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Newton polytope of good symmetric polynomials

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Abstract. We introduce a general class of symmetric polynomials that have saturated Newton polytope and their Newton polytope has integer decomposition property. The class covers numerous previously studied symmetric polynomials.

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1. Introduction

In combinatorics, if a convex polytope equals the convex hull of its integer points, we say that it is a lattice polytope. Studying lattice polytopes is important because of their connections in many other domains. For instance, in mathematical optimization, if a system of linear inequalities defines a polytope, then we can use linear programming to solve integer programming problems for this system (see [1]). In algebraic geometry, lattice polytopes are used to study projective toric varieties (see [4, 7]). The Newton polytope is a lattice polytope associated with a polynomial: it is the convex hull of exponent vectors. The Newton polytope is a central object in tropical geometry (see [9]), and they are used to characterizing Grobner bases (see [22]).

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Lattice polytopes are studied by Ehrhart polynomials (see [6]). Important properties of Ehrhart polynomials such as unimodality and log-concavity are related to the integer decomposition property (IDP) of the lattice polytope (see [3, 13, 15]). In [2], the authors studied the Newton polytope of inflated symmetric Grothendieck polynomials. The saturated property (SNP) of inflated symmetric Grothendieck polynomials in [2] generalizes the SNP of symmetric Grothendieck polynomials in [5]. The SNP of the inflated symmetric Grothendieck polynomials is an important point to derive the IDP of their Newton polytope.

In this paper, we introduce a general class of symmetric polynomials that has SNP with Newton polytope has IDP (see Theorem 7 and Corollary 8). Our class covers symmetric polynomials in [2, 5, 11, 12]: symmetric Grothendieck polynomials, inflated symmetric Grothendieck polynomials, Stembridge's symmetric polynomials associated with totally nonnegative matrices, cycle index polynomials, Reutenauer's symmetric polynomials, Schur P-polynomials and Schur Q-polynomials, Stanley's symmetric polynomials, chromatic symmetric polynomials of co-bipartite graphs, indifference graphs of Dyck paths, incomparability graphs of (3+1)-free posets. It also covers other symmetric polynomials, for instance, dual Grothendieck polynomials in [10].

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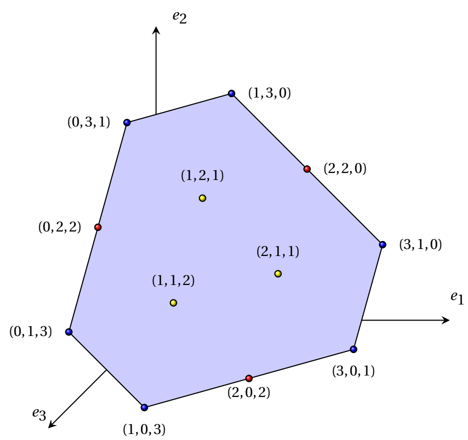
2. Newton polytope

A *polytope* $\mathscr{P}$ in $\mathbb{R}^m$ is the convex hull $\mathrm{Conv}(v_1,\dots,v_k)$ of finite many points $v_1,\dots,v_k \in \mathbb{R}^m$. The *vertex set* of $\mathscr{P}$ is the minimal set V in $\mathbb{R}^m$ such that $\mathscr{P} = \mathrm{Conv}(V)$. Algebraically, a point $v \in \mathscr{P}$ is a *vertex* if, $v = tw + (1-t)u$ for some $w, u \in \mathscr{P}$, $t \in (0,1)$ implies $w = u = v$. We say that $\mathscr{P}$ is a *lattice polytope* if V is a subset of $\mathbb{Z}^m$.

Example 1. The convex hull $\mathscr{P}$ of twelve points in $\mathbb{R}^3$ below is a lattice polytope.

$$(3,1,0),(3,0,1),(1,0,3),(0,1,3),(0,3,1),(1,3,0),$$
$$(2,2,0),(2,0,2),(0,2,2),$$
$$(2,1,1),(1,1,2),(1,2,1).$$

The permutations of $(3,1,0)$ are vertices of the polytope $\mathscr{P}$. In the picture below, $\mathscr{P}$ is the blue hexagon.



