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Gloire Grâce Bockondas and Samuel Boissière

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Gloire Grâce Bockondas[*, *a*] **and Samuel Boissière**[*b*]

[*a*] Département de Mathématiques, Université Marien Ngouabi, Brazzaville, Congo
URL: https://sites.google.com/view/gloiregbockondas/

[*b*] Samuel Boissière, Laboratoire de Mathématiques et Applications, UMR 7348 du CNRS, Bâtiment H3, Boulevard Marie et Pierre Curie, Site du Futuroscope, TSA 61125, 86073 Poitiers Cedex 9, France
URL: http://www-math.sp2mi.univ-poitiers.fr/~sboissie/

E-mails: gloire.bockondas@umng.cg (G. G. Bockondas),
samuel.boissiere@math.univ-poitiers.fr (S. Boissière)

Abstract. The present paper deals with lines contained in a smooth complex cubic threefold. It is well-known that the set of lines of the second type on a cubic threefold is a curve on its Fano surface. Here we give a description of the singularities of this curve.

1. Introduction

Let $X \subset \mathbb{P}^4$ be a smooth complex cubic threefold and $\ell \subset X$ a line. The variety that parametrizes the lines on X is a smooth surface of general type called the Fano surface $F(X)$ of X (see [4]). Cubic threefolds and their Fano surfaces of lines is an object of interest in algebraic geometry going back to Fano's work in 1904. Later on, their geometry has been further studied by Clemens and Griffiths who proved the irrationality of the cubic threefold by introducing the intermediate Jacobian as a principal tool [4], Murre studied the geometry of the Fano variety of lines of a smooth cubic threefold [7], and Roulleau gave the classification of the configurations of the elliptic curves on the Fano surface of a smooth cubic threefold [8]. We can also cite Altman and Kleiman work [1] and Tjurin papers [11, 12]. In this work we are interested in lines contained in X. They are either of the first type or of the second type depending on the decomposition of the normal bundle $\mathcal{N}_{\ell/X}$. Lines of the first type are generic while the locus of lines of the second type defines a curve $M(X)$ in the associated Fano surface $F(X)$ (see [7]). In the curve $M(X)$ of lines of the second type there may be particular lines $\ell \subset X$ for which there exists a unique 2-plane $P \supset \ell$ tritangent to X along all of ℓ called triple lines. But this only occurs for at most a finite number of lines (see [4]).

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[*] Corresponding author.

By using local computations explained in Remark 7, Murre claimed that the curve $\mathrm{M}(X)$ of lines of the second type is smooth [7]. However, it is well-known to the experts that this curve can be singular and its singularities correspond to triple lines on X (see [5, 6, 9]). Nevertheless, no complete proof does appear in the literature. The purpose of this paper is to fill this gap by giving a complete proof of Theorem 1 following the techniques of [7].

Theorem 1. *The triple lines on a cubic threefold are exactly the singular points of the curve of lines of the second type.*

Consider the Fermat cubic in $\mathbb{P}^4$ defined by

$$F_4 = \left\{ x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \right\}.$$

Proposition 2. *The Fermat cubic $F_4 \subset \mathbb{P}^4$ contains exactly 135 triple lines.*

2. Notation and preliminaries

For $X \subset \mathbb{P}^4$ a smooth complex cubic threefold, we denote by $\mathrm{F}(X)$ the Fano surface of lines and $\mathrm{M}(X)$ the curve of lines of the second type. Denote by $(x_0 : x_1 : x_2 : x_3 : x_4)$ the homogeneous coordinates on $\mathbb{P}^4$ and $p_{i,j}, 0 \le i < j \le 4$ the Plücker coordinates of the grassmannian of lines $\mathbb{G}(1,4) \subset \mathbb{P}^9$. Throughout this paper, we locally study the Fano surface of lines $\mathrm{F}(X)$. For instance when we work in the affine chart $p_{0,1} = 1$ of $\mathbb{G}(1,4)$ isomorphic to $\mathbb{C}^6$, a point $[\ell] \in \mathrm{F}(X) \subset \mathbb{G}(1,4)$ corresponds to a line ℓ on X spanned by two points $v_0 = (1 : 0 : -p_{1,2} : -p_{1,3} : -p_{1,4})$ and $v_1 = (0 : 1 : p_{0,2} : p_{0,3} : p_{0,4})$. We take $(p_{0,2}, p_{0,3}, p_{0,4}, p_{1,2}, p_{1,3}, p_{1,4})$ as the local coordinates of $\mathbb{G}(1,4)$ on the affine chart $p_{0,1} = 1$ (see [2]).

