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Adrien Boyer, Antoine Pinochet Lobos & Christophe Pittet

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RADIAL RAPID DECAY DOES NOT IMPLY RAPID DECAY

by Adrien BOYER,
Antoine PINOCHET LOBOS & Christophe PITTET (*)

Abstract. — We provide a new, dynamical criterion for the radial rapid decay property. We work out in detail the special case of the group $\Gamma := \mathrm{SL}_2(A)$, where $A := \mathbb{F}_q[X, X^{-1}]$ is the ring of Laurent polynomials with coefficients in $\mathbb{F}_q$, endowed with the length function coming from a natural action of Γ on a product of two trees, and show that it has the radial rapid decay (RRD) property and doesn't have the rapid decay (RD) property. We show that the criterion also applies to all irreducible lattices (uniform or not) in semisimple Lie groups with finite center endowed with a length function defined with the help of a Finsler metric. When the rank is greater or equal to two and the lattice is non-uniform, the lattice has RRD but not RD. These examples answer a question asked by Chatterji and moreover show that, unlike the RD property, the RRD property isn't inherited by open subgroups.

Résumé. — Nous établissons un nouveau critère dynamique entraînant la propriété de décroissance rapide radiale. Nous explicitons le cas particulier du groupe $\Gamma := \mathrm{SL}_2(A)$, où $A := \mathbb{F}_q[X, X^{-1}]$ est l'anneau des polynômes de Laurent à coefficients dans le corps fini $\mathbb{F}_q$, muni d'une fonction longueur provenant d'une action naturelle de Γ sur le produit de deux arbres. Nous prouvons que pour cette fonction longueur, ce groupe vérifie la propriété de décroissance rapide radiale (RRD), mais ne vérifie pas la propriété de décroissance rapide (RD). Nous prouvons aussi que notre critère s'applique à tout réseau irréductible (uniforme ou non), de tout groupe de Lie semi-simple à centre fini, muni d'une certaine fonction longueur définie à l'aide d'une métrique de Finsler. Lorsque le rang réel est supérieur ou égal à deux et que le réseau n'est pas uniforme, le réseau vérifie la propriété RRD, mais pas la propriété RD. Ces exemples répondent à une question de Chatterji et montrent que, contrairement à la propriété RD, la propriété RRD n'est pas héréditaire par passage à un sous-groupe ouvert.

Keywords: RD property, Koopman representation, semi-simple group, lattice, Harish-Chandra function, convolution operator norm, length function.
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1. Introduction

The rapid decay property (RD), which can be stated as an inequality between two different norms on the convolution algebra of a group, was first introduced in [14] and further developped in [15]. It became a subject of great importance since V. Lafforgue discovered its connection with the Baum–Connes conjecture [16]. Any connected, non-compact semisimple Lie group with finite center has RD [9], but it is an open question (asked in [25] and now known as Valette's conjecture) to know whether cocompact lattices inherit the rapid decay property.

The radial rapid decay property (RRD) is a weakening of RD, first studied in [24], and consists in restricting the RD inequality to the class of radial functions. The strategy of proof used in [9] for Lie groups, a reduction to radial functions, raised hope for a solution of Valette's conjecture when Perrone [19] managed to show that cocompact lattices have RRD. In this context, Chatterji asked for a group having RRD but not having RD [8, p. 57].

In this paper, we provide a sufficient, dynamical condition, for a group to have property RRD. We then study the case of the discrete group $\Gamma := \mathrm{SL}_2(A)$, where $A := \mathbb{F}_q[X, X^{-1}]$ is the ring of Laurent polynomials with coefficients in $\mathbb{F}_q$, that acts naturally on a product of trees, and on the product of the boundaries of these trees. Using the dynamical criterion, we prove that Γ has RRD. Moreover, noticing that Γ contains a lamplighter group as a subgroup, we prove that Γ doesn't have RD, and give therefore a negative answer to Chatterji's question; finally, this example shows that RRD isn't inherited by open subgroups, whereas RD is. At first sight, this example may look surprising, since containing an amenable subgroup with exponential growth is a well-known obstruction to having RD.

The criterion also applies to irreducible lattices (uniform or not) in semisimple Lie groups. We prove this in Section 2.2 below.

Notation. — Throughout the paper, we will use the $\ll$ notation, as an alternative to the big O notation. Precisely,

$$f(n) \ll g(n)$$

means that there is $M \in \mathbb{R}$ such that for every sufficiently large n, we have that $|f(n)| \leqslant M|g(n)|$.

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2. Statement of the results

2.1. Statement of the criterion

In this section, we give the necessary definitions and notation in order to state the dynamical criterion; we then investigate the range of application of the criterion.

Let G be a locally compact group. Let $e \in G$ denote the identity element.

Definition 2.1 (Length functions). — *A* length function *on G is a map $L : G \to \mathbb{R}_+$ such that*

(1) $L(e) = 0$*;*
(2) $\forall g \in G$, $L(g^{-1}) = L(g)$*;*
(3) $\forall g, h \in G$, $L(gh) \leqslant L(g) + L(h)$.

If $E \subset G$ is any subset, we define, for all $t \in \mathbb{R}_+$, $E_t = E \cap L^{-1}([0,t])$. A length function L is said to be proper *if G_t is compact, for all t.*

Notation 2.2. — When it is unambiguous, we use the notation

$$C_n \coloneqq \{g \in G \mid L(g) \in [n, n+1)\}.$$

If necessary, we add the reference to the group or the length function by writing C_n^G or C_n^L.

Definition 2.3 (Radial functions). — *Let $L : G \to \mathbb{R}_+$ be a proper length function. Denote by $C_c(G)$ the space of compactly-supported functions on G (if G is a discrete group, $C_c(G)$ is the space of finitely-supported functions and we denote it by $\mathbb{C}[G]$).*

We say that $f \in C_c(G)$ is radial *if*

$$\forall g_1, g_2 \in G, \quad L(g_1) = L(g_2) \Rightarrow f(g_1) = f(g_2).$$

Denote by $C_c^{\mathrm{rad}}(G)$ the set of radial functions (if G is a discrete group, $C_c^{\mathrm{rad}}(G)$ is the space of radial functions of finite support, and we denote it by $\mathbb{C}[G]^{\mathrm{rad}}$).

Let μ be a left Haar measure on G. Let $\mathbb{R}[X]$ denote the algebra of polynomial functions in one variable X and with real coefficients.

For $f \in C_c(G)$, let

$$L(f) \coloneqq \sup\{L(g) \mid g \in \mathrm{supp}(f)\},$$

and for $f \in C_c(G)$ and $\xi \in L^2(G, \mu)$, consider the convolution

$$f * \xi \coloneqq \left(g \mapsto \int_G \xi(h^{-1}g) f(h) \,\mathrm{d}\mu(h) \right).$$

Let us denote by $\|\cdot\|_{p\to q}$ the norm of a continuous operator between a L^p and a L^q space. Recall $\xi \mapsto f * \xi$ is a continuous linear operator on $L^2(G,\mu)$. For reasons of concision, let us denote

$$\|f\|_{\mathrm{op}} \coloneqq \|\xi \mapsto f * \xi\|_{2\to 2}.$$

We can now define the rapid decay property.

Definition 2.4 (Rapid decay). — *We say that G has* property RD *with respect to L if*

$$\exists P \in \mathbb{R}[X], \quad \forall f \in C_c(G), \quad \|f\|_{\mathrm{op}} \leqslant P(L(f))\|f\|_2$$

and we say that it has radial property RD *with respect to L if*

$$\exists P \in \mathbb{R}[X], \quad \forall f \in C_c^{\mathrm{rad}}(G), \quad \|f\|_{\mathrm{op}} \leqslant P(L(f))\|f\|_2.$$

For further information on property RD, see [8] and [12].

The dynamical criterion asserts, for short, that if there is a suitable action of the group on some probability space, then the group has RRD. In order to state it precisely, we will need additional definitions and notation. Let $(B,\mathcal{T})$ be a measurable space, $G \curvearrowright B$ be a measurable action.

Definition 2.5 (Quasi-invariant measure). — *We say that a measure ν on $(B,\mathcal{T})$ is* quasi-invariant *if the action preserves ν-null-sets, that is, for every $g \in G$, for every $C \in \mathcal{T}$ such that $\nu(C) = 0$, then $\nu(gC) = 0$.*

Let ν be a σ-finite quasi-invariant measure for the action $G \curvearrowright (B,\mathcal{T})$. We consider the following objects, which existence relies on the Radon–Nikodym theorem.

Definition 2.6 (Radon–Nikodym cocycles, Koopman representation and Harish-Chandra function). — *We call*

$$\begin{aligned} c : G \times B &\to \mathbb{R}_+^* \\ (g,b) &\mapsto \frac{\mathrm{d}{g^{-1}}_*\nu}{\mathrm{d}\nu}(b) \end{aligned}$$

the Radon–Nikodym cocycle*.*

The formula

$$\begin{aligned} \pi : G &\to \mathcal{U}(L^2(B,\nu)) \\ g &\mapsto \left(h \mapsto (b \mapsto c(g^{-1},b)^{\frac{1}{2}} h(g^{-1}b)\right) \end{aligned}$$

defines a unitary representation called the Koopman representation *associated to the action $\Lambda \curvearrowright (B,\nu)$. The associated* Harish-Chandra function *is defined as*

$$\Xi \coloneqq g \mapsto \int_B c(g^{-1},b)^{\frac{1}{2}}\,\mathrm{d}\nu(b) = \langle \pi(g)\mathbf{1}_B, \mathbf{1}_B\rangle.$$

The main result of this paper is the following theorem.