Let $\mathscr{P}$ be a lattice polytope. For a positive integer t, let $t\mathscr{P} = \{tv \mid v \in \mathscr{P}\}$. We say that $\mathscr{P}$ has *integer decomposition property (IDP)* if, for any positive integer t and $p \in t\mathscr{P} \cap \mathbb{Z}^m$, there are t points $v_1, \dots, v_t \in \mathscr{P} \cap \mathbb{Z}^m$ such that $p = v_1 + \cdots + v_t$.

Example 2. Let $\mathscr{P}$ be the lattice polytope in Example 1. It is known that $\mathscr{P}$ has IDP ([2, Proposition 11]). For instance, $3\mathscr{P}$ is the convex hull of six points

$$(9,3,0), (9,0,3), (3,0,9), (0,3,9), (0,9,3), (3,9,0).$$

We see that $(9,2,1) \in 3\mathscr{P} \cap \mathbb{Z}^3$ and is the sum of three points in $\mathscr{P} \cap \mathbb{Z}^3$.

$$(9,2,1) = (3,1,0) + (3,1,0) + (3,0,1).$$

Example 3. Let $\mathscr{G}$ be convex hull of four points

$$(0,0,0), (1,0,0), (0,0,1), (1,2,1).$$

The elements in $\mathscr{G} \cap \mathbb{Z}^3$ are

$$(0,0,0), (1,0,0), (0,0,1), (1,2,1).$$

We have $(1,1,1) \in 2\mathscr{G} \cap \mathbb{Z}^3$, but it can not be written as a sum of two points in $\mathscr{G} \cap \mathbb{Z}^3$. So $\mathscr{G}$ does not have IDP.

Let $f(x) = \sum_{\alpha \in \mathbb{Z}^m_{\geq 0}} c_\alpha x^\alpha \in \mathbb{C}[x_1, \dots, x_m]$. The *support* of f is defined by

$$\mathrm{Supp}(f) = \{\alpha \in \mathbb{Z}^m_{\geq 0} \mid c_\alpha \neq 0\}.$$

The *Newton polytope* of f is defined by

$$\mathrm{Newton}(f) = \mathrm{Conv}(\mathrm{Supp}(f)).$$

We say that f has *satured Newton polytope (SNP)* if $\mathrm{Newton}(f) \cap \mathbb{Z}^m = \mathrm{Supp}(f)$.

Example 4. Let $f(x_1, x_2, x_3)$ be the polynomial

$$x^{(3,1,0)} + x^{(3,0,1)} + x^{(1,0,3)} + x^{(0,1,3)} + x^{(0,3,1)} + x^{(1,3,0)} \\ + x^{(2,2,0)} + x^{(2,0,2)} + x^{(0,2,2)} + 2x^{(2,1,1)} + 2x^{(1,1,2)} + 2x^{(1,2,1)}.$$

The set $\mathrm{Supp}(f)$ contains twelve points in Example 1. Then $\mathrm{Newton}(f)$ is the polytope $\mathscr{P}$ in Example 1. Since $\mathrm{Newton}(f) \cap \mathbb{Z}^3 = \mathrm{Supp}(f)$, f has SNP.

