and parallel compositions in the lemma below. They can for example also be found in [66].

Lemma 2.5. *Let G_1 and G_2 be two two-terminal graphs and let us denote by K_2 an edge. Then we have the following identities:*

$$\begin{array}{llcl}
\text{(P1)} & Z^{\text{dif}}(G; q) & = & Z(G \parallel K_2; q), \\
\text{(P2)} & Z^{\text{same}}(G_1 \bowtie G_2; q) & = & Z(G_1 \parallel G_2; q), \\
\text{(P3)} & Z(G_1 \bowtie G_2; q) & = & \frac{1}{q} \cdot Z(G_1; q) \cdot Z(G_2; q), \\
\text{(P4)} & Z^{\text{same}}(G_1 \parallel G_2; q) & = & \frac{1}{q} \cdot Z^{\text{same}}(G_1; q) \cdot Z^{\text{same}}(G_2; q), \\
\text{(P5)} & Z^{\text{dif}}(G_1 \parallel G_2; q) & = & \frac{1}{q(q-1)} \cdot Z^{\text{dif}}(G_1; q) \cdot Z^{\text{dif}}(G_2; q), \\
\text{(P6)} & Z^{\text{same}}(G_1 \bowtie G_2; q) & = & \frac{1}{q} \cdot Z^{\text{same}}(G_1; q) \cdot Z^{\text{same}}(G_2; q) \\
 & & + & \frac{1}{q(q-1)} \cdot Z^{\text{dif}}(G_1; q) \cdot Z^{\text{dif}}(G_2; q), \\
\text{(P7)} & Z^{\text{dif}}(G_1 \bowtie G_2; q) & = & \frac{1}{q} \cdot Z^{\text{same}}(G_1; q) \cdot Z^{\text{dif}}(G_2; q) \\
 & & + & \frac{1}{q} \cdot Z^{\text{dif}}(G_1; q) \cdot Z^{\text{same}}(G_2; q) \\
 & & + & \frac{q-2}{q(q-1)} \cdot Z^{\text{dif}}(G_1; q) \cdot Z^{\text{dif}}(G_2; q).
\end{array}$$

An important tool in our analysis of absence/presence of complex zeros is the use of the *ratio* defined as

$$R(G; q) := \frac{Z^{\text{same}}(G; q)}{Z^{\text{dif}}(G; q)},$$

[1] This can be seen to be the deletion-contraction relation for $G \parallel K_2$ with $Z^{\text{dif}}(G; q) = Z(G \parallel K_2; q)$.

which we view as a rational function in q. We note that in case G contains an edge between s and t, the rational function $q \mapsto R(G;q)$ is constantly equal to 0. We observe that if $R(G;q) = -1$, then $Z(G;q) = 0$ and the converse holds provided $Z^{\mathrm{dif}}(G;q) \neq 0$.

The next lemma provides a certain strengthening of this observation for series-parallel graphs.

Lemma 2.6. *Let $q \in \mathbb{C} \setminus \{0,1,2\}$. Then the following are equivalent*

(i) $Z(G;q) = 0$ *for some* $G \in \mathcal{G}_{\mathrm{SP}}$*,*

(ii) $R(G;q) = -1$ *for some* $G \in \mathcal{G}^*_{\mathrm{SP}}$*,*

(iii) $R(G;q) \in \{0,-1,\infty\}$ *for some* $G \in \mathcal{G}^*_{\mathrm{SP}}$*.*

Proof. Throughout the proof we will refer to the properties stated in Lemma 2.5 without explicitly mentioning the lemma each time.

We start with '(i) $\Rightarrow$ (ii)'. Let q be as in the statement of the lemma such that $Z(G;q) = 0$ for some series-parallel graph $G \in \mathcal{G}_{\mathrm{SP}}$. Take such a graph G with as few edges as possible.

By the above we may assume that $Z^{\mathrm{dif}}(G;q) = 0$, for otherwise $R(G;q) = -1$ (and hence $G \in \mathcal{G}^*_{\mathrm{SP}}$). Then also $Z^{\mathrm{same}}(G;q) = 0$.

Suppose first that s,t are not connected by an edge. By minimality, (P3) and (P4), G must be the parallel composition of two series-parallel graphs G_1 and G_2 such that, say $Z^{\mathrm{same}}(G_1,q) = 0$ and G_1 is not 2-connected, or in other words such that G_1 is a series composition of two smaller series-parallel graphs G_1' and G_1''. By (P2) we have that $Z(G_1' \parallel G_1'';q) = 0$. This is a contradiction since $G_1' \parallel G_1''$ has fewer edges than G. We conclude that $R(G;q) = -1$ in this case.

Suppose next that s and t are connected by an edge. We shall show that we can find another series-parallel graph $\widehat{G} \in \mathcal{G}^*_{\mathrm{SP}}$, that is isomorphic to G as a graph (and hence has q as zero of its chromatic polynomial) but not as two-terminal graph. By the argument above we then have $R(\widehat{G};q) = -1$.

Let G' be obtained from G by removing the edge $\{s,t\}$. Then by (P1) $Z^{\mathrm{dif}}(G';q) = Z(G;q) = 0$. If $Z^{\mathrm{same}}(G';q) = 0$, then $Z(G';q) = 0$, contradicting the minimality of G. Therefore $Z^{\mathrm{same}}(G';q) \neq 0$. If G' is the parallel composition of G_1 and G_2, then by (P5),

$$Z^{\mathrm{dif}}(G_1;q)Z^{\mathrm{dif}}(G_2;q) = q(q-1)Z^{\mathrm{dif}}(G';q) = 0,$$

so there is a smaller graph, (namely $G_1 \parallel K_2$ or $G_2 \parallel K_2$), where q is a zero, contradicting our choice of G. Hence G' is the series composition of two graphs G_1 and G_2. The graphs G_1 and G_2 cannot both be single edges, for otherwise G would be a triangle and we excluded the values $q = 0,1,2$. So let us assume that

G_1 is not a single edge. We will now construct G in a different way as series-parallel graph. First switch the roles of s_2 and t_2 in G_2 and denote the resulting series-parallel graph by G_2^T. Then put G_2^T in series with a single edge, and then put this in parallel with G_1. In formulas this reads as $\widehat{G} := (K_2 \bowtie G_2^T) \parallel G_1$. The resulting graph $\widehat{G}$ is then isomorphic to G (but not equal to G as a two-terminal graph). In case $\widehat{G}$ is not contained in $\mathcal{G}^*_{SP}$, then G_1 is also not in $\mathcal{G}^*_{SP}$. In that case let G_2' be obtained from G_2 by first taking a series composition with an edge and then a parallel composition with an edge, that is, $G_2' = (K_2 \bowtie G_2^T) \parallel K_2$. We then have by (P1) and (P5),

$$\begin{aligned} Z(G;q) = Z(\widehat{G};q) = Z^{\text{dif}}(G;q) &= \tfrac{1}{q(q-1)} Z^{\text{dif}}(G_1;q) Z^{\text{dif}}(K_2 \bowtie G_2^T;q) \\ &= \tfrac{1}{q(q-1)} Z(G_1;q) Z(G_2';q), \end{aligned}$$

So q must be a zero of $Z(G_1;q)$, or of $Z(G_2';q)$. Because G_1 is not an edge, both G_1 and G_2' contain fewer edges than G contradicting the choice of G. Hence we conclude that $\widehat{G}$ is contained in $\mathcal{G}^*_{SP}$, finishing the proof of the first implication.

The implication '(ii) $\Rightarrow$ (iii)' is obvious. So it remains to show '(iii) $\Rightarrow$ (i)'.

To this end suppose that $R(G;q) \in \{-1, 0, \infty\}$ for some series-parallel graph $G \in \mathcal{G}^*_{\text{SP}}$. If the ratio equals -1, then clearly $Z(G;q) = 0$. So let us assume that the ratio equals 0. Then $Z^{\text{same}}(G;q) = 0$ and we may assume that $Z^{\text{dif}}(G;q) \neq 0$, for otherwise $Z(G;q) = Z^{\text{same}}(G;q) + Z^{\text{dif}}(G;q) = 0$. Let us take such a graph G with the smallest number of edges. By minimality, G cannot arise as the parallel composition of two series-parallel graphs G_1 and G_2 by (P4) and (P5). Therefore G must be equal to the series composition of two series-parallel graphs G_1 and G_2. Now, as in the proof of '(i) $\Rightarrow$ (ii)', identify vertices s and t of G to form a new series-parallel graph G', such that $Z(G';q) = Z^{\text{same}}(G;q) = 0$.

Let us finally consider the case that the ratio is equal to ∞. In this case $Z^{\text{dif}}(G;q) = 0$. Then by (P1), $Z(G \parallel K_2;q) = Z^{\text{dif}}(G;q) = 0$ and we are done. $\square$

We next provide a description of the behavior of the ratios under the series and parallel compositions. To simplify the calculations, we will look at the the modified ratio

$$y_G(q) := (q-1)R(G;q), \tag{2.2}$$

which, loosely following Sokal [66], we call the *effective edge interaction.*

Remark 2.7. Observe that $y_G(q)$ cannot be equal to any of the functions $q \mapsto -1, q \mapsto \infty$ and $q \mapsto 1-q$, since the numerator, $(q-1)Z^{\text{same}}(G;q)$, and the denominator, $Z^{\text{dif}}(G;q)$, have the same degree and leading coefficient, unless G has an edge connecting s and t, in which case $y_G(q)$ is the constant 0 function.

Given $q_0 \in \mathbb{C}$ define

$$\mathcal{E}(q_0) := \{y_G(q_0) \mid G \in \mathcal{G}_{SP}\}, \tag{2.3}$$

the set of all values of the effective edge interaction at q_0 for the family of series-parallel graphs as a subset of the Riemann sphere, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. As an example note that $0 \in \mathcal{E}(q_0)$ for any q_0, being the effective edge interaction of a single edge.

For any $q \neq 0$ define the following Möbius transformation[2]

$$f_q(z) := 1 + \frac{q}{z-1}$$

and note that f_q is an involution, i.e. $f_q(f_q(z)) = z$ for all z.

The next lemma captures the behavior of the effective edge interactions under series and parallel compositions and can be easily derived from Lemma 2.5.

Lemma 2.8. *Let G_1, G_2 be two two-terminal graphs. Then*

$$\begin{aligned} y_{G_1 \| G_2} &= y_{G_1} y_{G_2}, \\ y_{G_1 \bowtie G_2} &= f_q(f_q(y_{G_1}) f_q(y_{G_2})). \end{aligned}$$

Moreover, for any fixed $q_0 \in \mathbb{C}$, if $\{y_{G_1}(q_0), y_{G_2}(q_0)\} \neq \{0, \infty\}$, then

$$y_{G_1 \| G_2}(q_0) = y_{G_1}(q) y_{G_2}(q_0);$$

and if $\{y_{G_1}(q_0), y_{G_2}(q_0)\} \neq \{1, 1-q_0\}$ and $q_0 \neq 0$, then

$$y_{G_1 \bowtie G_2}(q_0) = f_{q_0}(f_{q_0}(y_{G_1}(q_0)) f_{q_0}(y_{G_2}(q_0))).$$

We include a proof of the lemma for convenience of the reader.

Proof. First of all we note that the product $y_{G_1} y_{G_2}$ is always a well-defined rational function. By Remark 2.7, $f_q(y_{G_i})$ cannot be constant 0, but could be constant ∞. Therefore the product $f_q(y_{G_1}) f_q(y_{G_2})$ could be constant ∞, but applying f_q once more to it results again in a well-defined rational function.

The statements for the parallel connections follow directly from (P4) and (P5) from Lemma 2.5 and the definition of the effective edge interaction. For the statements for the series connections let us denote $y_1 = y_{G_1}, y_2 = y_{G_2}$ and $y_{\text{ser}} = y_{G_1 \bowtie G_2}$. We use (P6) and (P7) from Lemma 2.5 to write $y_{\text{ser}} = \frac{y_1 y_2 + q - 1}{y_1 + y_2 + q - 2}$.

[2] Readers familiar with the Tutte polynomial will recognize this formula as expressing the x-coordinate from the y-coordinate (or the other way around) on the hyperbola $(x-1)(y-1) = q$, on which, for positive integer q, the Tutte polynomial corresponds to the q-state Potts model partition function.

It is then not difficult to see that $f_q(y_{\text{ser}}) = f_q(y_1)f_q(y_2)$. Therefore, since f_q is an involution,

$$y_{\text{ser}} = f_q(f_q(y_{\text{ser}})) = f_q(f_q(y_1)f_q(y_2)),$$

as desired. The statements for the evaluation at a fixed value $q_0 \in \mathbb{C}$ now follow directly. $\square$

Remark 2.9. Note that this lemma allows us to compute the effective edge interaction of any series-parallel graph. For example, the effective edge interaction of the path on three vertices, P_2, can be computed as

$$y_{P_2} = y_{K_2 \bowtie K_2} = f_q(f_q(0)^2) = f_q((1-q)^2) = \frac{q-1}{q-2}.$$

2.3 Absence of zeros near $(0, 32/27)$

In this section we prove Theorem 2.1. In the proof we will use the following condition that guarantees absence of zeros and check this condition in three different regimes. We first need a few quick definitions.

For a set $S \subseteq \mathbb{C}$, denote $S^2 := \{s_1 s_2 \mid s_1, s_2 \in S\}$. For subsets S, T of the complex plane, we use the notation $S \subsetneq T$ (and say S is strictly contained in T) to say that the closure of S is contained in the interior of T. For $r > 0$ we define $\overline{B}_r \subseteq \mathbb{C}$ to be the closed disk of radius r centered at 0.

Lemma 2.10. *Let $q \in \mathbb{C}\backslash\{0, 1, 2\}$ and assume there exists a set $V \subseteq \mathbb{C}$ satisfying: $0 \in V$, $1-q \notin V^2$, $V^2 \subseteq V$ and $f_q(f_q(V)^2) \subseteq V$. Then $Z(G; q) \neq 0$ for all series-parallel graphs G.*

Proof. By Lemma 2.6 it suffices to show that the ratios avoid the point -1. Or equivalently, since $q \neq 1$, that the effective edge interactions at q avoid the point $1 - q$.

We will do so by proving the following stronger statement:

$$\mathcal{E}(q) \subseteq V \text{ and } 1 - q \notin \mathcal{E}(q). \tag{2.4}$$

We show this by induction on the number of edges. The base case follows since $0 \in V$ and $q \neq 1$. Assume next that $y \in \mathcal{E}(q) \setminus \{0\}$ and suppose that y is the effective edge interaction of some series-parallel graph G. If G is the parallel composition of two series-parallel graphs G_1 and G_2 with effective edge interactions y_1 and y_2 respectively, then, by induction, $y_1, y_2 \in V$ and neither of them is equal to $1 - q$. By Lemma 2.8 and our assumption we have $y = y_1 y_2 \in V^2 \subseteq V$. Since $1-q \notin V^2$, we also have that $y \neq 1 - q$. If G is the series composition of two series-parallel graphs G_1 and G_2 with effective edge interactions y_1 and y_2 respectively, then, by induction, $y_1, y_2 \in V$ and neither of them is equal to $1-q$. Therefore $f_q(y_i) \neq 0$ for $i = 1, 2$. Then by Lemma 2.8 and our assumption, $y = f_q(f_q(y_1)f_q(y_2)) \in V$. Moreover, $f_q(1-q) = 0 \neq f_q(y_1)f_q(y_2) = f_q(y)$. Therefore $y \neq 1-q$. This shows (2.4) and finishes the proof.

$\square$

Below we prove three lemmas allowing us to apply the previous lemma to different parts of the interval $(0, 32/27)$. First we collect two useful tools. For two complex numbers a, b we denote by $C(a, b)$ the circle in the complex plane with the line segment between a and b as a diameter. In case $a = b$, $C(a, b)$ consists of the single point $\{a\}$.

Lemma 2.11. *Let $q, r \in \mathbb{R}$, then the circle $C(r, f_q(r))$ is f_q-invariant.*

Proof. First note that f_q maps the real line to itself, because q is real. Now let $C = C(r, f_q(r))$. Then C intersects the real line at right angles. The Möbius transformation f_q sends C to a circle through $f_q(r), f_q(f_q(r)) = r$, and because f_q is conformal the image must again intersect the real line at right angles. Therefore $f_q(C) = C$. $\square$

Proposition 2.12. *Let $V \subseteq \mathbb{C}$ be a disk. Then*

$$V^2 = \{y^2 \mid y \in V\}.$$

Proof. Obviously the second is contained in the first. The other inclusion is an immediate consequence of the Grace-Walsh-Szegő theorem. $\square$

Now we can get into the three lemmas mentioned.

Lemma 2.13. *For each $q \in (0, 1)$ there exists a closed disk $V \subseteq \mathbb{C}$ strictly contained in $\overline{B}_{\sqrt{1-q}}$, satisfying $0 \in V$, $f_q(V) = V$ and $V^2 \subsetneq V$.*

Proof. Let $r = \sqrt{1-q}$ and choose real numbers $a \in (r^2, r), b \in (-r, -r^2)$ with $f_q(a) = b$. They exist because $f_q(r) = -r$ and $f_q'(r) = \frac{-q}{(1-r)^2} < 0$. Let V be the closed disk with diameter the line segment between a and b. Clearly $V \subsetneq \overline{B}_r$ and $0 \in V$. From Lemma 2.11 it follows that the boundary of V is mapped to itself. Further, the interior point $0 \in V$ is mapped to $f_q(0) = 1 - q = r^2$ which is also an interior point of V. Therefore $f_q(V) = V$. Last, we see that $V^2 \subseteq \overline{B}_r^2 = \overline{B}_{r^2} \subsetneq V$, confirming all properties of V. $\square$

Lemma 2.14. *For each $q \in (1, 32/27)$ there exists a closed disk $V \subseteq \mathbb{C}$ strictly contained in $\overline{B}_{\sqrt{q-1}}$ satisfying $0 \in V$, $f_q(V) = V$ and $V^2 \subsetneq V$.*

Proof. The equation $f_q(z) = z^2$ has a solution in $(-1/3, 0)$, since $f_q(0) = 1-q < 0$ and $f_q(-1/3) = 1 - 3q/4 > 1/9$. Denote one such solution as r. Then we see that

$$f_q'(r) = \frac{-q}{(r-1)^2} = -r - 1 < 2r = [z^2]'_{z=r}, \tag{2.5}$$

and

$$q - 1 = r^3 - r^2 - r > -\tfrac{1}{3}r^2 - r^2 + 3r^2 > r^2. \tag{2.6}$$

Since $f_q(r) = r^2 < -r$, it follows that for $t \in (-1/3, r)$ close enough to r we have $f_q(t) < -t$, $t^2 < f_q(t)$ by (2.5) and $t > -\sqrt{q-1}$ by (2.6). Fix such a value of t and let V be the closed disk with diameter the line segment between t and $f_q(t)$. The exterior point ∞ is now mapped to the exterior point 1, so by Lemma 2.11 we then know that $f_q(V) = V$. By construction we have that

$$V^2 \subseteq B_t^2 = \overline{B}_{t^2} \subsetneq \overline{B}_{f_q(t)} \subseteq V$$

and so V satisfies the desired properties. $\square$

Lemma 2.15. *There exists an open neighborhood I around 1 such that for each $q \in I \setminus \{1\}$ there exists a disk $V \subseteq \mathbb{C}$, satisfying $0 \in V$, $1 - q \notin V^2$, $V^2 \subseteq V$ and $f_q(f_q(V)^2) \subseteq V$.*

Proof. Let $R = \sqrt{|1-q|}$. We claim that if R is sufficiently small, there exists an $0 < s < R$ such that $V = \overline{B}_s$ satisfies the required conditions. Actually, we will show this to be true with $R < 2 - \sqrt{3}$, thus giving for I the open disk $|q - 1| < 7 - 4\sqrt{3}$.

Trivially, $0 \in V, 1 - q \notin V^2$ and $V^2 \subseteq V$, so we only need to show that $f_q(f_q(V)^2) \subseteq V$, or equivalently $f_q(V)^2 \subseteq f_q(V)$.

We start with bounding the image of the disk $\overline{B}_s$:

$$\begin{aligned}
f_q(\overline{B}_s) &= \left\{ \frac{y+q-1}{y-1} \;\middle|\; y \in \overline{B}_s \right\} \\
&\subseteq \left\{ \frac{y+q'-1}{y'-1} \;\middle|\; y, y' \in \overline{B}_s, q' \in \overline{B}_{R^2}(1) \right\} \\
&\subseteq \left\{ \frac{z}{y'-1} \;\middle|\; y' \in \overline{B}_s, z \in \overline{B}_{R^2+s} \right\} \\
&\subseteq \left\{ z \;\middle|\; |z| \le \frac{R^2+s}{1-s} \right\}.
\end{aligned}$$

So if we define $\rho(s) = \frac{R^2+s}{1-s}$, then $f_q(\overline{B}_s) \subseteq \overline{B}_{\rho(s)}$. Since f_q is an involution, we have

$$\overline{B}_{\rho^{(-1)}(s)} \subseteq f_q(\overline{B}_s).$$

Now we claim that if $R < 2-\sqrt{3}$, then there exists $0 < s < R$ such that $\rho(s)^2 < \rho^{(-1)}(s)$. This is sufficient since for this value of s we have

$$f_q(\overline{B}_s)^2 \subseteq B^2_{\rho(s)} = \overline{B}_{\rho(s)^2} \subseteq \overline{B}_{\rho^{(-1)}(s)} \subseteq f_q(\overline{B}_s),$$

as desired.

We now prove the claim. As $0 < s < R < 1$, the inequality $\rho(s)^2 < \rho^{(-1)}(s) = \frac{s-R^2}{1+s}$ is equivalent to

$$(R^2+1)(3s^2+(R^2-1)s+R^2) < 0, \qquad\qquad 0 < s < R.$$

If we have a solution, then the quadratic polynomial in the variable s should have 2 real solutions, since its main coefficient is positive. Since the linear term is negative and the constant term is positive, both roots are positive. Thus it is sufficient to prove that the "smaller" real root is less then R, i.e.

$$\frac{(1-R^2)-\sqrt{(1-R^2)^2-12R^2}}{6} < R.$$

This indeed holds true for $R < 2-\sqrt{3}$. $\square$

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. For every $q \in (0, 32/27)$ we will now find an open U around q, such that $U \setminus \{1\}$ does not contain chromatic zeros of series-parallel graphs. For $q = 1$ this follows directly from Lemmas 2.15 and 2.10. For $q \in (0,1)$ and $q \in (1, 32/27)$ we appeal to Lemmas 2.13 and 2.14 respectively to obtain a closed disk V with $V \subsetneq \overline{B}_{\sqrt{|1-q|}}$, $f_q(V) = V$ and $V^2 \subsetneq V$. We then claim that there is an open U around q, for which this disk V still satisfies the requirements of Lemma 2.10 for all $q' \in U$.
Certainly $0 \in V$ and $V^2 \subseteq V$ remain true. Because $V \subsetneq \overline{B}_{\sqrt{|1-q|}}$ holds, we can take U small enough such that $V \subseteq \overline{B}_{\sqrt{|1-q'|}}$ still holds, which confirms $1-q' \notin V^2$. Lastly, we know that $f_q(f_q(V)^2) = f_q(V^2) \subsetneq f_q(V) = V$. Because V is compact, and the function $y \mapsto f_{q'}(f_{q'}(y)^2)$ depends continuously on q', the inclusion $f_{q'}(f_{q'}(V)^2) \subsetneq V$ remains true on a small enough open U around q. $\square$

2.4 Activity and zeros

In this section we prove Theorems 2.2 and 2.3. We start with a theorem that gives a concrete condition to check for presence of chromatic zeros. For any $q \neq 0$ we call any $y \in f_q(\mathcal{E}(q))$ a *virtual interaction*. For example, $f_q(0) = 1-q$ is a virtual interaction (obtained from the effective edge interaction of a single edge).

Theorem 2.16. *Let $q_0 \in \mathbb{C} \setminus \{0\}$. If there exists either an effective edge interaction $y \in \mathcal{E}(q_0)$ or a virtual interaction $y \in f_{q_0}(\mathcal{E}(q_0))$ such that $1 < |y| < \infty$, then there exist q arbitrarily close to q_0 and $G \in \mathcal{G}_{\mathrm{SP}}$ such that $Z(G;q) = 0$.*

We will provide a proof for this result in section 2.4.1. Actually some converse of this statement is true, but this is not needed for our main results. The statement and proof are therefore postponed to section 2.4.2.

First we consider some corollaries. The first corollary recovers a version of Sokal's result [66].

Corollary 2.17. *Let $q \in \mathbb{C}$ such that $|1-q| > 1$. Then there exists q' arbitrarily close to q and $G \in \mathcal{G}_{\mathrm{SP}}$ such that $Z(G;q') = 0$.*

Proof. First of all note that as mentioned above, $y = f_q(0) = 1 - q$, is a virtual interaction (since 0 is the effective edge interaction of a single edge). By assumption we thus have a finite virtual interaction y such that $|y| > 1$. The result now directly follows from Theorem 2.16. □

Remark 2.18. Recall that a generalized theta graph is the parallel composition of a number of equal length paths. Sokal [66] in fact showed that we can take G in the corollary above to be a generalized theta graph. Our proof of Theorem 2.16 in fact also gives this. We will elaborate on this in Corollary 2.25 after giving the proof.

Our second corollary gives us Theorem 2.2.

Corollary 2.19. *Let $q > 32/27$. Then there exists q' arbitrarily close to q and $G \in \mathcal{G}_{\mathrm{SP}}$ such that $Z(G;q') = 0$.*

Proof. Consider the map $g(z) = f_q(z^2)$. We claim that $g(z) < z$ for any $z \in (-1, 0]$. As $g(0) = 1 - q < 0$, it is sufficient to show that $g(z) \neq z$ for any $z \in (-1, 0)$. Or equivalently,

$$q \neq (z-1)^2(z+1).$$

The maximal value of $(z-1)^2(z+1)$ on the interval $(-1, 0]$ is $32/27$ (which is achieved at $-1/3$), thus the claim holds.

We next claim that there exists k such that $g^{\circ k}(0) \leq -1$. Suppose not, then since the sequence $\{g^{\circ k}(0)\}_{k \geq 0}$ is decreasing it must have a limit L. By construction, $L \in [-1, 0]$ and it must be a fixed point of the map g. Since $\lim_{z \to -1^+} g(z) = -\infty$, it follows that g has no fixed points in $[-1, 0]$, a contradiction.

We also claim that $g^{\circ k}(0)$ is an element of $\mathcal{E}(q) \cup f_q(\mathcal{E}(q))$ for any integer $k \geq 0$. Indeed this follows by induction, the base case being $k = 0$. Assuming that $g^{\circ i}(0) \in \mathcal{E}(q)$ for some $i \geq 0$, it follows that $g^{\circ i}(0)^2 \in \mathcal{E}(q)$ by Lemma 2.8

and therefore $g^{\circ i+1}(0) \in f_q(\mathcal{E}(q))$. And similarly, if $g^{\circ i}(0) \in f_q(\mathcal{E}(q))$ for some $i \geq 0$, it follows that $g^{\circ i}(0) = f_q(y)$ for some $y \in \mathcal{E}(q)$ and hence by Lemma 2.8, $g^{\circ i+1}(0) = f_q(f_q(y)^2) \in \mathcal{E}(q)$.

To finish the proof, we choose $k \in \mathbb{N}$ such that $g^{\circ k}(0) \leq -1$. If the inequality is actually strict, so $g^{\circ k}(0) < -1$, the result now directly follows from Theorem 2.16, since $g^{\circ k}(0)$ is an element of $\mathcal{E}(q) \cup f_q(\mathcal{E}(q))$. If on the other hand $g^{\circ k}(0) = -1$, then $g^{\circ k+1}(0) = \infty$. For even k, we see that $g^{\circ k}(0)$ is an effective interaction. As a rational function of q, it cannot be constant -1 by Remark 2.7. So the value of $g^{\circ k}(0)$ for some q' arbitrarily close to q is outside the unit disk and we again apply Theorem 2.16. For odd k we see that $g^{\circ k+1}(0)$ is an effective interaction and cannot be constant ∞, again by Remark 2.7. Hence there again exists q' arbitrarily close to q where the value is finite and outside the unit disk and we again can apply Theorem 2.16. $\square$

Our next corollary gives us Theorem 2.3.

Corollary 2.20. *Let $q \in \mathbb{C}$ such that $\Re(q) > 3/2$. Then there exists q' arbitrarily close to q and $G \in \mathcal{G}_{\mathrm{SP}}$ such that $Z(G; q') = 0$.*

Proof. Consider the path P_2 of length 2, which is the series composition of two single edges. Therefore, by Lemma 2.8 its effective edge interaction is given by

$$f_q(f_q(0)^2) = f_q((1-q)^2) = \frac{q-1}{q-2}.$$

Now the Möbius transformation $q \mapsto \frac{q-1}{q-2}$ maps the half plane $\{z \mid \Re(z) \geq 3/2\}$ to the complement of the unit disk, since $\infty \mapsto 1$, $3/2 \mapsto -1$ and the angle that the image of $\{z \mid \Re(z) = 3/2\}$ makes with $\mathbb{R}$ at -1 is 90 degrees and since $0 \mapsto 1/2$. The result now directly follows from Theorem 2.16. $\square$

2.4.1 Proof of Theorem 2.16

We first introduce some definitions inspired by [23]. Let $\mathcal{G}$ be a family of two-terminal graphs. Let $q_0 \in \widehat{\mathbb{C}}$. Then we call q_0 *passive for* $\mathcal{G}$ if there exists an open neighborhood U around q_0 such that the family of ratios $\{q \mapsto R(G; q) \mid G \in \mathcal{G}\}$ is a normal family on U, that is, if any infinite sequence of ratios contains a subsequence that converges uniformly on compact subsets of U to a holomorphic function $f : U \to \widehat{\mathbb{C}}$. We call q_0 *active for* $\mathcal{G}$ is q_0 is not passive for $\mathcal{G}$. We define the *activity locus of* $\mathcal{G}$ by

$$\mathcal{A}_{\mathcal{G}} := \{q_0 \in \widehat{\mathbb{C}} \mid q_0 \text{ is active for } \mathcal{G}\}. \tag{2.7}$$

Note that the activity locus is a closed subset of $\widehat{\mathbb{C}}$.

We next state Montel's theorem, see [19, 58] for proofs and further background.

Theorem 2.21 (Montel)**.** *Let $\mathcal{F}$ be a family of rational functions on an open set $U \subseteq \widehat{\mathbb{C}}$. If there exists three distinct points $a, b, c \in \widehat{\mathbb{C}}$ such that for all $f \in \mathcal{F}$ and all $u \in U$, $f(u) \notin \{a, b, c\}$, then $\mathcal{F}$ is a normal family on U.*

Montel's theorem combined with activity and Lemma 2.6 give us a very quick way to demonstrate the presence of chromatic zeros.

Lemma 2.22. *Let $q_0 \in \mathbb{C} \setminus \{0, 1, 2\}$ and suppose that q_0 is contained in the activity locus of $\mathcal{G}_{\mathrm{SP}}$. Then there exists q arbitrarily close to q_0 and $G \in \mathcal{G}_{\mathrm{SP}}$ such that $Z(G; q) = 0$.*

Proof. Suppose not. Then by Lemma 2.6, there must be an open neighborhood of q_0 on which family of ratios must avoid the points $-1, 0, \infty$. Montel's theorem then gives that the family of ratios must be normal on this neighborhood, contradicting the assumptions of the lemma. □

Lemma 2.23. *Let $q_0 \in \mathbb{C}$, and assume there exists an effective edge interaction $y \in \mathcal{E}(q_0)$ or a virtual interaction $y \in f_{q_0}(\mathcal{E}(q_0))$ such that $1 < |y| < \infty$. Then q_0 is contained in the activity locus of $\mathcal{G}_{\mathrm{SP}}$.*

Proof. We will show that for every open U' around q_0 there exists a family of series-parallel graphs $\mathcal{G}$ such that $\{q \mapsto y_G(q) \mid G \in \mathcal{G}\}$ is non-normal. This of course implies non-normality of the family $\{q \mapsto R(G; q) \mid G \in \mathcal{G}\}$ on U' and hence that q_0 is contained in the activity locus $\mathcal{A}_{\mathcal{G}_{\mathrm{SP}}}$.