THEOREM 2.7 (Dynamical criterion for RRD). — *Let Λ be a discrete group with a proper length function L. Let (B,ν) be a σ-finite probability space. Let $\pi : \Lambda \to \mathcal{U}(L^2(B))$ be the Koopman representation arising from a measurable action $\Lambda \curvearrowright (B,\mu)$ leaving ν quasi-invariant, and let Ξ be the corresponding Harish-Chandra function.*

Assume that there is $M \in \mathbb{R}$, $P \in \mathbb{R}[X]$, such that

(1) $\forall n \in \mathbb{N}, \sup_{\gamma \in C_n} \Xi(\gamma) \leqslant \dfrac{P(n)}{\sqrt{|C_n|}}$.

(2) $\forall n \in \mathbb{N}, \left\| \dfrac{1}{|C_n|} \displaystyle\sum_{\gamma \in C_n} \dfrac{\pi(\gamma)}{\Xi(\gamma)} \right\|_{2\to 2} \leqslant M$.

Then Λ has RRD with respect to L.

We will refer to the two hypotheses of the criterion as the *Harish-Chandra volume estimates condition* and the *uniform boundedness condition*. For a proof of Theorem 2.7, see Section 3.

The following theorem, which generalizes ideas from [6], gives sufficient conditions for the uniform boundedness condition to hold.

THEOREM 2.8 (Sufficient conditions for the uniform boundedness condition). — *Let G be a locally compact group endowed with a left Haar measure μ_G, and a proper length function $\mathcal{L}$. Let B be a compact space, ν be a Borel probability measure on B and consider a measurable action of G on B leaving ν quasi-invariant. Let Λ be a lattice in G, and let us endow it with the length function given by the restriction of $\mathcal{L}$.*

Let us assume that

- *the Radon–Nikodym cocycle and the Harish-Chandra function are continuous;*
- *the compact subgroup G_0 of G consisting of elements of length 0 acts transitively on B and leaves ν invariant;*
- *at least one of the following two conditions holds:*
  (i) *G_0 is open and we have the following growth condition:*
  $$\mu_G(C_n^G) \ll |C_n^\Lambda|;$$
  (ii) *the subset G_1 of G consisting of elements of length at most 1 is a neighborhood of e and we have the following growth condition:*
  $$\max\left\{\mu_G(C_{n-1}^G), \mu_G(C_n^G), \mu_G(C_{n+1}^G)\right\} \ll |C_n^\Lambda|.$$

Let us finally denote by π the Koopman representation of Λ associated with the (restricted) action $\Lambda \curvearrowright B$. Then there is $M \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N}, \left\| \frac{1}{|C_n^\Lambda|} \sum_{\gamma \in C_n^\Lambda} \frac{\pi(\gamma)}{\Xi(\gamma)} \right\|_{2\to 2} \leqslant M.$$

For a proof of Theorem 2.8, see Section 4.

The criterion allows us to prove the radial rapid decay property for two sets of examples:

- on the one hand, it applies to lattices in semisimple Lie groups (see Section 2.2 for a precise statement and a short proof which heavily relies on sharp estimates on the Harish-Chandra function);
- on the other hand, we present a self-contained, detailed proof that the criterion applies to some specific lattice in an algebraic semisimple group over a non-archimedean local field (see Section 2.3 for a precise statement and Section 5 for the proof).

2.2. Application of the criterion to lattices in semisimple Lie groups

In this section, we prove (using Theorem 2.7 and Theorem 2.8) that irreducible lattices, in semisimple connected Lie groups with no compact factors and a finite center, have RRD for a suitable length function. We need to start with some notation.

Let G be a connected semisimple Lie group, with no compact factors and a finite center. Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$. Let Σ be the root system of $(\mathfrak{g}, \mathfrak{a})$, let Σ^+ be a system of positive roots, let $\mathfrak{a}^+$ be the corresponding Weyl chamber, let Σ_0^+ be the set of indivisible positive roots. Let $G := KAN$ be the corresponding Iwasawa decomposition, let $M := K \cap Z(A)$, and let $P := MAN$ be the corresponding parabolic subgroup. Let ρ be the half-sum of positive roots. Let us denote $A^+ := \exp(\mathfrak{a}^+)$.

If $H \in \mathfrak{a}^+$, let us denote

$$\|H\| := 2\rho(H).$$

Let us endow G with the following so-called Finsler length function: if

$$g = k_1 \exp(H) k_2$$

is a KA^+K decomposition of any g in G, let us denote

$$L(g) := \|H\|.$$

Finally, consider the action $G \curvearrowright G/P$, a quasi-invariant measure ν on G/P and Ξ the associated Harish-Chandra function.

We then have the following corollary of Theorem 2.7 and Theorem 2.8.

Corollary 2.9. — *Let Λ be an irreducible lattice in G. There exists $\alpha > 0$ such that if we define L_α as*

$$L_\alpha : \gamma \mapsto \left\lceil \frac{L(\gamma)}{\alpha} \right\rceil,$$

then (Λ, L_α) has RRD.

Proof. — We will show the corollary by using the dynamical criterion, Theorem 2.7, and in order to do that, we just need two check two assumptions: the Harish-Chandra estimates and the uniform boundedness condition. Let us begin by checking the first assumption of the dynamical criterion, when Λ is endowed with the length function L (it obviously follows that the first assumption is satisfied when Λ is endowed with the length function L_α for any $\alpha > 0$).

From the work of Anker (see [2]), we have the following form of the Harish-Chandra estimate: for every $g \in G$, and any KA^+K-decomposition $g = k_1 \exp(H) k_2$, we have

$$\Xi(g) \asymp \prod_{\alpha \in \Sigma_0^+} (1 + \alpha(H))\, e^{-\rho(H)}.$$

We then have, by definition of L, the following estimate:

$$\forall g \in G, \quad \Xi(g) \ll (1 + L(g))^{|\Sigma_0^+|}\, e^{-\frac{L(g)}{2}}.$$

On the other hand, if C_n^Λ denotes the sphere with respect to L, we have the following estimate (by combining Prop 7.2 and 7.3 in [1, p. 25]):

$$|C_n^\Lambda| \ll n^{\operatorname{rk} G - 1} e^n.$$

Therefore, by combining these two results, we have

$$\sup_{\gamma \in C_n^\Lambda} \Xi(\gamma) \sqrt{|C_n^\Lambda|} \ll (1 + n)^{|\Sigma_0^+| + \frac{\operatorname{rk} G - 1}{2}},$$

so that condition (1) of the dynamical criterion is satisfied.

In order to check assumption (2) of the criterion, we use Theorem 2.8 by checking its assumptions.

We will check that condition (ii) is satisfied. First of all, the ball centered at e and of radius 1 with respect to L_α is a neighborhood of the identity if α is chosen large enough, since at least one ball has non-empty interior. It remains to prove the growth condition.

The following growth assertion

$$\mu_G(C_n^G) \ll |C_n^\Lambda|,$$

is, in fact, true for a large class of integer-valued proper length functions L on G, up to a rescaling, i.e. replacing L by $L' := \lceil \frac{L}{\alpha} \rceil$ for α large enough (see [13] or [21, Théorème 1.4.41] for details); let us shortly prove how to deduce from it the growth assumption we need. Remember that, for a subgroup H of G, we denote by H_n the intersection of H with the ball of radius n around the identity in G; assuming α is large enough so that the 1-ball generates G, we have, since G is non-amenable,

$$\mu_G(G_n) \ll \mu_G(C_n^G);$$

(see for example [20]) therefore,

$$\mu_G(C_{n-1}^G) \leqslant \mu_G(G_{n-1}) \leqslant \mu_G(G_n) \ll \mu_G(C_n^G)$$

and, using an argument of bounded geometry,

$$\mu_G(C_{n+1}^G) \leqslant \mu_G(G_{n+1}) \ll \mu_G(G_n) \ll \mu_G(C_n^G),$$

so that

$$\max\left\{\mu_G(C_{n-1}^G), \mu_G(C_n^G), \mu_G(C_{n+1}^G)\right\} \ll \mu_G(C_n^G)$$

and, since

$$\mu_G(C_n^G) \ll |C_n^\Lambda|,$$

we are done. □

2.3. Application of the criterion to a specific example

In this section, we apply Theorem 2.7 and Theorem 2.8 to a specific example and exhibit a group having RRD but not having RD for a natural length function. In order to do so, we need a substantial amount of notation.