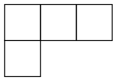
3. Schur polynomials

A *partition* with at most m parts is a sequence of weakly decreasing nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_m)$. The *size* of partition λ is defined by $|\lambda| = \sum_{i=1}^m \lambda_i$. Each partition λ is presented by a *Young diagram* $Y(\lambda)$ that is a collection of boxes such that the leftmost boxes of each row are in a column, and the numbers of boxes from the top row to bottom row are $\lambda_1, \lambda_2, \dots$, respectively. A *semistandard Young tableau* of shape λ with entries from $\{1, \dots, m\}$ is a filling of the Young diagram $Y(\lambda)$ by the ordered alphabet $\{1 < \cdots < m\}$ such that the entries in each column are strictly increasing from top to bottom, and the entries in each row are weakly increasing from left to right. A Young tableau T is said to have *content* $\alpha = (\alpha_1, \alpha_2, \dots)$ if α_i is the number of entries i in the tableau T. We write

$$x^T = x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots.$$

For each partition λ with at most m parts, the *Schur polynomial* $s_\lambda(x_1, \dots, x_m)$ is defined as the sum of x^T, where T runs over the semistandard Young tableaux of shape λ with filling from $\{1, \dots, m\}$.

Example 5. Vector $(3,1,0)$ is a partition. The Young diagram of $(3,1,0)$ is



The following filling is a semistandard tableau of shape $(3,1,0)$ and content $(1,2,1)$.

1	2	3
2		

Schur polynomial $s_{(3,1,0)}(x_1,x_2,x_3)$ is the polynomial f in Example 4.

4. Good symmetric polynomials

Let α and β be partitions with at most m parts. We say β is *bigger* than α and write $\beta \geq \alpha$ if and only if $\beta_i \geq \alpha_i$ for all i. If α, β are partitions of the same size, we say β *dominates* α and write $\beta \trianglerighteq \alpha$ if $\sum_{i=1}^{j} \beta_i \geq \sum_{i=1}^{j} \alpha_i$ for all $j \geq 1$.

Example 6. $(3,1,0) < (3,3,3)$ and $(3,2,0) \trianglerighteq (3,1,1)$.

Let $F(x_1,\dots,x_m)$ be a linear combination of Schur polynomials associated to partitions with at most m parts. We can collect Schur polynomials appearing in F associated with partitions of the same size to a bracket. We say that F is *good* if it satisfies the following conditions:

(a) The support of each bracket equals the union of supports of its Schur elements.

(b) Suppose that there are $l+1$ brackets in condition (a). In each bracket, there is a unique $\trianglerighteq$-maximum partition. These $\trianglerighteq$-maximum partitions have a form

$$\alpha = \lambda^0 < \dots < \lambda^l = \beta, \tag{1}$$

where $\alpha \leq \beta$ are fixed partitions and for each $i > 0$, λ^i is obtained from λ^{i-1} by adding a box in the northmost row of λ^{i-1} such that the addition gives a Young diagram, $\alpha < \lambda^i \leq \beta$.

Theorem 7. *Let F be a good linear combination of Schur polynomials. Then F has SNP and* Newton(F) *has IDP.*

Corollary 8. *Let F be a linear combination of Schur polynomials such that the condition* (a) *is replaced by* (a′) *or the condition* (b) *is replaced by* (b′) *below:*

(a′) *any two Schur polynomials in the same bracket of F have the same sign,*

(b′) *there exists partitions $\overline{\lambda}, \widehat{\lambda}$ so that s_μ appears in F if and only if $\overline{\lambda} \leq \mu \leq \widehat{\lambda}$.*

Then F is a good polynomial. In particular, F has SNP and Newton(F) *has IDP.*

Proof. The condition (a′), (b′) are particular cases of condition (a), (b), respectively. Moreover, the partitions α, β in (b′) are $\overline{\lambda}, \widehat{\lambda}$, respectively. □

Example 9. Let $F(x_1,x_2,x_3)$ be

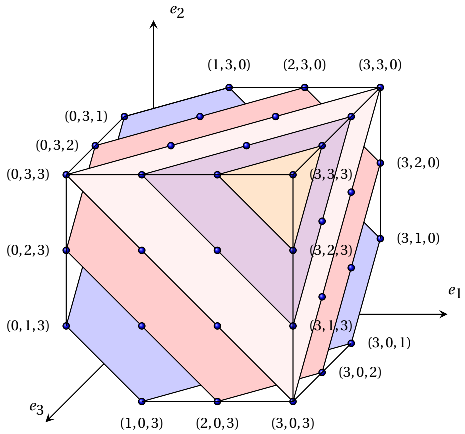
$$s_{(3,1,0)} - (3s_{(3,2,0)} + 6s_{(3,1,1)}) + (3s_{3,3,0} + 18s_{(3,2,1)}) - (18s_{(3,3,1)} + 4s_{(3,2,2)}) + 44s_{(3,3,2)} - 55s_{(3,3,3)}.$$