Let ℓ be a line on X. There are two types of lines in X: lines with normal bundle $\mathcal{N}_{\ell/X} \simeq \mathcal{O}_\ell \oplus \mathcal{O}_\ell$ called lines of the first type and those with $\mathcal{N}_{\ell/X} \simeq \mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell(-1)$ called lines of the second type [4]. Another description is as follows: consider the intersection $P \cap X$ of the cubic threefold X with a plane $P \simeq \mathbb{P}^2$ containing ℓ. We observe that $P \cap X = \ell \cup C$ where C is a conic. It may happen that the conic degenerates, that is $P \cap X = \ell \cup \ell' \cup \ell''$. If $P \cap X = 2\ell \cup \ell'$ then we say the plane P is tangent to X at every point of ℓ [11]. The line ℓ is called a multiple line and ℓ' the residual line. We have in particular two cases: if $\ell \ne \ell'$ then ℓ is called a double line, and if $\ell = \ell'$, that is $P \cap X = 3\ell$, then ℓ is called a triple line.

Proposition 3 ([4, Lemma 6.7]). *The line $\ell \subset X$ is of the second type if and only if there exists a unique 2-plane $P \supset \ell$ tangent to X at every point of ℓ. If $\ell \subset X$ is a line of the first type, then there is no 2-plane tangent to X in all points of ℓ.*

It is worth noting that the multiple lines on X are exactly the lines of the second type [7, Lemma 1.14].

3. Lines on a cubic threefold

Throughout this section we gather and extend some facts about lines on cubic threefolds.

Proposition 4 ([7, Corollary 1.9]). *The locus*

$$\mathrm{M}(X) = \left\{ [\ell] \in \mathrm{F}(X), \exists\, P \simeq \mathbb{P}^2 \,\middle|\, P \cap X = 2\ell \cup \ell' \right\}$$

of lines of the second type on X is a curve.

On the affine chart $p_{0,1} = 1$, the Fano surface F(X) is defined by the vanishing locus of the terms $\phi^{i,j}(\ell)$ coming from the decomposition $F(p) = \sum_{i+j=3} t_0^i t_1^j \phi^{i,j}(\ell)$ for any point $p \in \ell \subset X$ with coordinates $t_0 v_0 + t_1 v_1$, where $\phi^{i,j}(\ell)$ are functions of the local Plücker coordinates of the line ℓ. Any plane $P \simeq \mathbb{P}^2$ containing ℓ meets the plane $P_{0,1} = \{x_0 = 0, x_1 = 0\}$ at a unique point $v_2 = (0 : 0 : \alpha_2 : \alpha_3 : \alpha_4)$ so that $P = \text{span}(\ell, v_2)$. Then the plane cubic $P \cap X$ has equation $F(t_0 v_0 + t_1 v_1 + t_2 v_2) = 0$ where $(v_0 : v_1 : v_2)$ are the projective coordinates of the plane P. Expanding in t_2 we have:

$$0 = F(t_0 v_0 + t_1 v_1) + t_2 \sum_{i=2}^{4} \frac{\partial F}{\partial x_i}(t_0 v_0 + t_1 v_1) \alpha_i + \frac{1}{2} t_2^2 \sum_{2 \le i,j \le 4} \frac{\partial^2 F}{\partial x_j \partial x_i}(t_0 v_0 + t_1 v_1) \alpha_i \alpha_j + t_2^3 F(v_2).$$

The line $\ell \subset P$ of equation $t_2 = 0$ is a line of the second type on X if and only if $F(t_0 v_0 + t_1 v_1) = 0$ and the plane cubic equation is a multiple of t_2^2. For any point $p \in \ell \subset X$ with coordinates $t_0 v_0 + t_1 v_1$ on the affine chart $p_{0,1} = 1$, one can write

$$\frac{\partial F}{\partial x_i}(p) = \sum_{j+k=2} t_0^j t_1^k \phi_i^{j,k}(\ell) \tag{1}$$

where $\phi_i^{j,k}(\ell)$ are functions of the local Plücker coordinates. Then the line $\ell \subset X$ of equation $t_2 = 0$ is a second type line if and only if

$$\mathrm{m}(\ell) := \det \begin{pmatrix} \phi_2^{2,0}(\ell) & \phi_3^{2,0}(\ell) & \phi_4^{2,0}(\ell) \\ \phi_2^{1,1}(\ell) & \phi_3^{1,1}(\ell) & \phi_4^{1,1}(\ell) \\ \phi_2^{0,2}(\ell) & \phi_3^{0,2}(\ell) & \phi_4^{0,2}(\ell) \end{pmatrix} = 0. \tag{2}$$

The curve of lines of the second type on X is thus locally given by

$$\mathrm{M}(X) = \left\{ \phi^{3,0}(\ell) = 0, \phi^{2,1}(\ell) = 0, \phi^{1,2}(\ell) = 0, \phi^{0,3}(\ell) = 0, \mathrm{m}(\ell) = 0 \right\}.$$

For a generic cubic threefold $X \subset \mathbb{P}^4$ the locus $\mathrm{M}(X)$ of lines of the second type is smooth [5, Proposition 2.13].

3.1. Triple lines

We recall that a triple line is a second type line $\ell \subset X$ such that $P \cap X = 3\ell$. To our knowledge, the locus of triple lines on a cubic threefold is shortly mentioned in the literature. We have the following lemma.