2.3.1. The group $\mathrm{SL}_2(A)$ and its action on the product of two trees

Let q be a power of a prime number and $A := \mathbb{F}_q[X, X^{-1}]$ be the commutative ring of Laurent polynomials in the variable X with coefficients in $\mathbb{F}_q$, the field with q elements. It is the subring of $\mathbb{K} := \mathbb{F}_q(X)$ (the ring of rational fractions over the field $\mathbb{F}_q$) generated by the elements X and X^{-1} and its additive group is the $\mathbb{F}_q$-vector space with basis $\{X^n \mid n \in \mathbb{Z}\}$, so each element of A can be written as $\sum_{n\in\mathbb{Z}} a_n X^n$, such that $\forall n \in \mathbb{Z}$, $a_n \in \mathbb{F}_q$ and all but a finite number of the a_n being zero.

Let us now define on $\mathbb{K}$ two valuations: if $F \in \mathbb{K}$, there are $n \in \mathbb{Z}$, $P, Q \in \mathbb{F}_q[X]$ such that $F := X^n P/Q$ with $X \nmid P$ and $X \nmid Q$. The integer n only depends on F and is denoted by $v_0(F)$ (it is the valuation at the place 0). If $F = P/Q \in \mathbb{K}$, we define $v_\infty(F)$ to be the number $\deg P - \deg Q$ (it is the valuation at infinity). We denote, for $i \in \{0, \infty\}$, $|\cdot|_i := q^{-v_i(\cdot)}$ the associated norms, and we consider the corresponding completions $\mathbb{K}_0$ and $\mathbb{K}_\infty$. The elements of the ring A are called the $\{0,\infty\}$-integral elements of $\mathbb{K}$. Now consider the diagonal embedding $\Delta : \mathrm{SL}_2(A) \hookrightarrow \mathrm{SL}_2(\mathbb{K}_0) \times \mathrm{SL}_2(\mathbb{K}_\infty)$. According to [18, p. 1],

$$\Gamma := \Delta(\mathrm{SL}_2(A)) \subset G := \mathrm{SL}_2(\mathbb{K}_0) \times \mathrm{SL}_2(\mathbb{K}_\infty)$$

is a lattice in G.

Now, both $\mathrm{SL}_2(\mathbb{K}_0)$ and $\mathrm{SL}_2(\mathbb{K}_\infty)$ act simplicially and properly on their Bruhat–Tits trees T_0 and T_∞ (see [22, p. 69] for the detailed construction of the tree and the action on it) which both happen to be $(q+1)$-regular trees, so the group $G = \mathrm{SL}_2(\mathbb{K}_0) \times \mathrm{SL}_2(\mathbb{K}_\infty)$ (and therefore, Γ) acts cellularly on the product $\mathbb{I} := T_0 \times T_\infty$. Let us denote $V(T_0)$ and $V(T_\infty)$ the sets of vertices of these two trees. Let us denote by d_0 and d_∞ the distances giving length 1 to the edges of T_0 and T_∞. We define on $\mathbb{I} := T_0 \times T_\infty$, the so-called L^1 distance, that is,

$$\forall x, y \in T_0, \quad \forall x', y' \in T_\infty, \quad d_{\mathbb{I}}\left((x,y),(x',y')\right) := d_0(x,y) + d_\infty(x',y').$$

For this distance function, the group G (and, in particular, Γ) acts on $\mathbb{I}$ (and also on $V(T_0) \times V(T_\infty)$) by isometries.

Let $(v_0, v_\infty) \in V(T_0) \times V(T_\infty)$. We define two length functions as follows: for $i \in \{0, \infty\}$, let us denote

$$\forall g_i \in G_i, \quad L_i(g_i) := d_i(v_i, g_i v_i);$$

finally, we define a length function on G as follows: $\forall g := (g_0, g_\infty) \in G$,

$$L(g) := d_{\mathbb{I}}\left((v_0, v_\infty), g(v_0, v_\infty)\right) = L_0(g_0) + L_\infty(g_\infty).$$

According to [17], the length function L on Γ is quasi-isometric to any of the word-lengths on Γ.

We perform the necessary computations and apply the criterion to Γ and the action of Γ on B, the product of the boundaries of the Bruhat–Tits trees to deduce the following corollary.

Corollary 2.10. — *The group* $\mathrm{SL}_2(A)$ *has RRD with respect to* L.

The proof of Corollary 2.10 we present here is self-contained and elementary, but rather lengthy. Section 5 is devoted to collect all ingredients of the proof.

2.4. General consequences on RRD

The examples of the above sections allows us to show two general consequences on the radial rapid decay property, stated in the following corollary.

Corollary 2.11. — *There exists finitely generated groups endowed with lengths functions which are quasi-isometric to word-lengths*

(1) *which have RRD but do not have RD;*
(2) *which contain subgroups that do not have RRD for some word-lengths.*

2.5. Comments on the results

2.5.1. Comments on the criterion

The Harish-Chandra function is usually defined in the general theory of harmonic analysis on reductive groups, and the definition we give coincides with the general definition if one consider the action of a semisimple Lie group on its Poisson–Furstenberg boundary. Its asymptotics have been extensively studied and we make use of an estimate presented by Anker (in [2]) in Section 2.2.

The uniform boundedness condition has been studied in [3, 5, 6, 11] and [7] where authors investigate generalizations of the von Neumann ergodic theorem to the situation where the measure is only quasi-invariant. In several cases of interest, the relevant generalization of von Neumann means are the normalized averages

$$\frac{1}{|C_n|}\sum_{\gamma\in C_n}\frac{\pi(\gamma)}{\Xi(\gamma)}$$

which are shown to converge, in the weak-operator topology, to the orthogonal projector on the constant functions subspace, using the fact that the sequence of these means is uniformly operator-norm-bounded. In particular, the following are true:

Facts. — *The uniform boundedness condition is true (and has been investigated) for (Λ, L, B, ν) in the following situations:*

- [3] *if Λ is the fundamental group of a compact, negatively curved manifold X with universal cover $\widetilde{X}$, such that L is the length function associated to the action of Γ on $\widetilde{X}$ and a fixed base-point $x_0 \in \widetilde{X}$, and B is the Gromov boundary of $\widetilde{X}$ endowed with the Patterson–Sullivan measure ν associated to x_0;*

- [7] *if Λ is a free group over a finite set of generators, where L is the length function associated to the action of Λ on its Cayley tree (with respect to a generating basis) $\widetilde{X}$ and B is the Gromov boundary of $\widetilde{X}$ and ν is the Patterson–Sullivan measure;*
- [11] *if Λ is a non elementary hyperbolic group acting properly and cocompactly on a proper roughly geodesic hyperbolic space X, L is the length function associated to the action, B is the Gromov boundary and ν is the Patterson–Sullivan probability measure;*
- [5] *if Λ is a convex cocompact discrete group of isometries of a* CAT(-1)*-space X, and L is the length function associated to the action and B is the Gromov boundary endowed with the Patterson–Sullivan measure ν;*
- [6] *if G is a noncompact, connected, semisimple Lie group with finite center, with the notation from Section 2.5.1: Λ is a lattice in G, $B := G/P$, ν is a quasi-invariant probability measure and L is a length function defined as in Section 2.2.*

2.5.2. Comments on Corollary 2.9

In rank one, the content of Corollary 2.9 is already known; in fact, Chatterji and Ruane, in [10], show that such lattices have RD.

In [19], Perrone has shown that uniform lattices inherit RRD for length functions built from the Riemannian metric on the symmetric space of the ambient semisimple Lie group. We suspect the criterion does not apply to such length functions, in higher rank, because the rate of decay of the Harish-Chandra function critically depends on the direction in the Weyl chamber (see the estimation in [2] quoted in the proof of Corollary 2.9); it is in order to circumvent this difficulty that we investigated the Finsler metric.

No non-uniform lattice in a semisimple Lie group of rank at least two has RD, since such a lattice has U-elements [17]. The question of deciding if uniform lattices in higher rank semisimple Lie groups have RD is a difficult problem, known as Valette's conjecture, stated in [25].

2.5.3. Comments on Corollary 2.11

The strategy of proof used in [9], in order to prove that semisimple Lie groups have RD, is a reduction to radial functions and the question of comparing RRD and RD has therefore been raised by Chatterji in [8]. Our results show that finitely-generated groups, endowed with length functions quasi-isometric to word-lengths, can have RRD but can lack RD. It would be of great interest to have examples of finitely generated groups that have RRD with respect to some word length but do not have RD (we do not know if RRD is, in general, invariant by quasi-isometries); this problems looks very difficult and we do not even know if $\mathrm{SL}_3(\mathbb{Z})$ has RRD with respect to one of its word-lengths.

As mentioned in the introduction, one way to prove that a discrete group does not have RD is to prove that it contains an amenable subgroup of exponential growth, since an open subgroup of a group having RD has it as well, and since a finitely-generated amenable group has RD with respect to any of its word-lengths if and only if it is of polynomial growth. However, the example of $\mathrm{SL}_2(A)$ in Corollary 2.10 shows that the analogue obstruction for RRD does not hold, by noticing that $\mathrm{SL}_2(A)$ contains a lamplighter group as a subgroup. It would be interesting to further investigate conditions incompatible with RD; for example, we do not know if a group lacking RD must have a subgroup lacking RRD.