Schur polynomials in the same bracket have the same sign. The $\trianglerighteq$-maximum partitions λ^i for $i = 0,\dots,5$ chosen from brackets have form

$$\alpha = (3,1,0) < (3,2,0) < (3,3,0) < (3,3,1) < (3,3,2) < (3,3,3) = \beta.$$

Hence, F is a good symmetric polynomial. Newton(F) is the convex hull of six different color polygons in the picture below. Each polygon is the Newton polytope of each bracket. In fact,

F is the inflated symmetric Grothendieck polynomial $G_{2,(3,1,0)}$ in [2]. Hence, F has SNP and Newton(F) has IDP by [2, Proposition 21, Theorem 27].



The following examples tell us that when Theorem 7 does not apply, we may not have a definite affirmation of SNP and IDP.

Example 10. When the condition (a) fails, for instance:

- Let $F(x_1,x_2,x_3)$ be $s_{(3,1,0)} - s_{(2,2,0)}$. Then F does not have SNP because $(2,2,0) \notin \mathrm{Supp}(F)$, but $\mathrm{Newton}(F) = \mathrm{Newton}(s_{(3,1,0)})$ still has IDP.

When adding blocks to α in a wrong order in (b), for instance:

- Let choose $\alpha = (3,1,0) < (3,1,1) < (3,2,1) = \beta$ and let $F(x_1,x_2,x_3)$ be $s_{(3,1,0)} + s_{(3,1,1)} + s_{(3,2,1)}$. Then F has SNP.
- Let choose $\alpha = (6,4,0) < (6,4,1) < (6,4,2) < (6,4,3) < (6,5,3) < (6,6,3) = \beta$ and let $F(x_1,x_2,x_3)$ be $s_{(6,4,0)} + s_{(6,4,1)} + s_{(6,4,2)} + s_{(6,4,3)} + s_{(6,5,3)} + s_{(6,6,3)}$. Since $(6,5,2) \in \mathrm{Newton}(F) \cap \mathbb{Z}^3 \setminus \mathrm{Supp}(f)$, then F does not has SNP.

We are not sure if there exists a symmetric polynomial that has SNP, but its Newton polytope does not have IDP.

We need the following facts to prove Theorem 7.

Proposition 11 ([14, Proposition 2.5]). *Let α, β be partitions of the same size. Then,* $\mathrm{Newton}(s_\alpha) \subseteq \mathrm{Newton}(s_\beta)$ *if and only if $\alpha \trianglelefteq \beta$.*

Lemma 12 ([5, Theorem 0.1]). *Let α be a partition with at most m parts. Then s_α has SNP with Newton polytope being the convex hull of the S_m-orbit of α.*

Proof of Theorem 7. *We first prove that F has SNP.*

We use the trick from [5].

(1) Let $F = \sum_\mu C_\mu s_\mu$ with $C_\mu \neq 0$. By condition (a) of F, we have

$$\mathrm{Supp}(F) = \bigcup_\mu \mathrm{Supp}(s_\mu). \tag{2}$$

Then

$$\mathrm{Newton}(F) = \mathrm{Conv}\left(\bigcup_\mu \mathrm{Supp}(s_\mu)\right). \tag{3}$$

Let $\alpha = \lambda^0 < \lambda^1 < \cdots < \lambda^l = \beta$ be the $\trianglerighteq$-maximum partitions in condition (b) of F. By Proposition 11, the right-hand side of (2) is

$$\bigcup_{\mu} \mathrm{Supp}(s_\mu) = \bigcup_{i=0}^{l} \mathrm{Supp}(s_{\lambda^i}). \tag{4}$$