We will first assume that $y \in f_{q_0}(\mathcal{E}(q_0))$ and $1 < |y| < \infty$. Suppose $y = f_{q_0}(y_G(q_0)))$ for some series-parallel graph G. The virtual interaction is not a constant function of q, because at $q = \infty$ the virtual interaction is ∞, cf. Remark 2.7. Therefore any open neighborhood U' of q_0 is mapped to an open neighborhood U of y and we may assume that U' is small enough, such that U lies completely outside the closed unit disk. Now the pointwise powers $\{u^n \mid u \in U\}_{n \in \mathbb{N}}$ converge to ∞ and the complex argument of the powers $\arg(\{u^n \mid u \in U\}) = n \arg(U)$ cover the entire unit circle for n large enough.

Let us denote the unit circle by $C \subseteq \mathbb{C}$. Then $f_q(C)$ is a straight line for every q. Inside the Riemann sphere, $\widehat{\mathbb{C}}$, these lines are circles passing through ∞. Assuming U' is small enough, there is a neighborhood of ∞ such that the circles $f_q(C)$ will lie in two sectors for all $q \in U'$. More precisely, there exists R large enough such that the argument of the complex numbers in $\bigcup_{q \in U'} f_q(C) \cap \{z \in \mathbb{C} \mid |z| > R\}$ are contained in two small intervals. Therefore we can find two sectors S_1 and S_2 around ∞ such that $f_q(S_1)$ lies inside C for all $q \in U'$ and $f_q(S_2)$ lies outside of C for all $q \in U'$. Because the pointwise powers $\{u^n \mid u \in U\}$ converge towards ∞ and the argument of the complex numbers are spread over the entire unit circle, there must be an N for which $\{u^N \mid u \in U\}$ intersects with both S_1 and S_2. Then $\{f_q(f_q(y_G(q))^N) \mid q \in U'\}$ has points inside and outside the unit circle. Now the family $\{q \mapsto f_q(f_q(y_G(q))^N)^m \mid m \in \mathbb{N}\}$ is non-normal on U'. Indeed, the values inside the unit circle converge to 0, and the values outside the unit circle converge to ∞. So any limit function of any subsequence can therefore not be holomorphic. An easy induction argument, as in the proof of Corollary 2.19, shows that $f_q(f_q(y_G(q))^N)^m$ is the effective edge interaction of the parallel composition of m copies of the series composition of N copies of the graph G.

For the case $y \in \mathcal{E}(q_0)$ with $|y| > 1$, we note again that this interaction cannot be a constant function of q, because at $q = \infty$ the value must be 1, cf. Remark 2.7. If we perform the same argument as above, we obtain a non-normal family of virtual interactions on U'. Applying f_q to this family, produces a non-normal family on U' of effective edge interactions of series compositions of copies of parallel compositions of copies of the graph G. □

Remark 2.24. For later reference we record the family of graphs that provides the non-normal family of interactions/ratios. In the case that we have a virtual interaction $|f_{q_0}(y_G(q_0))| > 1$ for a graph G, the family consists of N copies of G in series, and m copies of this in parallel. For the case of an effective edge interaction $|y_G(q_0)| > 1$, we instead put N copies of G in parallel, and m copies of this in series.

Proof of Theorem 2.16. For $q_0 \in \mathbb{C} \setminus \{1, 2\}$ where either the interaction or the virtual interaction escapes the unit disk, the theorem is a direct consequence of Lemmas 2.22 and 2.23. If for $q_0 \in \{1, 2\}$ there is an interaction or virtual interaction escaping the unit disk, this holds for all q in a neighborhood as well. At these values, we already know that zeros accumulate, so they will accumulate at q_0 as well. □

We now explain how to strengthen Corollary 2.17 to generalized theta graphs. Let Θ denote the family of all generalized theta graphs.

Corollary 2.25. *Let $q \in \mathbb{C}$ such that $|1 - q| > 1$. Then there exists q' arbitrarily close to q and $G \in \Theta$ such that $Z(G; q') = 0$.*

Proof. Note that $y = f_q(0) = 1 - q$ is a virtual activity such that $|y| > 1$. From Lemma 2.23 and Remark 2.24 we in fact find that q is in the activity locus of Θ. By Theorem 2.21 (Montel's theorem) we may thus assume that there exists $G \in \Theta$ such that $R(G; q) \in \{-1, 0, \infty\}$. We claim that the ratio must in fact equal -1, meaning that q is in fact a zero of the chromatic polynomial of the generalized theta graph G.

The argument follows the proof of '(iii) $\Rightarrow$ (i)' in Lemma 2.6. Suppose that the ratio is ∞. Then we add an edge between the two terminals and realize that the resulting graph is equal to a number cycles glued together on an edge. Since chromatic zeros of cycles are all contained in $\overline{B}(1,1)$, this implies that the ratio could not have been equal to ∞. If the ratio equals 0, then we again obtain a chromatic zero of a cycle after identifying the start and terminal vertices. This proves the claim and hence finishes the proof. $\square$

2.4.2 Zeros imply activity

In this section we will prove the following converse to Theorem 2.16.

Theorem 2.26. *Let $q \in \mathbb{C} \setminus \{0,1,2\}$ such that $Z(G;q) = 0$ for some $G \in \mathcal{G}_{\mathrm{SP}}$. Then there exists a $G' \in \mathcal{G}_{\mathrm{SP}}$ such that $1 < |y_{G'}(q)| < \infty$.*

The idea of the proof is as follows. If we assume that G is the smallest such graph, we can even assume that the terminals of G are connected by an edge. Then $Z^{\mathrm{dif}}(G \setminus e; q) = Z(G;q) = 0$ and $Z^{\mathrm{same}}(G \setminus e; q) \neq 0$, so $y_{G \setminus e}(q) = \infty$. This is not yet what we want, because we want $y_{G'}(q)$ to be a finite number. So to obtain G' we will replace every edge of $G \setminus e$ with some gadget H such that $y_H(q)$ is very close to 0. The effect is that $y_{G'}(q)$ is close $y_{G\setminus e}(q) = \infty$, and this is what we want.

Lemma 2.27. *Let G be a 2-connected, series-parallel graph. There exists a series-parallel graph $\widehat{G}$ where the terminals are connected, such that G and $\widehat{G}$ are isomorphic as graphs.*

Proof. The proof is by induction on the distance between the terminals of G. The base case where the distance is 1, is of course trivial.

Because G is 2-connected, it means that G is a parallel composition. More specifically, we can write $G = (G_1 \bowtie G_2) \parallel G_3$, such that the distance between the terminals in G is the same as in $G_1 \bowtie G_2$. Again let G_2^T be the series-parallel graph G_2 where the roles of the terminals are reversed. Then $G_1 \parallel (G_3 \bowtie G_2^T)$ is isomorphic to G as a graph, but the distance between the terminals is smaller. By the induction hypothesis we are now done. $\square$

We can generalize the definition of the effective edge interaction to the random cluster model (see Chapter 3 as well), which will help us to compute some other effective edge interactions:

$$y_G(q,y) := (q-1)\frac{Z^{\mathrm{same}}(G;q,y)}{Z^{\mathrm{dif}}(G;q,y)}.$$

Lemma 2.28. *Let G and H be two-terminal graphs, and let G' be a graph where every edge of G is replaced with H. For any q we have*

$$y_{G'}(q) = y_G(q, y_H(q)).$$

Proof. We will first show that $Z(G';q) = Z(G;q,y_H(q)) \cdot \left(\frac{Z^{\mathrm{dif}}(H;q)}{q(q-1)}\right)^{|E|}$. Because it is an identity of rational functions, it is sufficient to prove it for all integers $q \geq 2$, so we interpret Z as the partition function of the Potts model. We concentrate on one term in the right-hand side, so one (not necessarily proper) colouring of G, and the factor contributed by one edge e to that term. If the edge is monochromatic, it contributes the factor $y_H(q) \cdot \frac{Z^{\mathrm{dif}}(H;q)}{q(q-1)} = \frac{Z^{\mathrm{same}}(H;q)}{q}$, while if it is not monochromatic, it contributes $\frac{Z^{\mathrm{dif}}(H;q)}{q(q-1)}$.

In the graph G', the edge e is replaced by a copy of H, and we have to consider all (proper) colourings of H which agree with the given colouring of G on the terminals. If the edge e is monochromatic, there are $\frac{Z^{\mathrm{same}}(H;q)}{q}$ compatible colourings on H, and if e is not monochromatic, there are $\frac{Z^{\mathrm{dif}}(H;q)}{q(q-1)}$ colourings, giving exactly the same contribution. This proves the relation.

By looking at the graph where the terminals of G are identified into a single vertex, we also find that

$$Z^{\mathrm{same}}(G';q) = Z^{\mathrm{same}}(G;q,y_H(q)) \cdot \left(\frac{Z^{\mathrm{dif}}(H;q)}{q(q-1)}\right)^{|E|},$$

and finally, taking the difference yields

$$Z^{\mathrm{dif}}(G';q) = Z^{\mathrm{dif}}(G;q,y_H(q)) \cdot \left(\frac{Z^{\mathrm{dif}}(H;q)}{q(q-1)}\right)^{|E|},$$

Dividing both quantities we obtain the result. □

Proof of Theorem 2.26. We can take G to be the minimal such series-parallel graph G, which then must be 2-connected. By Lemma 2.27 we can also assume that the terminals of G are connected by an edge e. As already mentioned, we see that $Z^{\mathrm{dif}}(G \setminus e;q) = Z(G;q) = 0$. Because G/e is smaller than G, we see that $Z^{\mathrm{same}}(G \setminus e;q) = Z(G/e;q) \neq 0$, and $y_{G\setminus e}(q) = \infty$.

Of course, if there exists a series-parallel graph H such that $1 < |y_H(q)| < \infty$, we are immediately done. Next assume that there exists a series-parallel graph H such that $0 < |y_H(q)| < 1$, so putting N copies of H in parallel, the effective edge interactions $y_H(q)^N$ converge to 0. Now consider the more general effective edge interaction $y_{G\setminus e}(q,y)$ as a function of y. Because at $y = 0$ and $y = 1$ the values are ∞ and 1 respectively, it is not a constant function and there exists an $\varepsilon > 0$ such that $1 < |y_{G\setminus e}(q,y)| < \infty$ for all y with $0 < |y| < \varepsilon$. This means that if we take N large enough, we have $1 < |y_{G\setminus e}(q, y_H(q)^N)| < \infty$. Now to get this as an effective edge interaction of the chromatic polynomial, we replace all edges of $G \setminus e$ with N copies of H in parallel, which by Lemma 2.28 has effective edge interaction $y_{G\setminus e}(q, y_H(q)^N)$. This is the graph we wanted.

Finally we consider the case that the value of $|y_H(q)|$ is 0, 1 or ∞ for all series-parallel graphs H. Now we look at the effective edge interactions $y_{P_2}(q) = f_q(f_q(0)^2) = \frac{q-1}{q-2}$ and $y_{P_4}(q) = f_q(f_q(0)^4) = \frac{q^3-4q^2+6q-3}{q^3-4q^2+6q-4}$. The first one cannot be 0 or ∞, so its absolute value is 1, which gives $\Re(q) = \frac{3}{2}$. With this restriction, the second one cannot have absolute value 1 or ∞, and the absolute value is 0 exactly when $q = \frac{3\pm i\sqrt{3}}{2}$. In this case $f_q(f_q(f_q(f_q(0)^2)^3)^2) = -\frac{3\pm 2i\sqrt{3}}{7}$, which is the effective edge interaction of a series-parallel graph, has absolute value $\sqrt{3/7}$, so we reach a contradiction. $\square$

Remark 2.29. The following generalization of Theorem 2.26 is also true, as can be seen from the proof: if $Z(G;q) = 0$ for a planar graph G, then there exists a planar, two-terminal graph G' with the terminals on a common face, such that $1 < |y_{G'}(q)| < \infty$.

Remark 2.30. This Theorem allows us to slightly strengthen Lemma 2.10. Instead of asking that $1 - q \notin V^2$, we only need that $V \subseteq B_1$.

2.5 Chromatic zeros of leaf joined trees from independence zeros

This section is devoted to the chromatic roots of leaf joined trees. The main goal is proving Theorem 2.4, but in section 2.5.3 we also find a region where the roots are dense for 'bounded degree' leaf joined trees.

Fix a positive integer $\Delta \geq 2$ and write $d = \Delta - 1$. Given a rooted tree (T, v) consider the two-terminal graph $\widehat{T}$ obtained from (T, v) by identifying all leaves (except v) into a single vertex u. We take v as the start vertex and u as the terminal vertex of $\widehat{T}$. Following Royle and Sokal [64], we call $\widehat{T}$ a *leaf joined tree.* We abuse notation and say that a leaf joined tree $\widehat{T}$ has maximum degree at most $\Delta = d + 1$ if all its vertices except possibly its terminal vertex have degree at most Δ. We denote by $\mathcal{T}_d$ the collection of leaf joined trees of maximum degree at most $d + 1$ for which the start vertex has degree at most d.

Our strategy will be to use Lemma 2.6 in combination with an application of Montel's theorem, much like in the previous section. To do so we make use of an observation of Royle and Sokal in the appendix of the arXiv version of [64] saying that ratios of leaf joined trees, where the the underlying tree is a Cayley tree, are essentially the *occupation ratios* (in terms of the independence polynomial) of the Cayley tree. We extend this relation here to all leaf-joined trees and make use of a recent description of the zeros of the independence polynomial on bounded degree graphs of large degree due to Bencs, Buys and Peters [6].

2.5.1 Ratios and occupation ratios

For a graph $G = (V, E)$ the independence polynomial in the variable λ is defined as

$$I(G; \lambda) = \sum_{\substack{I \subseteq V \\ I \text{ ind.}}} \lambda^{|I|}, \tag{2.8}$$

where the sum ranges over all sets of G. (Recall that a set of vertices $I \subseteq V$ is called *independent* if no two vertices in I form an edge of G.) We define the *occupation ratio* of G at $v \in V$ as the rational function

$$P_{G,v}(\lambda) := \frac{\lambda I(G \setminus N[v]; \lambda)}{I(G - v; \lambda)}, \tag{2.9}$$

where $G - v$ (resp. $G \setminus N[v]$) denotes the graph obtained from G by removing v (resp. v and all its neighbors). We define for a positive integer Δ, $\mathcal{G}_\Delta$ to be the collection of rooted graphs (G, v) of maximum degree at most Δ such that the root vertex, v, has degree at most $d := \Delta - 1$. We next define the relevant collection of occupation ratios,

$$\mathcal{P}_\Delta := \{P_{G,v} \mid (G, v) \in \mathcal{G}_\Delta\}.$$

A parameter $\lambda_0 \in \mathbb{C}$ is called *active for* $\mathcal{G}_\Delta$ if the family $\mathcal{P}_\Delta$ is not normal at λ_0.

We will use the following alternative description of $\mathcal{P}_\Delta$. Define

$$F_{\lambda,d}(z_1, \ldots, z_d) = \frac{\lambda}{\prod_{i=1}^d (1 + z_i)}$$

and let $\mathcal{R}_{\lambda,d}$ be the family of rational maps, parametrized by λ, and defined by

(i) the identify map $z \mapsto z$ is contained in $\mathcal{R}_{\lambda,d}$

(ii) if $r_1, \ldots, r_d \in \mathcal{R}_{d,\lambda}$, then $F_{\lambda,d}(r_1(z), \ldots, r_d(z)) \in \mathcal{R}_{\lambda,d}$.

Lemma 2.31 (Lemma 2.4 in [6]). *Let $\Delta \geq 2$ be an integer and write $d = \Delta - 1$. Then*

$$\mathcal{P}_\Delta = \{\lambda \mapsto r_\lambda(0) \mid r_\lambda \in \mathcal{R}_{\lambda,d}\}.$$

We will next show that, up to a simple factor, the occupation ratios of graphs of maximum degree at most Δ are contained in the family of chromatic ratios of leaf joined tree of maximum degree at most Δ. Define

$$\lambda(q, d) := \frac{(q-1)^d}{(q-2)^{d+1}}.$$

Proposition 2.32. *Let $\Delta \geq 2$ be a positive integer and write $d = \Delta - 1$. Then*
$$\left\{ q \mapsto \frac{q-2}{q-1} r_{\lambda(q,d)}(0) \mid r_\lambda \in \mathcal{R}_{\lambda,d} \right\} \subseteq \{ q \mapsto R(\widehat{T}; q) \mid (T, v) \in \mathcal{T}_d \}.$$

Proof. Suppose that $r_\lambda \in \mathcal{R}_{\lambda,d}$ and that $r_\lambda(z) = F_{\lambda,d}(r_{\lambda;1}(z), \ldots, r_{\lambda;d}(z))$ for certain $r_{\lambda;i} \in \mathcal{R}_{\lambda,d}$. We need to show that the map $q \mapsto \frac{q-2}{q-1} r_{\lambda(q,d)}(0)$ is equal to the ratio $R(\widehat{T}; q)$ for some rooted tree $(T, v) \in \mathcal{T}_d$. By induction we may assume that there are leaf joined trees $\widehat{T_1}, \ldots \widehat{T_d} \in \mathcal{T}_d$ such that there exists $r_{\lambda;i} \in \mathcal{R}_{\lambda,d}$ for each $i = 1, \ldots, d$ such that
$$q \mapsto \frac{q-2}{q-1} r_{\lambda(q,d);i}(0) = R(\widehat{T_i}; q). \tag{2.10}$$
Note that the base case is covered since the map $q \mapsto 0$ is the ratio of the edge $\{v, u\}$.

Let $(T_1, v_1), \ldots, (T_d, v_d)$ be the underlying rooted trees of the $\widehat{T_i}$. Let $\widehat{T}$ be the leaf joined tree whose underlying rooted tree (T, v) is obtained from $(T_1, v_1), \ldots, (T_d, v_d)$ by adding a new root vertex v and connecting it to all the v_i. We claim that
$$R(\widehat{T}; q) = q \mapsto \frac{q-2}{q-1} r_{\lambda(q,d),d}(0). \tag{2.11}$$
To prove this we will first compute the effective edge interaction of $\widehat{T}$. To do so observe that $\widehat{T}$ is obtained by first putting K_2 in series with $\widehat{T_i}$ for $i = 1, \ldots, d$ and then putting the resulting graphs in parallel. (Incidentally this shows that all leaf joined trees are series-parallel graphs). In formulas this reads as
$$\widehat{T} = (K_2 \bowtie \widehat{T_1}) \parallel (K_2 \bowtie \widehat{T_2}) \parallel \cdots \parallel (K_2 \bowtie \widehat{T_d}). \tag{2.12}$$
Suppose the graphs $\widehat{T_i}$ have effective edge interaction y_i ($i = 1, \ldots, d$), then by Lemma 2.8 $\widehat{T}$ has effective interaction y given by
$$y = \prod_{i=1}^{d} f_q(f_q(0) f_q(y_i)) = \left(\frac{q-1}{q-2} \right)^d \prod_{i=1}^{d} \frac{1}{1 + y_i/(q-2)}. \tag{2.13}$$
Recall that $R(\widehat{T}; q) = y_G(q)/(q-1)$. If we now define the *modified ratio* $\widetilde{R}(G; q) = \frac{q-1}{q-2} R(G; q)$ for any two-terminal graph G, we can write this relation as
$$\begin{aligned} \widetilde{R}(\widehat{T}; q) &= \frac{\lambda(q,d)}{\prod_{i=1}^{d} (1 + \widetilde{R}(\widehat{T_i}; q))} \\ &= F_{\lambda(q,d),d} \left(\widetilde{R}(\widehat{T_1}; q), \ldots, \widetilde{R}(\widehat{T_d}; q) \right) \\ &= r_{\lambda(d,q)}(0) \end{aligned}$$
by (2.10). This finishes the proof. $\square$

Corollary 2.33. *Let $\Delta \geq 2$ be an integer and write $d = \Delta - 1$. Let $q_0 \in \mathbb{C} \setminus \{1, 2, 1-d\}$. If $\lambda(q_0, d)$ is active for $\mathcal{G}_\Delta$, then q_0 is active for $\mathcal{T}_d$.*

Proof. Note that the derivative of $\lambda(q, d)$ with respect to q is given by

$$-(q+d-1)\frac{(q-1)^{d-1}}{(q-2)^{d+2}}.$$

Therefore the map $q \mapsto \lambda(q, d)$ is injective on a neighborhood of q_0 and the result follows from the previous proposition. □

2.5.2 Proof of Theorem 2.4

We are now ready to harvest some results from [6] and provide a proof of Theorem 2.4.

Let for an integer $\Delta \geq 2$ and $u \in \mathbb{C}$

$$\lambda_\Delta(u) := \frac{-(\Delta-1)^{\Delta-1}u}{(\Delta-1+u)^\Delta}$$

and define

$$\mathcal{C}_\Delta := \{\lambda_\Delta(u) \mid |u| < 1\}.$$

Define the following collection of active parameters

$$\mathcal{N}_\Delta := \{u \in \overline{B}(\tfrac{1}{2}, \tfrac{1}{2}) \mid \text{ the family } \mathcal{P}_\Delta \text{ is not normal at } \lambda_\Delta(-u)\}.$$

Theorem 2.34. *There exists $\Delta_0 > 0$ such that for all integers $\Delta \geq \Delta_0$ the set $\mathcal{N}_\Delta$ contains a nonempty open set and in particular is nonempty.*

Proof. This follows directly from [6, Theorem 1.2 and 1.3] combined with [23, Theorem 1] and the fact that the boundary of the set $\mathcal{U}_\infty$ (as defined in [6]) is not differentiable at e. Indeed, a close inspection of the function describing the part of the boundary with positive imaginary part near e shows that it in fact makes an angle of 120 degrees with the real axis. □

We now give a proof of Theorem 2.4.

Proof of Theorem 2.4. Let Δ_0 from the theorem above. Fix any integer $\Delta \geq \Delta_0$ and write $d = \Delta - 1$. Choose any non real $u_0 \in \mathcal{N}_\Delta$. Define $q_0 = 1 + d/u_0$ and observe that since the Möbius transformation $u \mapsto 1 + d/u$ maps the disk $\overline{B}(\frac{1}{2}, \frac{1}{2})$ onto the half plane $\{z \in \mathbb{C} \mid \Re(z) \geq d+1\}$, it follows that $\Re(q_0) > \Delta$. Furthermore,

$$\lambda(q_0, d) = \frac{(q_0-1)^d}{(q_0-2)^{d+1}} = \frac{(d/u_0)^d}{((d-u_0)/u_0)^{d+1}} = \frac{d^d u_0}{(d-u_0)^{d+1}} = \lambda_\Delta(-u_0).$$

Therefore, by Corollary 2.33 and Theorem 2.34, we obtain that $q_0 \in \mathcal{A}_{\mathcal{T}_d}$, the activity locus of the family of ratios of the leaf joined trees contained in $\mathcal{T}_d$. By Theorem 2.21 (Montel's theorem) we conclude that there must exist q such that $\Re(q) > \Delta$ and a leaf joined tree $\widehat{T} \in \mathcal{T}_d$ such that $R(\widehat{T}; q) \in \{0, -1, \infty\}$.

We now show that there exists a leaf joined tree of maximum degree Δ for which q is zero of its chromatic polynomial. We cannot directly invoke Lemma 2.6, but its proof will essentially give us what we need.

If the ratio, $R(\widehat{T}; q)$, is equal to -1 then $Z(\widehat{T}; q) = 0$. If the ratio equals ∞, then $Z(\widehat{T} \parallel K_2; q) = Z^{\mathrm{dif}}(\widehat{T}; q) = 0$, so q is a chromatic zero of the leaf joined tree $\widehat{T} \parallel K_2$, whose maximum degree is still Δ. Finally, suppose the ratio equals 0, then $Z^{\mathrm{same}}(\widehat{T}; q) = 0$. We know that $\widehat{T}$ is the parallel composition of d leaf joined trees $\widehat{T}_i$ each in series with K_2 (see (2.12)). We know by Lemma 2.5 that $Z(K_2 \parallel \widehat{T}_i; q) = Z^{\mathrm{same}}(K_2 \bowtie \widehat{T}_i; q) = 0$ for some i. So q is a chromatic zero of the leaf joined tree $K_2 \parallel \widehat{T}_i$, still with maximum degree Δ. This finishes the proof. □

2.5.3 Density of zeros for bounded degree leaf joined trees in an annulus

The result of Corollary 2.33 allows us to find chromatic roots of leaf joined trees, but it relies on knowing active parameters for the occupation ratios. In this section we prove a result which does not follow from this connection, namely that in the annulus $\{q \mid 1 < |1-q| < d\}$ the chromatic zeros of leaf joined trees with maximum degree $d+1$ are dense.

We will first prove a variant of Proposition 2.32, exhibiting some other ratios in the set $\{q \mapsto R(\widehat{T}; q) \mid (T, v) \in \mathcal{T}_d\}$. Define $\zeta_q := \frac{1}{q-2}$, and the functions

$$g_{q,d}(z) := F_{\lambda(q,d),d}(z, z, \ldots, z) = \frac{\lambda(q,d)}{(1+z)^d} = \frac{\zeta_q(1+\zeta_q)^d}{(1+z)^d},$$
$$\mu_q(z) := g_{q,1}(z) = \frac{\zeta_q(1+\zeta_q)}{1+z}.$$

Now we can define the family $\mathcal{M}_{q,d}$ recursively as follows:

(i) the identity map $z \mapsto z$ is contained in $\mathcal{M}_{q,d}$;

(ii) if $r \in \mathcal{M}_{q,d}$, then $g_{q,d} \circ r$ and $\mu_q \circ r$ are in $\mathcal{M}_{q,d}$.

Now we can state the new Proposition:

Proposition 2.35. *Let d be a positive integer, then*

$$\left\{ q \mapsto \frac{q-2}{q-1} r_q(0) \mid r_q \in \mathcal{M}_{q,d} \right\} \subseteq \{q \mapsto R(\widehat{T}; q) \mid (T, v) \in \mathcal{T}_d\}.$$

Proof. This proof is very similar to the proof of Proposition 2.32. We see that the function $g_{q,d}$ corresponds to the construction

$$(K_2 \bowtie \widehat{T}) \parallel \cdots \parallel (K_2 \bowtie \widehat{T}),$$

putting d copies in parallel. The special case μ_q then just corresponds to the construction $K_2 \bowtie \widehat{T}$. $\square$

Remark 2.36. To get a recursive description of the entire family $\{q \mapsto R(\widehat{T}; q) \mid (T, v) \in \mathcal{T}_d\}$, we should use the multi-variable recursion as in the family $\mathcal{R}_{q,d}$. That is, for every $r_1, \ldots, r_d$ in the family, we also have $F_{\lambda(q,d),d}(r_1, \ldots, r_d)$ in the family. The difference is that we take two starting points: the identity $z \mapsto z$ and the constant function $z \mapsto \zeta_q$.

Lemma 2.37. *Let d be a positive integer and let $q_0 \in \mathbb{C}$ such that $1 < |1-q_0| < d$. Then q_0 is active for $\mathcal{T}_d$.*

Proof. First note that ζ_q is a fixed point of both g_q and μ_q. We easily compute that $\mu_q'(\zeta_q) = \frac{-\zeta_q}{1+\zeta_q} = \frac{1}{1-q}$ and $g_q'(\zeta_q) = d \cdot \mu_q'(\zeta_q) = \frac{d}{1-q}$, so by the assumptions ζ_q is an attracting fixed point for μ_q, but a repelling fixed point for g_q. We are going to show that the family $\{q \mapsto r_q(0) \mid r_q \in \mathcal{M}_{q,d}\}$ is not normal at q_0, so by Proposition 2.35 we find that q_0 is active for $\mathcal{T}_d$.

First we will show, if h_q is an element of the family $\mathcal{M}_{q,d}$, then $h_q(0) = \zeta_q$ occurs for at most finitely many q. If actually $h_q(0) = \zeta_q$ for infinitely many values of q, then they are equal as meromorphic functions of q. This is impossible, because at $q = 1$ we have $h_1(0) = 0$ and $\zeta_1 = -1$.

Now we consider an open U around q_0, and we assume that U lies inside the annulus $\{q \mid 1 < |1-q| < d\}$. We will first find a $q_2 \in U$ such that $h_{q_2}(0) \neq \zeta_q$ for any $h_q \in \mathcal{M}_{q,d}$. This is possible because $\mathcal{M}_{q,d}$ is countable, for any $h_q \in \mathcal{M}_{q,d}$ there are only finitely many q such that $h_q(0) = \zeta_q$, and U is uncountable. Next we find $q_1 \in U$ such that $|\mu_{q_1}'(\zeta_{q_1})| < |\mu_{q_2}'(\zeta_{q_2})|$, and note that both are strictly between $\frac{1}{d}$ and 1. It is easy to see that there exist positive integers $k < m$ such that $d^k \cdot |\mu_{q_1}'(\zeta_{q_1})|^m < 1 < d^k \cdot |\mu_{q_2}'(\zeta_{q_2})|^m$. This tells us that the fixed point ζ_q of the function $h_q := g_q^{\circ k} \circ \mu_q^{\circ(m-k)}$ is attracting at $q = q_1$ and repelling at $q = q_2$. Now we claim that for some N large enough, the sequence $(q \mapsto h_q^{\circ n} \circ \mu_q^{\circ N}(0))_{n \in \mathbb{N}}$ is not normal on U.

Fix some $\alpha < 1$ such that $|h_{q_1}'(\zeta_{q_1})| < \alpha$. The function $h_q'(z)$ is continuous in both q and z (at least locally at $(q, z) = (q_1, \zeta_{q_1})$), this means there exist an open $U' \subseteq U$ around q_1 and an open ball $V_1 = B(\zeta_{q_1}, \eta)$ such that $|h_q'(z)| < \alpha$ for all $(q, z) \in U' \times V_1$. Recall that μ_{q_1} is a Möbius transformation with attracting fixed point ζ_{q_1} and repelling fixed point $-1 - \zeta_{q_1} = \frac{1-q_1}{q_1-2} \neq 0$, so there exists an N such that $\mu_{q_1}^{\circ N}(0) \in V_1$. Next we shrink the open U' to the open $U'' = \{q \in$

$U' \mid \mu_q^{\circ N}(0) \in V_1, |\zeta_q - \zeta_{q_1}| < \delta\}$, with $\delta = \frac{1-\alpha}{1+\alpha}\eta$. We claim that for any $q \in U''$, the sequence $(h_q^{\circ n} \circ \mu_q^{\circ N}(0))_{n\in\mathbb{N}}$ converges to ζ_q. Using the uniform bound on the derivative, we show for any $z \in V_1$ that $|h_q(z) - \zeta_q| \leq \alpha|z - \zeta_q|$. This implies that

$$|h_q(z) - \zeta_{q_1}| \leq |h_q(z) - \zeta_q| + |\zeta_q - \zeta_{q_1}| < \alpha(\eta + \delta) + \delta = \eta.$$

This shows that h_q maps V_1 into itself, and because the distance to ζ_q decreases with at least the constant factor α, we obtain the claimed convergence.

On the other hand, ζ_{q_2} is a repelling fixed point for h_{q_2}. This means there exist a $\beta > 1$ and an open ball V_2 around ζ_{q_2}, such that $\frac{|h_{q_2}(z)-\zeta_{q_2}|}{|z-\zeta_{q_2}|} > \beta > 1$ for all $z \in V_2 \setminus \{\zeta_{q_2}\}$. Then for any $z \in V_2 \setminus \{\zeta_{q_2}\}$, there exists an n such that $h_{q_2}^{\circ n}(z)$ is not in V_2. By definition of q_2, we know that the sequence $(h_{q_2}^{\circ n} \circ \mu_{q_2}^{\circ N}(0))_{n\in\mathbb{N}}$ does not contain ζ_{q_2}. Then the previous argument shows that there are infinitely many terms in this sequence that are not in V_2. Let I be the corresponding set of indices.