Lemma 5 ([4, Lemma 10.15]). *The locus $\{[\ell] \in \mathrm{F}(X), \exists\, P \simeq \mathbb{P}^2 \,|\, P \cap X = 3\ell\}$ of triple lines on X defines a finite set.*

The line of the second type $\ell \subset X$ of equation $t_2 = 0$ is a triple line if and only if the plane cubic equation is a multiple of t_2^3. For all point $p \in \ell \subset X$ with coordinates $t_0 v_0 + t_1 v_1$ one can write

$$\frac{\partial^2 F}{\partial x_i \partial x_j}(p) = \sum_{k+l=1} t_0^k t_1^l \phi_{i,j}^{k,l}(\ell) \tag{3}$$

where $\phi_{i,j}^{k,l}(\ell)$ are functions of the local Plücker coordinates on the affine chart $p_{0,1} = 1$. Then the line of the second type $\ell \subset X$ of equation $t_2 = 0$ is a triple line if and only if

$$\begin{pmatrix} \frac{1}{2}\phi_{2,2}^{1,0}(\ell) & \frac{1}{2}\phi_{3,3}^{1,0}(\ell) & \frac{1}{2}\phi_{4,4}^{1,0}(\ell) & \phi_{2,3}^{1,0}(\ell) & \phi_{2,4}^{1,0}(\ell) & \phi_{3,4}^{1,0}(\ell) \\[2mm] \frac{1}{2}\phi_{2,2}^{0,1}(\ell) & \frac{1}{2}\phi_{3,3}^{0,1}(\ell) & \frac{1}{2}\phi_{4,4}^{0,1}(\ell) & \phi_{2,3}^{0,1}(\ell) & \phi_{2,4}^{0,1}(\ell) & \phi_{3,4}^{0,1}(\ell) \end{pmatrix} \begin{pmatrix} \alpha_2^2 \\ \alpha_3^2 \\ \alpha_4^2 \\ \alpha_2 \alpha_3 \\ \alpha_2 \alpha_4 \\ \alpha_3 \alpha_4 \end{pmatrix} = 0 \tag{4}$$

holds.

4. Triple lines on a cubic threefold and the curve of lines of the second type

In this section we prove the following theorem by using the techniques of [7].

Theorem 6. *The triple lines on a cubic threefold are exactly the singular points of the curve of lines of the second type.*

Proof. Let us start by fixing the notations. We can assume that $[\ell_0] \in M(X)$ is a line of the second type given by

$$x_2 = 0, x_3 = 0, x_4 = 0.$$

On the affine chart $p_{0,1} = 1$ of $\mathbb{G}(1,4)$ the line $[\ell_0] \in M(X)$ is thus given by

$$p_{0,2} = 0, p_{0,3} = 0, p_{0,4} = 0, p_{1,2} = 0, p_{1,3} = 0, p_{1,4} = 0.$$

We may assume after a linear change of variables x_2, x_3 and x_4 that the 2-plane $P = \mathrm{span}(\ell_0, v_2)$ tangent to X at every point of ℓ_0 is such that $\alpha_2 = 0, \alpha_3 = 0$ and $\alpha_4 = 1$. Then from the following equation

$$\begin{pmatrix} \phi_2^{2,0}(\ell_0) & \phi_3^{2,0}(\ell_0) & \phi_4^{2,0}(\ell_0) \\ \phi_2^{1,1}(\ell_0) & \phi_3^{1,1}(\ell_0) & \phi_4^{1,1}(\ell_0) \\ \phi_2^{0,2}(\ell_0) & \phi_3^{0,2}(\ell_0) & \phi_4^{0,2}(\ell_0) \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we get

$$\phi_4^{2,0}(\ell_0) = 0, \phi_4^{1,1}(\ell_0) = 0, \phi_4^{0,2}(\ell_0) = 0. \tag{5}$$

The proof goes as follows: we first compute the affine tangent space $T_{[\ell_0]}F(X)$ of $F(X)$ at the point $[\ell_0]$. Then we compute the affine tangent space $T_{[\ell_0]}M(X)$ of $M(X)$ at the point $[\ell_0]$. Afterwards we prove that the triple lines on X are singular points of the curve $M(X)$. Finally we prove that the singular points of the curve $M(X)$ correspond to triple lines on X.

(1) *Tangent space of* $F(X)$ *at the point* $[\ell_0]$. On the affine chart $p_{0,1} = 1$, a point in $\mathbb{C}^6$ with coordinates $(p_{i,j})_{i,j}$ belongs to the tangent space $T_{[\ell_0]}F(X)$ if and only if

$$\sum_{i=2}^{4} \left(-p_{1,i} t_0 + p_{0,i} t_1 \right) \left(t_0^2 \phi_i^{2,0}(\ell_0) + t_0 t_1 \phi_i^{1,1}(\ell_0) + t_1^2 \phi_i^{0,2}(\ell_0) \right) = 0$$

for all $(t_0 : t_1) \in \mathbb{P}^1$. Setting the coefficients of $t_0^3, t_0^2 t_1, t_0 t_1^2$ and t_1^3 each equal to zero one can see that for such a point the equality