Therefore, by (2), (4),

$$\mathrm{Supp}(F) = \bigcup_{i=0}^{l} \mathrm{Supp}(s_{\lambda^i}). \tag{5}$$

By Proposition 11,

$$\mathrm{Conv}(\mathrm{Supp}(s_\mu)) = \mathrm{Newton}(s_\mu) \subseteq \mathrm{Newton}(s_{\lambda^i}) = \mathrm{Conv}(\mathrm{Supp}(s_{\lambda^i}))$$

for some i. It implies that the right-hand side of (3) is

$$\mathrm{Conv}\left(\bigcup_{\mu} \mathrm{Supp}(s_\mu)\right) = \mathrm{Conv}\left(\bigcup_{i=0}^{l} \mathrm{Newton}(s_{\lambda^i})\right). \tag{6}$$

Hence by (3), (6), we have

$$\mathrm{Newton}(F) = \mathrm{Conv}\left(\bigcup_{i=0}^{l} \mathrm{Newton}(s_{\lambda^i})\right). \tag{7}$$

(2) Let p be a point in $\mathrm{Newton}(F) \cap \mathbb{Z}^m$. By (7), p has form $p = \sum_{i=0}^{l} c_i v^i$ for some $v^i \in \mathrm{Newton}(s_{\lambda^i})$, and some $c_i \in \mathbb{R}_{\geq 0}$, $\sum_{i=1}^{l} c_i = 1$. We see that v^i is not a partition in general. However, if we denote the sum of its coordinates by $|v^i|$, then $|v^i| = |\lambda^i|$. Then $|p| = \sum_{i=0}^{l} c_i |\lambda^i|$ is between $|\lambda^0|$ and $|\lambda^l|$, because of (1). Thus $|p| = |\lambda^j|$ for some $j \in [0, l]$, because λ^i is obtained from λ^{i-1} by adding a box. Let $\bar{p}$ be $\sum_{i=0}^{l} c_i \lambda^i$ and $p^{\downarrow}$ be the rearrangement of the components of p into decreasing order. It was proven in [5] that $p^{\downarrow} \trianglelefteq (\bar{p})^{\downarrow}$ (Claim B) and $(\bar{p})^{\downarrow} \trianglelefteq \lambda^j$ (Claim C). So $p^{\downarrow} \trianglelefteq \lambda^j$. By Lemma 12, Proposition 11, p is a point in

$$\mathrm{Newton}(s_{p^{\downarrow}}) \cap \mathbb{Z}^m \subseteq \mathrm{Newton}(s_{\lambda^j}) \cap \mathbb{Z}^m = \mathrm{Supp}(s_{\lambda^j}) \subseteq \mathrm{Supp}(F). \tag{8}$$

Therefore we conclude that F has SNP.

Now we show that Newton(F) *has IDP.*
We use the trick from [2].

(1) We have proven that F has SNP. Then by (5), Lemma 12, we have

$$\mathrm{Newton}(F) \cap \mathbb{Z}^m = \mathrm{Supp}(F) = \bigcup_{i=0}^{l} \mathrm{Supp}(s_{\lambda^i}) = \bigcup_{i=0}^{l} \mathrm{Newton}(s_{\lambda^i}) \cap \mathbb{Z}^m. \tag{9}$$

(2) Suppose that $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$. For $i = 1, \ldots, m-1$, set $\lambda^{(i)} = (\beta_1, \ldots, \beta_i, \alpha_{i+1}, \ldots, \alpha_m)$. Set $\lambda^{(0)} = \alpha$, $\lambda^{(m)} = \beta$. Then $\alpha = \lambda^{(0)} < \cdots < \lambda^{(m)} = \beta$ is a subchain of (1). We have

$$\mathrm{Newton}(F) = \mathrm{Conv}\left(\bigcup_{i=0}^{m} \mathrm{Newton}(s_{\lambda^{(i)}})\right). \tag{10}$$

Indeed, Newton(F) is the convex hull of its vertex set. We can get (10) from (7) by showing that a partition λ^j not of form $\lambda^{(i)}$ is not a vertex of Newton(F). It is trivial because $\lambda^j = \frac{1}{2}(\lambda^{j-1} + \lambda^{j+1})$.