2.6. Structure of the paper

We prove the dynamical criterion (Theorem 2.7) in Section 3. We prove Theorem 2.8 in Section 4. Section 5 is devoted to the self-contained proof of the fact that $\mathrm{SL}_2(A)$ has RRD (Corollary 2.10) in which we prove that we can apply the dynamical criterion. In Section 6, we recall some facts on the RD property, exhibit the lamplighter subgroup H of $\mathrm{SL}_2(A)$ and prove that $\mathrm{SL}_2(A)$ does not have RD and that H does not have RRD (Corollary 2.11).

3. Proof of the criterion (Theorem 2.7)

Let Λ be a discrete group, and L be a proper integer-valued length function on Λ. Let us define

$$\forall \gamma \in \Lambda, \quad \mathbf{1}_n(\gamma) := \begin{cases} 1 & \text{if } L(\gamma) = n, \\ 0 & \text{else.} \end{cases}$$

The following proposition shows that for integer-valued length functions, rapid decay on spheres implies radial rapid decay.

Proposition 3.1. — *Let Γ be a discrete group, and L an integer-valued proper length function on Γ. Then if*

$$\exists P \in \mathbb{R}[X], \quad \forall n \in \mathbb{N}, \quad \|\mathbf{1}_n\|_{\mathrm{op}} \leqslant P(n)\|\mathbf{1}_n\|_2,$$

then Γ has RRD with respect to L.

Proof. — Let $f \in \mathbb{C}[\Gamma]^{\mathrm{rad}}$. We have $f = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_n$ where a_n is the common value of f on elements of length n. Notice that for all but finitely many n, $a_n = 0$. Assuming there is a $P \in \mathbb{R}[X]$ as in the hypotheses, choose $Q \in \mathbb{R}[X]$ positive and non-decreasing on $\mathbb{R}_+$ such that $(1+t)^2(P(t))^2 \leqslant Q(t)$ for all $t \in \mathbb{R}_+$. Now, we have

$$\begin{aligned}
\|f\|_{\mathrm{op}} &\leqslant \sum_{n\in\mathbb{N}} \|a_n\mathbf{1}_n\|_{\mathrm{op}} \\
&\leqslant \sum_{n\in\mathbb{N}} P(n)\|a_n\mathbf{1}_n\|_2 \\
\text{(Cauchy–Schwarz inequality)} &\leqslant \left(\sum_{n\in\mathbb{N}}(1+n)^2(P(n))^2\|a_n\mathbf{1}_n\|_2^2\right)^{\frac{1}{2}}\left(\sum_{n\in\mathbb{N}}(1+n)^{-2}\right)^{\frac{1}{2}} \\
&\leqslant C\cdot\left(\sum_{n\in\mathbb{N}} Q(n)\|a_n\mathbf{1}_n\|_2^2\right)^{\frac{1}{2}} \\
&\leqslant C\cdot\sup\left(\{Q(n)\mid a_n\neq 0\}\right)\left(\sum_{n\in\mathbb{N}}\|a_n\mathbf{1}_n\|_2^2\right)^{\frac{1}{2}} \\
&= C\cdot Q(L(f))\left(\sum_{n\in\mathbb{N}}\|a_n\mathbf{1}_n\|_2^2\right)^{\frac{1}{2}} \\
&= C\cdot Q(L(f))\|f\|_2
\end{aligned}$$

so Γ has radial property RD with respect to L. □

We are now ready to prove Theorem 2.7. Let us recall that we are given (B,ν), a σ-finite probability space, on which Λ acts measurably, and we assume that ν is quasi-invariant. We denote by $\pi : \Gamma \to \mathcal{U}(L^2(B))$ the associated Koopman representation and by Ξ be the corresponding Harish-Chandra function and that we assume that there is $M \in \mathbb{R}^+$, $P \in \mathbb{R}[X]$, such that

(1) $\forall n \in \mathbb{N}, \sup_{\gamma\in C_n} \Xi(\gamma) \leqslant \dfrac{P(n)}{\sqrt{|C_n|}}$.

(2) $\forall n \in \mathbb{N}, \left\| \dfrac{1}{|C_n|}\sum_{\gamma\in C_n}\dfrac{\pi(\gamma)}{\Xi(\gamma)}\right\|_{2\to 2} \leqslant M.$

Proof of Theorem 2.7. — Let us first notice that $\|\mathbf{1}_{C_n}\|_2 = \sqrt{|C_n|}$. According to Proposition 3.1, it is enough to prove

$$\exists P \in \mathbb{R}[X], \forall n \in \mathbb{N}, \quad \|\mathbf{1}_{C_n}\|_{\mathrm{op}} \leqslant P(n)\sqrt{|C_n|}.$$

We will in fact prove

$$\exists P \in \mathbb{R}[X], \forall n \in \mathbb{N}, \quad \|\pi(\mathbf{1}_{C_n})\|_{2\to 2} \leqslant P(n)\sqrt{|C_n|}.$$

This is enough, because according to [23, Lemma 2.3] (we can apply this lemma, since its hypotheses are satisfied, because $\mathbf{1}_B$ is an obvious positive vector), we have that

$$\forall n \in \mathbb{N}, \|\mathit{1}_{C_n}\|_{\mathrm{op}} \leqslant \|\pi(\mathit{1}_{C_n})\|_{2\to 2}.$$

We claim that

$$\forall n \in \mathbb{N}, \left\| \frac{1}{|C_n|} \sum_{\gamma \in C_n} \pi(\gamma) \right\|_{2\to 2} \leqslant \sup_{\gamma \in C_n} \Xi(\gamma) \left\| \frac{1}{|C_n|} \sum_{\gamma \in C_n} \frac{\pi(\gamma)}{\Xi(\gamma)} \right\|_{2\to 2}.$$

Notice that the claim, combined with (1) and (2), ends the proof of the Theorem.

Let us now prove the claim. If $h \in L^2(B)$, denote by h_r and h_i its real and imaginary parts and let, $\forall a \in \{r, i\}$, $h_a^+ := \max(h_a, 0)$ and $h_a^- := h_a^+ - h_a$. Then $h_r^\pm$ and $h_i^\pm$ are all positive L^2 functions, and

$$\max\{\|h_r^+\|_2, \|h_r^-\|_2, \|h_i^+\|_2, \|h_i^-\|_2\} \leqslant \|h\|_2.$$

Let us denote, until the end of the proof,

$$M_n := \frac{1}{|C_n|} \sum_{\gamma \in C_n} \pi(\gamma)$$

and

$$M_n^\Xi := \sup_{\gamma \in C_n} \Xi(\gamma) \frac{1}{|C_n|} \sum_{\gamma \in C_n} \frac{\pi(\gamma)}{\Xi(\gamma)}.$$

We then have

$$\begin{aligned} \|M_n(h)\|_2^2 &= \|M_n(h_r^+)\|_2^2 + \|M_n(h_r^-)\|_2^2 + \|M_n(h_i^+)\|_2^2 + \|M_n(h_i^-)\|_2^2 \\ &\leqslant \|M_n^\Xi(h_r^+)\|_2^2 + \|M_n^\Xi(h_r^-)\|_2^2 + \|M_n^\Xi(h_i^+)\|_2^2 + \|M_n^\Xi(h_i^-)\|_2^2 \\ &= \|M_n^\Xi h\|_2^2 \end{aligned}$$

where the inequality comes from the fact that for every nonnegative function f, we have

$$0 \leqslant M_n(f) \leqslant M_n^\Xi(f).$$

This proves our claim. □

4. Proof of Theorem 2.8

In this section, we prove Theorem 2.8. Let us recall the context. Let G be a locally compact group with left Haar measure μ_G and L be a proper integer-valued length function on G, B be a compact space, ν be a Borel probability measure on B, and Λ be a lattice in G. Let us denote, for every $n \in \mathbb{N}$,

$$C_n^G := \{g \in G \mid L(g) = n\}$$

and

$$C_n^\Lambda := \{g \in G \mid L(g) = n\}.$$

We consider a continuous action of G on B which leaves ν quasi-invariant.

We make the following assumptions:

- the Radon–Nikodym cocycle and the Harish-Chandra function are continuous ;
- the compact subgroup G_0 acts transitively on B by preserving ν;
- at least one of the following two conditions holds:
  - (i) G_0 is open and we have the following growth condition:

    $$\mu_G(C_n^G) \ll |C_n^\Lambda|;$$

  - (ii) the subset G_1 of G consisting of elements of length at most 1 is a neighborhood of e and we have the following growth condition:

    $$\max\left\{\mu_G(C_{n-1}^G), \mu_G(C_n^G), \mu_G(C_{n+1}^G)\right\} \ll |C_n^\Lambda|.$$

Remark 4.1. — As for the proofs, the only difference between assuming (i) and assuming (ii) appears in the proof of Proposition 4.7, where assumption (i) simplifies the situation.