$$\underbrace{\begin{pmatrix} 0 & 0 & \phi_2^{2,0}(\ell_0) & \phi_3^{2,0}(\ell_0) \\ \phi_2^{2,0}(\ell_0) & \phi_3^{2,0}(\ell_0) & \phi_2^{1,1}(\ell_0) & \phi_3^{1,1}(\ell_0) \\ \phi_2^{1,1}(\ell_0) & \phi_3^{1,1}(\ell_0) & \phi_2^{0,2}(\ell_0) & \phi_3^{0,2}(\ell_0) \\ \phi_2^{0,2}(\ell_0) & \phi_3^{0,2}(\ell_0) & 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} p_{0,2} \\ p_{0,3} \\ -p_{1,2} \\ -p_{1,3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{6}$$

holds (see Equation (5)). The determinant of the matrix A defined in Equation (6) can be expressed in terms of the resultant below

$$\det \begin{pmatrix} \phi_2^{2,0}(\ell_0) & 0 & \phi_3^{2,0}(\ell_0) & 0 \\ \phi_2^{1,1}(\ell_0) & \phi_2^{2,0}(\ell_0) & \phi_3^{1,1}(\ell_0) & \phi_3^{2,0}(\ell_0) \\ \phi_2^{0,2}(\ell_0) & \phi_2^{1,1}(\ell_0) & \phi_3^{0,2}(\ell_0) & \phi_3^{1,1}(\ell_0) \\ 0 & \phi_2^{0,2}(\ell_0) & 0 & \phi_3^{0,2}(\ell_0) \end{pmatrix} \tag{7}$$

after permutation of the columns. For any point $p_0 \in \ell_0 \subset X$ with coordinates $(t_0 : t_1 : 0 : 0 : 0)$ in the affine chart $p_{0,1} = 1$ the polynomials

$$\frac{\partial F}{\partial x_i}(p_0) = \sum_{j+k=2} t_0^j t_1^k \phi_i^{j,k}(\ell_0)$$

vanish for $i \in \{0, 1\}$, and using Equation (5) so also does

$$\frac{\partial F}{\partial x_4}(p_0) = \sum_{j+k=2} t_0^j t_1^k \phi_4^{j,k}(\ell_0).$$

Therefore the equations

$$\frac{\partial F}{\partial x_i}(p_0) = \sum_{j+k=2} t_0^j t_1^k \phi_i^{j,k}(\ell_0) = 0$$

do not have a common root for $i \in \{2, 3\}$, otherwise the cubic threefold X would be singular. The resultant (7) does not vanish and thus we have

$$\det(A) \neq 0. \tag{8}$$

The tangent space to $\mathrm{F}(X)$ at the point $[\ell_0]$ is then given by

$$T_{[\ell_0]}\mathrm{F}(X) = \{p_{0,2} = 0,\, p_{0,3} = 0,\, p_{1,2} = 0,\, p_{1,3} = 0\}$$

in the affine chart $p_{0,1} = 1$.

(2) *Tangent space of* $\mathrm{M}(X)$ *at the point* $[\ell_0]$. The tangent space $T_{[\ell_0]}\mathrm{M}(X)$ is given by the intersection of the tangent space $T_{[\ell_0]}\mathrm{F}(X)$ with the hypersurface defined by

$$\sum_{0 \leq i < 2 \leq j \leq 4} p_{i,j} \frac{\partial \mathrm{m}}{\partial p_{i,j}}(\ell_0) = 0.$$

We differentiate the determinant in (2) with respect to $p_{i,j}$ and by using Equation (5) we get

$$\det \begin{pmatrix} \phi_2^{2,0}(\ell_0) & \phi_3^{2,0}(\ell_0) & p_{0,4}\dfrac{\partial \phi_4^{2,0}}{\partial p_{0,4}}(\ell_0) + p_{1,4}\dfrac{\partial \phi_4^{2,0}}{\partial p_{1,4}}(\ell_0) \\ \phi_2^{1,1}(\ell_0) & \phi_3^{1,1}(\ell_0) & p_{0,4}\dfrac{\partial \phi_4^{1,1}}{\partial p_{0,4}}(\ell_0) + p_{1,4}\dfrac{\partial \phi_4^{1,1}}{\partial p_{1,4}}(\ell_0) \\ \phi_2^{0,2}(\ell_0) & \phi_3^{0,2}(\ell_0) & p_{0,4}\dfrac{\partial \phi_4^{0,2}}{\partial p_{0,4}}(\ell_0) + p_{1,4}\dfrac{\partial \phi_4^{0,2}}{\partial p_{1,4}}(\ell_0) \end{pmatrix} = 0. \tag{9}$$

Let us explicit Equation (9). For any point $p \in \ell \subset X$ with coordinates $t_0 v_0 + t_1 v_1$ in the affine chart $p_{0,1} = 1$ one can write $F(p) = F(t_0 v_0 + t_1 v_1)$. Differentiate Equation (1) with respect to $p_{u,4}$ for $u \in \{0, 1\}$ and using Equation (3) we have

$$\sum_{j+k=2} t_0^j t_1^k \frac{\partial \phi_4^{j,k}}{\partial p_{u,4}}(\ell) = (-1)^u t_0^u t_1^{1-u} \sum_{k+l=1} t_0^k t_1^l \phi_{4,4}^{k,l}(\ell).$$