Now we consider the subsequence $(q \mapsto h_q^{\circ n} \circ \mu_q^{\circ N}(0))_{n\in I}$, which converges to ζ_q on U''. If it would converge uniformly on compact sets, its limit function has to be ζ_q, and so the pointwise limit at $q = q_2$ has to be ζ_{q_2}. But we choose I such that the values at $q = q_2$ are bounded away from ζ_{q_2}, which is a contradiction. Therefore we conclude that indeed the family $\{q \mapsto r_q(0) \mid \mathcal{M}_{q,d}\}$ is not normal at q_0, as we wanted. $\square$

Corollary 2.38. *Let $d = \Delta - 1 \geq 2$ be an integer and $q \in \mathbb{C}$ such that $1 < |1 - q| < d$. There exists q' arbitrarily close to q, and a leaf joined tree $\widehat{T}$ of maximum degree Δ, such that $Z(\widehat{T}; q) = 0$.*

Proof. We use the previous lemma, together with Theorem 2.21, to show that there exist a $\widehat{T} \in \mathcal{T}_d$ and q' such that $R(\widehat{T}; q') \in \{0, -1, \infty\}$. Now refer back to the proof of Theorem 2.4, where we show that this implies there exists a leaf joined tree of maximum degree Δ with q' as chromatic zero. $\square$

2.6 Concluding remarks, questions and conjectures

In this chapter we embarked on the quest to determine the location of the chromatic zeros of the family of series-parallel graphs. While we have made several contributions, a complete characterization remains elusive, as is visible in Figure 2.1. Several concrete questions and conjectures arise in this regard.

First of all, it is important to note that Figure 2.1 is a pixel picture, and the color of a pixel only displays the behavior of the center point of the pixel. Potential features of the picture that are smaller than the resolution will therefore be invisible. We believe however that with a bit more effort one can create a more rigorous picture that looks exactly the same.

A pixel is colored blue, if for q at the center of the pixel, there exist integers $n_1, \ldots, n_k$ with $|f_q(f_q(\cdots f_q(f_q(0)^{n_1})^{n_2} \cdots)^{n_k})| > 1$ (within the search depth $n_1 \cdot \ldots \cdot n_k \leq 300$). This composition is either an effective edge interaction, or a virtual interaction, so Theorem 2.16 ensures that q is contained in the closure of the chromatic zeros of series parallel graphs. Conversely, Theorem 2.26 tells us that there is no zero if we do not escape the unit disk. The main inaccuracy in the blue pixels, is thus due to the search depth.

A pixel is colored orange in Figure 2.1, if for q at the center of the pixel, it is possible to find a disk V such that $f_q(V) = V$ and which satisfies the conditions of Lemma 2.10. There is a very explicit description of the disks V satisfying $f_q(V) = V$. This makes it easy to check $0 \in V$ and $1 - q \notin V^2$. The condition $V^2 \subseteq V$ is verified by checking that $\sup\{|z|^2 \mid z \in V\} < \inf\{|z| \mid z \in \mathbb{C} \setminus V\}$. Figure 2.1 directly motivates the following conjecture.

Conjecture 2.1. *For each q in the punctured disk $\overline{B}(1, 5/27) \setminus \{1\}$ and any series-parallel graph G, $Z(G; q) \neq 0$.*

Note that our proof of Lemma 2.15 gives a punctured disk of radius $(2-\sqrt{3})^2 \approx 0.072$ around 1, which is much less than $5/27 \approx 0.185$.

Another interesting question motivated by Theorem 2.4 is whether there exist chromatic zeros with real part larger than the second largest degree for all degrees. We have verified this question up to $\Delta \leq 45$, see Table 2.1 below. The values were obtained using the technique of Buys [17] to find zeros of the independence polynomial. First we find a family of spherically regular trees of degree $d_1 \geq d_2$ that are active at $\lambda \in \mathbb{C}$ for this family, using Appendix B of [17]. Therefore by Corollary 2.33 we obtain that q_0 is active for $\mathcal{T}_{d_1}$, where we choose q_0 to be the solution of $\lambda(q_0, d_1) = \lambda$ of the largest real part.

Figure 2.2 strongly supports the following conjecture. This is related to a question from [62, 17] on zeros of the independence polynomial of bounded degree graphs.

Conjecture 2.2. *Theorem 2.4 is true with $\Delta_0 = 3$.*

We end with a question on the possible extension of one of our result to a larger family of graphs to which our techniques do not seem to apply. Most interesting would be planar graphs, where Figure 2.1 definitely changes. Namely, the family of wheel graphs have chromatic zeros which are dense on the circle $\{q \mid |2-q| = 1\}$, so they accumulate at 1. Even more so: Remark 2.29 on Theorem 2.26 says that every chromatic zero of a wheel graph satisfies the condition of Theorem 2.16. But this is an open condition, so all chromatic zeros of the wheel graphs are surrounded by opens where planar chromatic zeros accumulate.

This raises the following question.

Question 2.3. Do (triangulated) planar graphs have an open regio which is free of chromatic zeros?

Δ	q	type	Δ	q	type
4	$4.027 + 0.783i$	$3, 2$	25	$25.204 + 5.925i$	$24, 16$
5	$5.088 + 0.836i$	$4, 3$	26	$26.269 + 5.986i$	$25, 17$
6	$6.132 + 0.881i$	$5, 4$	27	$27.158 + 6.596i$	$26, 17$
7	$7.058 + 1.521i$	$6, 4$	28	$28.227 + 6.658i$	$27, 18$
8	$8.120 + 1.577i$	$7, 5$	29	$29.293 + 6.719i$	$28, 19$
9	$9.012 + 2.194i$	$8, 5$	30	$30.182 + 7.329i$	$29, 19$
10	$10.084 + 2.256i$	$9, 6$	31	$31.251 + 7.392i$	$30, 20$
11	$11.147 + 2.314i$	$10, 7$	32	$32.317 + 7.453i$	$31, 21$
12	$12.038 + 2.928i$	$11, 7$	33	$33.206 + 8.063i$	$32, 21$
13	$13.109 + 2.990i$	$12, 8$	34	$34.274 + 8.125i$	$33, 22$
14	$14.173 + 3.049i$	$13, 9$	35	$35.340 + 8.187i$	$34, 23$
15	$15.063 + 3.662i$	$14, 9$	36	$36.229 + 8.796i$	$35, 23$
16	$16.133 + 3.724i$	$15, 10$	37	$37.298 + 8.859i$	$36, 24$
17	$17.197 + 3.784i$	$16, 11$	38	$38.364 + 8.920i$	$37, 25$
18	$18.087 + 4.395i$	$17, 11$	39	$39.252 + 9.530i$	$38, 25$
19	$19.157 + 4.457i$	$18, 12$	40	$40.321 + 9.592i$	$39, 26$
20	$20.222 + 4.518i$	$19, 13$	41	$41.387 + 9.654i$	$40, 27$
21	$21.111 + 5.129i$	$20, 13$	42	$42.276 + 10.263i$	$41, 27$
22	$22.180 + 5.191i$	$21, 14$	43	$43.344 + 10.326i$	$42, 28$
23	$23.246 + 5.252i$	$22, 15$	44	$44.411 + 10.387i$	$43, 29$
24	$24.135 + 5.862i$	$23, 15$	45	$45.299 + 10.997i$	$44, 29$

Table 2.1: Table of parameters q with real part bigger than Δ, such that q is active for the following family of leaf joined trees: construct trees where alternately every vertex has down degree exactly d_1 resp. d_2, add $d_1 - d_2$ leaves to the down vertices of degree d_2, and add one vertex connected to all leaves. The proof of Theorem 2.4 implies that chromatic zeros of leaf joined trees of maximum degree $d_1 + 1$ accumulate at q.

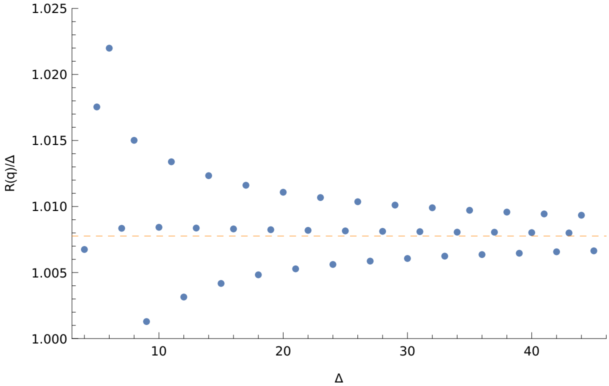

Figure 2.2: For each $\Delta = 4, \ldots, 45$ we record the value of $\Re(q)/\Delta$ from Table 2.1. The orange dashed line denotes the limiting value as $d_1 \to \infty$ and $d_2/d_1 \to 2/3$.

Chapter 3

Approximating the chromatic polynomial is as hard as computing it exactly

3.1 Introduction

The study of (approximately) computing the chromatic polynomial, or in fact the more general Tutte polynomial[1] was initiated by Jaeger, Vertigan and Welsh [47] over thirty years ago. Among other things they proved that evaluating the chromatic polynomial of a graph at any algebraic number q exactly is #P-hard except for $q = 0, 1, 2$. This was extended by Vertigan [72] who showed that the same is true when restricted to planar graphs. The next step was taken by Goldberg and Jerrum [36] who proved that it is NP-hard to approximate the chromatic polynomial at real values $q > 2$ (as part of a much larger result concerning inapproximability of the Tutte polynomial). As far as we know the complexity of approximating the chromatic polynomial at real q on *planar graphs* is open. See [37] for hardness result for evaluations of the Tutte polynomial on planar graphs 'close' to the chromatic polynomial.

Partly motivated by applications to quantum computing, Goldberg and Guo [35] proved the first inapproximability results for certain non-real evaluations of the Tutte polynomial, showing #P-hardness of approximating and not just NP-hardness. These results were recently extended by Galanis, Goldberg and Herrera [30] to a much larger family of evaluations and planar graphs.

So far, as far as we know, no inapproximability results were known for the chromatic polynomial at non-real values of q. For several graph polynomials, such as the independence polynomial and the partition function of the Ising model, recent developments in the study of approximate counting has indicated that approximating evaluations of these polynomials is computationally hard in the vicinity of the zeros of these polynomials [62, 10, 17, 21, 63, 18, 29]. Motivated by this connection and Sokal's famous result [66] saying that the zeros of of the

[1]Recall that the Tutte polynomial is a 2-variable polynomial that has the chromatic polynomial among its many specializations.

chromatic polynomial are dense in the complex plane, one might be tempted to conjecture that approximating the chromatic polynomial at non-real numbers should be hard. Our main result indeed confirms this.

3.1.1 Main results

Before we state our main result we first formally state the computational problems we are interested in and give the definition of the chromatic polynomial.

Recall that the chromatic polynomial of a graph $G = (V, E)$ is defined as

$$Z(G; q) := \sum_{A \subseteq E} (-1)^{|A|} q^{k(A)},$$

where $k(A)$ denotes the number of components of the graph (V, A). For a positive integer q, $Z(G; q)$ equal the number of proper q-colorings of G.

We will consider two types of approximation problems one for the norm of $Z(G; q)$ and one for its argument, for each algebraic number q separately. For a nonzero complex number ξ we will consider the argument $\arg(\xi)$ as an element of $\mathbb{R}/(2\pi\mathbb{Z})$. For $\overline{a} \in \mathbb{R}/(2\pi\mathbb{Z})$ we denote $|\overline{a}| := \min_{a' \in \overline{a}} |a'|$.

Let ξ be a complex number and $\eta > 0$. We call a number $r \in \mathbb{Q}$ an *η-abs-approximation* of ξ if $\xi \neq 0$ implies $e^{-\eta} \leq r/|\xi| \leq e^{\eta}$. We call a number $r \in \mathbb{Q}$ an *η-arg-approximation* of ξ if $\xi \neq 0$ implies that $|r - \arg(\xi)| \leq \eta$. Note that in both cases an approximation of 0 could be anything. Consider for an algebraic number q the following computational problems. In Subsection 3.2.3 below we will indicate how we will represent algebraic numbers. Throughout graphs may have multiple edges between any pair of vertices and loops, unless stated otherwise.

Name:	q-PLANAR-ABS-CHROMATIC
Input:	A planar graph G.
Output:	An 0.25-abs-approximation of $Z(G; q)$.

Name:	q-PLANAR-ARG-CHROMATIC
Input:	A planar graph G.
Output:	An 0.25-arg-approximation of $Z(G; q)$.

We define the problems q-ABS-CHROMATIC, q-ARG-CHROMATIC in the same way except that the input for both problems may now be any graph. We note that these problems do not change in complexity when restricting to simple graphs, since the chromatic polynomial of a graph with a loop is constantly equal to 0 and the chromatic polynomial of a graph with no loops is equal to the chromatic polynomial of its underlying simple graph.

Our main result is the following:

Theorem 3.1. *For each non-real algebraic number $q \in \mathbb{C}$ such that $|1-q| > 1$ or $\Re(q) > 3/2$, the problems q-*Planar-Abs-Chromatic *and q-*Planar-Arg-Chromatic *are #P-hard.*

Note that by planar duality, this result also applies to the flow polynomial.

As an immediate consequence of Theorem 3.1, we obtain hardness for approximately computing the chromatic polynomial on the entire complex plane except the real line for the family of all graphs:

Corollary 3.2. *For each non-real algebraic number $q \in \mathbb{C}$, the problems q-*Abs-Chromatic *and q-*Arg-Chromatic *are #P-hard.*

Proof. This follows the same argument as Sokal's density result [66]. We reduce the problems to their planar counterpart. Given a planar graph G. Clearly, we may assume that $|q-1| \leq 1$. Create a new graph $\widehat{G}$ by adding three new vertices pairwise connected by an edge and connect each of these three vertices to all original vertices of G. It is well known and easy to see that $Z(\widehat{G}; q) = q(q-1)(q-2)Z(G; q-3)$. Denote $q' = q-3$ and note that $|q'-1| = |(q-1)-3| > 1$. So a polynomial time algorithm that solves the problem q-Abs-Chromatic, respectively q-Arg-Chromatic, can be used to solve q'-Planar-Abs-Chromatic, respectively q'-Planar-Arg-Chromatic, in polynomial time. Since the latter two problems are #P-hard by Theorem 3.1, the same holds for the former two. □

While the main focus of this chapter is the chromatic polynomial, we also derive results for the more general partition function of the *random cluster model*

$$Z(G; q, y) := \sum_{A \subseteq E} (y-1)^{|A|} q^{k(A)},$$

and the associated problems of approximating the value for fixed q, y on input of a planar graph G. In Theorem 3.12 we find a sufficient condition such that these problems are #P-hard, and in Corollary 3.13 we record some explicit ranges for q, y where the problems are #P-hard, which includes the result of Theorem 3.1. Figure 3.1 shows a region of q-values for which we could verify the condition in Theorem 3.12 with a computer.

As is well known, $Z(G; q, y)$ is essentially equal to the Tutte polynomial $T(G; x, y)$ for $q = (x-1)(y-1)$. See e.g. [67, 27] and Section 1.3 for background. Thereby Theorem 3.12 immediately gives us several inapproximability results for the Tutte polynomial on planar graphs. However, the cases $x = 1$ or $y = 1$, corresponding to $q = 0$, are not covered by our approach. (The case $x = 1$ corresponds to the all terminal reliability polynomial.) We comment on how our

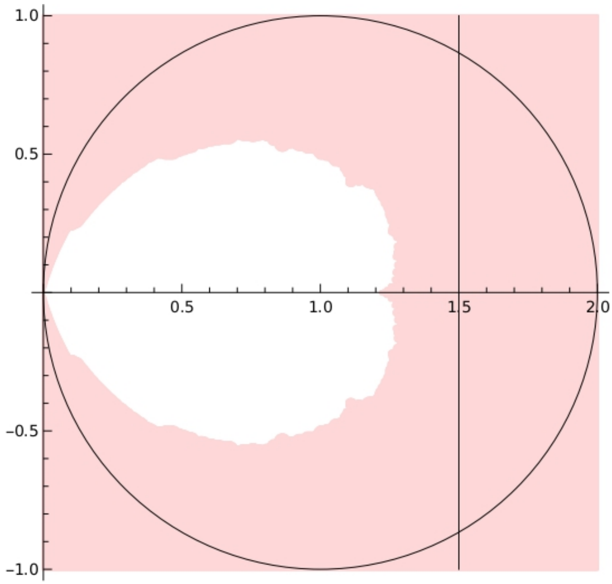

Figure 3.1: The red-shaded region represents values of q for which the problems q-Planar-Abs-Chromatic and q-Planar-Arg-Chromatic are #P-hard. This is a pixel-picture with a 1001×1001 resolution. The region depicted in the figure ranges from $-i$ to $2+i$.

approach could be used to show to hardness of approximation for these cases in Section 3.6.

3.1.2 Proof outline

We sketch a proof outline here (only for the chromatic polynomial), leaving most of the technical details and definitions to later sections. Our proof goes along similar lines as other inapproximability results for non-real parameters obtained recently [35, 10, 18, 30, 29], but it also differs from these at certain steps. For example, in the previous works just mentioned the problem of approximate evaluation is reduced from exact evaluation of the polynomial/partition function at different parameters (often real) causing extra work to be done. One of our novel contributions is that we reduce exact evaluation at q (which by [72] is #P-hard) to approximate evaluation at the same parameter, yielding a very clean reduction. Our approach for this is quite robust and could be applied to other partition functions/graph polynomials. We say a bit more about this below. First we describe our approach.

The goal is to show that for a given algebraic number q, assuming the existence of a polynomial time algorithm for q-Planar-Abs-Chromatic or q-Planar-Arg-Chromatic, we can design a polynomial time algorithm to compute the evaluation of the chromatic polynomial at q exactly. This is essentially done in two steps.

The first step is to replace an edge e of a given graph G by another graph H (with two marked vertices) also called a *gadget*. The chromatic polynomial of the resulting graph G' is then, up to some easily computable factor (if the graph H is series-parallel for example), given by

$$Z(G;q) + y_H Z(G/e;q),$$

where is y_H the *effective edge interaction* of H (to be defined in the next section). If one can determine the value y^* such that $Z(G;q) + y^* Z(G/e;q) = 0$, then this means that one can determine the ratio $r = \frac{Z(G;q)}{Z(G/e;q)}$, assuming $Z(G/e;q) \neq 0$. A potential problem is when both $Z(G/e;q)$ and $Z(G \setminus e;q)$ are equal to zero. Below we indicate how to overcome this difficulty.

In case q is real the value of y^* can be approximated very accurately by means of a binary search procedure due to [38]. Then, using that y^* is an algebraic number of polynomial size, this implies one can in fact determine y^* *exactly* in polynomial time. The binary search is done by applying the assumed polynomial time algorithm that approximately computes the absolute value or argument of $Z(G';q)$ and using the output of this algorithm for various values of y to steer the binary search. We extend and simplify this binary search strategy to a procedure we call 'box shrinking', see Theorem 3.6 below, so that it applies to non-real q. Moreover, we modify it in way so that it allows us to determine if $Z(G/e;q)$ is equal to 0 or not, provided not both $Z(G/e;q)$ and $Z(G \setminus e, q)$ are zero. Having this extra information allows us then to compute $Z(G;q)$ exactly, writing it as a telescoping product. See Theorem 3.5 below for the details.

The box shrinking procedure requires us to be able to generate graphs H such that their effective edge interactions approximate any given $y_0 \in \mathbb{Q}[i]$ with very high precision fast. This brings us to the second step. As in [30], we use series-parallel graphs to achieve this. The approach to do this is partly based on [10] and inspired by [23] and Lemma 2.23. See Theorem 3.8 below for the precise statement of what we obtain.

The novel parts of our approach, the box shrinking procedure (Theorem 3.6) and Theorem 3.5, are quite robust and can for example be applied to the independence polynomial. Doing so, one reduces the problem of exactly evaluating the independence polynomial at some non-real fugacity parameter, which is #P-hard by [53], to approximately evaluating it. This would shorten and simplify the reduction used in [10], where a reduction from evaluating the polynomial at 1 is used.

Organization In the next section we collect some definitions and results around series-parallel graphs and some notions regarding the representation of algebraic numbers. Then in Section 3.3 we state our two main technical contributions and combine these to give a proof of Theorem 3.1. Sections 3.4 and 3.5 contain our proofs of these two contributions. Finally in Section 3.6 we conclude with some questions and problems left open by our work.

3.2 Preliminaries

In this section we set up some notation and introduce some basic notions that we will use.

3.2.1 Series-parallel graphs

As mentioned in the introduction we will be using series-parallel graphs as gadgets. We introduce these here closely following Chapter 2 and thereby Royle and Sokal [64] in their use of notation.

Let G_1 and G_2 be two graphs with designated start- and endpoints s_1, t_1, and s_2, t_2 respectively, referred to as *two-terminal graphs*. The *parallel composition* of G_1 and G_2 is the graph $G_1 \parallel G_2$ with designated start- and endpoints s, t obtained from the disjoint union of G_1 and G_2 by identifying s_1 and s_2 into a single vertex s and by identifying t_1 and t_2 into a single vertex t. The *series composition* of G_1 and G_2 is the graph $G_1 \bowtie G_2$ with designated start- and endpoints s, t obtained from the disjoint union of G_1 and G_2 by identifying t_1 and s_2 into a single vertex and by renaming s_1 to s and t_2 to t. Note that the order matters here. A two-terminal graph G is called *series-parallel* if it can be obtained from a single edge using series and parallel compositions. Note that series-parallel graphs are automatically connected.

From now on we will implicitly assume the presence of the start- and endpoints when referring to a two-terminal graph G. We denote by $\mathcal{G}_{\mathrm{SP}}$ the collection of all series-parallel graphs.

3.2.2 Effective edge interactions

An important ingredient in our proof will be the notion of *effective edge interaction*. It requires a few preliminary definitions to define it. We extend these definitions from Chapter 2 to the partition function of the random cluster model.

Recall that for a positive integer q, any $y \in \mathbb{C}$ and a graph $G = (V, E)$ we have

$$Z(G; q, y) = \sum_{\varphi : V \to \{1,\dots,q\}} \prod_{uv \in E} (1 + (y-1)\delta_{\varphi(u),\varphi(v)}),$$

where $\delta_{i,j}$ denotes the Kronecker delta. For a positive integer q and a two-terminal graph G, we can thus write,

$$Z(G; q, y) = Z^{\text{same}}(G; q, y) + Z^{\text{dif}}(G; q, y), \tag{3.1}$$

where $Z^{\text{same}}(G; q, y)$ collects those contributions where s, t receive the same color and where $Z^{\text{dif}}(G; q, y)$ collects those contribution where s, t receive different colors. Since $Z^{\text{same}}(G; q, y)$ is equal to $Z(G'; q, y)$ where G' is obtained from G by identifying the vertices s and t, both these terms are polynomials in q and y. Therefore (3.1) also holds for any $q \in \mathbb{C}$.

For fixed $y \in \mathbb{C}$ the *effective edge interaction* is defined as

$$y_G(q, y) := (q-1)\frac{Z^{\text{same}}(G; q, y)}{Z^{\text{dif}}(G; q, y)},$$

which we view as a rational function in q. (It might be slightly confusing that both the function and one of its inputs are called y. This is because y_G will play a role similar as y.) We note that in case G contains an edge between s and t, the rational function $q \mapsto y_G(q, 0)$ is constantly equal to 0. If q, y are clear from the context we may occasionally just write y_G for the effective edge interaction.

For any $q \neq 0$ define the following Möbius transformation

$$f_q(z) := 1 + \frac{q}{z-1}$$

and note that f_q is an involution. For a two-terminal graph G with effective edge interaction y_G, we call $f_q(y_G)$ a *virtual interaction.*

We repeat Lemmas 2.5 and 2.8 and capture the behavior of the effective edge interactions under series and parallel compositions. Even though they were only stated for the chromatic polynomial in Chapter 2, the proofs automatically extends to the more general setting of the partition function of the random cluster model.

Lemma 3.3 (Lemma 2.5)**.** *Let G_1 and G_2 be two two-terminal graphs and let $q, y \in \mathbb{C}$. Then we have the following identities:*

$$\begin{array}{lrcl}
\bullet & Z^{\text{same}}(G_1 \parallel G_2; q, y) & = & \frac{1}{q} \cdot Z^{\text{same}}(G_1; q, y) \cdot Z^{\text{same}}(G_2; q, y), \\
\bullet & Z^{\text{dif}}(G_1 \parallel G_2; q, y) & = & \frac{1}{q(q-1)} \cdot Z^{\text{dif}}(G_1; q, y) \cdot Z^{\text{dif}}(G_2; q, y), \\
\bullet & Z(G_1 \bowtie G_2; q, y) & = & \frac{1}{q} \cdot Z(G_1; q, y) \cdot Z(G_2; q, y), \\
\bullet & Z^{\text{same}}(G_1 \bowtie G_2; q, y) & = & Z(G_1 \parallel G_2; q, y) \\
& & = & \frac{1}{q} \cdot Z^{\text{same}}(G_1; q, y) \cdot Z^{\text{same}}(G_2; q, y) \\
& & + & \frac{1}{q(q-1)} \cdot Z^{\text{dif}}(G_1; q, y) \cdot Z^{\text{dif}}(G_2; q, y), \\
\bullet & Z^{\text{dif}}(G_1 \bowtie G_2; q, y) & = & \frac{1}{q} \cdot Z^{\text{same}}(G_1; q, y) \cdot Z^{\text{dif}}(G_2; q, y) \\
& & + & \frac{1}{q} \cdot Z^{\text{dif}}(G_1; q, y) \cdot Z^{\text{same}}(G_2; q, y) \\
& & + & \frac{q-2}{q(q-1)} \cdot Z^{\text{dif}}(G_1; q, y) \cdot Z^{\text{dif}}(G_2; q, y).
\end{array}$$

Lemma 3.4 (Lemma 2.8). *Let G_1, G_2 be two two-terminal graphs and let $y \in \mathbb{C}$. Then the following identities hold as rational functions:*

$$y_{G_1 \parallel G_2} = y_{G_1} \cdot y_{G_2},$$
$$f_q(y_{G_1 \bowtie G_2}) = f_q(y_{G_1}) \cdot f_q(y_{G_2}).$$

Moreover, for any fixed $q_0, y_0 \in \mathbb{C}$, if $\{y_{G_1}(q_0, y_0), y_{G_2}(q_0, y_0)\} \neq \{0, \infty\}$, then

$$y_{G_1 \parallel G_2}(q_0, y_0) = y_{G_1}(q_0, y_0) \cdot y_{G_2}(q_0, y_0),$$

and if $\{y_{G_1}(q_0, y_0), y_{G_2}(q_0, y_0)\} \neq \{1, 1 - q_0\}$ and $q_0 \neq 0$, then

$$f_{q_0}(y_{G_1 \bowtie G_2}(q_0, y_0)) = f_{q_0}(y_{G_1}(q_0, y_0)) \cdot f_{q_0}(y_{G_2}(q_0, y_0)).$$

3.2.3 Representing algebraic numbers

We discuss here how we deal with the representation of algebraic numbers.

An algebraic number a is by definition a complex number that is the zero of some polynomial with integer coefficients. The minimal polynomial of a is the unique polynomial $p \in \mathbb{Z}[x]$ of smallest degree such that $p(a) = 0$, whose coefficients have no common prime factors, and whose leading coefficient is positive when $a \neq 0$. Following [29, 30] we represent an algebraic number a by its minimal polynomial together with an open rectangle in the complex plane (defined by two rational intervals parallel to the real and imaginary axis respectively) such that a is the only zero of the polynomial in that rectangle. There is some ambiguity here, as many rectangles will do the job. However one can check whether two representations represent the same number, by checking if the polynomials are equal and checking if the two rectangles intersect and by counting the number of zeros in the intersection (using for example the algorithm of Wilf [74]). When referring to an algebraic number we thus implicitly assume we have its minimal polynomial and a rectangle as above. The *size* of an algebraic number will thus be the number of bits needed to represent the minimal polynomial and the rectangle.

It is described in [68] how to execute all basic operations, i.e., addition, subtraction, multiplication and inversion in terms of this representation. Combining a result from Mahler [56], lower bounding the distance between roots of a polynomial, a result on using resultants to find polynomials with a prescribed zero [55], the famous factoring algorithm of Lenstra, Lenstra and Lovász [54], a result of Mignotte [57] bounding the coefficients of factors of polynomials, and an algorithm of Wilf [74] finding the number of zeros of a polynomial in a rectangle, it can be seen that these operations can be executed in polynomial time in the representation sizes. It follows that we can compare and compute absolute values of algebraic numbers and test whether two algebraic numbers are equal in polynomial time in their representation size.

3.3 Proof of the main results

In this section we state our main technical contributions, which we will prove in the subsequent sections. These results allow us to prove our main results at the end of this section. Let us first define for $q, y \in \mathbb{C}$ the more general computational problems (q, y)-Planar-Abs-RC and (q, y)-Planar-Arg-RC, which on input of a planar graph G ask for an 0.25-abs-approximation resp. an 0.25-arg-approximation to $Z(G; q, y)$. The problems q-Planar-Abs-Chromatic and q-Planar-Arg-Chromatic are the particular cases $(q, 0)$-Planar-Abs-RC and $(q, 0)$-Planar-Arg-RC.

3.3.1 Telescoping

For a graph G and an edge e of G we denote by $G \setminus e$ the graph obtained from e by removing the edge e and by G/e the graph obtained from G by contracting the edge e Recall that we allow multiple edges between two vertices and loops. We note here that the family of planar graphs is closed under deletion and contraction of edges. We recall the well known deletion contraction recurrence for a graph G and an edge e of G (contrary to the Tutte polynomial, this recurrence also holds when e is a bridge or a loop):

$$Z(G; q, y) = Z(G \setminus e; q, y) + (y - 1)Z(G/e; q, y). \tag{3.2}$$

If one can determine for any graph G and an edge $e \in E(G)$ one of the ratios $\frac{Z(G;q,y)}{Z(G/e;q,y)}$, or $\frac{Z(G;q,y)}{Z(G\setminus e;q,y)}$ exactly, then one can compute $Z(G; q, y)$ exactly by writing it as a telescoping product. A potential catch is that both $Z(G; q, y)$ and

$Z(G/e;q,y)$ (and hence $Z(G \setminus e;q,y)$ by (3.2)) could equal zero. Under suitable assumptions, our next result is able to deal with this issue. In what follows we denote for a graph H by size(H) the sum of the number of vertices of H and the number of edges of H.

Theorem 3.5. *Let q, y be fixed algebraic numbers, with $q \neq 0$. Suppose that we have access to an algorithm that on input of a planar graph G and an edge $e \in E(G)$ outputs an algebraic number r and a number $b \in \{0,1\}$ in polynomial time in* size(G) *such that*

- *If $Z(G/e;q,y) \neq 0$, then $b = 1$ and $r = \frac{Z(G;q,y)}{Z(G/e;q,y)}$;*
- *if $Z(G/e;q,y) = 0$ and $Z(G \setminus e;q,y) \neq 0$, then $b = 0$ and $r = 1$;*
- *if both $Z(G/e;q,y)$ and $Z(G\setminus e;q,y)$ are zero, then the algorithm may output any algebraic number r and bit b.*

Then there is an algorithm to compute $Z(G;q,y)$ in polynomial time in size(G).

Proof. We construct a sequence of planar graphs $G_0, G_1, \ldots, G_m$, where G_m is a graph with no edges, as follows. We let $G_0 = G$. Now for $i \geq 0$, we apply the assumed algorithm to the graph G_i and an edge e_i of G_i. Assume the algorithm outputs the pair (r_i, b_i). If $b_i = 1$ we set $G_{i+1} = G_i/e_i$ and if $b_i = 0$ we set $G_{i+1} = G_i \setminus e_i$. Let n denote the number of vertices of G_m. The output of our algorithm will be the number $q^n \prod_{i=0}^{m-1} r_i$. Since m is at most the number of edges of G and since size$(G_i) \leq$ size(G_0) for each i, this clearly takes polynomial time in size(G) to compute.

What remains to show is that

$$Z(G;q,y) = q^n \prod_{i=0}^{m-1} r_i. \tag{3.3}$$

To prove (3.3), let us first assume that $Z(G_0;q,y) = 0$. Then we do not know whether or not we can trust the output of the algorithm, as possibly both $Z(G/e_0;q,y)$ and $Z(G \setminus e_0;q,y)$ could be zero. However, there is a smallest index i such that $Z(G_i;q,y) \neq 0$ (since $Z(G_m;q,y) = q^n \neq 0$). In this case we know that $Z(G_{i-1};q,y) = 0$ and hence $b_{i-1} = 1, G_i = G_{i-1}/e_i$ and thus $r_{i-1} = 0$ as well. Therefore (3.3) holds in this case.