(3) For a positive integer t, we construct a chain of form (1)

$$t\alpha = \Lambda^0 < \cdots < \Lambda^L = t\beta. \tag{11}$$

Set $F_t = \sum_{i=0}^{L} s_{\Lambda^i}$. Then F_t is a good linear combination of Schur polynomials and $\Lambda^{(i)} = t\lambda^{(i)}$ for each $i = 0,\dots,m$. By (10), we have

$$\begin{aligned} \mathrm{Newton}(F_t) &= \mathrm{Conv}\left(\bigcup_{i=0}^{m} \mathrm{Newton}(s_{\Lambda^{(i)}})\right) \\ &= \mathrm{tConv}\left(\bigcup_{i=0}^{m} \mathrm{Newton}(s_{\lambda^{(i)}})\right) \\ &= t\,\mathrm{Newton}(F). \end{aligned} \tag{12}$$

(4) Let p a point in $t\,\mathrm{Newton}(F)\cap\mathbb{Z}^m$. By (12), p is a point in $\mathrm{Newton}(F_t)\cap\mathbb{Z}$. Since F_t has SNP, by (9), it is a point in $\mathrm{Newton}(s_{\Lambda^i})\cap\mathbb{Z}$ for some Λ^i in (11). Hence, p is the content of some semistandard tableau T of shape Λ^i with filling from $\{1,\dots,m\}$. For $j = 1,\dots,t$, let T_j be the semistandard tableau obtained by taking j'-th column of T for $j' \equiv j \mod t$. Let $\theta(j)$ be the shape of tableau T_j. Let v_j be the content of tableau T_j. Then $p = v_1 + \cdots + v_t$. We also have $\alpha \le \theta(j) \le \beta$. So there is a unique partition λ^k in chain (1) such that $\theta(j) \trianglelefteq \lambda^k$. Then by Proposition 11, v_j is a point in

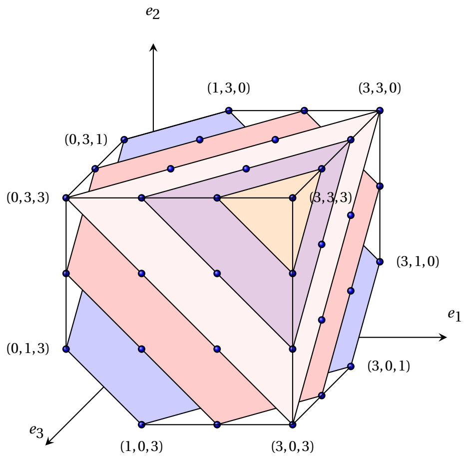
$$\mathrm{Newton}(s_{\theta(j)}) \cap \mathbb{Z}^m \subseteq \mathrm{Newton}(s_{\lambda^k}) \cap \mathbb{Z}^m.$$

So by (9), v_j is a point of $\mathrm{Newton}(F)\cap\mathbb{Z}^m$. Therefore we conclude that $\mathrm{Newton}(F)$ has IDP. □

Example 13. In Example 9, the subchain $\lambda^{(i)}$ for $i = 0,\dots,3$ in the proof of Theorem 7 is

$$\alpha = (3,1,0) = (3,1,0) < (3,3,0) < (3,3,3) = \beta.$$

In this case, $\lambda^{(0)} = \lambda^{(1)}$. The vertex set of $\mathrm{Newton}(F)$ is the union of S_3-orbits of partitions $(3,1,0), (3,3,0), (3,3,3)$.