By identifying the homogeneous components in t_0 and t_1 we get

$$\frac{\partial \phi_4^{j,k}}{\partial p_{u,4}}(\ell) = \phi_{4,4}^{j,k-1}(\ell), \quad \frac{\partial \phi^{j,k}}{\partial p_{u,4}}(\ell) = -\phi_{4,4}^{j-1,k}(\ell)$$

with $\phi_{4,4}^{2,-1}(\ell) = 0, \phi_{4,4}^{-1,2}(\ell) = 0$. Then Equation (9) becomes

$$\phi(p_{0,4}, p_{1,4}) := \det \begin{pmatrix} \phi_2^{2,0}(\ell_0) & \phi_3^{2,0}(\ell_0) & -p_{1,4}\phi_{4,4}^{1,0}(\ell_0) \\ \phi_2^{1,1}(\ell_0) & \phi_3^{1,1}(\ell_0) & p_{0,4}\phi_{4,4}^{1,0}(\ell_0) - p_{1,4}\phi_{4,4}^{0,1}(\ell_0) \\ \phi_2^{0,2}(\ell_0) & \phi_3^{0,2}(\ell_0) & p_{0,4}\phi_{4,4}^{0,1}(\ell_0) \end{pmatrix} = 0.$$

Therefore the tangent space of the curve $\mathrm{M}(X)$ at the point $[\ell_0]$ is given by

$$T_{[\ell_0]}\mathrm{M}(X) = \{p_{0,2} = 0, p_{0,3} = 0, p_{1,2} = 0, p_{1,3} = 0, \phi(p_{0,4}, p_{1,4}) = 0\}$$

in the affine chart $p_{0,1} = 1$. We now prove that the triple lines on X are singular points of $\mathrm{M}(X)$.

(3) *Triple lines and the curve* $\mathrm{M}(X)$ *of lines of the second type.* From Equation (4) the double line $[\ell_0] \in \mathrm{M}(X)$ is a triple line if and only if

$$\begin{cases} \phi_{4,4}^{1,0}(\ell_0) = 0 \\ \phi_{4,4}^{0,1}(\ell_0) = 0 \end{cases}$$

which implies that $\phi(p_{0,4}, p_{1,4}) = 0$. Hence for $[\ell_0] \in \mathrm{M}(X)$ a triple line, the tangent space of the curve $\mathrm{M}(X)$ at the point $[\ell_0]$ is given by

$$T_{[\ell_0]}\mathrm{M}(X) = \{p_{0,2} = 0, p_{0,3} = 0, p_{1,2} = 0, p_{1,3} = 0\}$$

in the affine chart $p_{0,1} = 1$. Its dimension is strictly greater than one. Therefore the triple lines on X are singular points of $\mathrm{M}(X)$.

(4) *Singular points of the curve* $\mathrm{M}(X)$ *of lines of the second type.* We have seen that the curve of lines of the second type is given by $\mathrm{M}(X) = \{\phi^{3,0}(\ell) = 0, \phi^{2,1}(\ell) = 0, \phi^{1,2}(\ell) = 0, \phi^{0,3}(\ell) = 0, \mathrm{m}(\ell) = 0\}$ in the affine chart $p_{0,1} = 1$. We are going to compute the Jacobian matrix of this curve at the point $[\ell_0]$. Keeping notation as above, we have for any point $p \in \ell \subset X$ with coordinates $t_0 v_0 + t_1 v_1$:

$$F(p) = F(t_0 v_0 + t_1 v_1) = \sum_{i+j=3} t_0^i t_1^j \phi^{i,j}(\ell). \tag{10}$$

Differentiating Equation (10) with respect to $p_{u,v}$ for $u \in \{0,1\}$ and $v \in \{2,3,4\}$ we get

$$\sum_{i+j=3} t_0^i t_1^j \frac{\partial \phi^{i,j}}{\partial p_{u,v}}(\ell) = (-1)^u t_0^u t_1^{1-u} \sum_{j+k=2} t_0^j t_1^k \phi_v^{j,k}(\ell).$$