Under these assumptions, we are going to prove that there is a constant $M \in \mathbb{R}_+$ such that

$$\forall n \in \mathbb{N}, \left\| \frac{1}{|C_n^\Lambda|} \sum_{\gamma \in C_n^\Lambda} \frac{\pi(\gamma)}{\Xi(\gamma)} \right\|_{2\to 2} \leqslant M.$$

Let us define the averages on Λ and on G:

$$M_n^G := \frac{1}{\mu(C_n^G)} \int_{C_n^G} \frac{\pi(g)}{\Xi(g)} \, \mathrm{d}\mu_G(g) \in \mathcal{B}(L^2(B))$$

and

$$M_n^\Lambda := \frac{1}{|C_n^\Lambda|} \sum_{\gamma \in C_n^\Lambda} \frac{\pi(\gamma)}{\Xi(\gamma)} \in \mathcal{B}(L^2(B)).$$

The goal is to prove that the family $(M_n^\Lambda)_{n\in\mathbb{N}}$ is bounded in $\mathcal{B}(L^2(B))$ and the strategy is to prove the following chain of inequalities:

$$\|M_n^\Lambda\|_{2\to2} \overset{(1)}{\leqslant} \|M_n^\Lambda\|_{\infty\to\infty} \overset{(2)}{=} \|M_n^\Lambda \mathit{1}_B\|_\infty \overset{(3)}{\ll} \sum_{j=-1}^{1} \|M_{n+j}^G \mathit{1}_B\|_\infty \overset{(4)}{=} 3.$$

Remark 4.2. — As a careful examination of the proof of Proposition 4.7 shows, under assumption (i), it is in fact possible to strenghthen inequality (3) and equality (4) and prove

$$\|M_n^\Lambda \mathbf{1}_B\|_\infty \leqslant \|M_n^G \mathbf{1}_B\|_\infty = 1.$$

This fact will not be used in the paper.

Let us first state a version of the Riesz–Thorin theorem we need (a proof of the reduction of the lemma to the general Riesz–Thorin theorem can be found in [7, Proposition 2.8]).

Lemma 4.3. — *Let (X,m) be a probability space. Let T be a continuous operator $L^1(X,m)\to L^1(X,m)$ such that*

- *the restriction of T to $L^2(X,m)$ induces a continuous self-adjoint operator on $L^2(X,m)$;*
- *the restriction of T to $L^\infty(X,m)$ induces a continuous operator on $L^\infty(X,m)$.*

Then we have

$$\|T\|_{2\to2} \leqslant \|T\|_{\infty\to\infty}.$$

Lemma 4.4. — *We have that $\|M_n^\Lambda\|_{\infty\to\infty} = \|M_n^\Lambda \mathbf{1}_B\|_\infty$.*

Proof. — Let $h \in L^\infty(B)$. From the pointwise inequality (valid almost everywhere)

$$-\|h\|_\infty \mathbf{1}_B \leqslant h \leqslant \|h\|_\infty \mathbf{1}_B$$

we deduce the pointwise inequality (valid almost everywhere)

$$-\|h\|_\infty M_n^\Lambda \mathbf{1}_B \leqslant M_n^\Lambda h \leqslant \|h\|_\infty M_n^\Lambda \mathbf{1}_B.$$

So, we get $\|M_n^\Lambda h\|_\infty \leqslant \|h\|_\infty \|M_n^\Lambda \mathbf{1}_B\|_\infty$, so $\|M_n^\Lambda\|_{\infty\to\infty} \leqslant \|M_n^\Lambda \mathbf{1}_B\|_\infty$. □

The following elementary lemma states stability properties satisfied by the Radon–Nikodym cocycle and the Harish-Chandra function (which are well-known in the case of the Harish-Chandra function of a semi-simple Lie group).

Lemma 4.5. — *Let G be a locally compact group, acting on a probability space (B,ν) such that the Radon–Nikodym cocycles are continuous. Let us recall that Λ is a discrete subgroup of G. Then, there exist a relatively compact neighborhood U of e in G and a non-zero constant C such that*

(1) $\Lambda \cap U = \{e\}$;
(2) $\forall u \in U$, $\forall g \in G$, $\forall b \in B$,

$$\pi(g)\mathbf{1}_B(b) \leqslant C\pi(gu)\mathbf{1}_B(b) \ ;$$

(3) $\forall u \in U$, $\forall g \in G$,

$$\Xi(gu) \leqslant C\Xi(g).$$

Proof. — Let V be a symmetric compact neighborhood of e in G. Since c is continuous, it reaches its minimum C_1 and its maximum C_2 on the compact $V \times B$. Let $C := \sqrt{C_2}$, and let W be a symmetric neighborhood of the identity in G which trivially intersects Γ and take $U := V \cap W$. We now prove that C and U satisfy the required properties.

Let us recall the cocycle identity:

$$\forall g, h \in G,\ \forall b \in B, \quad c(gh, b) = c(g, hb)c(h, b).$$

Let $v \in V$ and $b \in B$. Since V is symmetric, $v^{-1} \in V$ and therefore, using the cocycle identity, we have that

$$\begin{aligned} 1 &= c(v^{-1}v, b) \\ &= c(v^{-1}, vb)c(v, b) \\ &\leqslant C_2 c(v, b) \end{aligned}$$

from which we deduce that $1 \leqslant C_1C_2$. One shows similarly that $C_1C_2 \leqslant 1$, so $C_1C_2 = 1$, and, in particular, that C_2, and, therefore, C, is non-zero. Moreover, for any $g \in G$, $v \in V$ and $b \in B$, we have that

$$C_1 c(g, b) \leqslant c(v, gb)c(g, b) = c(vg, b)$$

and, similarly,

$$c(vg, b) \leqslant C_2 c(g, b).$$

We now use the first of these two inequalities in order to prove (2). Let $v \in V$, $g \in G$ and $b \in B$. We have

$$\begin{aligned} \pi(g)\mathbf{1}_B(b) &= \sqrt{c(g^{-1}, b)} \\ &\leqslant \sqrt{C_2 c(u^{-1}g^{-1}, b)} \\ &= \sqrt{C_2 c((gu)^{-1}, b)} \\ &= \sqrt{C_2}\pi(gu)\mathbf{1}_B(b)) \\ &= C\pi(gu)\mathbf{1}_B(b)). \end{aligned}$$

The second inequality allows us to prove (3). Let $v \in V$, $g \in G$. We have that

$$\begin{aligned}\Xi(gv) &= \Xi(v^{-1}g^{-1}) \\ &= \int_B c(v^{-1}g^{-1}, b)^{\frac{1}{2}} \,\mathrm{d}\mu(b) \\ &\leqslant \int_B \sqrt{C_2} c(g^{-1}, b)^{\frac{1}{2}} \,\mathrm{d}\mu(b) \\ &= \sqrt{C_2}\Xi(g) \\ &= C\Xi(g). \end{aligned}$$ □

Proposition 4.6. — *There exists a relatively compact neighborhood U of e in G and a constant C such that for every finite subset $A \subset \Lambda$,*

$$\sum_{\gamma \in A} \frac{\pi(\gamma)\mathbf{1}_B}{\Xi(\gamma)} \leqslant C \int_{AU} \frac{\pi(g)\mathbf{1}_B}{\Xi(g)} \,\mathrm{d}\mu_G(g).$$

Proof. — Let U be a compact neighborhood of e in G and C be as in Lemma 4.5.

Let $\gamma \in \Lambda$ and $b \in B$. Then

$$\begin{aligned}\frac{\pi(\gamma)\mathbf{1}_B(b)}{\Xi(\gamma)} &= \frac{1}{\mu(U)} \int_U \frac{\pi(\gamma)\mathbf{1}_B(b)}{\Xi(\gamma)} \,\mathrm{d}u \\ &\leqslant \frac{1}{\mu(U)} \int C \frac{\pi(\gamma u)\mathbf{1}_B(b)}{\Xi(\gamma)} \,\mathrm{d}u \\ &\leqslant \frac{C}{\mu(U)} \int_U \frac{\pi(\gamma u)\mathbf{1}_B(b)}{\frac{\Xi(\gamma u)}{C}} \,\mathrm{d}u \\ &\leqslant \frac{C^2}{\mu(U)} \int_{\gamma U} \frac{\pi(u)\mathbf{1}_B(b)}{\Xi(u)} \,\mathrm{d}u \end{aligned}$$

Now, a summation over all $\gamma \in A$ ends the proof. □

Proposition 4.7. — *There exists $C \in \mathbb{R}^+$ such that for all sufficiently large n and for all $b \in B$, we have*

$$0 \leqslant M_n^\Lambda \mathbf{1}_B(b) \leqslant C \left(M_{n-1}^G \mathbf{1}_B(b) + M_n^G \mathbf{1}_B(b) + M_{n+1}^G \mathbf{1}_B(b)\right).$$

In particular, we have

$$\|M_n^\Lambda \mathbf{1}_B\|_\infty \ll \|M_{n-1}^G \mathbf{1}_B\|_\infty + \|M_n^G \mathbf{1}_B\|_\infty + \|M_{n+1}^G \mathbf{1}_B\|_\infty.$$

Proof. — Take a neighborhood U of e in G and C given by Proposition 4.6.

We now split the proof in two parts, one for each of the two assumptions, (i) and (ii).