Let us next assume that $Z(G_0;q,y) \neq 0$. In that case one of the two values $Z(G/e_0;q,y)$, $Z(G \setminus e_0;q,y)$ must be non-zero. So in this case it holds that $Z(G_1;q,y) \neq 0$ and by induction it follows that for all i, $Z(G_i,q,y) \neq 0$ and thus $r_i \neq 0$ for all i. Next observe that

$$r_i = \frac{Z(G_i;q,y)}{Z(G_{i+1};q,y)}.$$

Indeed, if $b_i = 1$, then this follows by construction and if $b_i = 0$ we have $Z(G_{i+1}; q, y) = Z(G_i \setminus e_i; q, y) = Z(G_i; q, y)$ by (3.2) since $Z(G_i/e_i; q, y) = 0$. Therefore,

$$\frac{Z(G; q, y)}{q^n} = \frac{Z(G_0; q, y)}{Z(G_1; q, y)} \cdot \frac{Z(G_1; q, y)}{Z(G_2; q, y)} \cdots \frac{Z(G_{m-1}; q, y)}{Z(G_m; q, y)} = \prod_{i=0}^{m-1} r_i.$$

This finishes the proof. □

This result and its proof are fairly general and do not really rely on specific properties of the partition function of the random cluster model, but could equally well be applied to other polynomials and partition functions that satisfy some recurrence relation such as the independence polynomial for example.

3.3.2 Implementing gadgets

In the previous subsection we indicated that if one can test whether $Z(G/e; q, y)$ is zero or not and compute the ratios $\frac{Z(G;q,y)}{Z(G/e;q,y)}$ exactly this gives rise to an algorithm to exactly compute $Z(G; q, y)$.

The idea from [38] for real valued parameters is to use approximations to $\widehat{y}Z(G/e; q, y) + Z(G; q, y)$ to steer a binary search procedure to obtain a very precise approximation to the value y^* which satisfies $y^* Z(G/e; q, y) + Z(G; q, y) = 0$, or in other words the ratio $-\frac{Z(G;q,y)}{Z(G/e;q,y)}$. The following result describes the outcome of our box shrinking procedure, which extends this binary search procedure to complex parameters and moreover simplifies it. (It may be helpful to think of $A = Z(G/e; q, y)$ and $B = Z(G; q, y)$ in the statement of the theorem below.)

For $r > 0$ and $m \in \mathbb{C}$ we denote $B(m, r) = \{z \in \mathbb{C} \mid |z - m| < r\}$ and $B_\infty(m, r) = \{z \in \mathbb{C} \mid |\Re(z - m)| < r, |\Im(z - m)| < r\}$. Note that we can view B_∞ as an open ball of radius r centered at m in the ℓ^∞-metric on $\mathbb{R}^2 = \mathbb{C}$.

Theorem 3.6. *Let A, B complex numbers and let $C > 0$ be a rational number such that $|A|$ and $|B|$ are both at most C, and both are either 0 or at least $1/C$. Assume one of the following:*

- *there exists a* $\mathrm{poly}(\mathrm{size}(y_0, \varepsilon))$*-time algorithm to compute on input of $y_0 \in \mathbb{Q}[i]$ and a rational number $\varepsilon > 0$ an 0.25-abs-approximation of $A\widehat{y} + B$ for some algebraic number $\widehat{y} \in B(y_0, \varepsilon)$, or,*

- *there exists a* $\mathrm{poly}(\mathrm{size}(y_0, \varepsilon))$*-time algorithm to compute on input of $y_0 \in \mathbb{Q}[i]$ and a rational number $\varepsilon > 0$ an 0.25-arg-approximation of $A\widehat{y} + B$ for some algebraic number $\widehat{y} \in B(y_0, \varepsilon)$.*

Then there exists an algorithm that on input of a rational $\delta > 0$ and $C > 0$ as above that outputs "$A = 0$" when $A = 0$ and $B \neq 0$, and that outputs "$A \neq 0$" and a number $\overline{y} \in \mathbb{Q}[i]$ such that $-B/A \in B_\infty(\overline{y}, \delta/2)$ when $A \neq 0$. When $A = B = 0$ it is allowed to output anything. The running time is $\mathrm{poly}(\mathrm{size}(C, \delta))$.

We will prove Theorem 3.6 in Section 3.5. To utilize it we must be able to generate for any given value y_0 a number $\widehat{y}$ that approximates y_0 with arbitrary precision in a way that we can approximate the norm or argument of $\widehat{y}Z(G/e; q, y) + Z(G; q, y)$ efficiently. The idea, going back to Goldberg and Jerrum [38], is to use certain series-parallel gadgets for this task.

For fixed values of q, y, we call a series-parallel graph H a *series-parallel gadget for* (q, y) if $Z^{\mathrm{dif}}(H; q, y) \neq 0$. Let G be a graph with a designated edge $e = \{u, v\}$, and let H be a series-parallel gadget for (q, y). Then we construct the graph G' obtained from G and H by removing the edge e from G and identifying the start vertex of H with u and the terminal vertex with v. Note that by flipping u and v this may result in a different graph. We call any such graph an *implementation of H in G on e*. See Figure 3.2 for an illustration.

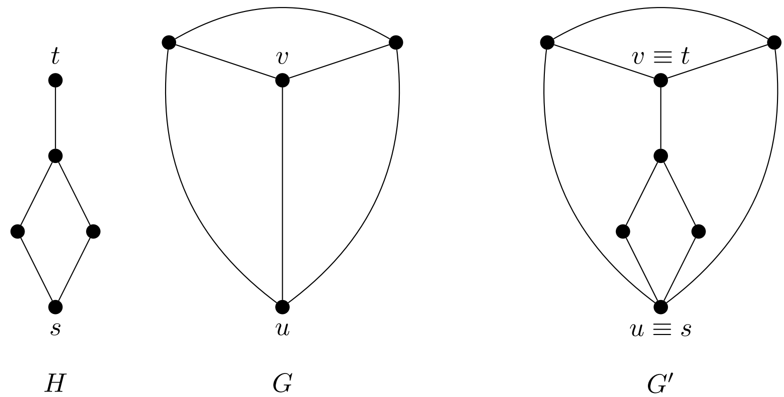

Figure 3.2: Implementation of H in G on the edge $\{u, v\}$.

The next result says that with access to an algorithm that efficiently solves any of the problems (q, y)-Planar-Arg-RC or (q, y)-Planar-Abs-RC, we can efficiently approximate $\widehat{y}Z(G/e; q, y) + Z(G; q, y)$ for any value $\widehat{y} + y$ that is the effective edge interaction of a series-parallel gadget.

Lemma 3.7. *Let $(q, y) \in \mathbb{C}^2$ be algebraic numbers, such that $q \notin \{0, 1\}$. Assume there exists a* $\mathrm{poly}(\mathrm{size}(G))$*-time algorithm to find an 0.25-abs-approximation (resp. 0.25-arg-approximation) of $Z(G; q, y)$ for planar graphs G. Then there exists a* $\mathrm{poly}(\mathrm{size}(G, H))$*-time algorithm to find an 0.25-abs-approximation (resp.*

0.25-arg-approximation) of $(y_H(q,y)-y)Z(G/e;q,y)+Z(G;q,y)$ *on input of any planar graph* G*, edge* e *of* G*, series-parallel gadget* H *and the number* $Z^{\mathrm{dif}}(H;q,y)$*.*

Proof. We will see G as a two-terminal graph, with the endpoints of e being the terminals. In this case, we have the relations

$$\begin{aligned}
Z^{\mathrm{same}}(G\setminus e;q,y) &= Z(G/e;q,y),\\
Z^{\mathrm{dif}}(G\setminus e;q,y) &= Z^{\mathrm{dif}}(G;q,y) = Z(G;q,y) - Z^{\mathrm{same}}(G;q,y)\\
&= Z(G;q,y) - yZ(G/e;q,y).
\end{aligned}$$

Let G' denote an implementation of H in G on e and observe that G' is planar. We compute using Lemma 3.3,

$$\begin{aligned}
Z(G';q,y) &= \tfrac{1}{q}\cdot Z^{\mathrm{same}}(G\setminus e;q,y)\cdot Z^{\mathrm{same}}(H;q,y)\\
&\quad + \tfrac{1}{q(q-1)}\cdot Z^{\mathrm{dif}}(G\setminus e;q,y)\cdot Z^{\mathrm{dif}}(H;q,y)\\
&= \tfrac{Z^{\mathrm{dif}}(H;q,y)}{q(q-1)}\cdot\big(y_H(q,y)\cdot Z^{\mathrm{same}}(G\setminus e;q,y) + Z^{\mathrm{dif}}(G\setminus e;q,y)\big)\\
&= \tfrac{Z^{\mathrm{dif}}(H;q,y)}{q(q-1)}\left((y_H(q,y)-y)Z(G/e;q,y) + Z(G;q,y)\right).
\end{aligned}$$

As H is a gadget we know that $Z^{\mathrm{dif}}(H;q,y)\neq 0$, and we have its exact value as input of the algorithm.

We then use the assumed algorithm to compute an 0.25-abs-approximation of $Z(G';q,y)$. Multiplying the output by $\left|\frac{q(q-1)}{Z^{\mathrm{dif}}(H;q,y)}\right|$ produces an 0.25-abs-approximation of $(y_H(q,y)-y)Z(G/e;q,y)+Z(G;q,y)$. Because $\mathrm{size}(G') = \mathrm{size}(G)+\mathrm{size}(H)-3$ this will be done in poly(size(G,H))-time, as desired.

Similarly, if we use the assumed algorithm to compute an 0.25-arg-approximation of $Z(G';q,y)$ and add $\arg\left(\frac{q(q-1)}{Z^{\mathrm{dif}}(H;q,y)}\right)$ to the output, we obtain an 0.25-arg-approximation of $(y_H(q,y)-y)Z(G/e;q,y)+Z(G;q,y)$ in poly(size(G,H))-time. □

In order to obtain a desired algorithm for Theorem 3.6 using Lemma 3.7 we need to be able to quickly approximate any desired value y_0 with effective edge interactions of gadgets. This final ingredient is Theorem 3.8, which will be proved in Section 3.4.

To do this precisely, we limit ourselves to a slightly smaller family of series-parallel graphs. Define $\mathcal{H}^*_{q,y}$ to be the family of all series-parallel graphs H that satisfy $Z^{\mathrm{dif}}(H;q,y)\neq 0$ and $y_H(q,y)\notin\{1,\infty\}$.

Theorem 3.8. *Let* $(q,y)\in\mathbb{C}^2\setminus\mathbb{R}^2$ *be algebraic numbers such that* $q\notin\{0,1\}$*. Assume there exists* $G\in\mathcal{H}^*_{q,y}$ *for which* $y_G(q,y)$ *or* $f_q(y_G(q,y))$ *is finite and has absolute value strictly bigger than* 1*. There exists an algorithm that for every*

$y_0 \in \mathbb{Q}[i]$ and rational $\varepsilon > 0$ outputs a series-parallel graph $H \in \mathcal{H}^*_{q,y}$ such that $y_H(q,y) \in B(y_0,\varepsilon)$, and outputs the number $Z^{\mathrm{dif}}(H;q,y)$. Both the running time of the algorithm and the size of the graph are $\mathrm{poly}(\mathrm{size}(\varepsilon, y_0))$.

3.3.3 Proof of Theorem 3.1

In this section we combine all ingredients mentioned above to provide a proof of Theorem 3.1

To utilize these ingredients, we need to introduce some concepts regarding algebraic numbers. We closely follow [30] in doing so.

We will introduce these concepts for polynomials with integer coefficients, and then automatically obtain definitions for algebraic numbers by applying the definition to their minimal polynomial. In particular, we can talk about the degree $d(\alpha)$ of an algebraic number α. We further define several variants of the height of a polynomial $p \in \mathbb{Z}[x]$: the usual height $H(p)$ is the largest absolute value among the coefficients, and the length $L(p)$ is the sum of the absolute value of the coefficients (similarly we define these notions for multi-variable polynomials). If we let d be the degree of p, let a_d be the leading coefficient of p and let α_i be the roots of p, we define the Mahler measure $M(p) := |a_d| \prod_{i=1}^{d} \max(1, |\alpha_i|)$ and finally the absolute logarithmic height $h(p) := \frac{1}{d}\log(M(p))$.

From the definitions we can deduce for any algebraic number α that $d(1/\alpha) = d(\alpha)$, $h(1/\alpha) = h(\alpha)$, and for every rational function $f \in \mathbb{Q}(x)$ that $d(f(\alpha)) \leq d(\alpha)$. The next lemma records some more intricate results about these heights, and gives some bounds for algebraic numbers of bounded height. Note in particular that item (b) yields the special case $h(\alpha_1\alpha_2) \leq h(\alpha_1) + h(\alpha_2)$ for any algebraic numbers α_1, α_2.

Lemma 3.9. *(a) For any non-zero algebraic number α we have $-d(\alpha)h(\alpha) \leq \log|\alpha| \leq d(\alpha)h(\alpha)$.*

(b) Let $p \in \mathbb{Z}[x_1,\dots,x_t]$ be a polynomial, $\alpha_1,\dots,\alpha_t$ algebraic numbers, and define the algebraic number $\beta = p(\alpha_1,\dots,\alpha_t)$. Write $d_i(p)$ for the degree of p as polynomial in x_i, then $h(\beta) \leq \log(L(p)) + \sum_{i=1}^{t} d_i(p)h(\alpha_i)$.

(c) For any polynomial $p \in \mathbb{Z}[x]$ we have

$$\tfrac{1}{d(p)}\log(H(p)) - \log(2) \leq h(p).$$

Proof. All parts can be found in [73]: parts (a) and (b) are respectively in Section 3.5.1 and Lemma 3.7; part (c) is in Lemma 3.11 for the special case that p is irreducible, but the proof works for any polynomial p. □

Corollary 3.10. *Let q, y be algebraic numbers and let G be a graph with n vertices and m edges. Write d, h_q, h_y for respectively $d(q)\cdot d(y), h(q), h(y)$.*

(a) If $Z(G;q,y) \neq 0$*, then* $\big|\log|Z(G;q,y)|\big| \leq (nh_q + mh_y + 2m\log(2))\,d$.

(b) Let e *be an edge of* G *and assume that* $Z(G/e;q,y) \neq 0$*. Then*

$$\log\left(H\left(\frac{Z(G;q,y)}{Z(G/e;q,y)}\right)\right) \leq (2nh_q + 2mh_y + 4m\log(2))\,d.$$

Proof. Let $G = (V,E)$ be our graph, recall that $Z(G;q,y) = \sum_{A\subseteq E} q^{k(A)}(y-1)^{|A|}$. From this we see that $Z(G;q,y)$ is a polynomial of degree n in q and degree m in $y-1$. In the variables q and $y-1$, we also see that the sum of the absolute value of the coefficients is 2^m. Note that part (b) of the previous lemma gives $h(y-1) \leq h(y) + \log(2)$, and applying it again to Z gives $h(Z(G;q,y)) \leq m\log(2) + nh_q + m(h_y + \log(2))$. Further $d(Z(G;q,y)) \leq d$. Then by part (a) when $Z(G;q,y) \neq 0$, we see that $\big|\log|Z(G;q,y)|\big| \leq (nh_q + mh_y + 2m\log(2))d$, proving part (a).

For part (b) we first look at the absolute logarithmic height of the ratio. We have

$$\begin{aligned} h\left(\frac{Z(G;q,y)}{Z(G/e;q,y)}\right) &\leq h(Z(G;q)) + h(Z(G/e;q)) \\ &\leq (2n-1)h_q + (2m-1)h_y + (4m-2)\log(2). \end{aligned}$$

Again the ratio has degree at most d, so part (c) of the previous lemma gives

$$\begin{aligned} \log\left(H\left(\frac{Z(G;q,y)}{Z(G/e;q,y)}\right)\right) &\leq \left(h\left(\frac{Z(G;q,y)}{Z(G/e;q,y)}\right) + \log(2)\right) d \\ &\leq (2nh_q + 2mh_y + 4m\log(2))\,d. \qquad \square \end{aligned}$$

The following result follows from a slight modification of a result of Kannan, Lenstra and Lovász [51].

Proposition 3.11. *Let* $d, H \in \mathbb{N}_{\geq 2}$ *and* $\overline{\alpha} \in \mathbb{Q}[i]$*. There exists an algorithm that on input* d, H *and* $\overline{\alpha}$ *outputs a polynomial* p *such that if there exists an algebraic number* α *of degree at most* d*, height at most* H *such that* $\log|\alpha - \overline{\alpha}| \leq -(d^2 + 5d + 2d\log(H))$*, then* p *is the minimal polynomial of* α*. The running time is bounded by* $\mathrm{poly}(d, \log(H), \mathrm{size}(\overline{\alpha}))$*.*

Proof. We first describe the algorithm and then prove its correctness.

We first assume $|\overline{\alpha}| \leq 1$. We will apply Algorithm 1.16 from [51]. If the algorithm terminates within d steps and outputs a polynomial p, we output p. If after d steps the algorithm does not return anything we output 0. If $|\overline{\alpha}| \geq 1$ we replace $\overline{\alpha}$ by $1/\overline{\alpha}$ and we refer to the proof [51, Theorem 1.19] how to modify the output of Algorithm 1.16 in this case.

To prove correctness, we may assume $|\overline{\alpha}| \leq 1$ and we may further assume there exist an algebraic number α of degree at most d and height at most H such that

$$|\alpha - \overline{\alpha}| \leq e^{-(d^2+5d)} \cdot H^{-2d}. \tag{3.4}$$

Otherwise there is nothing to prove.

We next note that as $d \geq 2$ we have

$$d^2 + 5d \geq \log(48d(d+1)^{(3d+4)/2} 2^{d^2/2}),$$

therefore there exists a non-negative integer $s \geq 2$ such that

$$\tfrac{1}{12d} e^{d^2+5d} \cdot H^{2d} \geq 2^s \geq 2(d+1)^{(3d+4)/2} 2^{d^2/2} \cdot H^{2d}. \tag{3.5}$$

If $|\alpha| \leq 1$ the conclusion follows immediately from the correctness of [51, Algorithm 1.16]. Unfortunately we do not have this information, but we will argue that even in the case $|\alpha| > 1$ the algorithm is still correct. We will show that by assuming that a bound of $1+1/(2d)$ on $|\alpha|$ is sufficient. By (3.4) and (3.5) this bound is clearly satisfied and it implies that for any $k = 1, \dots, d$, $|\alpha|^k \leq 2$. With this bound on $|\alpha|$ [51, Proposition 1.6] remains true if we multiply the right-hand side by $1/2$. By our slightly stronger lower bound on s we see that also the conclusion of [51, Lemma 1.9] is also valid for this bound on $|\alpha|$. Finally as in [51, Explanation 1.17] the conditions of [51, Theorem 1.15] are still met for this bound on $|\alpha|$. Therefore the output of [51, Algorithm 1.16] is indeed the minimal polynomial of α, as desired

The running time bound follows from (the proof of) Theorem 1.19 from [51] where we note that [51] in fact does not require the input of the number $\overline{\alpha}$, but only a certain number of bits of it and therefore the size of $\overline{\alpha}$ does not appear in the running time of the statement of the theorem. To avoid dealing with how to feed $\overline{\alpha}$ to the algorithm we just allow the algorithm to 'read' it completely. ☐

Now we are ready to prove the following result, from which we will derive Theorem 3.1 after giving the proof.

Theorem 3.12. *Let $(q, y) \in \mathbb{C}^2 \setminus \mathbb{R}^2$ be algebraic numbers, such that $q \notin \{0, 1, 2\}$. Assume there exists $G \in \mathcal{H}^*_{q,y}$ for which $y_G(q, y)$ or $f_q(y_G(q, y))$ is finite and has absolute value strictly bigger than 1. Then the problems (q, y)-*Planar-Abs-RC *and (q, y)-*Planar-Arg-RC *are #P-hard.*

Proof. We will assume there exists a polynomial-time algorithm for either the problem (q, y)-Planar-Abs-RC or (q, y)-Planar-Arg-RC, and then we will find a poly(size(G))-time algorithm to compute $Z(G; q, y)$ exactly for planar graphs G. Since the latter problem is #P-hard by [72], apart from several exceptional values, this implies that (q, y)-Planar-Abs-RC and (q, y)-Planar-Arg-RC are #P-hard as well. The exceptional values are $y = 1$, $q = 0, 1, 2$,

$(q,y)=(3,e^{\pm\frac{2\pi i}{3}})$ and $(q,y)=(4,-1)$. We explicitly excluded most of them in the statement, while for $y=1$ the family $\mathcal{H}^*_{q,y}$ is empty, and for $(q,y)=(3,e^{\pm\frac{2\pi i}{3}})$ we can show that $y_G(q,y)$ is contained in $\{e^{\frac{2\pi i}{3}},e^{-\frac{2\pi i}{3}}\}$ for any $G\in\mathcal{H}^*_{q,y}$ (see the proof of Corollary 3.13 below).

The proof essentially consists of linking together Theorem 3.8, Lemma 3.7, Theorem 3.6, Corollary 3.10, Proposition 3.11 and Theorem 3.5, roughly in that same order.

To keep track of the running times of all the separate algorithms we describe and analyze the resulting algorithm in one go and prove its correctness afterwards.

The assumption in the theorem means that we can apply Theorem 3.8 to find a gadget F with effective edge interaction y_F close to y_0+y for any y_0 (note that $\text{size}(y_0+y)=O(\text{size}(y_0))$). We use F and $Z^{\text{dif}}(F;q,y)$ in Lemma 3.7 to approximate $(y_F-y)Z(G/e;q,y)+Z(G;q,y)$. Combined we obtain an algorithm that on input of $y_0\in\mathbb{Q}[i]$, rational $\varepsilon>0$, planar graph G and an edge e, will compute an 0.25-abs-approximation (resp. 0.25-arg-approximation) to $\widehat{y}Z(G/e;q,y)+Z(G;q,y)$ for some algebraic number $\widehat{y}\in B(y_0,\varepsilon)$, in $\text{poly}(\text{size}(G,y_0,\varepsilon))$-time.

As in Corollary 3.10, we write h_q,h_y for the absolute logarithmic height of q and y, and $d=d(q)d(y)$. We now want to apply Theorem 3.6 with $A=Z(G/e;q,y)$ and $B=Z(G;q,y)$. To do so we must find an common upper bound for them. Note that by Corollary 3.10, we have that $|\log|A||$ and $|\log|B||$ are both bounded by $(nh_q+mh_y+2m\log(2))d$, where n resp. m denote the number of vertices resp. edges of G. Now taking C such that $\log(C)=(nh_q+mh_y+2m\log(2))d$ we satisfy the assumption of Theorem 3.6. We thus have that $C=O(\text{size}(G))$ (since q,y and hence h_q,h_y,d are considered to be constant).

Take H such that $\log(H)=2(nh_q+mh_y+2m\log(2))d$. We now apply the algorithm from Theorem 3.6 with $\log(\delta^{-1})=(d^2+5d)+2d\log(H)$ and as output we get either the statement that "$A=0$", or "$A\neq 0$" and a number $\overline{y}$ such that $Z(G;q,y)/Z(G/e;q,y)\in B_\infty(-\overline{y},\delta/2)$. The running time is bounded by $\text{poly}(\text{size}(C,\delta))=\text{poly}(\text{size}(G))$ (using that h_q,h_y,d are constants).

We now turn this algorithm into an algorithm as required in Theorem 3.5. In case we get "$A=0$", we output the pair $(0,1)$. In case we get "$A\neq 0$", we run the algorithm of Proposition 3.11 on input of d,H and $-\overline{y}$ and let p be the output of this algorithm. We output the pair $(1,(p,B_\infty(-\overline{y},\delta/2)))$. The running time is $\text{poly}(\text{size}(G))$, using that h_q,h_y,d are constants and the size of $\overline{y}$ is $\text{poly}(\text{size}(G))$. This implies that the overall running time is $\text{poly}(\text{size}(G))$, as desired.

We next turn to proving correctness of our algorithm. What remains is to show that the algorithm as required in Theorem 3.5 is correct. If both $Z(G/e;q,y)$ and $Z(G\setminus e;q,y)$ are 0, there is nothing to prove. So let us first assume that $Z(G/e;q,y)=0$ and $Z(G\setminus e;q,y)\neq 0$. Then by correctness of Theorem 3.6 we know that our algorithm gives the desired output, namely the pair $(0,1)$. Similarly, if $Z(G/e;q,y)\neq 0$, then by Corollary 3.10(b) we know that $y^*=$

$-Z(G;q,y)/Z(G/e;q,y)$ is an algebraic number of degree at most d and height at most H and by correctness of Theorem 3.6 it is contained in $B_\infty(\overline{y},\delta/2)$). Therefore, by our choice of δ, Proposition 3.11 implies that p is indeed the minimal polynomial of $-y^*$ in this case. By a result of Mahler [56], the absolute value of the logarithm of the distance between any two distinct zeros of p is upper bounded by

$$-\log(\sqrt{3})+\tfrac{d+2}{2}\log(d)+(d-1)\log((d+1)H) \le \tfrac{3}{2}d\log(d+1)+d\log(H) \le \log(\delta^{-1}).$$

Therefore the rational rectangle $B_\infty(-\overline{y},\delta/2)$ together with p is a representation of the algebraic number $-y^*$, and so the output of our algorithm is as desired.

This finishes the proof. □

In the following corollary, we indicate several regions of q,y-parameters for which Theorem 3.12 applies. Part (a) includes the (complex-valued) ferromagnetic Potts model. Theorem 3.1 is included in part (b). Note that part (b) also contains Theorem 1.5 in [30], our methods merely give a different proof. Note however that part (b) is not tight, as Figure 3.1 shows a larger region of q-values where both problems are #P-hard.

Corollary 3.13. *Let $(q,y) \in \mathbb{C}^2\backslash\mathbb{R}^2$ be algebraic numbers, such that $q \notin \{0,1,2\}$. In both of the following cases, the problems (q,y)-*Planar-Abs-RC *and (q,y)-*Planar-Arg-RC *are #P-hard:*

(a) $|y| > 1$;

(b) $|1-q| > 1$ or $\Re(q) > 3/2$, except when $y = 1$ or $(q,y) = (3, e^{\pm\frac{2\pi i}{3}})$;

(c) $Z(G;q) = 0$ for a planar graph G, except when $y = 1$ or $(q,y) = (3, e^{\pm\frac{2\pi i}{3}})$.

Proof. For part (a), we simply have to note that $K_2 \in \mathcal{H}^*_{q,y}$ and $y_{K_2}(q,y) = y$, so the graph K_2 satisfies the requirement of Theorem 3.12.

For part (b), the assumption yields that either $f_q(0) = 1-q$ or $f_q(f_q(0)^2) = \frac{q-1}{q-2}$ will have absolute value strictly bigger than 1. This means that if we find a y_G or a $f_q(y_G)$ close to these values, we can apply Theorem 3.12.

In what follows we implicitly use Lemma 3.4 and Proposition 3.14 below to see that certain numbers are the virtual or effective edge interaction of some series-parallel graph in $\mathcal{H}^*_{q,y}$. We assume $y \neq 1$, which ensures that the family $\mathcal{H}^*_{q,y}$ is non-empty. If there now exists a graph G in this family with $|y_G| > 1$, we are done. However, if $|y_G| < 1$, its powers converge to 0. This means that for N large enough, one of $f_q(y_G^N)$ and $f_q(f_q(y_G^N)^2)$ will have absolute value strictly bigger than 1, and both are the (virtual) interaction of a series-parallel graph in $\mathcal{H}^*_{q,y}$. So we are left with the case that $|y_G| = 1$ for all $G \in \mathcal{H}^*_{q,y}$. Applying the same argument to $f_q(y_G)$, we may even assume that $|f_q(y_G)| = 1$ for all $G \in \mathcal{H}^*_{q,y}$.

Geometrically, the equations $|z| = 1$ and $|f_q(z)| = 1$ give a circle and a line in the complex plane, so there can be at most two common solutions z. This means that the sets $\{y_G(q, y) \mid G \in \mathcal{H}^*_{q,y}\}$ and $\{f_q(y_G(q, y)) \mid G \in \mathcal{H}^*_{q,y}\}$ both contain at most two elements. But from Lemma 3.4 and Proposition 3.14 below it follows that both sets are closed under taking products, as long as the product is not 1. Therefore the only options for both sets are $\{-1\}$ and $\{e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\}$, corresponding to $(q, y) \in \{(4, -1), (3, e^{\frac{2\pi i}{3}}), (3, e^{-\frac{2\pi i}{3}})\}$. All these options are excluded, so we conclude that this special case cannot occur.

For part (c), we will consider the family $\mathcal{P}^*_{q,y}$ which generalizes $\mathcal{H}^*_{q,y}$. This consists of all planar two-terminal graphs H for which the terminals are on the same face, and for which $Z^{\text{dif}}(H; q, y) \neq 0$ and $y_H(q, y) \notin \{1, \infty\}$. This family is still more or less closed under series and parallel composition, in the sens that Proposition 3.14 is still true for this family. Therefore Theorem 3.8 remains true for this family $\mathcal{P}^*_{q,y}$ with virutally the same proof.

We will first find a graph $G' \in \mathcal{P}^*_{q,0}$ with $1 < |y_{G'}(q, 0)| < \infty$, and we will closely follow the proof of Theorem 2.26 in doing this. If we can find a graph $H \in \mathcal{P}^*_{q,0}$ with $0 < |y_H(q, 0)| < 1$, we can put many copies of H in parallel, so we can assume that $y_H(q, 0)$ is arbitrarily close to 0. We construct G' as follows: choose any edge of G, take $G \setminus e$ with the endpoints of e as terminals (which are automatically on the same face), and replace all of its edges with H. As in the proof of Theorem 2.26 we see that $1 < |y_{G'}(q, 0)| < \infty$ (possibly by assuming G is the minimal graph with q as chromatic zero), so we only need to show that $Z^{\text{dif}}(G'; q) \neq 0$. From the proof of Lemma 2.28 we see that

$$Z^{\text{dif}}(G'; q) = Z^{\text{dif}}(G \setminus e; q, y_H(q, 0)) \cdot \left(\frac{Z^{\text{dif}}(H; q)}{q(q-1)}\right)^{|E(G \setminus e)|}.$$

The assumption that $H \in \mathcal{P}^*_{q,0}$, implies that the second factor is non-zero. The polynomial $Z^{\text{dif}}(G \setminus e; q, t)$ equals $Z(G; q) = 0$ at $t = 0$, and is non-zero at $t = 1$. So for $t = y_H(q, 0)$ close enough to zero, $Z^{\text{dif}}(G \setminus e; q, y_H(q, 0))$ must be non-zero, and then $Z^{\text{dif}}(G'; q)$ is also non-zero.

Of course we are immediately done if there exists a graph $H \in \mathcal{P}^*_{q,0}$ with $1 < |y_H(q, 0)| < \infty$, so we are left with the case where $|y_H(q, 0)|$ is 0 or 1 for all graphs $H \in \mathcal{P}^*_{q,0}$.

In case $q \notin \{1 \pm i, \frac{3 \pm i\sqrt{3}}{2}\}$, the paths P_2 and P_4 are in $\mathcal{P}^*_{q,0}$, and just as in the proof of Theorem 2.26 we see that either $|y_{P_2}(q, 0)|$ or $|y_{P_4}(q, 0)|$ is not in $\{0, 1\}$. If $q = \frac{3 \pm i\sqrt{3}}{2}$, we still find a (series-parallel) graph in $\mathcal{P}^*_{q,0}$ with effective interaction $f_q(f_q(f_q(f_q(0)^2)^3)^2)$ of absolute value $\sqrt{3/7}$. Finally, if $q = 1 \pm i$, the path P_4 is not in $\mathcal{P}^*_{q,0}$, but P_2 is, and its effective interaction has absolute value $\sqrt{1/2}$. This finishes the case $y = 0$.

To also find the required graph in $\mathcal{P}^*_{q,y}$ for other values of y, we can essentially copy the proof of part (b). If we find a graph $H \in \mathcal{P}^*_{q,y}$ such that $|y_H(q,y)| < 1$, we can put many copies of H in parallel, so we assume that y_H is arbitrarily close to 0. Then we replace all the edges of G' with H to find a graph in $\mathcal{P}^*_{q,y}$ with effective interaction $y_{G'}(q, y_H)$ (follows from Lemma 2.28) which then has absolute value bigger than 1. All other cases remain unchanged. $\square$

3.4 Constructing series-parallel gadgets

In this section we will prove Theorem 3.8.

3.4.1 A family of potential gadgets

We start by exhibiting a family of series-parallel graphs that can serve as gadgets.