By identifying the homogeneous components in t_0 and t_1 we get

$$\frac{\partial \phi^{i,j}}{\partial p_{0,v}}(\ell) = \phi_v^{i,j-1}(\ell), \quad \frac{\partial \phi^{i,j}}{\partial p_{1,v}}(\ell) = -\phi_v^{i-1,j}(\ell)$$

with $\phi_v^{3,-1}(\ell) = 0, \phi_v^{-1,3}(\ell) = 0$. The Jacobian matrix of $\mathrm{M}(X)$ at the point $[\ell_0]$ is thus

$$\begin{pmatrix} 0 & 0 & 0 & -\phi_2^{2,0}(\ell_0) & -\phi_3^{2,0}(\ell_0) & 0 \\ \phi_2^{2,0}(\ell_0) & \phi_3^{2,0}(\ell_0) & 0 & -\phi_2^{1,1}(\ell_0) & -\phi_3^{1,1}(\ell_0) & 0 \\ \phi_2^{1,1}(\ell_0) & \phi_3^{1,1}(\ell_0) & 0 & -\phi_2^{0,2}(\ell_0) & -\phi_3^{0,2}(\ell_0) & 0 \\ \phi_2^{0,2}(\ell_0) & \phi_3^{0,2}(\ell_0) & 0 & 0 & 0 & 0 \\ \frac{\partial \mathrm{m}}{\partial p_{0,2}}(\ell_0) & \frac{\partial \mathrm{m}}{\partial p_{0,3}}(\ell_0) & \frac{\partial \mathrm{m}}{\partial p_{0,4}}(\ell_0) & \frac{\partial \mathrm{m}}{\partial p_{1,2}}(\ell_0) & \frac{\partial \mathrm{m}}{\partial p_{1,3}}(\ell_0) & \frac{\partial \mathrm{m}}{\partial p_{1,4}}(\ell_0) \end{pmatrix}$$

using again Equation (5). Denote by M_i its 5×5 minors where i corresponds to the omitted column. The minors M_1, M_2, M_4, M_5 vanish and the singular points of the curve $\mathrm{M}(X)$ are the points $[\ell_0] \in \mathrm{M}(X)$ for which $M_3 = 0, M_6 = 0$ with

$$M_3 = \frac{\partial \mathrm{m}}{\partial p_{1,4}}(\ell_0) \det \begin{pmatrix} 0 & 0 & -\phi_2^{2,0}(\ell_0) & -\phi_3^{2,0}(\ell_0) \\ \phi_2^{2,0}(\ell_0) & \phi_3^{2,0}(\ell_0) & -\phi_2^{1,1}(\ell_0) & -\phi_3^{1,1}(\ell_0) \\ \phi_2^{1,1}(\ell_0) & \phi_3^{1,1}(\ell_0) & -\phi_2^{0,2}(\ell_0) & -\phi_3^{0,2}(\ell_0) \\ \phi_2^{0,2}(\ell_0) & \phi_3^{0,2}(\ell_0) & 0 & 0 \end{pmatrix}$$

$$M_6 = \frac{\partial \mathrm{m}}{\partial p_{0,4}}(\ell_0) \det \begin{pmatrix} 0 & 0 & -\phi_2^{2,0}(\ell_0) & -\phi_3^{2,0}(\ell_0) \\ \phi_2^{2,0}(\ell_0) & \phi_3^{2,0}(\ell_0) & -\phi_2^{1,1}(\ell_0) & -\phi_3^{1,1}(\ell_0) \\ \phi_2^{1,1}(\ell_0) & \phi_3^{1,1}(\ell_0) & -\phi_2^{0,2}(\ell_0) & -\phi_3^{0,2}(\ell_0) \\ \phi_2^{0,2}(\ell_0) & \phi_3^{0,2}(\ell_0) & 0 & 0 \end{pmatrix}.$$

Since the determinant appearing in the expression of the minors M_3 and M_6 cannot vanish (see (8)) then one can see that the singular points of $\mathrm{M}(X)$ are the points $[\ell_0]$ that satisfy

$$\begin{cases} \dfrac{\partial \mathrm{m}}{\partial p_{1,4}}(\ell_0) = \det \begin{pmatrix} \phi_2^{2,0}(\ell_0) & \phi_3^{2,0}(\ell_0) & -\phi_{4,4}^{1,0}(\ell_0) \\ \phi_2^{1,1}(\ell_0) & \phi_3^{1,1}(\ell_0) & -\phi_{4,4}^{0,1}(\ell_0) \\ \phi_2^{0,2}(\ell_0) & \phi_3^{0,2}(\ell_0) & 0 \end{pmatrix} = 0 \\[20pt] \dfrac{\partial \mathrm{m}}{\partial p_{0,4}}(\ell_0) = \det \begin{pmatrix} \phi_2^{2,0}(\ell_0) & \phi_3^{2,0}(\ell_0) & 0 \\ \phi_2^{1,1}(\ell_0) & \phi_3^{1,1}(\ell_0) & \phi_{4,4}^{1,0}(\ell_0) \\ \phi_2^{0,2}(\ell_0) & \phi_3^{0,2}(\ell_0) & \phi_{4,4}^{0,1}(\ell_0) \end{pmatrix} = 0. \end{cases}$$