- Under assumption (i), we can assume that $\forall g \in U$, $L(g) = 0$, and therefore, we have
$$C_n^\Lambda U \subseteq C_n^G$$
so, by Proposition 4.6, we have, for all n,
$$\begin{aligned}\sum_{\gamma \in C_n^\Lambda} \frac{\pi(\gamma)\mathbf{1}_B}{\Xi(\gamma)} &\leqslant C_1 \int_{C_n^G} \frac{\pi(g)\mathbf{1}_B}{\Xi(g)} \,\mathrm{d}\mu_G(g) \\ &\leqslant C_1 \mu_G(C_n^G) M_n^G \mathbf{1}_B(b).\end{aligned}$$
Now, we also know that there exists $C_2 \in \mathbb{R}$ such that for every sufficiently large n,
$$\mu_G(C_n^G) \leqslant C_2 |C_n^\Lambda|$$
so
$$M_n^\Lambda \mathbf{1}_B(b) = \frac{1}{|C_n^\Lambda|} \sum_{\gamma \in C_n^\Lambda} \frac{\pi(\gamma)\mathbf{1}_B}{\Xi(\gamma)} \leqslant C_1 C_2 M_n^G \mathbf{1}_B(b).$$
Since $M_{n-1}^G \mathbf{1}_B$ and $M_{n+1}^G \mathbf{1}_B$ are non-negative, the claim follows.
- Under assumption (ii), since G_1 is a neighborhood of e, we can assume that $\forall g \in U$, $L(g) \leqslant 1$. We have
$$C_n^\Lambda U \subseteq C_{n-1}^G \cup C_n^G \cup C_{n+1}^G,$$
so, by Proposition 4.6, we have, for all sufficiently large n,
$$\begin{aligned}\sum_{\gamma \in C_n^\Lambda} \frac{\pi(\gamma)\mathbf{1}_B}{\Xi(\gamma)} &\leqslant C_1 \int_{C_{n-1}^G \cup C_n^G \cup C_{n+1}^G} \frac{\pi(g)\mathbf{1}_B}{\Xi(g)} \,\mathrm{d}\mu_G(g) \\ &\leqslant C_1 \max\left\{\mu_G(C_{n-1}^G), \mu_G(C_n^G), \mu_G(C_{n+1}^G)\right\} \\ &\qquad \left(M_{n-1}^G \mathbf{1}_B(b) + M_n^G \mathbf{1}_B(b) + M_{n+1}^G \mathbf{1}_B(b)\right).\end{aligned}$$
Now, we assumed that there exists $C_2 > 0$ such that for all sufficiently large n,
$$\max\left\{\mu_G(C_{n-1}^G), \mu_G(C_n^G), \mu_G(C_{n+1}^G)\right\} \leqslant C_2 |C_n^\Lambda|,$$
so we have
$$\begin{aligned}M_n^\Lambda \mathbf{1}_B(b) &= \frac{1}{|C_n^\Lambda|} \sum_{\gamma \in C_n^\Lambda} \frac{\pi(\gamma)\mathbf{1}_B}{\Xi(\gamma)} \\ &\leqslant C_1 C_2 \left(M_{n-1}^G \mathbf{1}_B(b) + M_n^G \mathbf{1}_B(b) + M_{n+1}^G \mathbf{1}_B(b)\right)\end{aligned}$$
which gives the claim. □

Lemma 4.8. — *The function $M_n^G \mathbf{1}_B$ is constant and equal to 1.*

Proof. — Let $k \in G_0$. First of all, let us notice that we have, for all $b \in B$,

$$M_n^G \mathbf{1}_B(b) = M_n^G \mathbf{1}_B(kb).$$

To prove it, let us make the following computation using the cocycle property and the fact that G_0 preserves μ, so that $c(k^{-1}, \cdot) \equiv 1$:

$$\begin{aligned}
M_n^G(\mathbf{1}_B)(kb) &= \int_{C_n^G} \left(\pi(g)\mathbf{1}_B\right)(kb)\,\mathrm{d}\mu_G(g) \\
&= \int_{C_n^G} c(g^{-1}, kb)^{\frac{1}{2}} \mathbf{1}_B(kb)\,\mathrm{d}\mu_G(g) \\
&\overset{h=k^{-1}g}{=} \int_{C_n^G} c(h^{-1}k^{-1}, kb)^{\frac{1}{2}} \mathbf{1}_B(h^{-1}b)\,\mathrm{d}\mu_G(g) \\
&= \int_{C_n^G} c(h^{-1}, k^{-1}kb)^{\frac{1}{2}} c(k^{-1}, b)^{\frac{1}{2}} \mathbf{1}_B(h^{-1}b)\,\mathrm{d}\mu_G(g) \\
&= \int_{C_n^G} c(h^{-1}, b)^{\frac{1}{2}} \mathbf{1}_B(h^{-1}b)\,\mathrm{d}\mu_G(g) \\
&= M_n^G(\mathbf{1}_B)(b).
\end{aligned}$$

Moreover, recall that, by assumption, the action of G_0 on B is transitive, so the function $M_n^G \mathbf{1}_B$ is constant and equal to $c \in \mathbb{R}$.

We integrate and use Fubini's theorem to prove that c is in fact 1:

$$\begin{aligned}
c &= \int_B M_n^G \mathbf{1}_B \\
&= \frac{1}{\mu(C_n^G)} \int_B \int_{C_n^G} \frac{c(g^{-1}, b)^{\frac{1}{2}}}{\Xi(g)} \mathbf{1}_B(g^{-1}b)\,\mathrm{d}\mu_G(g)\,\mathrm{d}\mu(b) \\
&= \frac{1}{\mu(C_n^G)} \int_{C_n^G} \int_B \frac{c(g^{-1}, b)^{\frac{1}{2}}}{\Xi(g)}\,\mathrm{d}\mu(b)\,\mathrm{d}\mu_G(g) \\
&= \frac{1}{\mu(C_n^G)} \int_{C_n^G} \frac{\Xi(g^{-1})}{\Xi(g)}\,\mathrm{d}\mu_G(g) \\
&= 1
\end{aligned}$$

□

Proof of Theorem 2.8. — Let us recall the chain of inequalities we want to prove:

$$\|M_n^\Lambda\|_{2\to 2} \overset{(1)}{\leqslant} \|M_n^\Lambda\|_{\infty\to\infty} \overset{(2)}{=} \|M_n^\Lambda 1_B\|_\infty \overset{(3)}{\ll} \sum_{j=-1}^{1} \|M_{n+j}^G 1_B\|_\infty \overset{(4)}{=} 3.$$

Inequality (1) is a simple application of the Riesz–Thorin theorem (Lemma 4.3). Equality (2) is proved in Lemma 4.4. Inequality (3) is proved in Proposition 4.7 and equality (4) is proved in Lemma 4.8. □

5. Proof of Corollary 2.10

5.1. Outline of the section

This section is devoted to the proof of Corollary 2.10. Precisely, let us recall that in Section 2.3.1, we defined a ring A, fields $\mathbb{K}_0$ et $\mathbb{K}_\infty$, two $(q+1)$-regular trees T_0 and T_∞, a distance $d_{\mathbb{I}}$ on the product of these trees, an action of $G \coloneqq \mathrm{SL}_2(\mathbb{K}_0) \times \mathrm{SL}_2(\mathbb{K}_\infty)$ on $T_0 \times T_\infty$, a length function L on G, and Γ to be the image, under the diagonal embedding, of $\mathrm{SL}_2(A)$ in G. Corollary 2.10 asserts that Γ has RRD with respect to L.

In order to prove Corollary 2.10, we just need to check that, in this particular setting, the two hypotheses of the dynamical criterion stated in Theorem 2.7 (the Harish-Chandra estimates condition and the uniform boundedness condition) are satisfied. To that end, we build a probability space on which Γ acts in Section 5.2, we perform growth estimates on Γ we need in Section 5.3, we perform estimates on the Harish-Chandra function in Section 5.4, and collect all the results and give the proof of Corollary 2.9 in Section 5.5.

5.2. The Koopman representation on the product of the boundaries

Here we recall the construction of the boundary of a tree in order to build a useful compact space B on which G acts and endow it with a quasi-invariant Borel probability measure.

Let T be a d-regular tree. Let us recall a few facts from the theory of CAT(-1) spaces and measures at infinity (see [4] for details).

Let us denote by ∂T the set of equivalence classes of asymptotic rays in T (the equivalence class of r is denoted by $r(+\infty)$). Fixing a point $x \in T$, we can consider $\partial_x T$, the set of geodesic rays starting at x. The quotient map $\partial_x T \to \partial T$ can be shown to be a bijection. The image of the topology of uniform convergence on compact sets on $\partial_x T$ is a topology on ∂T that, in fact, does not depend on x. For this topology, ∂T is homeomorphic to a Cantor set.

Now, the set $\overline{T} = T \cup \partial T$ can be endowed with a natural topology which makes $\overline{T}$ a compactification of T and induces on ∂T the topology defined above. This compactification has the important property that every isometry γ of T extends to a homeomorphism of $\overline{T}$, the restriction of which to ∂T being denoted by $\partial\gamma$.