Let $(q, y) \in \mathbb{C}^2$ and recall that we denote by $\mathcal{H}^*_{q,y}$ the family of all series-parallel graphs H that satisfy $Z^{\text{dif}}(H; q, y) \neq 0$ and $y_H(q, y) \notin \{1, \infty\}$. Note that if $y = 1$, the family $\mathcal{H}^*_{q,y}$ is always empty; while for $y \neq 1$ and $q \notin \{0, 1\}$, the edge K_2 is contained in $\mathcal{H}^*_{q,y}$. The next result says that this family is more or less closed under taking parallel and series composition.

Proposition 3.14. *Let $(q, y) \in \mathbb{C}^2$ such that $q \notin \{0, 1\}$, and let $H_1, H_2 \in \mathcal{H}^*_{q,y}$.*

- *If $y_{H_1 \parallel H_2}(q, y) \notin \{1, \infty\}$, then $H_1 \parallel H_2 \in \mathcal{H}^*_{q,y}$.*
- *If $y_{H_1 \bowtie H_2}(q, y) \notin \{1, \infty\}$, then $H_1 \bowtie H_2 \in \mathcal{H}^*_{q,y}$.*

Proof. Write $H = H_1 \parallel H_2$. Lemma 3.3 yields

$$Z^{\text{dif}}(H; q, y) = \tfrac{1}{q(q-1)} \cdot Z^{\text{dif}}(H_1; q, y) \cdot Z^{\text{dif}}(H_2; q, y).$$

If $Z^{\text{dif}}(H; q, y) = 0$, then $Z^{\text{dif}}(H_i; q, y) = 0$ for some $i \in \{1, 2\}$, which is a contradiction. So we conclude that $Z^{\text{dif}}(H; q, y) \neq 0$, and hence that $H \in \mathcal{H}^*_{q,y}$.

The proof for $H = H_1 \bowtie H_2$ is similar, we first recall from Lemma 3.3:

$$\begin{aligned} Z(H; q, y) &= \tfrac{1}{q} \cdot Z(H_1; q, y) \cdot Z(H_2; q, y), \\ Z^{\text{same}}(H; q, y) &= \tfrac{1}{q} \cdot Z^{\text{same}}(H_1; q, y) \cdot Z^{\text{same}}(H_2; q, y) \\ &\quad + \tfrac{1}{q(q-1)} \cdot Z^{\text{dif}}(H_1; q, y) \cdot Z^{\text{dif}}(H_2; q, y). \end{aligned}$$

Suppose again that $Z^{\text{dif}}(H; q, y) = 0$. This implies that $Z^{\text{same}}(H; q, y) = 0$, since $y_H \neq \infty$. Therefore $Z(H; q, y) = 0$, and it follows without loss of generality that $Z(H_2; q, y) = 0$. So $Z^{\text{same}}(H_2; q, y) = -Z^{\text{dif}}(H_2; q, y)$, and they are non-zero by assumption. Plugging this into the second equation, together with

$Z^{\mathrm{same}}(H;q,y) = 0$, yields $(q-1)Z^{\mathrm{same}}(H_1;q,y) = Z^{\mathrm{dif}}(H_1;q,y)$. We also assumed that $Z^{\mathrm{dif}}(H_1;q,y) \neq 0$, leading to $y_{H_1}(q,y) = 1$, which is a contradiction. Therefore we can again conclude that $H \in \mathcal{H}^*_{q,y}$. □

3.4.2 Values of q, y with dense set of interactions

In this subsection we determine for which values of q, y the effective edge interactions of members of $\mathcal{H}^*_{q,y}$ are dense in the complex plane. The next subsection deals with making this algorithmic yielding a proof of Theorem 3.8.

The following lemma will turn out to be useful to get density results. We say that a set $S \subseteq \mathbb{C}$ is ε-dense if for every $z \in \mathbb{C}$ there exists a $z' \in S$ such that $|z - z'| < \varepsilon$.

Lemma 3.15. *Let $\varepsilon > 0$ and let $a, b, c \in B(0, \varepsilon)$ such that the convex cone spanned by a, b, c is $\mathbb{C}$. Then the set $a\mathbb{N} + b\mathbb{N} + c\mathbb{N}$ is ε-dense in $\mathbb{C}$.*

Proof. For any $z \in \mathbb{C}$ we can write $z = ra + sb + tc$ with $r, s, t \geq 0$. We can even assume that one of r, s, t is zero, say $t = 0$. Now we round r, s to the nearest integers R, S. Then $Ra + Sb \in a\mathbb{N} + b\mathbb{N}$ and

$$|Ra + Sb - z| \leq |R - r||a| + |S - s||b| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \qquad \square$$

Recall the definition $f_q(z) = 1 + \frac{q}{z-1}$. Also when G is a two-terminal graph, we call $f_q(y_G(q,y))$ its *virtual interaction.*

Proposition 3.16. *Let $(q,y) \in \mathbb{C}^2 \setminus \mathbb{R}^2$, such that $q \notin \{0,1\}$. Assume there exists a series-parallel graph $G \in \mathcal{H}^*_{q,y}$ such that $y_G(q,y)$ or $f_q(y_G(q,y))$ is finite and has absolute value strictly bigger than 1, then the set $\{y_H(q,y) \mid H \in \mathcal{H}^*_{q,y}\}$ is dense in $\mathbb{C}$.*

Proof. We start by showing that we may assume that $|f_q(y_G)| > 1$ and $f_q(y_G)$ is not real for some $G \in \mathcal{H}^*_{q,y}$. In what follows we implicitly use Lemma 3.4 and Proposition 3.14 to conclude that certain numbers are the virtual or effective edge interaction of some series-parallel graph in $\mathcal{H}^*_{q,y}$.

First assume that $|y_G| > 1$ and y_G is real. Let $\xi_n = y_G^n$ for $n \in \mathbb{N}$. If $y \notin \mathbb{R}$, then $y\xi_n$ is non-real and it is converges to infinity as $n \to \infty$. If $y \in \mathbb{R}$, then $q \notin \mathbb{R}$ and $f_q(f_q(y)^2) = 1 + \frac{(y-1)^2}{2y+q-2}$ is also non-real (note that $y \neq 1$ because $\mathcal{H}^*_{q,y}$ is non-empty). This means that $f_q(f_q(y)^2)\xi_n$ is non-real and converges to infinity as $n \to \infty$. In either case we find a non-real term in the sequence with absolute value bigger than 1, which is the effective edge interaction of a graph in $\mathcal{H}^*_{q,y}$. The precise graphs is a parallel composition of sufficiently many copies of G, together with either one edge or one path of length 2.

For the second case, assume that $|y_G| > 1$ and y_G is not real. Then powers of y_G converge to ∞ and moreover the arguments (modulo 2π) of these powers take on at least 3 values. If the number of these values is unbounded, it is clear that for some n, $f_q(y_G^n)$ is non real and $|f_q(y_G^n)| > 1$. Otherwise, if these values take on precisely $k \geq 3$ values, the values y_G^n converge towards ∞ on the k rays from 0 through $e^{2\pi j/k}$, $j = 0, \ldots, k-1$. The images of these rays under f_q are circular arcs from $f_q(\infty) = 1$ to $f_q(0) = 1 - q$ making pairwise angles of $2\pi/k$ at 1. Since $k \geq 3$, at least one of these arcs contains a open segment starting at 1 that does not intersect the closed unit disk. Any element from this segment has absolute value bigger than 1, thus there exists a series-parallel graph $G' \in \mathcal{H}^*_{q,y}$ such that $|f_q(y_{G'})| > 1$, that is the parallel composition of sufficiently many copies of G.

Let τ be a non-real element of $\{f_q(y), f_q(y^2)\} = \{1 + \frac{q}{y-1}, 1 + \frac{q}{(y-1)(y+1)}\}$. Then $\tau \cdot \xi^n$ is not real, converges to infinity, and all are virtual interactions of graphs in $\mathcal{H}^*_{q,y}$ corresponding to series composition of copies of G with either an edge or a digon (two parallel edges). Therefore we will find one of absolute value more than 1, as desired.

From now on we thus assume that we have $G \in \mathcal{H}^*_{q,y}$ such that $f_q(y_G)$ is not real and $|f_q(y_G)| > 1$. Consider next the Möbius transformation

$$g(z) := f_q(f_q(z)f_q(y_G)) = \frac{q - 1 + zy_G}{q - 2 + z + y_G}. \tag{3.6}$$

Note that $z = 1$ is a fixed point of g and that $g'(1) = 1/f_q(y_G)$. So by assumption $|g'(1)| < 1$.

Now consider the sequence defined by $y_1 = y_G$ and $y_{i+1} = g(y_i)$, which converges to 1. The only reason that this could not be true is if y_G would be equal to the other (repelling) fixed point of g. The other fixed point of g is given by $1 - q$. Now $y_G \neq 1 - q$ since $f_q(1 - q) = 0$, while $|f_q(y_G)| > 1$.

We next claim that taking finite products of terms in this sequence, produces a dense subset of $\mathbb{C}$. Indeed, note that

$$\frac{\log(y_{i+1})}{\log(y_i)} = \frac{\log(g(y_i)) - \log(g(1))}{\log(y_i) - \log(1)} \to g'(1) = 1/f_q(y_G)$$

as $i \to \infty$. Since the number $1/f_q(y_G)$ is non-real, at some point the difference between the arguments of the $\log(y_i)$ lie in a small interval around $\arg(1/f_q(y_G))$, which is nonzero $\mod \pi$. In particular, this means that for any k the argument of the sequence $(\log(y_i))_{i \geq k}$ cannot be contained in a half plane. Then for every $\varepsilon > 0$, we can find three terms in the sequence that satisfy the requirements of Lemma 3.15. This lemma then implies that products of the values y_i are dense in $\mathbb{C}$. These products correspond exactly to parallel compositions of series connections of G by Lemma 3.4. By omitting any effective edge interactions equal to 1 possibly obtained by taking products of the y_i we still obtain density and since the corresponding 2-terminal graphs are contained in $\mathcal{H}^*_{q,y}$, this finishes the proof. $\square$

3.4.3 Implementing a dense set fast

In the previous subsection we saw for which values of q, y we can obtain density. We will show here how to exploit this algorithmically, by proving Theorem 3.8.

First we record the following lemma from [29], which in fact is essentially Lemma 2.8 from [10] adapted to algebraic numbers.

Lemma 3.17. *Let $m \in \mathbb{Q}[i]$ and $r > 0$ rational. Further suppose that we have Möbius transformations with algebraic coefficients $\Phi_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ for $i \in [t]$, satisfying the following with $U = B(m, r)$:*

1. *for each $i \in [t]$, Φ_i is contracting on U;*
2. *$U \subseteq \bigcup_{i=1}^t \Phi_i(U)$.*

There is an algorithm which, on input of algebraic numbers $x_0, x_1 \in U$ (respectively the target and starting point) and rational $\varepsilon > 0$, outputs in $\mathrm{poly}(\mathrm{size}(x_0, x_1, \varepsilon))$*-time an algebraic number $\widehat{x} \in B(x_0, \varepsilon)$ and a sequence $i_1, \ldots, i_k \in [t]$ such that*

$$\begin{aligned} &\widehat{x} = \Phi_{i_k}(\cdots \Phi_{i_1}(x_1)\cdots), && k = O(\log(\varepsilon^{-1})), \\ &\textit{and } \Phi_{i_j}(\cdots \Phi_{i_1}(x_1)\cdots) \in U && \textit{for all } j = 1, \ldots, k. \end{aligned}$$

Technically the last requirement in the lemma is not stated as such in [29] nor [10], but it follows immediately from the proof.

To describe the algorithm for Theorem 3.8, we first precompute some data which only depends on q, y (and not on y_0 or ε), so this counts as constant time. In the precomputation step, we find the following:

- a rational number $r > 0$ so that $U := B(1, r)$ is an open around 1;
- maps Φ_i as in the above lemma.

By the proof of Proposition 3.16, we may assume that $f_q(y_G(q, y))$ is non-real and has absolute value more than 1 for some $G \in \mathcal{H}^*_{q,y}$.

Recall the definition of the Möbius transformation g in (3.6) and recall from the proof of Proposition 3.16 that $g(1) = 1$ and $|g'(1)| < 1$. We can therefore choose an α such that $|g'(1)| < \alpha < 1$. Then take U to be a ball $B(1, r)$ contained in the open set

$$\{u \in \mathbb{C} \mid \alpha < |u| < 1/\alpha, |g'(u)| < \alpha\}.$$

This ensures that for any $u \in U$, the map $z \mapsto g(z)u$ is a contraction on U.

Restricting to effective edge interactions $y_H(q, y)$ contained in U for $H \in \mathcal{H}^*_{q,y}$, we see that by Proposition 3.16 the open sets $g(U)y_H$ cover the compact set $\overline{U}$. By the proof of Proposition 3.16 we can select a finite set $\{y_i \mid i \in I\}$ in finite time such that $\cup_{i \in I} g(U)y_i$ already covers $\overline{U}$. Now the Φ_i are defined to be the maps $z \mapsto g(z)y_i$. Let us fix for each $i \in I$ a series-parallel graph $H_i \in \mathcal{H}^*_{q,y}$ whose effective edge interaction is equal to y_i. Using Lemma 3.3, we also compute $Z^{\text{same}}(H_i; q, y)$ and $Z^{\text{dif}}(H_i; q, y)$.

We now give a proof of Theorem 3.8.

Proof of Theorem 3.8. We first consider the case where $y_0 \in U$. We take as starting point $y_1 = 1$ in U and run the algorithm of Lemma 3.17. This yields in poly(size(y_0, ε)) time a sequence $i_1, \ldots, i_k$ with $\widehat{y} = \Phi_{i_k}(\cdots \Phi_{i_1}(y_1) \cdots) \in B(y_0, \varepsilon)$. (Recall that $k = O(\log(\varepsilon^{-1}))$.) We may assume that $\Phi_{i_j}(\cdots \Phi_{i_1}(y_1) \cdots) \neq 1$ for all $j = 1, \ldots, k$. Otherwise we replace the sequence by $i_{j+1}, \ldots, i_k$ with j the largest index for which $\Phi_{i_j}(\cdots \Phi_{i_1}(y_1) \cdots) = 1$.

From the sequence $i_1, \ldots, i_k$ we can determine a sequence of series-parallel graphs $G_1, \ldots, G_k$. The sequence starts with $G_1 = H_{i_1}$, which has effective edge interaction $\Phi_{i_1}(1) = y_i$. Recall that every map Φ_i is of the form $z \mapsto y_i g(z)$, which by Lemma 3.4 corresponds to a series composition with G, and a parallel composition with a graph in $\{H_i \mid i \in I\}$. So we let $G_j = (G_{j-1} \bowtie G) \parallel H_{i_j}$, then $H := G_k$ has effective edge interaction $\widehat{y}$. Since G and H_i only depend on q, y they have constant size, and therefore the size of H is $O(\log(\varepsilon^{-1}))$.

Using Lemma 3.3 we inductively compute Z^{same} and Z^{dif} of G_j in poly(k)-time, so we can output $Z^{\text{dif}}(H; q, y)$ along with H.

By construction $\widehat{y} \in B(y_0, \varepsilon)$, so to prove correctness of the algorithm in this case it suffices to show that H is indeed a gadget, that is, $Z^{\text{dif}}(H; q, y) \neq 0$. To do so we will inductively show that $G_j \in \mathcal{H}^*_{q,y}$ using Proposition 3.14 and thereby in particular that $Z^{\text{dif}}(G_j; q, y) \neq 0$. Note that $G_1 = H_{i_1} \in \mathcal{H}^*_{q,y}$ by assumption. Continuing inductively, if $G_{j-1} \in \mathcal{H}^*_{q,q}$, then the series composition of G_{j-1} with G is in $\mathcal{H}^*_{q,y}$ since its effective edge interaction is not equal to 1 as $g(z) = 1$ if and only if $z = 1$. Its effective edge interaction is contained in U and therefore not equal to ∞. Next taking the parallel composition with H_{i_j} results in the series-parallel graph G_j contained in $\mathcal{H}^*_{q,y}$, as its effective edge interaction is contained in U and therefore not equal to ∞; it is not equal to 1 by construction.

Now we turn to the second case where $y_0 \notin U$ and $y_0 \neq 0$. The algorithm first determines a positive integer n such that $y_0 = u_0^n$ for some $u_0 \in U$. Then it runs the algorithm for the first part on input of u_0 and error parameter

$$\delta = \min\left(\frac{|u_0|}{n-1}, \frac{|u_0|}{e \cdot n|y_0|} \cdot \varepsilon\right),$$

to obtain a series-parallel graph $H' \in \mathcal{H}^*_{q,y}$ with effective edge interaction $\widehat{u} \in B(u_0, \delta)$. We then output the series-parallel graph H obtained as the n-fold parallel composition of H' with itself, which has effective edge interaction $\widehat{u}^n$, and we output $Z^{\text{dif}}(H; q, y)$ computed using Lemma 3.3. Clearly $H \in \mathcal{H}^*_{q,y}$.

To prove that the algorithm is also correct in this case, first note that $n = O(|\log|y_0||)$: the open U contains a set of the form $\{z \in \mathbb{C} \mid -a \leq \arg(z) \leq a, b^{-1} \leq |z| \leq b\}$. Now it suffices to take $n = \max(\lceil \pi/a \rceil, \lceil \frac{|\log|y_0||}{\log(b)} \rceil)$. We can thus compute u_0 in time poly(size(y_0)). Note that size(u_0) $= O(n \cdot \text{size}(y_0))$.

The output of first procedure, $\widehat{u}$, satisfies

$$|\widehat{u}^n - u_0^n| \leq |\widehat{u} - u_0| \cdot n \max(|u_0|, |\widehat{u}|)^{n-1} \leq \delta n(|u_0| + \delta)^{n-1}$$
$$\leq \delta n|u_0|^{n-1} e^{(n-1)\delta/|u_0|} = \delta n \frac{|y_0|}{|u_0|} e^{(n-1)\delta/|u_0|}.$$

This means that with our choice of δ we have

$$|\widehat{u}^n - y_0| \leq \delta n \frac{|y_0|}{|u_0|} e^{(n-1)\delta/|u_0|} \leq \delta n \frac{|y_0|}{|u_0|} e \leq \varepsilon.$$

The computation of $\widehat{u}$ will have a running time of poly(size(u_0, δ)) which by construction is poly(size(y_0, ε)). The series-parallel graph H corresponding to $\widehat{u}^n$ is the parallel composition of n copies of the graph corresponding to $\widehat{u}$. The number of edges in the gadget is thus $O(n \log(\delta^{-1})) = \text{poly}(\text{size}(\varepsilon, y_0))$.

Finally for $y_0 = 0$ we simply run the algorithm to find a gadget with an effective edge interaction in $B(\varepsilon/2, \varepsilon/2) \subset B(0, \varepsilon)$, where $\varepsilon/2$ is a non-zero target. This finishes the proof. $\square$

3.5 Box shrinking: a proof of Theorem 3.6

Recall that we consider a linear function $f(y) = Ay + B$, with A, B complex numbers. Note that in this section we will use y as a variable in this function f, and not as a variable in the partition function of the random cluster model.

Our goal is to approximate the root $y^* = -B/A$ of this function. We will assume initially that y^* lies within a box of 'radius' D (when $A \neq 0$), and every step of the algorithm will shrink this box with a constant factor. Recall the notation $B_\infty(m, r) = \{z \in \mathbb{C} \mid |\Re(z - m)| < r, |\Im(z - m)| < r\}$ for a square box in the complex plane with radius r and center m.

Also recall that we either have an algorithm at our disposal to find an 0.25-abs-approximation or to find an 0.25-arg-approximation to f. We will in this section write $\tilde{f}_{\text{abs}}(y)$ for any 0.25-abs-approximation to $f(y)$, and similarly $\tilde{f}_{\text{arg}}(y)$ for any 0.25-arg-approximation to $f(y)$. More concretely, the algorithm takes as input a rational number $y_0 \in \mathbb{Q}[i]$ and a rational number $\varepsilon > 0$, and outputs $\tilde{f}_{\text{abs}}(\widehat{y})$ or respectively $\tilde{f}_{\text{arg}}(\widehat{y})$ for some $\widehat{y} \in B(y_0, \varepsilon)$.

We now first describe two variants of our box shrinking procedure, one for the abs-approximator and one for the arg-approximator.

Definition 3.18. (a) Given a square box $B_\infty(m,D)$ with center m and radius D. With $\varepsilon = 0.1D$, we use the algorithm to compute for any $\widehat{y}_1 \in B(m - \frac{5}{4}D, \varepsilon)$, $\widehat{y}_2 \in B(m + \frac{5}{4}D, \varepsilon)$, $\widehat{y}_3 \in B(m - \frac{5}{4}Di, \varepsilon)$ and $\widehat{y}_4 \in B(m + \frac{5}{4}Di, \varepsilon)$ the values $\tilde{f}_{\mathrm{abs}}(\widehat{y}_1), \ldots, \tilde{f}_{\mathrm{abs}}(\widehat{y}_4)$. If $\tilde{f}_{\mathrm{abs}}(\widehat{y}_1) \leq \tilde{f}_{\mathrm{abs}}(\widehat{y}_2)$ remove the strip of width $\frac{1}{4}D$ at the right side of the box and at the left side otherwise (see Figure 3.3). If $\tilde{f}_{\mathrm{abs}}(\widehat{y}_3) \leq \tilde{f}_{\mathrm{abs}}(\widehat{y}_4)$ remove the strip of width $\frac{1}{4}D$ at the top of the box and and the bottom of the box otherwise. Denote the resulting box by $\mathcal{S}_{\mathrm{abs}}(B_\infty(m,D)) = B_\infty(m', D')$.

(b) Given a square box $B_\infty(m,D)$ with center m and radius D. With $\varepsilon = 0.1D$, we use the algorithm to compute for any $\widehat{y}_1 \in B(m - \frac{5}{4}D, \varepsilon)$, $\widehat{y}_2 \in B(m + \frac{5}{4}D, \varepsilon)$, $\widehat{y}_3 \in B(m - \frac{5}{4}Di, \varepsilon)$ and $\widehat{y}_4 \in B(m + \frac{5}{4}Di, \varepsilon)$ the values $\tilde{f}_{\mathrm{arg}}(\widehat{y}_1), \ldots, \tilde{f}_{\mathrm{arg}}(\widehat{y}_4)$. If $\tilde{f}_{\mathrm{arg}}(\widehat{y}_1) - \tilde{f}_{\mathrm{arg}}(\widehat{y}_2)$ is in the interval $(0,\pi)$ we remove the top strip of width $\frac{1}{4}D$ of the box, remove the bottom strip otherwise (see Figure 3.4). If $\tilde{f}_{\mathrm{arg}}(\widehat{y}_3) - \tilde{f}_{\mathrm{arg}}(\widehat{y}_4)$ is in the interval $(0,\pi)$, we remove the left strip of width $\frac{1}{4}D$, remove the right strip otherwise. Denote the resulting box by $\mathcal{S}_{\mathrm{arg}}(B_\infty(m,D)) = B_\infty(m', D')$.

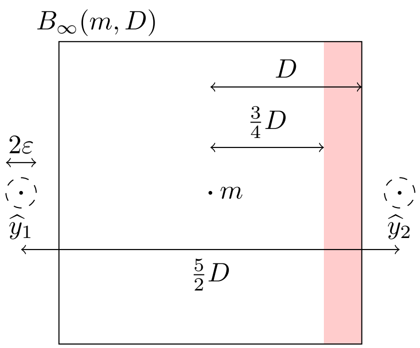

Figure 3.3: Box shrinking procedure $\mathcal{S}_{\mathrm{abs}}$.
If $\tilde{f}_{\mathrm{abs}}(\widehat{y}_1) \leq \tilde{f}_{\mathrm{abs}}(\widehat{y}_2)$, the red-shaded strip is removed.

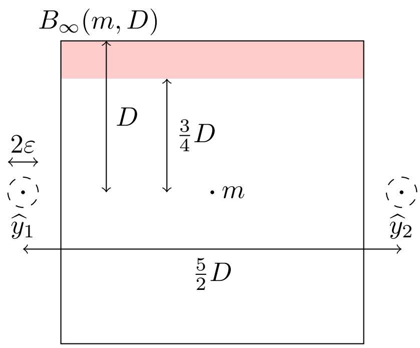

Figure 3.4: Box shrinking procedure $\mathcal{S}_{\mathrm{arg}}$.
If $\tilde{f}_{\mathrm{arg}}(\widehat{y}_1) - \tilde{f}_{\mathrm{arg}}(\widehat{y}_2)$ is in the interval $(0,\pi)$, the red-shaded strip is removed.

In the next two lemmas we record important observations about this procedure. The first says we have a good understanding of the midpoint and radius of the resulting box, and the second says we are guaranteed to retain the zero y^* in our box.

Lemma 3.19. *The resulting radius D' satisfies $D' = \frac{7}{8}D$ and the resulting center m' satisfies $m' = m + (\frac{\pm 1}{8} + \frac{\pm i}{8})D$.*

Lemma 3.20. *Suppose that $A \neq 0$ and $\mathcal{S} \in \{\mathcal{S}_{\text{abs}}, \mathcal{S}_{\text{arg}}\}$. If $y^* \in B_\infty(m, D)$, then $y^* \in \mathcal{S}(B_\infty(m, D))$.*

Proof. We prove the cases $\mathcal{S} = \mathcal{S}_{\text{abs}}$ and $\mathcal{S} = \mathcal{S}_{\text{arg}}$ separately.
Case 1 ($\mathcal{S} = \mathcal{S}_{\text{abs}}$) In this case we write $\tilde{f}$ for $\tilde{f}_{\text{abs}}$. Suppose that $y^* \notin \mathcal{S}(B_\infty(m, D))$, we will reach a contradiction and thereby prove the lemma. We may assume wlog that $\tilde{f}(\widehat{y}_1) \leq \tilde{f}(\widehat{y}_2)$, but that y^* is in the strip of width $\frac{1}{4}D$ at the right side of the box. Note that $\widehat{y}_1$ and $\widehat{y}_2$ are certainly not equal to y^*.

Then we see that

$$\frac{\tilde{f}(\widehat{y}_1)}{\tilde{f}(\widehat{y}_2)} > e^{-0.5}\frac{|A\widehat{y}_1 + B|}{|A\widehat{y}_2 + B|} = e^{-0.5}\frac{|\widehat{y}_1 - y^*|}{|\widehat{y}_2 - y^*|}.$$

We will now proceed by bounding $|\widehat{y}_1 - y^*|^2 - |\widehat{y}_2 - y^*|^2$ and $|\widehat{y}_2 - y^*|$. For this denote $y_1 = m - \frac{5}{4}D$ and $y_2 = m + \frac{5}{4}D$. See Figure 3.3 for a sketch of the situation.

We easily find the upper bounds $|y_1 - y^*| \leq \frac{\sqrt{97}}{4}D$ and $|y_2 - y^*| \leq \frac{\sqrt{5}}{2}D$ (the maxima are reached when y^* is in a corner of the strip), so $|\widehat{y}_2 - y^*| \leq \frac{\sqrt{5}}{2}D + \varepsilon < \frac{3}{2}D$.

Next we see, using Pythagoras, that

$$\begin{aligned}
|y_1 - y^*|^2 - |y_2 - y^*|^2 &= \left[\Re(y_1 - y^*)^2 - \Re(y_2 - y^*)^2\right] + \left[\Im(y_1 - y^*)^2 - \Im(y_2 - y^*)^2\right] \\
&= (\Re(y_1 - y^*) - \Re(y_2 - y^*)) \cdot (\Re(y_1 - y^*) + \Re(y_2 - y^*)) \\
&\geq \tfrac{5}{2}D \cdot \tfrac{3}{2}D = \tfrac{15}{4} \cdot D^2.
\end{aligned}$$

Including the uncertainty in $\widehat{y}_i$ versus y_i yields that

$$\begin{aligned}
|\widehat{y}_1 - y^*|^2 - |\widehat{y}_2 - y^*|^2 &\geq (|y_1 - y^*| - \varepsilon)^2 - (|y_2 - y^*| + \varepsilon)^2 \\
&= |y_1 - y^*|^2 - |y_2 - y^*|^2 - 2\varepsilon(|y_1 - y^*| + |y_2 - y^*|) \\
&\geq \tfrac{15}{4}D^2 - 0.2D\left(\tfrac{\sqrt{97}}{4}D + \tfrac{\sqrt{5}}{2}D\right) > 3D^2.
\end{aligned}$$

Putting this together yields

$$\frac{|\widehat{y}_1 - y^*|^2}{|\widehat{y}_2 - y^*|^2} = 1 + \frac{|\widehat{y}_1 - y^*|^2 - |\widehat{y}_2 - y^*|^2}{|\widehat{y}_2 - y^*|^2} > 3 > e,$$

which proves that $\tilde{f}(\widehat{y}_1)/\tilde{f}(\widehat{y}_2) > 1$. This contradicts $\tilde{f}(\widehat{y}_1) \leq \tilde{f}(\widehat{y}_2)$, completing the proof of Case 1.

Case 2 ($\mathcal{S} = \mathcal{S}_{\mathrm{arg}}$) In this case we write $\tilde{f}$ for $\tilde{f}_{\mathrm{arg}}$. Suppose that $y^* \notin \mathcal{S}(B_\infty(m, D))$, we will again reach a contradiction. This time we may assume that $\tilde{f}(\widehat{y}_1) - \tilde{f}(\widehat{y}_2)$ is in the interval $(0, \pi)$, but that y^* is in the strip of width $\frac{1}{4}D$ at the top of the box.