So we have

$$\underbrace{\begin{pmatrix} -M_{13} & M_{23} \\ -M_{23} & M_{33} \end{pmatrix}}_{B} \begin{pmatrix} \phi_{4,4}^{1,0}(\ell_0) \\ \phi_{4,4}^{0,1}(\ell_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

where M_{13}, M_{23} and M_{33} are the 2×2 minors of the matrix

$$\begin{pmatrix} \phi_2^{2,0}(\ell_0) & \phi_3^{2,0}(\ell_0) & \phi_4^{2,0}(\ell_0) \\ \phi_2^{1,1}(\ell_0) & \phi_3^{1,1}(\ell_0) & \phi_4^{1,1}(\ell_0) \\ \phi_2^{0,2}(\ell_0) & \phi_3^{0,2}(\ell_0) & \phi_4^{0,2}(\ell_0) \end{pmatrix}.$$

This matrix is of rank two for if it was of rank one then one could find a common root of the equations $\frac{\partial F}{\partial x_i}(p_0) = \sum_{j+k=2} t_0^j t_1^k \phi_i^{j,k}(\ell_0) = 0$ for $i \in \{2,3,4\}$ with $p_0 \in \ell_0$ a point with coordinates $(t_0 : t_1 : 0 : 0 : 0)$ in the affine chart $p_{0,1} = 1$. Let us consider the matrix B defined in Equation (11). If $\det(B) = 0$ we can assume that a column is a multiple of the other one. Then one could find a common root of $\frac{\partial F}{\partial x_i}(p_0) = \sum_{j+k=2} t_0^j t_1^k \phi_i^{j,k}(\ell_0) = 0$ for $i \in \{2,3\}$ and X would be singular using Equation (5). Hence $\det(B) \neq 0$ and Equation (11) holds if and only if $(\phi_{4,4}^{1,0}(\ell_0), \phi_{4,4}^{0,1}(\ell_0))$ vanishes, which are the necessary and sufficient conditions for the line $[\ell_0] \in \mathrm{M}(X)$ to be a triple line. Therefore the singular points of $\mathrm{M}(X)$ are the triple lines on X. □

As a consequence of Theorem 1 one can deduce that the curve $\mathrm{M}(X)$ of lines of the second type is reduced.

Remark 7. Let ℓ_0 be a line on X given by $x_2 = 0, x_3 = 0, x_4 = 0$. Then the equation of X may be written

$$F(x_0, \ldots, x_4) = x_2 q_2(x_0, \ldots, x_4) + x_3 q_3(x_0, \ldots, x_4) + x_4 q_4(x_0, \ldots, x_4) = 0$$

where $q_i(x_0, \ldots, x_4)$ are homogeneous polynomials of degree two. If moreover ℓ_0 is a line of the second type so that the plane $P = \mathrm{span}(\ell_0, v_2)$ tangent to X in all points of ℓ_0 is such that $\alpha_2 = 0, \alpha_3 = 0$ and $\alpha_4 = 1$ then the equation of X may take the form

$$F(x_0, \ldots, x_4) = x_2 q_2(x_0, \ldots, x_4) + x_3 q_3(x_0, \ldots, x_4) + x_4^2 l(x_0, \ldots, x_4) = 0$$

where $l(x_0, \ldots, x_4)$ is a linear polynomial. Set $l(x_0, \ldots, x_4) = a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$. Using the techniques of [7] we get

$$a_0 = \frac{1}{2}\phi_{4,4}^{1,0}(\ell_0) \quad \text{and} \quad a_1 = \frac{1}{2}\phi_{4,4}^{0,1}(\ell_0).$$

So for ℓ_0 a triple line on X we have $a_0 = 0, a_1 = 0$. As corrected in [6, p. 17], Murre erroneously wrote $l(x_0, x_1) = a_0 x_0 + a_1 x_1$ for the linear polynomial instead in equation (13) of [7, p. 167], implying that for $\ell_0 \subset X$ a triple line this polynomial vanishes and X contains the plane $\{x_2 =$

$0, x_3 = 0$}: the cubic threefold X would therefore be singular. Whereas for ℓ_0 a triple line on X the equation of X may be written

$$F(x_0, \ldots, x_4) = x_2 q_2(x_0, \ldots, x_4) + x_3 q_3(x_0, \ldots, x_4) + k x_4^3 = 0$$

with $k \neq 0$ and the cubic threefold X remains smooth.

5. Counting triple lines on the Fermat cubic

In this section we prove the following proposition.

Proposition 8. *The Fermat cubic $F_4 \subset \mathbb{P}^4$ contains exactly 135 triple lines.*

Proof. In the affine chart $p_{0,1} = 1$ the set $M(F_4)$ is a non smooth curve given by the following equations:

$$p_{1,2}^3 + p_{1,3}^3 + p_{1,4}^3 - 1 = 0$$
$$p_{0,2} p_{1,2}^2 + p_{0,3} p_{1,3}^2 + p_{0,4} p_{1,4}^2 = 0$$
$$p_{0,2}^2 p_{1,2} + p_{0,3}^2 p_{1,3} + p_{0,4}^2 p_{1,4} = 0$$
$$p_{0,2}^3 + p_{0,3}^3 + p_{0,4}^3 + 1 = 0$$
$$\left(p_{0,4} p_{1,3} - p_{0,3} p_{1,4}\right)\left(p_{0,4} p_{1,2} - p_{0,2} p_{1,4}\right)\left(p_{0,3} p_{1,2} - p_{0,2} p_{1,3}\right) = 0$$