If $x, y, z \in T$, we denote $(y \mid z)_x := \frac{1}{2}\left(d(x,y) + d(x,z) - d(y,z)\right)$ and we call it the *Gromov product*. If $x \in T$, then $(. \mid .)_x$ can be extended in a continuous manner to $\overline{T}^2$, which we again call the Gromov product, and if we set, for all $b, b' \in \partial T$, $d_x(b,b') := e^{-(b|b')_x}$, then d_x is a distance on ∂T which also induces the topology defined above. However, the distances d_x do depend on x, but in a conformal manner. That is, we have

$$\lim_{b' \to b} \frac{d_y(b,b')}{d_x(b,b')} = e^{\beta_b(x,y)}$$

for a certain number $\beta_b(x,y)$ defined in the following way: if r is a geodesic ray in T, then we denote by $\beta_r(x,y)$ the limit of $d(x,r(t)) - d(y,r(t))$ as t tends to infinity, which exists and only depends on $r(+\infty)$. We call it the *horospheric distance* between x and y with respect to $r(+\infty)$.

We can now define measures on ∂X. The Hausdorff dimension of $(\partial T, d_x)$ can easily be calculated, and is $\ln(d-1)$. Moreover, if we denote by μ_x the normalized $\ln(d-1)$-dimensional Hausdorff measure, isotropy around x implies that $\mu_x(B) = \frac{1}{d(d-1)^{i-1}}$ if B is a ball of radius e^{-i} for the distance d_x. The map

$$\begin{aligned} \mu : T &\to \mathcal{M}^1(\partial T) \\ x &\mapsto \mu_x \end{aligned}$$

is $\mathrm{Isom}(T)$ equivariant in the sense that $\forall \gamma \in \mathrm{Isom}(T)$, $(\partial\gamma)_*\mu_x = \mu_{\gamma(x)}$. Adding everything up, we get:

Fact. — *The action* $\mathrm{Isom}(T) \curvearrowright (\partial T, \mu_{x_0})$ *is a quasi-invariant action, and we have the formula*

$$\forall \gamma \in \mathrm{Isom}(T), \quad \forall b \in \partial T, \quad \frac{\mathrm{d}(\partial\gamma)_*\mu_{x_0}}{\mathrm{d}\mu_{x_0}}(b) = (d-1)^{\beta_b(x_0, \gamma^{-1}(x_0))} =: c_T(\gamma, b).$$

Let us recall that $G := \mathrm{SL}_2(\mathbb{K}_0) \times \mathrm{SL}_2(\mathbb{K}_\infty)$ acts componentwise on the product $T_0 \times T_\infty$ of the Bruhat–Tits trees, which are $(q+1)$-regular. Now, fix two vertices v_0 and v_∞ in T_0 and T_∞ and consider the product action $G \curvearrowright (\partial T_0 \times \partial T_\infty, \mu_{v_0} \otimes \mu_{v_\infty})$.

From the above discussion, the following proposition is obvious.

Proposition 5.1. — *In this setting, the product measure $\mu_{v_0} \otimes \mu_{v_\infty}$ is quasi-invariant under the action $G \curvearrowright \partial T_0 \times \partial T_\infty$, and the Radon–Nikodym cocycle is continuous.*

Let us denote π the Koopman representation associated to this action, and Ξ the Harish-Chandra function.

5.3. Estimates on the growth of Γ

In this section, we provide the estimates on $|C_n^\Gamma|$ we need. To do so, it is useful to estimate the cardinal of sets of vertices inside balls in a product of two trees. We keep notation from Section 2.3.1, so G_n denotes the ball of radius n (that is, the set of elements in G of length at most n with respect to L defined in Section 2.3.1) whereas C_n^H denotes similarly the sphere of radius n in the subgroup H.

We will now compute the number of elements of

$$B_n \coloneqq \{(x,y) \in V(T_0) \times V(T_\infty) \mid d_{\mathbb{I}}\left((v_0, v_\infty),(x,y)\right) \leqslant n\}.$$

If $a \in \{0,\infty\}$, $i \in \mathbb{N}$, $x \in T_a$, let us denote $S_a(x,i) \coloneqq \{y \in T_a \mid d_a(x,y) = i\}$ and $B_a(x,i) \coloneqq \bigcup_{j=0,\dots,i} S_a(x,j)$.

Lemma 5.2 (Ball counting). — *There are $A, B, C \in \mathbb{R}$ such that $A \neq 0$ and*

$$\forall n \in \mathbb{N}, \quad |B_n| = (An+B)\,(d-1)^n + C.$$

Proof. — To make the calculation more readable, we set $D \coloneqq \frac{d}{d-2}$. We first observe that

$$B_n = \bigsqcup_{i=0}^{n} \bigsqcup_{x \in S_0(v_0,i)} \{x\} \times B_\infty(v_\infty, n-i)$$

so that $|B_n| = \sum_{i=0}^n s_i b_{n-i}$ where we denote, for $i,j \in \mathbb{N}$, $s_i \coloneqq |S_0(v_0,i)|$ and $b_j \coloneqq |B_\infty(v_\infty, j)|$.

We have that $\forall i,j \in \mathbb{N}$,

$$\forall i \in \mathbb{N}, \quad s_i = \begin{cases} d(d-1)^{i-1} & \text{if } i \geqslant 1, \\ 1 & \text{if } i = 0, \end{cases}$$

$$\forall j \in \mathbb{N}, \quad b_j = \sum_{i=0}^{j} s_i = 1 + D\left((d-1)^j - 1\right)$$

$$\forall i \geqslant 1, \quad s_i b_{n-i} = D\left(d(d-1)^{n-1} - 2(d-1)^{i-1}\right)$$

and we get, $\forall n \in \mathbb{N}$,

$$\begin{aligned}|B_n| &= \sum_{i=0}^{n} s_i b_{n-i} \\ &= b_n + \sum_{i=1}^{n} s_i b_{n-i} \\ &= 1 + D((d-1)^n - 1) + \sum_{i=1}^{n} D\left[d(d-1)^{n-1} - 2(d-1)^{i-1}\right] \\ &= 1 - D + D(d-1)^n + Ddn(d-1)^{n-1} - 2D\sum_{i=1}^{n}(d-1)^{i-1} \\ &= 1 - D + D(d-1)^n + Ddn(d-1)^{n-1} - \frac{2D}{d-2}(d-1)^n + \frac{2D}{d-2} \\ &= (d-1)^n\left[n\frac{Dd}{d-1} + D - \frac{2D}{d-2}\right] + 1 - D + \frac{2D}{d-2}\end{aligned}$$

so we choose

$$\begin{aligned}A &\coloneqq \frac{Dd}{d-1} = \frac{d^2}{(d-2)(d-1)} \\ B &\coloneqq D - \frac{2D}{d-2} = \frac{d(d-4)}{(d-2)^2} \\ C &\coloneqq 1 - D + \frac{2D}{d-2} = 1 - B.\end{aligned}$$ □

Proposition 5.3. — *We have that*

$$\mu(G_n) \ll n(d-1)^n.$$

Proof. — Consider the map

$$\begin{aligned}\theta : G &\to T_0 \times T_\infty \\ g &\mapsto g(v_0, v_\infty).\end{aligned}$$

Then

$$G_n = \theta^{-1}(B_n) = \bigsqcup_{y \in B_n} \theta^{-1}(\{y\}).$$

Each of these fibers, if it is nonempty, is a G_0-left coset, so it has measure $\mu_G(G_0)$. So, according to Lemma 5.2,

$$\mu_G(G_n) \leqslant |B_n|\mu_G(G_0) \ll n(d-1)^n.$$ □

Proposition 5.4. — *We have*

$$\mu\left(C_n^G\right) \ll \left|C_n^\Gamma\right|.$$

Proof. — Let us recall that Γ is an irreducible lattice in G, according to [18, p. 1]. Since the action of G on G/Γ is mixing, the mean ergodic theorem holds, and therefore, we can apply [13, Lemma 6.7, p. 79]. □

5.4. Estimates on the Harish-Chandra function

In order to apply the criterion, we need to compute the Harish-Chandra function associated to the quasi-invariant action $\Gamma \curvearrowright \partial T_0 \times \partial T_\infty$. Since this is a product action, is is enough to calculate the Harish-Chandra functions on the factors:

Lemma 5.5. — *The Harish-Chandra function of a product of actions is the product of the Harish-Chandra functions on the factors. In particular,*

$$\forall (g_0, g_\infty) \in G,$$
$$\Xi(g_0, g_\infty) = \int_{\partial T_0} c_{T_0}(g_0^{-1}, b)^{\frac{1}{2}} \,\mathrm{d}\mu_{v_0}(b) \int_{\partial T_\infty} c_{T_\infty}(g_\infty^{-1}, b)^{\frac{1}{2}} \,\mathrm{d}\mu_{v_\infty}(b).$$

Proof. — Just apply Fubini's theorem. □

Our goal is now to compute $\int_{\partial T} c_T(\gamma^{-1}, b)^{\frac{1}{2}} \,\mathrm{d}\mu_{x_0}(b)$ for T a d-regular tree, $\gamma \in \mathrm{Isom}(T)$, x_0 a vertex of T and μ_{x_0} the boundary measure on ∂T associated to x_0. As we shall see, $b \mapsto c_T(\gamma, b)$ is piecewise constant, so we will suitably partition ∂T.

Let us define $S_{n,\gamma} \coloneqq \{y \in T_d \mid d(x_0, y) = n,\ \ l([x_0, y] \cap [x_0, \gamma^{-1}(x_0)]) = l([x_0, y]) - 1\}$, and, for $y \in T$, $\mathcal{O}_y \coloneqq \{\xi \in \partial T \mid y \in [x_0, \xi)\}$.