We setup some notation

$$\begin{aligned} d_1 &= |\widehat{y}_1 - y^*|, & d_2 &= |\widehat{y}_2 - y^*|, & \alpha &= \arg(\widehat{y}_2 - y^*) - \arg(\widehat{y}_1 - y^*), \\ x &= d_1^2 - d_2^2, & y &= 2d_1d_2, & z &= d_1^2 + d_2^2, \\ y_1 &= m - \tfrac{5}{4}D, & y_2 &= m + \tfrac{5}{4}D, & w &= y\sin(\alpha). \end{aligned}$$

Note that

$$\begin{aligned} \arg(f(\widehat{y}_2)) - \arg(f(\widehat{y}_1)) &= \arg(A\widehat{y}_2 + B) - \arg(A\widehat{y}_1 + B) \\ &= \arg(A\widehat{y}_2 + B - (Ay^* + B)) - \arg(A\widehat{y}_1 + B - (Ay^* + B)) \\ &= \alpha, \end{aligned}$$

and we can interpret this geometrically as the angle between the segments $\widehat{y}_1$ to y^* and $\widehat{y}_2$ to y^*, measured clockwise from $\widehat{y}_2$ to $\widehat{y}_1$. By definition of the arg-approximation, the difference $\tilde{f}(\widehat{y}_2) - \tilde{f}(\widehat{y}_1)$ will differ at most $0.5 < \frac{\pi}{6}$ from α. Also observe that x, y, z satisfy the relation $x^2 + y^2 = z^2$. The cosine rule in the triangle formed by $\widehat{y}_1, \widehat{y}_2, y^*$ yields $z = y\cos(\alpha) + |\widehat{y}_1 - \widehat{y}_2|^2$. Together this implies the following relation:

$$\begin{aligned} x^2 &= z^2 - y^2 \\ &= \left(y\cos(\alpha) + |\widehat{y}_1 - \widehat{y}_2|^2\right)^2 - y^2 \\ &= -y^2\sin^2(\alpha) + 2y\sin(\alpha) \cdot |\widehat{y}_1 - \widehat{y}_2|^2 \cot(\alpha) + |\widehat{y}_1 - \widehat{y}_2|^4 \\ &= -w^2 + 2w \cdot |\widehat{y}_1 - \widehat{y}_2|^2 \cot(\alpha) + |\widehat{y}_1 - \widehat{y}_2|^4, \end{aligned}$$

and hence

$$\cot(\alpha) = \frac{x^2 + w^2 - |\widehat{y}_1 - \widehat{y}_2|^4}{2w \cdot |\widehat{y}_1 - \widehat{y}_2|^2}. \tag{3.7}$$

The distance $|y_1 - y_2|$ is exactly $\frac{5}{2}D$, so we see that

$$2.3D = \tfrac{5}{2}D - 2\varepsilon < |\widehat{y}_1 - \widehat{y}_2| < \tfrac{5}{2}D + 2\varepsilon = 2.7D.$$

We can also bound $|y_i - y^*| \leq \frac{\sqrt{97}}{4}D$ for both $i = 1, 2$. Now we continue to bound x and w. Using Pythagoras, we compute

$$\begin{aligned} |y_1 - y^*|^2 - |y_2 - y^*|^2 &= \left[\Re(y_1 - y^*)^2 - \Re(y_2 - y^*)^2\right] + \left[\Im(y_1 - y^*)^2 - \Im(y_2 - y^*)^2\right] \\ &= (\Re(y_1 - y^*) - \Re(y_2 - y^*)) \cdot (\Re(y_1 - y^*) + \Re(y_2 - y^*)) \\ &= \tfrac{5}{2}D \cdot (\Re(y_1 - y^*) + \Re(y_2 - y^*)). \end{aligned}$$

The last factor has absolute value at most $2D$, so in total the absolute value is at most $5D^2$. With the perturbations $\widehat{y}_i$ from y_i this gives the bound

$$\begin{aligned}
|x| &= \left|d_1^2 - d_2^2\right| \\
&\leq \max\left\{\left|(|y_1-y^*|+\varepsilon)^2-(|y_2-y^*|-\varepsilon)^2\right|, \left|(|y_1-y^*|-\varepsilon)^2-(|y_2-y^*|+\varepsilon)^2\right|\right\} \\
&= \left||y_1-y^*|^2-|y_2-y^*|^2\right| + 2\varepsilon(|y_1-y^*|+|y_2-y^*|) \\
&\leq 5D^2 + 0.2D\cdot 2\tfrac{\sqrt{97}}{4}D < 6D^2.
\end{aligned}$$

We note that w is four times the area of the triangle formed by the points $\widehat{y}_1, \widehat{y}_2, y^*$. To obtain a lower bound, we may assume that $\widehat{y}_1 = y_1 + (\varepsilon + \varepsilon i)$, because this point is closer the the line through y^* and $\widehat{y}_2$ then any point in $B(y_1, \varepsilon)$. Similarly we may assume that $\widehat{y}_2 = y_2 + (-\varepsilon + \varepsilon i)$, and then the minimal area is attained when $\Im(y^*) = \Im(m) + \frac{3}{4}D$. For the upper bound, we assume that $\widehat{y}_1 = y_1 + (-\varepsilon - \varepsilon i)$ and $\widehat{y}_2 = y_2 + (\varepsilon - \varepsilon i)$, and the maximum is when $\Im(y^*) = \Im(m) + D$. This yields the bounds

$$2.99D^2 = 2(\tfrac{5}{2}D - 2\varepsilon)(\tfrac{3}{4}D - \varepsilon) \leq w \leq 2(\tfrac{5}{2}D + 2\varepsilon)(D+\varepsilon) = 5.94D^2.$$

Plugging in all these bounds into (3.7) yields

$$-\sqrt{3} < -1.40 < \cot(\alpha) < 1.37 < \sqrt{3}.$$

We see that $\arg(\widehat{y}_1 - y^*) \in (\pi, \frac{3}{2}\pi)$ and $\arg(\widehat{y}_2 - y^*) \in (\frac{3}{2}\pi, 2\pi)$, so that $\alpha \in (0, \pi)$. Then the bounds on $\cot(\alpha)$ imply $\alpha \in \left(\frac{1}{6}\pi, \frac{5}{6}\pi\right)$. Including the approximation error, we find that $\tilde{f}(\widehat{y}_2) - \tilde{f}(\widehat{y}_1)$ is in the interval $(0, \pi)$, contradicting the assumption that the opposite difference $\tilde{f}(\widehat{y}_1) - \tilde{f}(\widehat{y}_2)$ is in the interval $(0, \pi)$. This completes the proof. □

We now use the box shrinking procedure to prove Theorem 3.6.

Proof of Theorem 3.6. We will first decide whether $A = 0$ or $A \neq 0$.

If we are using the abs-approximation algorithm we do the following: compute $\tilde{f}_{\text{abs}}(\widehat{y})$ for any $\widehat{y}$ such that $|\widehat{y}| > 5C^2$. If $\tilde{f}_{\text{abs}}(\widehat{y}) < 2C$, we output "$A = 0$" and terminate, else we output "$A \neq 0$" and continue with the rest of the algorithm.

Instead if we are using the arg-approximation algorithm: with $\varepsilon = 0.1D$, we take $\widehat{y}_1 \in B(m - 5D, \varepsilon)$, $\widehat{y}_2 \in B(m + 5D, \varepsilon)$ and compute $\tilde{f}_{\text{arg}}(\widehat{y}_1)$ and $\tilde{f}_{\text{arg}}(\widehat{y}_2)$. If $\tilde{f}_{\text{arg}}(\widehat{y}_2) - \tilde{f}_{\text{arg}}(\widehat{y}_1)$ is in the interval $(-\frac{1}{4}\pi, \frac{1}{4}\pi)$, we output "$A = 0$" and terminate, else we output "$A \neq 0$" and continue with the rest of the algorithm.

Next, the idea is to apply the box shrinking procedure, with $\mathcal{S} = \mathcal{S}_{\text{abs}}$ or $\mathcal{S} = \mathcal{S}_{\text{arg}}$ depending on the algorithm at our disposal. We take as starting box the square box with center 0 and 'radius' $D = C^2$. We apply the box shrinking procedure

$$n := \lceil \log(2D/\delta)/\log(8/7) \rceil$$

many times. Finally the algorithm outputs the center m_n of $B_\infty(m_n, D_n) = \mathcal{S}^{\circ n}(B_\infty(0, D))$.

We first argue correctness of the algorithm and deal with the running time after that. We may assume that not both A and B are equal to 0, otherwise the algorithm is allowed to output anything anyway.

If $A = 0$ and $B \neq 0$ we have $f(y) = B \neq 0$ for any y. For the abs-approximator we see that $\tilde{f}_{\text{abs}}(\widehat{y}) \leq e^{0.25}|B| < 2C$ and then the algorithm indeed outputs "$A = 0$". On the other hand, for the arg-approximator we have that $\tilde{f}_{\text{arg}}(\widehat{y}_2) - \tilde{f}_{\text{arg}}(\widehat{y}_1)$ is at most $0.5 < \frac{\pi}{6}$ away from $\arg(f(\widehat{y}_2)) - \arg(f(\widehat{y}_1)) = 0$, so again the algorithm correctly outputs "$A = 0$".

If on the other hand $A \neq 0$, we see for the abs-approximator that $|f(\widehat{y})| \geq |A||\widehat{y}| - |B| > C^{-1} \cdot 5C^2 - C = 4C$. This is in particular non-zero, so $\tilde{f}(\widehat{y}) \geq e^{-0.25}|f(\widehat{y})| > 2C$ and indeed the algorithm outputs "$A \neq 0$". For the arg-approximator, we again adopt the notation from the proof of Lemma 3.20, and we will prove that $\cos(\alpha) < 0$. First we see that $|\widehat{y}_1 - \widehat{y}_2|^2 \geq (10D - 2\varepsilon)^2 = 96.04D^2$. We can bound d_i^2 by using Pythagoras, to see that $d_i^2 \leq (6D + \varepsilon)^2 + (D + \varepsilon)^2 = 38.42D^2$, yielding $z \leq 76.84D^2$. Then

$$y\cos(\alpha) = z - |\widehat{y}_1 - \widehat{y}_2|^2 \leq 76.84D^2 - 96.04D^2 < 0.$$

This means that $\alpha \in (\frac{1}{2}\pi, \frac{3}{2}\pi)$, and $\tilde{f}(\widehat{y}_2) - \tilde{f}(\widehat{y}_1)$ is in the interval $(\frac{1}{3}\pi, \frac{5}{3}\pi)$. Then the algorithm indeed outputs "$A \neq 0$".

When $A \neq 0$, we also see that $y^* \in B_\infty(0, D)$ by our choice of D. By Lemma 3.20 we then know that $y^* \in B_\infty(m_n, D_n)$. Because $D_n = (7/8)^n D \leq \delta/2$, this box is small enough. We thus conclude that our algorithm is correct and move on to the analysis of the running time.

The running time of the algorithm is dominated by the applications of the box shrinking. First of all we note that $n = O(\log(C/\delta))$. The smallest ε that we encounter is 0.05δ. By Lemma 3.19 and induction, it follows that after k steps, the diagonal of the current box, D_k, is of the form $(7/8)^k D$ and its center, m_k, is of the form 0 plus a rational multiple (of size $O(k)$) of D and hence is itself rational. The values of y_i that we encounter are of the form $m_k \pm \frac{5}{4}D_k$ and $m_k \pm \frac{5}{4}D_k i$. Therefore the y_i that are used as input for the assumed algorithm have their sizes bounded by $O(\log(D)) + O(n) = O(\log(C/\delta))$. Hence we obtain a running time of

$$O\big(\log(C/\delta)\big)\,\text{poly}(\log(C/\delta)) = \text{poly}(\log(C/\delta)). \qquad \square$$

3.6 Concluding remarks

Planar graphs: inside the disk $|q - 1| < 1$ An interesting question left open by our results is whether approximately computing the chromatic polynomial of planar graphs is #P-hard for all non-real algebraic q. We refer to Figure 3.1 for a figure displaying the region for which we can prove hardness with the aid of a computer; here we use the computer to try and verify the condition in Theorem 3.12. The family $\mathcal{H}^*_{q,y}$, appearing in this condition, is now restricted to series-parallel graphs. In Corollary 3.13(c) we extend this to a family $\mathcal{P}^*_{q,y}$ of planar graphs, and show that if q is a planar chromatic zero, the problems q-Planar-Abs-Chromatic and q-Planar-Arg-Chromatic are #P-hard. In Section 2.6 we showed that there exist chromatic zeros arbitrarily close to 1, which improves Figure 3.1, but is not yet enough to resolve the question completely.

Real evaluations of the chromatic polynomial for planar graphs The focus in this chapter has been on non-real evaluations. Using our techniques we can also obtain results for real evaluations of the chromatic polynomial. It suffices for $q \in \mathbb{R}$ to be able to get density on the real line with effective edge interactions of planar graphs (where, as in $\mathcal{P}^*_{q,y}$, the two terminals are on the same face). Having this, the machinery of this chapter can be adapted in a straightforward way to prove hardness of approximating the absolute value at q. To obtain density of the effective edge interactions at q it suffices to find an effective interaction that is finite and less than -1. When restricting to the family $\mathcal{H}^*_{q,0}$, this is possible if and only if $q \in (32/27, 2)$; for $q > 2$ the effective interactions are always positive, for $q < 0$ they are trapped in the interval $(0, 1)$, and for $q \in (0, 2)$ this follows from Lemma 2.13, 2.14 and Corollary 2.19. When we replace the family $\mathcal{H}^*_{q,0}$ by the family of two-terminal planar graphs with the two terminals on the same face, one can get a negative effective interaction, thus density in $\mathbb{R}$, for q inside the union of three intervals $(2, 3) \cup (3, t_1) \cup (t_2, 4)$, where $t_1 \approx 3.618032$ and $t_2 \approx 3.618356$ by results of Thomassen [69] and Perrett and Thomassen [61]. Consequently, it is then #P-hard to approximate the absolute value of the chromatic polynomial for planar graphs for any algebraic q in any of these intervals.

Interestingly, there is the value $\tau + 2$ (where τ is the golden ratio) between t_1 and t_2 at which the chromatic polynomial of any planar graphs is positive by a result of Tutte [71], see also [61]. This suggests that the computational complexity of approximating the absolute value of the chromatic polynomial at $\tau + 2$ is an intriguing problem.[2]

Finally, we ask about the complexity of approximating the absolute value of the chromatic polynomial at large values of q on planar graphs (computing the sign is hard for general graphs [38]). Woodall [75] showed that planar graphs have no chromatic roots larger than 5 indicating that the constructions that we employ

[2] In a recent seminar talk (see `https://homepages.dcc.ufmg.br/~gabriel/AGT/wp-content/uploads/2021/02/30_Gordon_Royle.pdf`) it was announced that Gordon Royle and Melissa Lee proved that t_1 can be chosen to be $\tau + 2$.

in this chapter are not possible. This suggests that determining the complexity of approximating the chromatic polynomial for planar graphs in this regime is an interesting problem.

Almost bounded degree (planar) graphs Combining our techniques with some ingredients from Section 2.5.3 and some tools from complex dynamics we expect that for $\Delta \geq 3$ and non-real algebraic q such that $1 < |q-1| < \Delta - 1$, approximating the chromatic polynomial for planar graphs of maximum degree at most Δ with one vertex of potentially unbounded degree is #P-hard. We leave the details for follow up work. This relates to a result of Galanis, Štefankovič and Vigoda [32], who showed that for even positive integer q, corresponding to proper q-colorings, proved that it is NP-hard to approximate the evaluation of the chromatic polynomial at q on all graphs of maximum degree Δ when $q < \Delta$. This should be contrasted with a result from [60] which shows that for all graphs of maximum degree at most Δ and any q such that $|q| > 6.91\Delta$ there exists an efficient algorithm to approximate the chromatic polynomial. This algorithm is based on Barvinok's interpolation method [4] and a zero-freeness result for the chromatic polynomial for bounded degree graphs [28, 46]. The zero-free region can be extended to the family graphs of maximum degree at most Δ where one vertex may have unbounded degree at the cost of replacing 6.91Δ by $7.97\Delta + 1$ using [65, Corollary 6.4]. Using Sokal's representation of the chromatic polynomial of a graph with one vertex of potentially unbounded degree as an evaluation of the partition function of a (multivariate) random cluster model with external fields of a bounded degree graph [65], the algorithm from [60] can be adapted to run in polynomial time for this class of graphs as well.

The reliability polynomial Recall that $T(G; x, y)$ denotes the Tutte polynomial of a graph G. If G is a connected graph and $x = 1$ we define

$$C(G; y) := (y-1)^{|V|-1} T(G; 1, y) = \sum_{\substack{A \subseteq E \\ (V,A) \text{ connected}}} (y-1)^{|A|} = \lim_{q \to 0} \frac{1}{q} Z(G; q, y).$$

This is up to a transformation the reliability polynomial, i.e. $(1-p)^{|E|} C(G; \frac{1}{1-p})$ gives the probability that the graph G remains connected if edges are independently selected with probability p, and deleted with probability $1 - p$ (see e.g. [67]). Clearly, the approach for proving density in this chapter does not apply directly. However, a variation of our methods can be applied in this setting. We will leave this for future work.

Chapter 4

Sampling from the low temperature Potts model through a Markov chain on flows

4.1 Introduction

Let $G = (V, E)$ be a graph and let $[q] \coloneqq \{1, \ldots, q\}$ be a set of spins or colours for an integer $q \geq 2$. A function $\sigma : V \to [q]$ is called a q-spin configuration or colouring. The Gibbs measure of the q-state Potts model on $G = (V, E)$ is a probability distribution on the set of all q-spin configurations $\{\sigma : V \to [q]\}$. For an *interaction parameter* $y > 0$, the Gibbs distribution $\mu_{\text{Potts}} \coloneqq \mu_{\text{Potts},G;q,y}$ is defined by

$$\mu_{\text{Potts}}[\sigma] \coloneqq \frac{y^{m(\sigma)}}{\sum_{\tau:V\to[q]} y^{m(\tau)}}, \tag{4.1}$$

where, for a given q-spin configuration τ, $m(\tau)$ denotes the number of edges $\{u, v\}$ of G for which $\tau(u) = \tau(v)$. The denominator of the fraction (4.1) is called the *partition function of the Potts model* and is denoted by $Z_{\text{Potts}}(G; q, y)$.

The regime $y \in (0, 1)$ is known as the anti-ferromagnetic Potts model, and $y \in (1, \infty)$ as the ferromagnetic Potts model. Furthermore, values of y close to 1 are referred to as high temperature, whereas values close to 0 or infinity are referred to as low temperature. This comes from the physical interpretation in which one writes $y = e^{J\beta}$ with $J > 0$ being the interaction energy between same spin sites and β the inverse temperature.

We will be concerned with the algorithmic problem of approximately sampling from μ_{Potts} as well as approximately computing $Z = Z_{\text{Potts}}(G; q, y)$ for y close to infinity (that is in the low temperature ferromagnetic regime). Given error parameters $\varepsilon, \delta \in (0, 1)$, an ε-approximate counting algorithm for Z outputs a number Z' so that $(1 - \varepsilon) \leq Z/Z' \leq (1 + \varepsilon)$, and a δ-approximate sampling algorithm for $\mu = \mu_{\text{Potts}}$ outputs a random sample I with distribution $\widehat{\mu}$ so that the total variation distance satisfies $\|\mu - \widehat{\mu}\|_{\text{TV}} \leq \delta$.

It was shown in [33] that, for graphs of a fixed maximum degree $\Delta \geq 3$, there is a critical parameter $y_\Delta > 1$, corresponding to a phase transition of the model on the infinite Δ-regular tree, such that approximating the partition function is computationally hard[1]. This result indicates that it might be hard to compute the partition function of the ferromagnetic Potts model for large values of y. However, recently several results emerged, showing that for certain finite subgraphs of $\mathbb{Z}^d$ [5, 42, 14] as well as Δ-regular graphs satisfying certain expansion properties [48, 41, 20] it is in fact possible to approximate the partition function of the ferromagnetic Potts model for y large enough. In fact the algorithms in [14, 41] even work for all values $y \geq 1$ under the assumption that the number of colours, q, is suitably large in terms of the maximum degree. The running times of all these aforementioned algorithms are polynomial in the number of vertices of the underlying graph, but typically with a large exponent. The exception is [20], in which the cluster expansion techniques of [48] for expander graphs are extended to a Markov chain setting giving running times of the form $O(n^2 \log n)$ for approximating the partition function, where n is the number of vertices of the input graph.

In this chapter we present Markov chain based algorithms for approximating the partition function of the ferromagnetic Potts model at sufficiently low temperatures with similar running times as [20]. While most results in this area focus on graphs of bounded maximum degree, the graph parameters of interest for us are different and so our methods, as well as being able to handle subgraphs of the grid $\mathbb{Z}^d$ (although not for all temperatures), can also handle certain graphs classes of unbounded degree (cf. Lemma 4.9). The parameters of interest for us are in fact similar to those in [5]; here we achieve better running times for our algorithms, while [5] achieves better parameter dependencies.

We show how to efficiently generate a sample from the Potts model using a rapidly mixing Markov chain and then use this to approximate the partition function. The Markov chain however is not supported on q-spin configurations[2] but on flows taking values in $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$. For planar graphs, this Markov chain on flows may be interpreted as Glauber dynamics of q-spin configurations on the dual graph; see Section 4.6 for an example of this. We use this Markov chain on flows together with another trick to show that we can efficiently approximate a certain partition function on flows at high temperatures, which in turn can be used to approximate the Potts partition function at low temperatures. Below we state our main results.

---

[1]Technically they showed that the problem is #BIS hard, a complexity class introduced in [24] and known to be as hard as #BIS, that is the problem of counting the number of independent sets in a bipartite graph. The exact complexity of #BIS is unknown, but it is believed that no fully polynomial time randomised approximation scheme exists for #BIS, but also that #BIS is not #P-hard

[2]See e.g. [13] for an analysis of the usual Glauber dynamics for the ferromagnetic Potts model (at high temperatures).

4.1.1 Main results

To state our main results, we need some definitions. In the present chapter we deal with multigraphs and the reader should read multigraph whenever the word graph is used. A graph is called *even* if all of its vertices have even degree. In what follows we often identify a subgraph of a given graph with its edge set.

Given a graph G, fix an arbitrary orientation of its edges. For any even subgraph C of G, we can associate to it a signed indicator vector $\chi_C \in \mathbb{Z}^E$ as follows: choose an Eulerian orientation of (each of the components of) C. Then for $e \notin C$ we set $\chi_C(e) = 0$ and for $e \in C$, we set $\chi_C(e) = 1$ if e has the same direction in both C and G, and we set $\chi_C(e) = -1$ otherwise. We often abuse notation and identify the indicator vector χ_C with the set of edges in C. A $\mathbb{Z}$*-flow*, is a map $f : E \to \mathbb{Z}$ satisfying

$$\sum_{e\colon e \text{ directed into } v} f(e) = \sum_{e\colon e \text{ directed out of } v} f(e) \quad \text{for all } v \in V.$$

We denote the collection of $\mathbb{Z}$-flows by $\mathcal{F}(G)$; note that $\mathcal{F}(G)$ with the obvious notion of addition is known as the first homology group of G, and also as the cycle space of G. Clearly, when viewing χ_C as a function on E, we have $\chi_C \in \mathcal{F}(G)$ for any even subgraph C. It is well known that $\mathcal{F}(G)$ has a generating set (as a $\mathbb{Z}$-module) consisting of indicator vectors of even subgraphs; see e.g. [34, Section 14].[3] We call such a generating set an *even generating set* for the cycle space.

Let $\mathcal{C}$ be an even generating set of $\mathcal{F}(G)$; we define some parameters associated to $\mathcal{C}$ (see below for some examples of even generating sets and associated parameters). For $C \in \mathcal{C}$, let $d(C) := |\{D \in \mathcal{C} \setminus \{C\} \mid C \cap D \neq \emptyset\}|$, and let

$$d(\mathcal{C}) := \max\{d(C) \mid C \in \mathcal{C}\}. \tag{4.2}$$

We write

$$\iota(\mathcal{C}) := \max\{|C_1 \cap C_2| \mid C_1, C_2 \in \mathcal{C} \text{ with } C_1 \neq C_2\}. \tag{4.3}$$

Define

$$\ell(\mathcal{C}) := \max\{|C| \mid C \in \mathcal{C}\}. \tag{4.4}$$

Finally, for an edge $e \in E$, define $s(e)$ to be the number of even subgraphs $C \in \mathcal{C}$ that e is contained in and

$$s(\mathcal{C}) := \max\{s(e) \mid e \in E\}. \tag{4.5}$$

We now present our approximate sampling and counting results. All of our results are based on randomised algorithms that arise from running Markov chains.

[3] In fact there is even a basis consisting of indicator functions of cycles. For later purposes we however need to work with even subgraphs.

For us, simulating one step of these Markov chains always includes choosing a random element from a set of t elements with some (often uniform) probability distribution, where t is at most polynomial in the size of the input graph. We take the time cost of such a random choice to be $O(1)$ as in the (unit-cost) RAM model of computation; see e.g. [59].

Our main sampling results read as follows.

Theorem 4.1. *Fix a number of spins $q \in \mathbb{N}_{\geq 2}$.*

(i) Fix integers $d \geq 2$ and $\iota \geq 1$ and let $\mathcal{G}$ be the set of graphs $G = (V, E)$ for which we have an even generating set $\mathcal{C}$ for G of size $O(|E|)$ such that $d(\mathcal{C}) \leq d$ and $\iota(\mathcal{C}) \leq \iota$. For any $y > \frac{(d+1)\iota}{2}q - (q-1)$ and $\delta \in (0,1)$, there exists a δ-approximate sampling algorithm for $\mu_{\mathrm{Potts},G;q,y}$, on all m-edge graphs $G \in \mathcal{G}$ with running time $O(m^2 \log(m\delta^{-1}))$.

(ii) Fix integers $\ell \geq 3$ and $s \geq 2$ and let $\mathcal{G}$ be the set of graphs $G = (V, E)$ for which we have an even generating set $\mathcal{C}$ for G of size $O(|E|)$ such that $\ell(\mathcal{C}) \leq \ell$ and $s(\mathcal{C}) \leq s$. For any $y > (q-1)(\ell s - 1)$ and $\delta \in (0,1)$ there exists a δ-approximate sampling algorithm for $\mu_{\mathrm{Potts},G;q,y}$ on all m-edge graphs $G \in \mathcal{G}$ with running time $O(m \log(m\delta^{-1}))$.

While parts (i) and (ii) are not directly comparable, we note that when $\iota = 1$, part (i) has a better range for y.

Our main approximate counting results read as follows.

Theorem 4.2. *Fix a number of spins $q \in \mathbb{N}_{\geq 2}$.*

(i) Fix integers $d \geq 2$ and $\iota \geq 1$ and let $\mathcal{G}$ be the set of graphs $G = (V, E)$ for which we have an even generating set $\mathcal{C}$ for G of size $O(|E|)$ such that $d(\mathcal{C}) \leq d$ and $\iota(\mathcal{C}) \leq \iota$. For $y > \frac{(d+1)\iota}{2}q - (q-1)$ and $\varepsilon \in (0,1)$, there exists a randomised ε-approximate counting algorithm for $Z_{\mathrm{Potts}}(G; q, y)$ on all n-vertex and m-edge graphs $G \in \mathcal{G}$ that succeeds with probability at least $3/4$ and has running time $O(n^2 m^2 \varepsilon^{-2} \log(nm\varepsilon^{-1}))$.

(ii) Fix integers $\ell \geq 3$ and $s \geq 2$ and let $\mathcal{G}$ be the set of graphs $G = (V, E)$ for which we have an even generating set $\mathcal{C}$ for G of size $O(|E|)$ such that $\ell(\mathcal{C}) \leq \ell$ and $s(\mathcal{C}) \leq s$. For any $y > (q-1)(\ell s - 1)$ and $\varepsilon \in (0,1)$ there exists a randomised ε-approximate counting algorithm for $Z_{\mathrm{Potts}}(G; q, y)$ on all n-vertex and m-edge graphs $G \in \mathcal{G}$ that succeeds with probability at least $3/4$ and has running time $O(n^2 m \varepsilon^{-2} \log(nm\varepsilon^{-1}))$.

Remark 4.3. We note that the dependence of the Potts model parameter y on the parameters $\ell(\mathcal{C})$ and $s(\mathcal{C})$ is similar as in [5], except there the dependence on s is order $\sqrt{s}$, which is better than our linear dependence. This of course raises the question whether our analysis can be improved to get the same dependence.

We now give a few examples of applications of our results.

Example 4.4. (i) Let G_1 and G_2 be two graphs that both contain a connected graph H as induced subgraph. Let $G_1 \cup_H G_2$ be the graph obtained from G_1 and G_2 by identifying the vertices of the graph H in both graphs. If both G_1 and G_2 have an even generating set consisting of cycles of length at most ℓ for some ℓ, then the same holds for $G_1 \cup_H G_2$.

Now use this procedure to build a subgraph $G = (V, E)$ of $\mathbb{Z}^d$, $d \geq 2$ from the union of finitely many copies of elementary cubes $((\{0,1\}^d)$. Since an elementary cube has a generating set consisting of 4-cycles, as is seen by induction on d, and so the resulting graphs has an even generating set $\mathcal{C}$ consisting only of 4-cycles. The relevant parameters of $\mathcal{C}$ are $d(\mathcal{C}) = 8(d-1) - 4$, $\iota(\mathcal{C}) = 1$, $\ell(\mathcal{C}) = 4$, $s(\mathcal{C}) = 2(d-1)$, and $|\mathcal{C}| \leq (d-1)|E|/2$.

(ii) In a similar manner as in (i) one can also construct graphs with concrete parameters from lattices such as the triangular lattices (here $d = 3$, $\iota = 1$, $\ell = 3$ and $s = 2$) or its dual lattice, the honeycomb lattice (here $d = 6$, $\iota = 1$, $\ell = 6$, and $s = 2$).

(iii) For any (multi)graph $G = (V, E)$ with even generating set $\mathcal{C}$, the graph G/e obtained by contracting some edge $e \in E$ has an even generating set $\mathcal{C}/e :=$ $\{C/e : C \in \mathcal{C}\}$. One can check that $\mathcal{C}$ and $\mathcal{C}/e$ have the same parameters d, ι, ℓ, s (see Lemma 4.9). This allows us to apply our algorithms to many graph classes of unbounded degree e.g. any graph that can be obtained from $\mathbb{Z}^d$ by a series of contractions.

As we shall see in the next subsection, the Markov chains on flows that we introduce are a natural means of studying the ferromagnetic Potts model at low temperatures. The examples above show that it is easy to generate many graphs (also of unbounded degree) for which these chains mix rapidly and therefore for which our results above apply. With the work in this chapter, we begin the analysis of these Markov chains on flows, but we believe there is a lot of scope for further study of these chains to obtain better sampling and counting algorithms for the ferromagnetic Potts model at low temperature.

4.1.2 Approach and discussion

The key step in our proof of Theorems 4.1 and 4.2 is to view the partition function of the Potts model as a generating function of flows taking values in an abelian group of order q. Although well known to those acquainted with the Tutte polynomial and its many specializations, this perspective has not been exploited in the sampling/counting literature (for $q \geq 3$) to the best of our knowledge. For the special case of the Ising model, that is, $q = 2$, this perspective is known as the even 'subgraphs world' and has been key in determining an efficient sampling/counting algorithm for the Ising model (with external field) by Jerrum and Sinclair [50], although the Markov chain used there is defined on the collection of all subsets of the edge set E rather than on just the even sets. We however define a Markov chain on a state space which, for $q = 2$, is supported only on the even sets. For $q = 2$ one could interpret our Markov chain as Glauber dynamics with respect to a fixed basis of the space of even sets (which forms a vector space over $\mathbb{F}_2$), that is, we move from one even subgraph to another by adding/subtracting elements from the basis. In the general case ($q \geq 3$) the even subgraphs need to be replaced by flows, but, aside from some technical details, our approach remains the same. We analyse the Markov chain using the well known method of path coupling [16, 49] to obtain our first sampling result Theorem 4.1(i), and the proof of Theorem 4.2(i) then follows by standard arguments after a suitable self-reducibility trick.

Another well known way of representing the partition function of the Potts model is via the random cluster model. Only recently, it was shown that a natural Markov chain called random cluster dynamics is rapidly mixing for the Ising model [40], yielding another way of obtaining approximation algorithms for the partition function of the Ising model. In the analysis a coupling due to Grimmet and Jansson[39] between the random cluster model and the even subgraphs world was used. We extend this coupling to the level of flows and we analyse the Glauber dynamics on the joint space of flows and clusters to obtain a proof of part (ii) of Theorems 4.1 and 4.2.

Organization In the next section we introduce the notion of flows and the flow partition function, showing the connection to the Potts model and the random cluster model. We also give some preliminaries on Markov chains. In Section 4.3, we introduce and analyse the flow chain and prove Theorem 4.1(i). In Section 4.4 we introduce and analyse the joint flow-random cluster Markov chain, which allows us to prove Theorem 4.1(ii). In Section 4.5 we examine the subtleties involved in showing that our sampling algorithms imply corresponding counting algorithms: we deduce Theorem 4.2 from Theorem 4.1 in this section. Finally in Section 4.6, we use the duality between flows and Potts configurations to deduce a slow mixing result for our flow chain (on $\mathbb{Z}^2$) from existing results about slow mixing for the Potts model.

4.2 Preliminaries

4.2.1 The flow partition function

Let $G = (V, E)$ be a graph. Throughout, if it is unambiguous, we will take $n := |V|$ and $m := |E|$. In order to define a flow on G, we first orient the edges of G. (We will assume from now on that the edges of graphs have been given a fixed orientation even if this is not explicitly stated). For an abelian group Γ, a Γ-flow (on G) is an assignment $f : E \to \Gamma$ of a value of Γ to every edge of G such that, for every vertex, the sum (in Γ) at the incoming edges is the same as the sum (in Γ) at the outgoing edges. For a positive integer q, the *flow partition function* is defined as [4]

$$Z_{\text{flow}}(G; q, z) = \sum_{f: E \to \mathbb{Z}_q \text{ flow}} z^{\#\text{non-zero edges in } f}.$$

Note that Z_{flow} only depends on the underlying graph and not on the orientation of G. It is moreover well known that in the definition of the partition function we can replace the group $\mathbb{Z}_q$ by any abelian group Γ of order q, without changing the partition function. We will however make no use of this and solely work with the group $\mathbb{Z}_q$.

Recall from the introduction that $\mathcal{F}(G)$ denotes the set of $\mathbb{Z}$-flows. We write $\mathcal{F}_q(G)$ for the set of $\mathbb{Z}_q$-flows (namely the set of all flows $f : E \to \mathbb{Z}_q$), and for $F \subseteq E$ we denote by $\mathcal{F}_q(V, F)$ the set of all flows $f : F \to \mathbb{Z}_q$. The *support* of a flow f is the collection of edges that receive a nonzero flow value and is denoted by $\text{supp}(f)$. We denote by $\text{nwz}(F; q)$ the number of flows $f : F \to \mathbb{Z}_q$ such that $\text{supp}(f) = F$ (where nwz stands for nowhere zero). Finally, for positive z, there is a natural probability measure μ_{flow} on $\mathcal{F}_q(V, E)$, defined by

$$\mu_{\text{flow}}(f) := \frac{z^{|\,\text{supp}(f)|}}{Z_{\text{flow}}(G; q, z)} \tag{4.6}$$

for each $f \in \mathcal{F}_q(V, E)$.