with

$$\left(p_{0,4} p_{1,3} - p_{0,3} p_{1,4}\right)\left(p_{0,4} p_{1,2} - p_{0,2} p_{1,4}\right)\left(p_{0,3} p_{1,2} - p_{0,2} p_{1,3}\right) = \mathrm{m}(\ell)$$

its local equation in the Fano surface $F(F_4)$. Consider the quadrics

$$Q_1 = p_{0,4} p_{1,3} - p_{0,3} p_{1,4}, \; Q_2 = p_{0,4} p_{1,2} - p_{0,2} p_{1,4}, \; Q_3 = p_{0,3} p_{1,2} - p_{0,2} p_{1,3}.$$

We note that the intersection of the Fano surface $F(F_4)$ with Q_1, Q_2 and Q_3, denoted by $M_1(F_4), M_2(F_4)$ and $M_3(F_4)$ respectively, are smooth curves that correspond to the irreducible components of $M(F_4)$.
The intersection points $[\ell]$ of the curves $M_1(F_4)$ and $M_2(F_4)$ are given by the equations:

$$p_{0,2} p_{1,3}^3 - p_{0,2}, \; p_{1,3}^4 - p_{1,3}, \; p_{0,2}^3 + p_{1,3}^3, \; p_{0,3}^3 - p_{1,3}^3 + 1, \; p_{1,2}^3 + p_{1,3}^3 - 1,$$
$$p_{0,2} p_{0,3}, p_{0,2} p_{1,2}, \; p_{0,3} p_{1,3}, \; p_{1,2} p_{1,3}, \; p_{0,4}, \; p_{1,4}.$$

If $p_{1,3} = 0$ then we have 9 intersection points given by the following equations:

$$p_{0,3}^3 = -1, \; p_{1,2}^3 = 1, \; p_{0,2} = 0, \; p_{0,4} = 0, \; p_{1,4} = 0.$$

We get $v_2 = (0 : 0 : 0 : 0 : 1)$. We have thus $f(t_0 v_0 + t_1 v_1 + t_2 v_2) = t_2^3$ implying that ℓ is a triple line on the Fermat cubic. Also the Jacobian matrix of the curve $M(F_4)$ at the point $[\ell]$ is not of rank 5; $[\ell]$ is a singular point of $M(F_4)$. We get 9 triple lines with coordinates $(0, p_{0,3}, 0, p_{1,2}, 0, 0)$ such that $p_{0,3}^3 = -1$ and $p_{1,2}^3 = 1$.
 If $p_{1,3} \neq 0$ then we have 9 intersection points given by the equations:

$$p_{0,2}^3 = -1, \; p_{1,3}^3 = 1, \; p_{0,3} = 0, \; p_{0,4} = 0, \; p_{1,2} = 0, \; p_{1,4} = 0.$$

They correspond to triple lines with coordinates $(p_{0,2}, 0, 0, 0, p_{1,3}, 0)$ with $p_{0,2}^3 = -1$ and $p_{1,3}^3 = 1$.
 Similarly we find the intersection points $[\ell]$ of the curves $M_1(F_4)$ and $M_3(F_4)$, $M_2(F_4)$ and $M_3(F_4)$. We get 18 triple lines at each intersection. We note that the intersection of the three curves $M_1(F_4), M_2(F_4)$ and $M_3(F_4)$ is empty. Therefore there are 54 triple lines in the affine chart $p_{0,1} = 1$.
 We do the same computations to count the triple lines on the Fermat cubic in the other affine charts of $\mathbb{G}(1,4)$ by using SageMath [10]. However a triple line on F_4 can be viewed in many affine charts and then counted several times. In order to avoid such a situation we use the Plücker stratification (see [3]). In the stratum $p_{0,1} = 0, p_{0,2} = 1$ we get 36 triple lines, we get 18 triple lines

in the stratum $p_{0,1} = p_{0,2} = 0, p_{0,3} = 1$ and in the stratum $p_{0,1} = \cdots = p_{0,4} = 0, p_{1,2} = 1$, we get 9 triple lines in the stratum $p_{0,1} = \cdots = p_{1,2} = 0, p_{1,3} = 1$ whereas the other strata contain no triple line. So there are exactly 135 triple lines on the Fermat cubic $F_4 \subset \mathbb{P}^4$. $\qquad\square$

Remark 9. It is clear that the curve $\mathrm{M}(F_4)$ of lines of the second type of the Fermat cubic $F_4 \subset \mathbb{P}^4$ is reducible. The Fano surface $\mathrm{F}(F_4)$ of the Fermat cubic contains 30 elliptic curves that intersect in 135 points [9]. These intersection points are exactly the triple lines on the Fermat cubic. We wish to point out that the elliptic curves of the Fano surface $\mathrm{F}(F_4)$ are exactly the irreducible components of the curve $\mathrm{M}(F_4)$ of lines of the second type. However, whether the singularities of $\mathrm{M}(X)$ are always nodal is still an open question.

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