5. Applications

Theorem 7, Corollary 8 cover the following cases. Known results are:

- SNP and IDP of inflated symmetric Grothendieck polynomials $G_{h,\lambda}$ (see [5, Theorem 0.1], [2, Proposition 21, Theorem 27]). Indeed, by definition

$$G_{h,\lambda} = \sum_{\mu} (-1)^{|\mu/\lambda|} b_{h,\lambda\mu} s_\mu,$$

where $b_{h,\lambda\mu}$ is the number of fillings satisfying certain conditions. So, all Schur elements in the same bracket with s_μ have the same sign $(-1)^{|\mu/\lambda|}$, and then the condition (a) is valid. By [2, Lemma 18 (c)], $b_{h,\lambda\mu}$ is nonzero if and only if $\lambda \le \mu \le \lambda^{(N)}$. Hence, by Corollary 8, the condition (b) is valid with $\alpha = \lambda$ and $\beta = \lambda^{(N)}$.

- SNP and IDP of the following symmetric polynomials in [12]: Stembridge's symmetric polynomials associated with totally nonnegative matrices (Theorem 2.28), cycle index polynomials (Theorem 2.30), Reutenauer's symmetric polynomials (Theorem 2.32), Schur P-polynomials and Schur Q-polynomials (Proposition 3.5), Stanley's symmetric polynomials (Theorem 5.8). They are particular cases of [12, Prositions 2.5 (III)]. The proposition considers homogenous symmetric polynomials of degree d
$$f = \sum_{|\mu|=d} c_\mu s_\mu$$
with suppose that there exists λ so that $c_\lambda \neq 0$, $c_\mu \neq 0$ only if $\mu \trianglelefteq \lambda$, and $c_\mu \ge 0$ for all μ. So, condition (a) is valid. The condition (b) is valid with $\alpha = \beta = \lambda$. More precisely, the Schur expansion of those polynomials have nonnegative coefficients by [21], [18, p. 396], [12, p. 12], [20], [16, Theorems 3.2, 4.1], respectively. The condition (b) is valid with $\alpha = \beta$ and they can be found in the proofs of corresponding theorems in [12].
- SNP and IDP of the following symmetric polynomials in [11]: chromatic symmetric polynomials of co-bipartite graphs (Proposition 3.1), indifference graphs of Dyck paths (Proposition 4.1), incomparability graphs of (3+1)-free posets (Theorem 5.7). They are also particular cases of [12, Proposition 2.5 (III)] above. More precisely, the Schur-expansion of those polynomials have nonnegative coefficients by [17, Corollary 3.6], [19], [8], respectively. Hence, condition (a) is valid. The condition (b) is valid with $\alpha = \beta$ and they are $\lambda(G), \lambda^{gr}(d), \lambda^{gr}(P)$, respectively.

Unknown results are:

- SNP and IDP of dual Grothendieck polynomials g_λ in [10]. Indeed, [10, Theorem 9.8] states that
$$g_\lambda = \sum_{\mu} f_\lambda^\mu s_\mu,$$
where f_λ^μ is the number of semistandard tableaux of the skew shape λ/μ with entries of the i-th row lie in $[1, i-1]$. So, all nonzero coefficients f_λ^μ have same sign, and then the condition (a) is valid. Moreover, f_λ^μ is nonzero if and only if $(\lambda_1) \le \mu \le \lambda$. Hence, by Corollary 8, the condition (b) is valid with $\alpha = (\lambda_1)$ and $\beta = \lambda$.

Remark 14. Though Theorem 7 covers [2, Theorem 27], inside the proofs we do not need to choose F_t as a generalization of $G_{th,t\lambda}$. The key point is to choose a set-up for F_t so that it has SNP and $\mathrm{Newton}(F_t) = t\,\mathrm{Newton}(F)$ for any t. For this purpose, there are many choices for F_t, for instance $\sum_{i=0}^{L} s_{\Lambda^i}$, or $\sum_{i=0}^{L} (-1)^i s_{\Lambda^i}$, or $G_{th,t\lambda}$ when $F = G_{h,\lambda}$, etc. Our first choice $F_t = \sum_{i=0}^{L} s_{\Lambda^i}$ is the simplest.