The following fact is well known and goes back to Tutte [70], and follows from Proposition 1.1

Lemma 4.5. *Let $q \in \mathbb{N}_{\geq 1}$ and let $z \in \mathbb{C} \setminus \{1\}$. Let $G = (V, E)$ be a graph. Then*

$$q^{|V|} Z_{\text{flow}}(G; q, z) = (1 - z)^{|E|} Z_{\text{Potts}}\left(G; q, 1 + \frac{qz}{1 - z}\right). \tag{4.7}$$

This lemma also follows by combining (4.9) and (4.11) below: it illustrates a useful coupling between random flows and the random cluster model. We remark that the function $z \mapsto 1 + \frac{qz}{1-z}$ (seen as a function from $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$) has the property that it sends 0 to 1, 1 to ∞ and the interval $[0, 1]$ to $[1, \infty]$ in

[4] Note that this definition differs slightly from the flow partition function as defined in Chapter 1. Calling that one $Z_{\text{flow,zero}}$, they are related as

$$Z_{\text{flow,zero}}(G; q, x) = x^{|E|} Z_{\text{flow}}(G; q, x^{-1})$$

an orientation preserving way. So approximating the partition function of the q-state ferromagnetic Potts model at low temperatures ($1 \ll y$) is equivalent to approximating the flow partition function for values $z \in (0,1)$ close to 1.

4.2.2 The random cluster model and a useful coupling

We view the *partition function of the random cluster model* for a fixed positive integer q as a polynomial in a variable y. It is defined for a graph $G = (V, E)$ as follows:

$$Z_{\mathrm{RC}}(G; q, y) := \sum_{F \subseteq E} q^{k(F)}(y-1)^{|F|}, \tag{4.8}$$

where $k(F)$ denotes the number of components of the graph (V, F). For $y \geq 1$, we denote the associated probability distribution on the collection of subsets of the edges $\{F \mid F \subseteq E\}$ by μ_{RC}, i.e., for $F \subseteq E$ we have

$$\mu_{\mathrm{RC}}(F) = \frac{q^{k(F)}(y-1)^{|F|}}{Z_{\mathrm{RC}}(G; q, y)}.$$

It is well known, see e.g. [26] and Section 1.3, that

$$Z_{\mathrm{Potts}}(G; q, y) = Z_{\mathrm{RC}}(G; q, y). \tag{4.9}$$

To describe the connection between Z_{RC} and Z_{flow} and a coupling between the associated probability distributions, it will be useful to consider the following partition function for a graph $G = (V, E)$:

$$Z(G; q, z) := (1-z)^{|E|} q^{|V|} \sum_{A \subseteq E} \mathrm{nwz}(A; q, z) \sum_{\substack{F \subseteq E \\ A \subseteq F}} \left(\frac{z}{1-z}\right)^{|F|}. \tag{4.10}$$

The associated probability distribution μ is on pairs (f, F) such that f is a $\mathbb{Z}_q$-flow on G with $\mathrm{supp}(f) \subseteq F$. By (4.9), the next lemma directly implies Lemma 4.5; the lemma and the coupling it implies extend [39].

Lemma 4.6. *Let $q \in \mathbb{N}_{\geq 1}$ and let $z \in \mathbb{C} \setminus \{1\}$. Let $G = (V, E)$ be a graph.*

$$q^{|V|} Z_{\mathrm{flow}}(G; q, z) = Z(G; q, z) = (1-z)^{|E|} Z_{\mathrm{RC}}(G; q, 1 + \tfrac{qz}{1-z}). \tag{4.11}$$

Proof. The first equality follows by the following sequence of identities:

$$\begin{aligned}
q^{-|V|}Z(G;q,z) &= (1-z)^{|E|}\sum_{A\subseteq E}\mathrm{nwz}(A;q)\sum_{\substack{F\subseteq E\\ A\subseteq F}}\left(\frac{z}{1-z}\right)^{|F|}\\
&= \sum_{A\subseteq E}\mathrm{nwz}(A;q)\sum_{\substack{F\subseteq E\\ A\subseteq F}} z^{|F|}(1-z)^{|E\setminus F|}\\
&= \sum_{A\subseteq E}\mathrm{nwz}(A;q)z^{|A|}\sum_{\substack{F\subseteq E\\ A\subseteq F}} z^{|F\setminus A|}(1-z)^{|E\setminus F|}\\
&= \sum_{A\subseteq E}\mathrm{nwz}(A;q)z^{|A|} = Z_{\text{flow}}(G;q,z).
\end{aligned}$$

For the second equality we use the well known fact that $|\mathcal{F}_q(V,F)|$ (the number of all flows on the graph (V,F) taking values in an abelian group of order q), satisfies

$$|\mathcal{F}_q(V,F)| = Z_{\text{flow}}((V,F);q,1) = q^{|F|-|V|+k(F)}. \tag{4.12}$$

To see it, note first that we may assume (V,F) is connected since both sides of the identity are multiplicative over components. Fix a spanning tree $T\subseteq F$ and assign values from $\mathbb{Z}_q$ to $F\setminus T$. It is not hard to see that these values can be uniquely completed to a flow by iteratively 'removing' a leaf from T.

We then have the following chain of equalities:

$$\begin{aligned}
(1-z)^{-|E|}Z(G;q,z) &= q^{|V|}\sum_{A\subseteq E}\mathrm{nwz}(A;q)\sum_{\substack{F\subseteq E\\ A\subseteq F}}\left(\frac{z}{1-z}\right)^{|F|}\\
&= \sum_{F\subseteq E}\left(\frac{z}{1-z}\right)^{|F|} q^{|V|}\sum_{A\subseteq F}\mathrm{nwz}(A;q)\\
&= \sum_{F\subseteq E}\left(\frac{z}{1-z}\right)^{|F|} q^{|V|}|\mathcal{F}_q(V,F)|\\
&= \sum_{F\subseteq E}\left(\frac{z}{1-z}\right)^{|F|} q^{|F|+k(F)} = Z_{\text{RC}}(G;q,1+\tfrac{qz}{1-z}). \qquad \square
\end{aligned}$$

The previous lemma in fact gives a coupling between the probability measures μ_{flow} and μ_{RC} (with the same parameters as in the lemma). More concretely, given a random flow f drawn from μ_{flow} let A be the support of f. Next select each edge $e\in E\setminus A$ independently with probability z. The resulting set F is then a sample drawn from μ_{RC}. To see this, observe that the probability of selecting the set F is given by

$$\sum_{A\subseteq F} \frac{\mathrm{nwz}(A;q)z^{|A|}}{Z_{\mathrm{flow}}(G;q,z)} z^{|F\setminus A|}(1-z)^{|E\setminus F|} = \frac{|\mathcal{F}_q(V,F)|}{Z_{\mathrm{flow}}(G;q,z)} z^{|F|}(1-z)^{|E\setminus F|} = \mu_{\mathrm{RC}}(F),$$

where the last equality follows by the lemma above and (4.12) and the definition of μ_{RC}. Conversely (by a similar calculation), given a sample F drawn from μ_{RC} one can obtain a random flow drawn from μ_{flow} by choosing a uniform flow on (V,F).

For any $\delta > 0$, this procedure transforms a δ-approximate sampler $\widehat{\mu_{\mathrm{flow}}}$ for μ_{flow} with parameters q and $z \in (0,1)$ into a δ-approximate sampler $\widehat{\mu_{\mathrm{RC}}}$ for μ_{RC} with parameters $q, 1+\frac{qz}{1-z}$ in time bounded by $O(|E|)$. Indeed, denoting for a flow f, $\delta_{\mathrm{flow}}(f) := \widehat{\mu_{\mathrm{flow}}}(f) - \mu_{\mathrm{flow}}(f)$, we have by the triangle inequality

$$\begin{aligned}
\sum_{F\subseteq E} |\widehat{\mu_{\mathrm{RC}}}(F) - \mu_{\mathrm{RC}}(F)| &= \sum_{F\subseteq E} \left| \sum_{f\in\mathcal{F}_q(V,F)} \delta_{\mathrm{flow}}(f) z^{|F\setminus \mathrm{supp}(f)|}(1-z)^{|E\setminus F|} \right| \\
&\leq \sum_{F\subseteq E} \sum_{f\in\mathcal{F}_q(V,F)} |\delta_{\mathrm{flow}}(f)| z^{|F\setminus \mathrm{supp}(f)|}(1-z)^{|E\setminus F|} \\
&= \sum_{f\in\mathcal{F}_q(V,F)} |\delta_{\mathrm{flow}}(f)| \sum_{F\supseteq \mathrm{supp}(f)} z^{|F\setminus \mathrm{supp}(f)|}(1-z)^{|E\setminus F|} \\
&= \sum_{f\in\mathcal{F}_q(V,F)} |\delta_{\mathrm{flow}}(f)| \leq 2\delta.
\end{aligned}$$

The Edwards-Sokal coupling [26] allows us to generate a sample from the Potts model (with parameters q, y), given a sample F from the random cluster model (with parameters q, y): for each component of (V,F) uniformly and independently choose a colour $i \in [q]$ and colour each of the vertices in this component with this colour. Again if we have a δ-approximate sampler $\widehat{\mu_{\mathrm{RC}}}$ for μ_{RC} this will be transformed into a δ-approximate sampler $\widehat{\mu_{\mathrm{Potts}}}$ for μ_{Potts} in time bounded by $O(|E|)$. We summarize the discussion above in a proposition.

Proposition 4.7. *Let $G = (V,E)$ be a graph and let $q \in \mathbb{N}_{\geq 2}$ and $z \in (0,1)$. Let $\delta > 0$. Given an approximate δ-approximate sampler $\widehat{\mu_{\mathrm{flow}}}$ for μ_{flow} with parameters q and z, we can obtain δ-approximate approximate samplers from*

- μ_{RC} *with parameters q and $1+\frac{qz}{1-z}$ in time $O(|E|)$,*
- μ_{Potts} *with parameters q and $1+\frac{qz}{1-z}$ in time $O(|E|)$.*

4.2.3 Generating sets and bases of flows

In this subsection we give some useful properties of the set of flows and their even generating sets that will allow us to define Markov chains for sampling from μ_{flow} in the next section. In particular we show that an even generating set for the cycle space also generates the collection of $\mathbb{Z}_q$-flows in an appropriate sense to be made precise below.

Let $G = (V, E)$ be a connected graph and recall (from the Introduction) that $\mathcal{F}(G)$ is the set of $\mathbb{Z}$-flows on G and let $\mathcal{C}$ be an even generating set of $\mathcal{F}(G)$. We already mentioned that $\mathcal{F}(G)$ forms a $\mathbb{Z}$-module; in fact it is a free-module of dimension $|E| - |V| + 1$, cf. [34, Section 14]. Similarly, the collection of $\mathbb{Z}_q$-flows on G is closed under adding two flows and multiplying a flow by an element of $\mathbb{Z}_q$, making the space of $\mathbb{Z}_q$-flows into a $\mathbb{Z}_q$-module; it is also a free module of dimension $|E|-|V|+1$ by the same argument as for $\mathbb{Z}$, cf. [34, Section 14]. (Note that this fact also implies (4.12).)

Lemma 4.8. *Let $\mathcal{C}$ be an even generating set for $\mathcal{F}(G)$. Then $\mathcal{C}$ is a generating set for $\mathcal{F}_q(G)$ for any positive integer q.*

Proof. Let $f \in \mathcal{F}_q(G)$ be a flow, we will construct a $\mathbb{Z}$-flow f' which reduces modulo q to f. Just as in the proof of Lemma 4.6, fix a spanning tree $T \subset E$, and now assign to every edge $e \in E \setminus T$ an integer from the residue class $f(e)$. These assignments can be completed iteratively into the flow f' by choosing the edge towards a leaf, assigning a value to satisfy the flow condition in the leaf, and removing the edge from T. These new values are also in the residue class prescribed by f, because f itself satisfies the flow condition in every leaf encountered. Writing f' as a linear combination of χ_C for $C \in \mathcal{C}$ and reducing modulo q, we obtain f as a $\mathbb{Z}_q$-linear combination of χ_C. □

Finally, we will require the following lemma for our reduction of sampling to counting in Section 4.5. For a graph $G = (V, E)$, a subgraph H of G and an edge $e \in E$, H/e denotes the graph obtained from H by contracting the edge e. (If e is not an edge of H, then H/e is just H.)

Lemma 4.9. *Let $G = (V, E)$ be a graph and let $q \in \mathbb{N}$. Let $\mathcal{C} = \{C_1, \ldots, C_r\}$ be an even generating set for the space of $\mathbb{Z}_q$-flows. Let $e \in E$ be a non-loop edge. Then $\mathcal{C}' := \{C_1/e, \ldots, C_r/e\}$ is an even generating set for the space of $\mathbb{Z}_q$-flows of the graph G/e satisfying $d(\mathcal{C}') \le d((\mathcal{C})$, $\iota(\mathcal{C}') \le \iota(\mathcal{C})$, $\ell(\mathcal{C}') \le \ell(\mathcal{C})$ and $s(\mathcal{C}') \le s(\mathcal{C})$.*

Proof. This follows from the fact that any flow f' on G/e uniquely corresponds to a flow f on G. The value on the edge e for f can be read off from the values of the edges incident to the vertex in G/e corresponding to the two endpoints of the edge e. So, writing $f = \sum_{i=1}^{r} a_i \chi_{C_i}$ for certain $a_i \in \mathbb{Z}_q$, we get $f' = \sum_{i=1}^{r} a_i \chi_{C_i/e}$, proving the claim. The claimed inequalities for the parameters are clear. □

Remark 4.10. Suppose q is a prime in which case $\mathbb{Z}_q$ is a field and $\mathcal{F}_q(G)$ is a vector space over $\mathbb{Z}_q$. Then given an even generating set $\mathcal{C}$ for $\mathcal{F}_q(G)$ there exists a basis $\mathcal{C}'$ consisting only of cycles for which the parameters d, ι, ℓ and s are all not worse. To see this note that if $\mathcal{C}$ is a generating set and not a basis, we can always remove elements from it to make it into a basis. If $\mathcal{C}$ forms a basis and some $C \in \mathcal{C}$ is the edge disjoint union of two nonempty even subgraphs K_1 and K_2, we have that either $(\mathcal{C} \setminus \{C\}) \cup \{K_1\}$ or $(\mathcal{C} \setminus \{C\}) \cup \{K_2\}$ forms a basis. This is generally not true for composite q and therefore we work with even generating sets.

4.2.4 Preliminaries on Markov chains

To analyse the mixing time of our Markov chains, we will use the path coupling technique. We briefly recall the following results from Section 2 in [25].

Let $\mathcal{M} = (Z_t)_{t=0}^{\infty}$ be an ergodic, discrete-time Markov chain on a finite state space Ω with transition matrix P. Let μ_t be the distribution of Z_t and let μ be the (unique) stationary distribution of $\mathcal{M}$. Two distributions on Ω are said to be *δ-close* if the total variation distance between them is at most δ. The *δ-mixing time* of $\mathcal{M}$ is the minimum number of steps after which $\mathcal{M}$ is δ-close to its stationary distribution (i.e. the smallest t such that $\|\mu_t - \mu\|_{\mathrm{TV}} \leq \delta$).

A *coupling* for $\mathcal{M}$ is a stochastic process (X_t, Y_t) on Ω^2, such that each of X_t and Y_t, considered independently, transition according to P. More precisely, the coupling can be defined by its transition matrix P': given (x, y) and $(x', y') \in \Omega^2$, $P'((x, y), (x', y'))$ is the probability that $(X_{t+1}, Y_{t+1}) = (x', y')$ given that $(X_t, Y_t) = (x, y)$. For P' to describe a valid coupling, it must satisfy for each $(x, y) \in \Omega^2$, that

$$\begin{aligned} \sum_{y' \in \Omega} P'((x, y), (x', y')) &= P(x, x') \quad \text{for all } x' \in \Omega; \\ \sum_{x' \in \Omega} P'((x, y), (x', y')) &= P(y, y') \quad \text{for all } y' \in \Omega. \end{aligned} \tag{4.13}$$

For our use of path coupling, we require an integer-valued distance function d on Ω such that between any two states $x, y \in \Omega$ there exists a sequence $x = x_0, x_1, \ldots, x_s = y$ in which consecutive states are at distance 1. If we can define a coupling on the set of pairs $(x, y) \in \Omega^2$ for which $d(x, y) = 1$. (that is, we define transition probabilities $P'((x, y), (x', y'))$ for all (x, y) such that $d(x, y) = 1$, and $(x', y') \in \Omega^2$ that satisfy equations (4.13)) then this can be extended to a complete coupling on Ω^2. We can use such a (partial) coupling to bound the mixing time of $\mathcal{M}$ via the following result:

Theorem 4.11 (Theorem 2.2 in [25]). *Let $\mathcal{M}$ be a Markov chain on Ω and d an integer-valued distance on Ω as above with maximum distance D. Assume there is a coupling $(X_t, Y_t) \mapsto (X_{t+1}, Y_{t+1})$ defined for all pairs with $d(X_t, Y_t) = 1$ (as described above) such that*

$$\mathbb{E}(d(X_{t+1}, Y_{t+1}) \mid (X_t, Y_t)) \leq 1 - \alpha$$

for some $\alpha > 0$. Then the Markov chain $\mathcal{M}$ has δ-mixing time at most $\frac{\log(D\delta^{-1})}{\alpha}$.

4.3 Flow Markov chain

In this section, we introduce and analyse the flow Markov chain and use it to prove Theorem 4.1(i).

Definition 4.12. Let $G = (V, E)$ be a graph and $\mathcal{C}$ an even generating set of $\mathcal{F}_q(V, E)$ of size r. The *flow Markov chain* for $(G, \mathcal{C})$ is a Markov chain on the state space $\mathcal{F}_q(V, E)$. For every flow $f \in \mathcal{F}_q(V, E)$, $t \in \mathbb{Z}_q \setminus \{0\}$ and $C \in \mathcal{C}$, the transition probabilities of the Markov chain are given by:

$$\begin{aligned} P_{\text{flow}}(f, f + t\chi_C) &= \frac{1}{r} \frac{\mu_{\text{flow}}(f + t\chi_C)}{\sum_{u \in \mathbb{Z}_q} \mu_{\text{flow}}(f + u\chi_C)}, \\ P_{\text{flow}}(f, f) &= \frac{1}{r} \sum_{C \in \mathcal{C}} \frac{\mu_{\text{flow}}(f)}{\sum_{u \in \mathbb{Z}_q} \mu_{\text{flow}}(f + u\chi_C)}, \end{aligned}$$

and all other transition probabilities are zero.

We see easily that the measure μ_{flow} satisfies the detailed balance equation

$$\mu_{\text{flow}}(f) P_{\text{flow}}(f, f + t\chi_C) = \mu_{\text{flow}}(f + t\chi_C) P_{\text{flow}}(f + t\chi_C, f),$$

so μ_{flow} is the stationary distribution of the flow Markov chain.

We can simulate one step of this Markov efficiently by first selecting $C \in \mathcal{C}$ uniformly at random, and for $t \in \mathbb{Z}_q$, selecting $f + t\chi_C$ with probability proportional to

$$\mu_{\text{flow}}(f + t\chi_C)/\mu_{\text{flow}}(f) = z^{\#\{e \in C \mid f(e) = 0\} - \#\{e \in C \mid f(e) + t\chi_C(e) = 0\}}.$$

For fixed q, simulating one step of the Markov chain requires $O(\ell)$ time (where $\ell = \max_{C \in \mathcal{C}} |C|$) in order to compute $f + t\chi_C$ and its support. We bound this by $O(m)$.

4.3.1 Rapid mixing of flow Markov chain

Theorem 4.13. *Let $q, d \geq 2, \iota \geq 1$ be integers and $1 > z > 1 - \frac{2}{(d+1)\iota}$. Write $\zeta = z - \left(1 - \frac{2}{(d+1)\iota}\right)$ and let $\delta > 0$. Now let $G = (V, E)$ be a graph and $\mathcal{C}$ an even generating set of $\mathcal{F}_q(G)$ of size r satisfying $d(\mathcal{C}) \leq d$ and $\iota(\mathcal{C}) \leq \iota$, then the δ-mixing time of the flow Markov chain for $(G, \mathcal{C})$ with parameter z is at most $\frac{4r}{d\iota} \log(r\delta^{-1})\zeta^{-1}$.*

Remark 4.14. Because $\zeta < \frac{2}{(d+1)\iota} \leq \frac{2}{d\iota}$, the upper bound in this Theorem is always at least $2r\log(r\delta^{-1})$. This shows the upper bound doesn't get better with larger d and ι, even though they are in the denominator.

For the given range of z, the flow Markov chain therefore gives an efficient, randomised algorithm for approximately sampling flows according to μ_{flow}. Combining this with Proposition 4.7, we obtain the following Corollary; it directly implies Theorem 4.1(i) by Lemma 4.8.

Corollary 4.15. *Fix integers $q, d \geq 2$ and $\iota \geq 1$. For any $y > \frac{(d+1)\iota}{2} q - (q-1)$ and $\delta > 0$, there exists an algorithm that on input of an m-edge graph G and even generating set $\mathcal{C}$ of $\mathcal{F}_q(G)$ of size r satisfying $d(\mathcal{C}) \leq d$ and $\iota(\mathcal{C}) \leq \iota$ outputs a q-state Potts colouring $\sigma : V \to [q]$ within total variation distance δ of the q-state Potts-measure μ_{Potts} with parameter y. This is obtained by running the flow Markov chain for at most $O(r\log(r\delta^{-1}))$ steps where each step takes $O(m)$ time.*

The following technical lemma will be used in the proof of Theorem 4.13. Note that the lower bound is actually attained in the limit case $(a_1, \ldots, a_q) = (\iota, 0, -\infty, \ldots, -\infty), (b_1, \ldots, b_q) = (0, \iota, -\infty, \ldots, -\infty)$. The proof is postponed to the end of this section.

Lemma 4.16. *Let $z \in (0, 1)$ be a real number, and $\iota \geq 0$ and $a_1, \ldots, a_q, b_1, \ldots, b_q$ integers satisfying the following constraints:*

- $\sum_i a_i = \sum_i b_i$;
- $\sum_i |a_i - b_i| \leq 2\iota$.

Then

$$S := \sum_i \min\left(\frac{z^{-a_i}}{\sum_j z^{-a_j}}, \frac{z^{-b_i}}{\sum_j z^{-b_j}}\right) \geq 1 - \frac{1 - z^\iota}{1 + z^\iota}.$$

Proof of Theorem 4.13. To prove the theorem we determine an upper bound for the mixing time of the flow Markov chain by using path coupling. For this we define the distance between two flows as the minimal number of steps the flow

Markov chain needs to go from one to the other. By Theorem 4.11 it is now enough to define a coupling for states at distance 1. If the expected distance after one step of this coupling is at most $1-\alpha$, the mixing time of the Markov chain is at most $T := \frac{\log(r\delta^{-1})}{\alpha}$. (The maximal distance in $\mathcal{F}_q(V,E)$ is at most r, because in r steps the coefficients of every even set in $\mathcal{C}$ can be adjusted to the desired value.)

We will construct a coupling on states at distance 1 for which $\alpha = \frac{(d+1)z^\iota-(d-1)}{2r} \geq \frac{d\iota}{4r}\zeta$. Therefore the running time of the sampler is bounded by $T \leq \frac{4r}{d\iota}\log(r\delta^{-1})\zeta^{-1}$ steps of the flow Markov chain.

Consider a pair of flows (f,g) which differ by a multiple of χ_C. To construct the coupling we first select u.a.r. an even set $D \in \mathcal{C}$. We will separate three cases, and define the transition probabilities in each of these cases. The cases are (a) when $C = D$, (b) when C and D have no common edges, and (c) when C and D do have common edges, but $C \neq D$.

(a) We get a valid coupling by making the transition $(f,g) \to (f+t\chi_D, f+t\chi_D)$ with probability $\frac{\mu_{\text{flow}}(f+t\chi_D)}{\sum_{u\in\mathbb{Z}_q}\mu_{\text{flow}}(f+u\chi_D)}$. Then the distance will always drop from 1 to 0.

(b) Now the edges of D have the same values in f and g, and we see that $\mu_{\text{flow}}(f+t\chi_D)/\mu_{\text{flow}}(f) = \mu_{\text{flow}}(g+t\chi_D)/\mu_{\text{flow}}(g)$ for all t. Therefore we get a valid coupling by making the transition $(f,g) \to (f+t\chi_D, g+t\chi_D)$ with probability $\frac{\mu_{\text{flow}}(f+t\chi_D)}{\sum_{u\in\mathbb{Z}_q}\mu_{\text{flow}}(f+u\chi_D)} = \frac{\mu_{\text{flow}}(g+t\chi_D)}{\sum_{u\in\mathbb{Z}_q}\mu_{\text{flow}}(g+u\chi_D)}$. In this case the distance between the two new states remains 1.

(c) The coupling in this case is more complicated, as the values of f and g on D are different. Below we prove the following:

Claim. *There is a coupling where the total probability for all transitions $(f,g) \to (f+t\chi_D, g+t\chi_D)$ is at least $1-\frac{1-z^\iota}{1+z^\iota}$.*

In all these transitions the distance remains 1, and therefore the probability of the distance increasing to 2 is at most $\frac{1-z^\iota}{1+z^\iota}$.

We can now calculate the expected distance after one step of this coupling. Case (a) occurs with probability $1/r$, and case (c) with probability at most d/r.

Hence the expected distance is at most

$$\begin{aligned}
&1 - \frac{1}{r} + \frac{d}{r} \cdot \frac{1 - z^\iota}{1 + z^\iota} \\
&= 1 - \frac{1 + z^\iota - d(1 - z^\iota)}{r(1 + z^\iota)} \\
&= 1 - \frac{(d+1)z^\iota - (d-1)}{r(1 + z^\iota)} \\
&\leq 1 - \frac{(d+1)z^\iota - (d-1)}{2r} = 1 - \alpha.
\end{aligned}$$

We see that α is positive for $z > 1 - \frac{2}{(d+1)\iota} > \sqrt[\iota]{1 - \frac{2}{d+1}} = \sqrt[\iota]{\frac{d-1}{d+1}}$. Further, we see for these z that the derivative of α with respect to z satisfies,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}z} = \frac{(d+1)\iota z^{\iota-1}}{2r} \geq \frac{(d+1)\iota z^\iota}{2r} \geq \frac{(d-1)\iota}{2r} \geq \frac{d\iota}{4r}.$$

Hence we find that $\alpha \geq \frac{d\iota}{4r}\zeta$.

We finish by proving the Claim in case (c). Explicitly the transition probabilities in this case are given by (writing $p_t = \frac{\mu_{\text{flow}}(f+t\chi_D)}{\sum_{u\in\mathbb{Z}_q}\mu_{\text{flow}}(f+u\chi_D)}$ and $q_t = \frac{\mu_{\text{flow}}(g+t\chi_D)}{\sum_{u\in\mathbb{Z}_q}\mu_{\text{flow}}(g+u\chi_D)}$)

$$(f, g) \to (f + t\chi_D, g + t\chi_D) \text{ with probability } \min(p_t, q_t),$$

and for $s \neq t$

$$\begin{aligned}
(f, g) \to (f + s\chi_D, g + t\chi_D) \text{ with probability}& \\
&\frac{(p_s - \min(p_s, q_s))(q_t - \min(p_t, q_t))}{\sum_{u\in\mathbb{Z}_q}(p_u - \min(p_u, q_u))} \\
&= \frac{(p_s - \min(p_s, q_s))(q_t - \min(p_t, q_t))}{\sum_{u\in\mathbb{Z}_q}(q_u - \min(p_u, q_u))}.
\end{aligned}$$

It is easily checked that this yields a valid coupling, i.e. that the first coordinate has transition probabilities p_t, and similary q_t for the second coordinate.

Now we wish to bound the sum of the diagonal entries. To do this we have to take a closer look at the weights occurring in this table. We define a_i to be the number of edges in D with value 0 in the flow $f + iD$. This ensures that $\mu_{\text{flow}}(f + i\chi_D) \propto z^{-a_i}$ and $p_t = \frac{z^{-a_t}}{\sum_u z^{-a_u}}$. Similarly, we define b_i as the number of edges in D with value 0 in the flow $g + i\chi_D$.

We derive some boundary conditions on the a_i's and b_i's. Ranging i over $\mathbb{Z}_q$, every edge of D will get value 0 in exactly one of $f + i\chi_D$. So $\sum_i a_i$ is the length $|D|$. The same holds for the b_i's, so in particular we find that $\sum_i a_i = \sum_i b_i$.

Second we will bound $\sum_i |a_i - b_i|$. If an edge is counted in a_i, but not in b_i, it must be an edge of C. For every such edge it can happen once that it is counted in a_i and not b_i, and once vice versa. Hence the total absolute difference $\sum_i |a_i - b_i|$ is bounded by $2|C \cap D| \leq 2\iota$.

Now the sum of all the probabilities on the diagonal is

$$\sum_i \min\left(\frac{z^{-a_i}}{\sum_j z^{-a_j}}, \frac{z^{-b_i}}{\sum_j z^{-b_j}}\right),$$

and the numbers a_i, b_i satisfy the conditions of Lemma 4.16, so the sum is bounded below by $1 - \frac{1-z^\iota}{1+z^\iota}$. □

Proof of Lemma 4.16. First of all, let us introduce a little terminology: an index i is called b-minimal if the minimum of the i-term in S is not equal to the a-term. Also assume that $\sum_j z^{-a_j} \geq \sum_j z^{-b_j}$. And note that the two conditions imply

$$2\iota \geq \sum_i |a_i - b_i| \geq |a_j - b_j| + \left|\sum_{i \neq j} a_i - b_i\right| = |a_j - b_j| + |b_j - a_j| = 2|a_j - b_j|.$$

Hence the absolute difference between a_j and b_j is always at most ι.

The proof contains two steps. In the first step, we change the numbers a_i in such a way that the conditions still hold and S does not increase. After the first step there will be at most one b-minimal index i. This allows us to eliminate the minima from the expression for S. In the second step, we give a lower bound for this new obtained expression.

For the first step, assume that two different indices t, u are b-minimal, and assume also that $a_t \geq a_u$. Now we increase a_t by 1, and decrease a_u by 1, i.e. define the new sequence

$$a'_i = \begin{cases} a_t + 1 & i = t, \\ a_u - 1 & i = u, \\ a_i & \text{otherwise.} \end{cases}$$

First we note that $\sum_j z^{-a'_j} > \sum_j z^{-a_j}$, simply because

$$z^{-a'_t} - z^{-a_t} = z^{-(a_t+1)}(1-z) > z^{-a_u}(1-z) = z^{-a_u} - z^{-a'_u}.$$

Now we will show for every i, that the term $\min(z^{-a_i}/\sum_j z^{-a_j}, z^{-b_i}/\sum_j z^{-b_j})$ does not increase. For $i \neq t, u$ this is easy, because z^{-a_i} does not change and the sum in the denominator increases. Hence the first term in the minimum decreases and the minimum cannot increase. We also assumed that both t, u were b-minimal, and because we don't change the b_i's, the minimum cannot increase.

Further, we have to check that the new sequence still satisfies all the conditions. It is clear that $\sum_i a'_i = \sum_i a_i = \sum_i b_i$ and $\sum_j z^{-a'_j} > \sum_j z^{-a_j} \geq \sum_j z^{-b_j}$. Further we see for $i = t, u$ that

$$\frac{z^{-a_i}}{\sum_j z^{-a_j}} > \frac{z^{-b_i}}{\sum_j z^{-b_j}} \geq \frac{z^{-b_i}}{\sum_j z^{-a_j}},$$

hence $a_i > b_i$ for $i = t, u$. Therefore $|a'_t - b_t| = |a_t - b_t + 1| = |a_t - b_t| + 1$ and $|a'_u - b_u| = |a_u - b_u - 1| = |a_u - b_u| - 1$ (because $a_u - b_u$ is a positive integer), so the sum of the absolute values remains the same.

After repeating this adjustment with the same indices, eventually one of them will stop being b-minimal. Now repeat with two new b-minimal indices, as long as they exist. In the end there must be at most one b-minimal index.

Now we are ready for step two. If there are no b-minimal indices, the sum is equal to 1 and the result holds. Hence we assume wlog that 1 is the only b-minimal index and we can write

$$S = \frac{z^{-b_1}}{\sum_j z^{-b_j}} + \sum_{i \neq 1} \frac{z^{-a_i}}{\sum_j z^{-a_j}} = \frac{z^{-b_1}}{\sum_j z^{-b_j}} + 1 - \frac{z^{-a_1}}{\sum_j z^{-a_j}}.$$

Note that for positive p, q, the function $\frac{-p}{p+q}$ is increasing in q and decreasing in p. Because $z^{-a_1} \leq z^{-(b_1+\iota)}$ and $\sum_{j \geq 2} z^{-a_j} \geq \sum_{j \geq 2} z^{-(b_j - \iota)}$, we can thus estimate that

$$S \geq \frac{z^{-b_1}}{\sum_j z^{-b_j}} + 1 - \frac{z^{-\iota} z^{-b_1}}{z^{-\iota} z^{-b_1} + z^{\iota} \sum_{j \geq 2} z^{-b_j}}.$$

Now write $X = z^{-\iota}$, $B_1 = z^{-b_1}$ and $B_2 = \sum_{j \geq 2} z^{-b_j}$, so that the lower bound for S can be written as $\frac{B_1}{B_1+B_2} + 1 - \frac{X^2 B_1}{X^2 B_1 + B_2}$. By AM-GM we can estimate that

$$\begin{aligned} (B_1 + B_2)(X^2 B_1 + B_2) &= X^2 B_1^2 + B_2^2 + (X^2+1) B_1 B_2 \\ &\geq 2X B_1 B_2 + (X^2+1) B_1 B_2 = (X+1)^2 B_1 B_2, \end{aligned}$$

so that we find:

$$\begin{aligned} S \geq 1 + \frac{B_1}{B_1 + B_2} - \frac{X^2 B_1}{X^2 B_1 + B_2} &= 1 + \frac{B_1(X^2 B_1 + B_2) - X^2 B_1 (B_1 + B_2)}{(B_1 + B_2)(X^2 B_1 + B_2)} \\ &= 1 - \frac{(X-1)(X+1) B_1 B_2}{(B_1 + B_2)(X^2 B_1 + B_2)} \\ &\geq 1 - \frac{(X-1)(X+1) B_1 B_2}{(X+1)^2 B_1 B_2} = 1 - \frac{X-1}{X+1}. \square \end{aligned}$$

4.4 Joint flow-random cluster Markov chain

In this section we will consider a different chain that allows us to sample flows. We will again prove rapid mixing by using path coupling, and this holds for roughly the same range of parameters z.

To describe the chain let $q \geq 2$ be an integer and let $G = (V, E)$ be a graph m edges. Let $\mathcal{C}$ be an even generating set for the flow space $\mathcal{F}_q(G)$ of size r and let $\ell = \ell(\mathcal{C})$.

Definition 4.17. Let $\Omega_{\text{flow}-\text{RC}}$ be the set of pairs (f, F) with $F \subset E$ a set of edges and f a flow on (V, F). The *joint flow-RC Markov chain* is a Markov chain on the state space $\Omega_{\text{flow}-\text{RC}}$ depending on two parameters $z, p \in (0, 1)$. The transition probabilities are as follows:
For $e \in E \setminus F$:

$$P_{\text{flow}-\text{RC}}[(f, F), (f, F \cup \{e\})] = \frac{(1-p)z}{m}.$$

For $e \in F$ such that $f(e) = 0$:

$$P_{\text{flow}-\text{RC}}[(f, F), (f, F \setminus \{e\})] = \frac{(1-p)(1-z)}{m}.$$

And for $t \in \{1, \ldots, q-1\}$, $C \in \mathcal{C}$ an even set such that $C \subseteq F$:

$$P_{\text{flow}-\text{RC}}[(f, F), (f + t\chi_C, F)] = \frac{p}{qr}.$$

All other transition probabilities are zero, except for the stationary probabilities $P_{\text{flow}-\text{RC}}[(f, F), (f, F)]$.

Simulating one step of this Markov chain starting in the state (f, F) can be done as follows. We first select either 'flow' or 'edges' with probabilities resp. p and $1 - p$.

- If we select 'flow', we will update the flow f. We choose $C \in \mathcal{C}$ and $t \in \mathbb{Z}_q$ uniformly at random. If the flow $f + t\chi_C$ is supported on F (for $t \neq 0$ this is equivalent to $C \subseteq F$), we make the transition $(f, F) \to (f + t\chi_C, F)$. Otherwise the chain stays in (f, F).

- If we select 'edges', we will update the set of edges F. We choose an edge $e \in E$ uniformly at random. If e is not contained in F, we make with probability z the transition $(f, F) \to (f, F \cup \{e\})$. If e is contained in F and $f(e) = 0$, we make with probability $1-z$ the transition $(f, F) \to (f, F \setminus \{e\})$. Otherwise the chain stays in (f, F).

The total cost of simulating one step of this Markov chain is $O(\ell)$ for checking whether $C \subseteq F$ in the first case.

Further this Markov chain has stationary distribution $\mu_{\text{flow}-\text{RC}}\colon (f, F) \mapsto \frac{1}{Z_{\text{flow}}} z^{|F|}(1-z)^{|E \setminus F|}$. (From Lemma 4.6 it follows that the sum over all states is 1.) This follows easily from checking the detailed balance equation.

4.4.1 Rapid mixing of joint flow-RC Markov chain

Theorem 4.18. *Let $\ell \geq 3, q, s \geq 2$ be integers and $1 > z > 1 - \frac{q}{(q-1)\ell s}$. Write $\zeta = z - \left(1 - \frac{q}{(q-1)\ell s}\right)$ and let $\delta > 0$. Let $G = (V, E)$ be a graph and $\mathcal{C}$ an even generating set of $\mathcal{F}_q(V, E)$ of size r satisfying $\ell(\mathcal{C}) \leq \ell$ and $s(\mathcal{C}) \leq s$, then there is a value of p for which the joint flow-RC Markov chain for $(G, \mathcal{C})$ comes δ-close to $\mu_{\text{flow}-\text{RC}}$ with parameter z in at most $\frac{2(m+r)}{\ell} \log((2m+r)\delta^{-1})\zeta^{-1}$ steps.*

Remark 4.19. An exact value for p in the theorem above can be obtained from equation (4.14) below.

Remark 4.20. Note again that $\zeta > \frac{q}{(q-1)\ell s} > \frac{1}{\ell s}$, and hence the required number of calls in the above theorem is at least $2s(m+r)\log((2m+r)\delta^{-1})$. Again this means the bound does not get better with larger ℓ, even though it appears in the denominator, and even gets worse with larger s.

It would be interesting to see if the theorem could be used to say anything about possible rapid mixing of the Glauber dynamics for the random cluster model at low temperatures cf. [40].

The following corollary is immediate by Proposition 4.7 and directly implies Theorem 4.1(ii) by Lemma 4.8.

Corollary 4.21. *Fix integers $\ell \geq 3$ and $q, s \geq 2$. Let $y > (q-1)(\ell s - 1)$ and $\delta > 0$, then there exists an algorithm that on input an m-edge graph G and an even generating set $\mathcal{C}$ for $\mathcal{F}_q(G)$ of size r satisfying $\ell(\mathcal{C}) \leq \ell$ and $s(\mathcal{C}) \leq s$, outputs a q-state Potts colouring $\sigma : V \to [q]$ within total variation distance δ of the q-state Potts-measure μ_{Potts} with parameter y. This is obtained by running the joint flow-RC Markov chain for $O((m+r)\log((m+r)\delta^{-1}))$ steps, where each step takes $O(1)$ time (since ℓ is fixed).*

Proof of Theorem 4.18. We will again use path coupling to deduce rapid mixing of the above defined Markov chain. The distance we use on the state space is defined as the least number of steps required in the Markov chain to go from one state to the another. A crude upper bound on the diameter is given by $2m+r$. There are two kinds of pairs of states at distance one, which we will treat separately. Just as in the proof of Theorem 4.13, we will prove that the expected distance after one step of the coupling is at most $1-\alpha$ for some α, and therefore the mixing time is at most $\log((2m+r)\delta^{-1})\alpha^{-1}$.

Consider the states (f, F) and $(f, F\cup\{e\})$. We will make a coupling on them. The transition probabilities of this coupling are as follows:

$$\begin{pmatrix} f & F \\ f & F\cup\{e\} \end{pmatrix} \to \begin{cases} \begin{pmatrix} f & F\cup\{e\} \\ f & F\cup\{e\} \end{pmatrix} & \frac{(1-p)z}{m}, \\ \begin{pmatrix} f & F \\ f & F \end{pmatrix} & \frac{(1-p)(1-z)}{m}, \\ \begin{pmatrix} f & F\cup\{e'\} \\ f & F\cup\{e,e'\} \end{pmatrix} & \frac{(1-p)z}{m} \text{ if } e' \notin F\cup\{e\}, \\ \begin{pmatrix} f & F\setminus\{e'\} \\ f & F\setminus\{e'\}\cup\{e\} \end{pmatrix} & \frac{(1-p)(1-z)}{m} \text{ if } e' \in F \text{ and } f(e') = 0, \\ \begin{pmatrix} f+t\chi_C & F \\ f+t\chi_C & F\cup\{e\} \end{pmatrix} & \frac{p}{qr} \text{ if } t \neq 0 \text{ and } C \subseteq F, \\ \begin{pmatrix} f & F \\ f+t\chi_C & F\cup\{e\} \end{pmatrix} & \frac{p}{qr} \text{ if } t \neq 0,\ e \in C \text{ and } C \subseteq F\cup\{e\}. \end{cases}$$

The first two cases each occur exactly once and decrease the distance by one. The last case occurs at most $s(q-1)$ times and increases the distance by one. Therefore the expected distance after one step of the coupling is at most

$$1 - \frac{1-p}{m} + \frac{(q-1)sp}{qr}$$

in this case.

Next is the coupling on the neighbouring states (f, F) and $(f+t\chi_C, F)$ (with

$t \neq 0$). The transition probabilities are as follows:

$$\begin{pmatrix} f & F \\ f + t\chi_C & F \end{pmatrix} \to \begin{cases} \begin{pmatrix} f & F \cup \{e\} \\ f + t\chi_C & F \cup \{e\} \end{pmatrix} & \frac{(1-p)z}{m} \text{ if } e \notin F, \\ \begin{pmatrix} f & F \setminus \{e\} \\ f + t\chi_C & F \setminus \{e\} \end{pmatrix} & \frac{(1-p)(1-z)}{m} \text{ if } e \notin F \text{ and } e \notin C, \\ \begin{pmatrix} f & F \setminus \{e\} \\ f + t\chi_C & F \end{pmatrix} & \frac{(1-p)(1-z)}{m} \text{ if } e \in C \text{ and } f(e) = 0, \\ \begin{pmatrix} f & F \\ f + t\chi_C & F \setminus \{e\} \end{pmatrix} & \frac{(1-p)(1-z)}{m} \begin{array}{l} \text{if } e \in C \text{ and} \\ f(e) + t\chi_C(e) = 0, \end{array} \\ \begin{pmatrix} f + t'\chi_C & F \\ f + t'\chi_C & F \end{pmatrix} & \frac{p}{qr}, \\ \begin{pmatrix} f + t'\chi_{C'} & F \\ f + t\chi_C + t'\chi_{C'} & F \end{pmatrix} & \frac{p}{qr} \begin{array}{l} \text{if } t' \neq 0,\ C' \neq C \text{ and} \\ C' \subseteq F. \end{array} \end{cases}$$

The third and fourth case occur together at most ℓ times and increase the distance with one. The fifth case occurs exactly q times and decreases the distance with one. Therefore the expected distance after one step of the coupling is at most

$$1 - \frac{p}{r} + \frac{\ell(1-z)(1-p)}{m}.$$

To find a useful coupling, both expected distances will have to be smaller than one and we have to solve the following equations (for p and α):

$$\begin{aligned} 1 - \frac{1-p}{m} + \frac{(q-1)sp}{qr} &= 1 - \frac{p}{r} + \frac{\ell(1-z)(1-p)}{m} = 1 - \alpha, \\ \text{i.e. } \frac{1-p}{m} - \frac{(q-1)sp}{qr} &= \frac{p}{r} - \frac{\ell(1-z)(1-p)}{m} = \alpha. \end{aligned} \tag{4.14}$$

For $p = 0$, the first term is positive while the second is negative, and vice versa for $p = 1$. Therefore the solution for p lies indeed in $(0, 1)$ and we will not calculate it explicitly. Instead we eliminate p to only calculate the value of α:

$$\begin{aligned}
&\frac{1}{qrm}\left(qr + (q-1)sm + qm + qr\ell(1-z)\right)\alpha \\
&= \left(\frac{1}{m} + \frac{(q-1)s}{qr}\right)\left(\frac{1}{r}p + \frac{\ell(1-z)}{m}p - \frac{\ell(1-z)}{m}\right) \\
&+ \left(\frac{1}{r} + \frac{\ell(1-z)}{m}\right)\left(\frac{1}{m} - \frac{1}{m}p - \frac{(q-1)s}{qr}p\right) \\
&= -\frac{(q-1)\ell s(1-z)}{qrm} + \frac{1}{rm},
\end{aligned}$$

reducing to

$$\alpha = \frac{q - (q-1)\ell s(1-z)}{qr + (q-1)sm + qm + qr\ell(1-z)}.$$

Since $z > 1 - \frac{q}{(q-1)\ell s}$, this value of α is positive. Plugging in $1 - z = \frac{q}{(q-1)\ell s} - \zeta$, we continue to find a bound on α^{-1}:

$$\begin{aligned}
\alpha^{-1} &= \frac{qr + (q-1)sm + qm + qr\ell(1-z)}{q - (q-1)\ell s(1-z)} = \frac{qr + (q-1)sm + qm + qr\ell(1-z)}{(q-1)\ell s\zeta} \\
&< \frac{qr + (q-1)sm + qm + \frac{q^2 r}{(q-1)s}}{(q-1)\ell s\zeta} \leq \frac{2(m+r)}{\ell}\zeta^{-1}.
\end{aligned}$$

This finishes the proof. □

4.5 Computing the partition function using the Markov chain sampler

In this section we prove Theorem 4.2. We will do this with a self-reducibility argument, making use of a connection between removing and contracting edges.

We have the following result.

Proposition 4.22. *Let $z \in [1/3, 1]$ and let $q \in \mathbb{N}_{\geq 2}$. Let $\mathcal{G}$ be a family of graphs which is closed under contracting edges. Assume we are given an algorithm that for n-vertex and m-edge graph $G \in \mathcal{G}$ and any $\delta > 0$ computes a random $\mathbb{Z}_q$-flow with distribution δ-close to μ_{flow} in time bounded by $T(\delta, n, m)$. Then there is an algorithm that given an n-vertex and m-edge graph $G \in \mathcal{G}$ and any $\varepsilon > 0$ computes a number ξ such that with probability at least $3/4$*

$$e^{-\varepsilon} \leq \frac{\xi}{Z_{\text{flow}}(G; q, z)} \leq e^{\varepsilon}$$

in time $O(n^2\varepsilon^{-2}T(\varepsilon/n, n, m))$.

Before proving the proposition, let us show how it implies Theorem 4.2.

Proof of Theorem 4.2. We prove part (i): part (ii) follows in exactly the same way. Fix positive integers ι and d with d at least 2. Consider the class of graphs $\mathcal{G}$ that have a basis for the cycle space $\mathcal{C}$ consisting of even sets satisfying $\iota(\mathcal{C}) \leq \iota$ and $d(\mathcal{C}) \leq d$. By Lemma 4.9 this class is closed under contracting edges. By Theorem 4.13 we have an algorithm that for each m-edge graph $G \in \mathcal{G}$ and any $\delta > 0$ computes a random $\mathbb{Z}_q$-flow with distribution within total variation distance δ from μ_{flow} in time bounded by $T(\delta, n, m) = O(m^2 \log(m\delta^{-1}))$ provided $z > 1 - \frac{2}{(d+1)\iota} \geq 1/3$; see Remark 4.14). The theorem now follows from the previous proposition combined with the fact that $Z_{\text{flow}}(G; q, z) = (1 - z)^{|E|} q^{-|V|} Z_{\text{Potts}}(G; q, \frac{1+(q-1)z}{1-z})$ by Lemma 4.5. The running time is given by $O(n^2 m^2 \varepsilon^{-2} \log(nm\varepsilon^{-1}))$. $\square$

We now turn to the proof of Proposition 4.22.

Proof of Proposition 4.22. As already mentioned above the proof relies on a self-reducibility argument.

The flow partition function satisfies the following well known deletion-contraction relation: for a graph $G = (V, E)$ and $e \in E$ not a loop, we have

$$Z_{\text{flow}}(G; q, z) = (1 - z) Z_{\text{flow}}(G \setminus e; q, z) + z Z_{\text{flow}}(G/e; q, z). \tag{4.15}$$

This holds because the collection of all flows on G and on G/e are in bijection with each other, while the flows on $G \setminus e$ correspond to the flows on G that take value 0 on e.

We rewrite (4.15) as

$$\frac{Z_{\text{flow}}(G/e; q, z)}{Z_{\text{flow}}(G; q, z)} = \frac{1}{z} - \frac{1-z}{z} \cdot \frac{Z_{\text{flow}}(G \setminus e; q, z)}{Z_{\text{flow}}(G; q, z)}, \tag{4.16}$$

and we interpret the fraction

$$\frac{Z_{\text{flow}}(G \setminus e; q, z)}{Z_{\text{flow}}(G; q, z)}$$

as the probability that e is assigned the value $0 \in \mathbb{Z}_q$ when a flow is sampled from μ_{flow}. This probability can be estimated using the assumed sampler. Hence we can use the sampler to estimate (4.16).

From $G = (V, E)$. we now construct a series of graphs $G = G_0, G_1, \ldots G_t$ where in each step we contract one edge (which is not a loop). We can do this, until every component has been contracted to a single vertex, possibly with some loops attached to it. This takes $t = |V| - c(G) \leq |V|$ steps, where $c(G)$ denotes the number of components of G. In the end we have $|E|-|V|+c(G) \leq |E|$ edges (loops) left and the resulting graph G_t thus has flow partition function $Z_{\text{flow}}(G_t;q,z) = (1+(q-1)z)^{|E|-|V|+c(G)}$. Therefore

$$\frac{(1+(q-1)z)^{|V|-c(G)}}{Z_{\text{flow}}(G;q,z)} = \frac{Z_{\text{flow}}(G_t;q,z)}{Z_{\text{flow}}(G_0;q,z)} = \frac{Z_{\text{flow}}(G_1;q,z)}{Z_{\text{flow}}(G_0;q,z)} \cdots \frac{Z_{\text{flow}}(G_t;q,z)}{Z_{\text{flow}}(G_{t-1};q,z)}. \tag{4.17}$$

Note that for each i and any non-loop edge $e \in E(G_i)$ we have by (4.16),

$$1 \leq \frac{Z_{\text{flow}}(G_i/e;q,z)}{Z_{\text{flow}}(G_i;q,z)} \leq 1/z \leq 3, \tag{4.18}$$

since $z \geq 1/3$.

We can now estimate each individual probability on the right-hand side of (4.17) to get an estimate for $Z_{\text{flow}}(G;q,y)$. This is rather standard and can be done following the approach in [49] for matchings. We therefore only give a sketch of the argument, leaving out technical details.

For each i, let

$$p_i := \frac{Z_{\text{flow}}(G_i \setminus e;q,z)}{Z_{\text{flow}}(G_i;q,z)}.$$

To estimate p_i we run our sampler $M = O(\varepsilon^{-2}t)$ times with $\delta = O(\varepsilon/t)$ to generate independent random flows f_j ($j = 1,\ldots,M$). Denote by X_j the random variable that is equal to 1 if e is not contained in $\text{supp}(f_j)$ and 0 otherwise. We are in fact not interested in p_i, but rather in

$$\widehat{p}_i := \frac{Z_{\text{flow}}(G_i/e;q,z)}{Z_{\text{flow}}(G_i;q,z)} = \frac{1}{z} - \frac{1-z}{z}p_i.$$

We therefore define the random variable $Y_j := \frac{1}{z} - \frac{1-z}{z}X_j$ and $Y^i := 1/M\sum_{j=1}^M Y_j$. Note that $\mathbb{E}[Y^i] = \mathbb{E}[Y_j] = \widehat{p}_i$ and it is easy to check that $\text{Var}[Y^i] = 1/M\text{Var}[Y_j] = 1/M(\mathbb{E}(Y_j)-1)(1/z-\mathbb{E}(Y_j))$ for any $j = 1,\ldots,M$. We note that, by definition of the total variation distance, the fact that $z \geq 1/3$, and (4.18), we have

$$\widehat{p}_i(1-2\delta) \leq \widehat{p}_i - \frac{1-z}{z}\delta \leq \mathbb{E}[Y^i] = \mathbb{E}[Y_j] \leq \widehat{p}_i + \frac{1-z}{z}\delta \leq (1+2\delta)\widehat{p}_i. \tag{4.19}$$

This implies that

$$\frac{\text{Var}[Y^i]}{\mathbb{E}[Y^i]^2} = \frac{1}{M}\frac{(\mathbb{E}(Y_j)-1)(1/z-\mathbb{E}(Y_j))}{\mathbb{E}[Y^i]^2} \leq O(\varepsilon^2/t).$$

Consider next the random variable $Y := \prod_{i=1}^t Y^i$. This will, up to a multiplicative factor (cf.(4.17)), give us the desired estimate. Since the Y^i are independent we have

$$\frac{\mathrm{Var}[Y]}{\prod_{i=1}^t \mathbb{E}[Y^i]^2} = \prod_{i=1}^t \frac{\mathbb{E}[(Y^i)^2]}{\mathbb{E}[Y^i]^2} - 1 = \prod_{i=1}^t \left(1 + \frac{\mathrm{Var}[Y^i]}{\mathbb{E}[Y^i]^2}\right) - 1 \leq O(\varepsilon^2).$$

Then by Chebychev's inequality Y does not deviate much from $\prod_{i=1}^t \mathbb{E}[Y^i]$ with high probability, which by (4.19) and our choice of δ does not deviates much from $\prod_{i=1}^t \widehat{p}_i$. More precisely, Y will not deviate more than an $\exp(O(\varepsilon))$ multiplicative factor from $\prod_{i=1}^t \widehat{p}_i$ with high probability, as desired.

We need to access the sampler $O(t/\varepsilon^2)$ many times with $\delta = O(\varepsilon/t)$ to compute each Y^i. So this gives a total running time of $O(n^2\varepsilon^{-2}T(\varepsilon/n, n, m))$. This concludes the proof sketch. $\square$

4.6 Slow mixing of the flow chain

In this section we show that the flow Markov chain cannot mix rapidly for all $z \in (0,1)$. We do this by using the duality of our Markov chain on flows and Glauber dynamics of the Potts model on the planar grid (although the duality holds more generally on planar graphs). A result of Borgs, Chayes, and Tetali [15] for slow mixing of the Glauber dynamics of the Potts model on the grid (below a critical temperature) then immediately implies slow mixing of our flows Markov chain at the same temperature.

Given a graph $G = (V, E)$, let $\mathcal{F}_q(G)$ be the set of $\mathbb{Z}_q$-flows on G and let $\Omega_q(G)$ be the set of $\tau : V \to [q]$ of q-spin configurations on G. Clearly $|\Omega_q(G)| = q^{|V|}$ and, as noted earlier, $|\mathcal{F}_q(G)| = q^{|E|-|V|-1}$.

Recall that the Glauber dynamics for the q-state Potts model for a graph G and parameter z is the following Markov chain with state space $\Omega_q(G)$. Given that we are currently at state $\sigma \in \Omega_q(G)$, we pick a vertex $v \in V$ uniformly at random and update its state as follows: we choose the new state to be i with probability $z^{m(i)}/Z_v$, where $m(i)$ is the number of neighbours of v that have state i in σ, and $Z_v = \sum_i z^{m(i)}$.

Let $G = (V, E)$ be the $((L+1) \times (L+1))$-grid and $H = (V', E')$ the $(L \times L)$-grid. One can easily check that $|V'| = |E| - |V| + 1$ and so $|\Omega_q(H)| = |\mathcal{F}_q(G)|$. There is a natural bijection $\varphi : \Omega_q(H) \to \mathcal{F}_q(G)$ defined as follows. First note that H is the planar dual of G (ignoring the outer face of G). Using this, write $v_1, \ldots, v_{L^2}$ for the vertices of H and $C_1, \ldots, C_{L^2}$ for the corresponding faces (i.e. 4-cycles) of G. Given $\sigma \in \Omega_q(H)$, let $\varphi(\sigma) = \sum_{i=1}^{L^2} \sigma(v_i)\chi_{C_i}$. We see that φ is injective since the C_i form a basis of the cycle space of G, and hence φ must be bijective.

Now it is easy to check that the q-state Potts Glauber dynamics on H is equivalent to the $\mathbb{Z}_q$-flow Markov chain on G (where both chains have the same interaction parameter, say z) via the correspondence φ between their state spaces. In other words if P and Q are their respective transition matrices then $P_{\sigma_1\sigma_2} = Q_{\varphi(\sigma_1)\varphi(\sigma_2)}$ for all $\sigma_1, \sigma_2 \in \Omega_q(H)$.

Borgs, Chayes, and Tetali [15] showed that the mixing time of the Glauber dynamics of the q-state Potts model on the $L \times L$ grid with interaction parameter $z = e^{-\beta}$ is bounded below by z^{CL} for some constant C when β is above the critical threshold for the grid, i.e. $\beta \geq \beta_0(\mathbb{Z}^2) = \frac{1}{2}\log q + O(q^{-1/2})$. In particular this shows the same exponential lower bound on the mixing time for the $\mathbb{Z}_q$-flow Markov chain (for the same interaction parameter z) on the $(L+1)\times(L+1)$-grid.

Summary

Chromatic polynomials: zeros, algorithms and computational complexity

The protagonist of this thesis is the chromatic polynomial, which is defined for any graph $G = (V, E)$ as

$$Z(G; q) \coloneqq \sum_{A \subseteq E} q^{k(A)} (-1)^{|A|},$$

where $k(A)$ is the number of connected components in the graph (V, A). This is a special case of the partition function of the random cluster model (also known as the partition function of the Potts model), which is defined as

$$Z(G; q, y) \coloneqq \sum_{A \subseteq E} q^{k(A)} (y - 1)^{|A|}.$$

The main motivating questions for this thesis are

Where are the zeros of $Z(G; q)$?

For which values of q and y is it easy to approximate $Z(G; q, y)$?

In Chapter 2 we look at the zeros of $Z(G; q)$, also called chromatic zeros, for the family of *series-parallel graphs*. The main results are Theorem 2.3 and 2.1. The first shows that series-parallel chromatic zeros are dense in $\{q \mid \Re(q) > 3/2\}$, while the second shows existence of an open $U \subseteq \mathbb{C}$ around $(0, 32/27)$ such that $U \setminus \{1\}$ is zero-free. Using a computer we can push these results further. This is summarized in Figure 2.1.

With the tools developed in this chapter, we also disprove a conjecture of Sokal [67]: for every Δ large enough, we find a (series-parallel) graph where all vertices but one have degree at most Δ, and which has a chromatic zero with real part bigger than Δ (this is Theorem 2.4).

Chapter 3 establishes a relation between the zeros of $Z(G; q)$ and the computational complexity of approximating $Z(G; q, y)$. In particular, we consider the

problem of approximating $|Z(G;q,y)|$ within a factor $e^{\frac{1}{4}}$ for planar graphs G, and algebraic numbers q,y. The main result Theorem 3.1 (and the more general Corollary 3.13) mirrors very much Theorem 2.3, it shows that for all non-real q in the region $\{q \mid \Re(q) > 3/2 \text{ or } |1-q| > 1\}$ it is #P-hard to approximate $|Z(G;q,y)|$. The reason is that both results ask for virtually the same condition. Even more, Corollary 3.13(c) shows that for any non-real, planar, chromatic zero q it is #P-hard to approximate $|Z(G;q,y)|$.

Finally in Chapter 4 we find an efficient randomised approximation algorithm for $Z(G;q,y)$ when y is large, and q is a positive integer. For this we consider an equivalent form of $Z(G;q,y)$, which is a partition function of flows on G. We choose a generating set for the set of flows, use them to define a variant of the Glauber dynamics, and analyze this Markov chain with path coupling. When y satisfies the bounds in Theorem 4.2, the chain is rapidly mixing and we use this to construct an efficient approximation algorithm for $Z(G;q,y)$.

Samenvatting

Chromatische polynomen: nulpunten, algoritmes en computationele complexiteit

De hoofdrolspeler van dit proefschrift is het chromatisch polynoom, dat voor een graaf $G = (V, E)$ gedefinieerd is als

$$Z(G; q) \coloneqq \sum_{A \subseteq E} q^{k(A)} (-1)^{|A|},$$

waar $k(A)$ het aantal componenten is in de graaf (V, A). Dit is een speciaal geval van de partitiefunctie van het 'random cluster model' (ook bekend als het Potts model), gedefinieerd als

$$Z(G; q, y) \coloneqq \sum_{A \subseteq E} q^{k(A)} (y - 1)^{|A|}.$$

De drijvende vragen in dit proefschrift zijn

Waar liggen de nulpunten van $Z(G; q)$?

Voor welke waarden van q en y is het makkelijk om $Z(G; q, y)$ te benaderen?

In Hoofdstuk 2 bekijken we de nulpunten van $Z(G; q)$, ookwel chromatische nulpunten genoemd, van de familie *serie-parallele grafen.* De hoofdresultaten zijn Stellingen 2.3 en 2.1. De eerste laat zien dat serie-parallele chromatische nulpunten dicht liggen in $\{q \mid \Re(q) > 3/2\}$, terwijl de tweede zegt dat er een open $U \subseteq \mathbb{C}$ rond $(0, 32/27)$ bestaat zodat $U \setminus \{1\}$ nulpuntsvrij is. Met de computer kunnen we deze resultaten nog iets verbeteren. Dit is samengevat in Figuur 2.1.

Met het gereedschap onwikkeld in dit hoofdstuk, ontkrachten we ook een vermoeden van Sokal [67]: voor elke groot genoege Δ, vinden we een (serie-parallele) graaf waar alle knopen op één na hooguit graad Δ hebben, die een chromatisch nulpunt heeft met reëel deel groter dan Δ (dit is Stelling 2.4).

Hoofdstuk 3 stelt een relatie vast tussen de nulpunten van $Z(G; q)$ en de computationele complexiteit van het benaderen van $Z(G; q, y)$. Het precieze computationele probleem is al volgt: gegeven een vlakke graaf G en algebraïsche getallen q, y, wordt er gevraagd om een benadering van $|Z(G; q, y)|$ binnen een factor $e^{\frac{1}{4}}$. Het hoofdresultaat Stelling 3.1 (en de generalisatie Gevolg 3.13) heeft een sterke gelijkenis met Stelling 2.3, het laat zien dat voor alle niet-reële q in het gebied $\{q \mid \Re(q) > 3/2 \text{ of } |1 - q| > 1\}$ dat het #P-moeilijk is om $|Z(G; q, y)|$ te benaderen. De reden voor deze gelijkenis is dat beide resultaten om min of meer dezelfde voorwaarden vragen. Sterker nog, Gevolg 3.13(c) laat zien dat voor elk niet-reëel, chromatisch nulpunt q van een vlakke graaf, het #P-moeilijk is om $|Z(G; q, y)|$ te benaderen.

Tot slot, in Hoofdstuk 4 vinden we een efficiënt algoritme (dat gebruik maakt van willekeur) om $Z(G; q, y)$ te benaderen wanneer y groot is, en q een positief geheel getal. Hiervoor beschouwen we een equivalente vorm van $Z(G; q, y)$, die een partitiefunctie is van stromingen op G. We kiezen een genererende verzameling voor alle stromingen, gebruiken die om een variant op de Glauber dynamica te definiëren, en analyseren deze markovketen met 'path coupling'. Wanneer y aan de voorwaarden van Stelling 4.2 voldoet, convergeert de markovketen snel naar de stabiele verdeling en dit gebruiken we om een efficiënt algoritme te construeren om $Z(G; q, y)$ te benaderen.

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