References

[1] A. Barvinok, "Lattice points and lattice polytopes", in *Handbook of discrete and computational geometry*, CRC Press, 2017, p. 185-210.

[2] M. Bayer, B. Goeckner, S. J. Hong, T. McAllister, M. Olsen, C. Pinckney, J. Vega, M. Yip, "Lattice polytopes from Schur and symmetric Grothendieck polynomials", *Electron. J. Comb.* **28** (2021), no. 2, article no. P2.45 (36 pages).

[3] W. Bruns, T. Römer, "h-vectors of Gorenstein polytopes", *J. Comb. Theory, Ser. A* **114** (2007), no. 1, p. 65-76.

[4] D. A. Cox, J. B. Little, H. K. Schenck, *Toric Varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, 2011.

[5] L. Escobar, A. Yong, "Newton polytopes and symmetric Grothendieck polynomials", *C. R. Math. Acad. Sci. Paris* **355** (2017), no. 8, p. 831-834.

[6] E. Eugène, "Sur les polyèdres rationnels homothétiques à n dimensions", *C. R. Math. Acad. Sci. Paris* **254** (1962), p. 616-618.

[7] W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, 2016.
[8] V. Gasharov, "Incomparability graphs of $(3+1)$-free posets are s-positive", *Discrete Math.* **157** (1996), no. 1-3, p. 193-197.
[9] B. Y. Kazarnovskii, A. G. Khovanskii, A. I. Esterov, "Newton polytopes and tropical geometry", *Russ. Math. Surv.* **76** (2021), no. 1, p. 91-175.
[10] T. Lam, P. Pylyavskyy, "Combinatorial Hopf algebras and K-homology of Grassmanians", *Int. Math. Res. Not.* **2007** (2007), no. 9, article no. RMN125 (48 pages).
[11] J. P. Matherne, A. H. Morales, J. Selover, "The Newton polytope and Lorentzian property of chromatic symmetric functions", 2022, https://arxiv.org/abs/2201.07333.
[12] C. Monical, N. Tokcan, A. Yong, "Newton polytopes in algebraic combinatorics", *Sel. Math., New Ser.* **25** (2019), no. 5, article no. 66 (37 pages).
[13] H. Ohsugi, T. Hibi, "Special simplices and Gorenstein toric rings", *J. Comb. Theory, Ser. A* **113** (2006), no. 4, p. 718-725.
[14] R. Rado, "An inequality", *J. Lond. Math. Soc.* **27** (1952), no. 1, p. 1-6.
[15] J. Schepers, L. Van Langenhoven, "Unimodality questions for integrally closed lattice polytopes", *Ann. Comb.* **17** (2013), no. 3, p. 571-589.
[16] R. P. Stanley, "On the number of reduced decompositions of elements of Coxeter groups", *Eur. J. Comb.* **5** (1984), no. 4, p. 359-372.
[17] ———, "A symmetric function generalization of the chromatic polynomial of a graph", *Adv. Math.* **111** (1995), no. 1, p. 166-194.
[18] ———, *Enumerative Combinatorics. Volume 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, 1999.
[19] R. P. Stanley, J. R. Stembridge, "On immanants of Jacobi–Trudi matrices and permutations with restricted position", *J. Comb. Theory, Ser. A* **62** (1993), no. 2, p. 261-279.
[20] J. R. Stembridge, "Shifted tableaux and the projective representations of symmetric groups", *Adv. Math.* **74** (1989), no. 1, p. 87-134.
[21] ———, "Immanants of totally positive matrices are nonnegative", *Bull. Lond. Math. Soc.* **23** (1991), no. 5, p. 442-428.
[22] B. Sturmfels, *Grobner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, 1996.