Lemma 5.6. — *With the above notation, let $n \coloneqq d(x_0, \gamma^{-1}(x_0))$. The following properties hold true.*

(1) *Assume $i < n$, $y \in S_{i,\gamma}$, $b \in \mathcal{O}_y$, $b' \in \partial T$. Then*
$$b' \in \mathcal{O}_y \Leftrightarrow (b|b')_{x_0} > i - 1 \Leftrightarrow d_{x_0}(b, b') < e^{-i+1}.$$
(2) $\partial T = \mathcal{O}_{\gamma^{-1}(x_0)} \sqcup \bigsqcup_{i=1}^{n} \left(\bigsqcup_{y \in S_{i,\gamma}} \mathcal{O}_y \right)$,
(3) $\forall i \in \{2, ..., n\}$, $|S_{i,\gamma(x_0)}| = d - 2$, *and* $|S_{1,\gamma(x_0)}| = d - 1$,
(4) $\forall i \in \{1, ..., n\}, \forall y \in S_{i,\gamma(x_0)}, \forall \xi \in \mathcal{O}_y, \beta_\xi(x_0, \gamma^{-1}(x_0)) = 2(i-1) - n$,
(5) $\forall \xi \in \mathcal{O}_{\gamma^{-1}(x_0)}$, $\beta_\xi(x_0, \gamma^{-1}(x_0)) = n$.

Proof.

(1) We have that $b' \notin \mathcal{O}(y)$ if and only if $l([x_0, b') \cap [x_0, b)) \leqslant l([x_0, y]) - 1 = i - 1$, which proves (1).

(2) The sets in the union are clearly disjoint, so it is enough to show that ∂T is the mentioned union. Let $b \in \partial T$, and $r : \mathbb{R}_+ \to T$ be the geodesic joining x_0 to b. Let

$$t \coloneqq \max\{t \in \mathbb{R}_+ \mid r(t) \in [x_0, \gamma^{-1}(x_0)]\}.$$

Then $y \coloneqq r(t+1) \in S_{t+1,\gamma}$ and $b \in \mathcal{O}_y$.

(3) (4) and (5) are straightforward. □

Proposition 5.7. — *Let $\gamma \in \mathrm{Isom}(T_d)$ and denote $n \coloneqq d(x_0, \gamma(x_0))$. Let $q = d-1$. We have*

$$\int_{\partial T} c(\gamma, b)^{\frac{1}{2}}\, \mathrm{d}\mu(b) = \left(1 + \frac{q-1}{q+1}\, n\right) q^{-\frac{n}{2}}.$$

Proof. — Using the information collected in the above lemma, we do the following calculation:

$$\begin{aligned}
&\int_{\partial T} c(\gamma, b)^{\frac{1}{2}}\, \mathrm{d}\mu(b) \\
&\quad = \int_{\mathcal{O}_{\gamma^{-1}x_0}} c(\gamma, b)^{\frac{1}{2}}\, \mathrm{d}\mu(b) + \sum_{i=1}^{n} \sum_{y \in S_{i,\gamma}} \int_{\mathcal{O}_y} c(\gamma, b)^{\frac{1}{2}}\, \mathrm{d}\mu(b) \\
&\quad = \mu\left(\mathcal{O}_{\gamma^{-1}x_0}\right)(d-1)^{\frac{n}{2}} + \sum_{i=1}^{n} \sum_{y \in S_{i,\gamma}} \mu\left(\mathcal{O}_y\right)(d-1)^{i-1-\frac{n}{2}} \\
&\quad = \frac{1}{d(d-1)^{n-1}}(d-1)^{\frac{n}{2}} + \sum_{i=1}^{n} \sum_{y \in S_{i,\gamma}} \frac{1}{d(d-1)^{i-1}}(d-1)^{i-1-\frac{n}{2}} \\
&\quad = \frac{1}{d}\left[(d-1)^{-\frac{n}{2}+1} + \sum_{i=1}^{n} \sum_{y \in S_{i,\gamma}} (d-1)^{-\frac{n}{2}}\right] \\
&\quad = \frac{1}{d}(d-1)^{-\frac{n}{2}}\left[d-1 + \sum_{i=1}^{n} |S_{i,\gamma}|\right] \\
&\quad = \frac{1}{d}(d-1)^{-\frac{n}{2}}\left[d-1+d-1+(n-1)(d-2)\right] \\
&\quad = \left(1 + \frac{d-2}{d} n\right)(d-1)^{-\frac{n}{2}}
\end{aligned}$$

□

Proposition 5.8 (The Harish-Chandra estimate). — *Let Ξ be the Harish-Chandra function Ξ associated to the action $G \curvearrowright (\partial T_0 \times \partial T_\infty, \mu_{x_0} \otimes \mu_{x_\infty})$. We then have*

(1) *for all* $g := (g_0, g_\infty) \in G$,

$$\Xi(g) = \left(1 + \frac{q-1}{q+1} L(g) + \left(\frac{q-1}{q+1}\right)^2 L_0(g_0) L_\infty(g_\infty)\right) q^{-\frac{L(g)}{2}};$$

(2) $\Xi : G \to \mathbb{R}$ *is continuous ;*

(3) *we have the following estimate: for all* $g \in G$ *(and therefore, for every* $g \in \Gamma$*),*

$$\Xi(g) \ll L(g)^2 (d-1)^{-\frac{L(g)}{2}}.$$

Proof. — The last two claims follow immediately from the first. We apply together Proposition 5.7 and Lemma 5.5: let $g := (g_0, g_\infty) \in G$. We then have

$$\begin{aligned}\Xi(g) &= \left(1 + \frac{q-1}{q+1} L_0(g_0)\right) q^{-\frac{L_0(g_0)}{2}} \left(1 + \frac{q-1}{q+1} L_\infty(g_\infty)\right) q^{-\frac{L_\infty(g_\infty)}{2}} \\ &= \left(1 + \frac{q-1}{q+1}(L_0(g_0) + L_\infty(g_\infty)) + \left(\frac{q-1}{q+1}\right)^2 L_0(g_0) L_\infty(g_\infty)\right) q^{-\frac{L(g)}{2}}. \square\end{aligned}$$

Using the above proposition, we can now perform the estimates needed in order to apply the criterion.

Proposition 5.9. — *We have that*

$$\sup_{\gamma \in C_n^\Gamma} \Xi(\gamma) \sqrt{|C_n^\Gamma|} \ll n^{5/2}.$$

Proof. — On the one hand, we apply Proposition 5.8 and we obtain:

$$\sup_{\gamma \in C_n} \Xi(\gamma) \leqslant \left(1 + \frac{d-2}{d}\, n + \frac{1}{2}\left(\frac{d-2}{d}\right)^2 n^2\right) (d-1)^{-\frac{n}{2}}.$$

Therefore, we have

$$\sup_{\gamma \in C_n} \Xi(\gamma) \ll n^2 (d-1)^{-\frac{n}{2}}.$$

On the other hand, since $C_n^\Gamma \subset \Gamma_n \subset G_n$, applying Lemma 5.2, we obtain (recall that $q = d-1$)

$$\sqrt{|C_n^\Gamma|} \ll \sqrt{n(d-1)^n} = n^{\frac{1}{2}} (d-1)^{\frac{n}{2}}.$$

So, we have,

$$\sup_{\gamma \in C_n^\Gamma} \Xi(\gamma) \sqrt{|C_n^\Gamma|} \ll n^{5/2}. \qquad \square$$

5.5. Proof of Corollary 2.10

In this section, we gather all the results of the section in order to prove Corollary 2.10.

Proof of Corollary 2.10. — Let B be the compact space, ν be the Borel probability measure on B, defined in Section 5.2, which also defines a measurable action $G \curvearrowright B$ leaving ν-invariant. Let us denote by π the Koopman representation of G associated to this action.

Proposition 5.9 asserts that there is a polynomial P such that

$$\sup_{\gamma \in C_n^\Gamma} \Xi(\gamma)\sqrt{|C_n^\Gamma|} \leqslant P(n),$$

so the first condition in the dynamical criterion is satisfied.

In order to prove that the second condition is also satisfied, we use Theorem 2.8; and, in order to do so, let us check its hypotheses: according to Proposition 5.1 and Proposition 5.8, the Radon–Nikodym and the Harish-Chandra function are continuous; finally, we check assumption (i): the subgroup G_0 is open, and according to Proposition 5.3, the growth estimate assumption is satisfied. We deduce then from Theorem 2.8 that

$$\left\| \frac{1}{|C_n^\Gamma|} \sum_{\gamma \in C_n^\Gamma} \frac{\pi(\gamma)}{\Xi(\gamma)} \right\|_{2\to 2} \ll 1.$$

Therefore, the two assumptions of the dynamical criterion are satisfied; so Γ has RRD with respect to L. ☐

6. Proof of Corollary 2.11

6.1. RD and amenable subgroups of exponential growth

Let us recall three easy lemmas, proved in [12], which hold for any finitely generated group Λ:

Lemma 6.1.

(1) *If a finitely generated group Λ has RD with respect to some proper, discrete length function, then it has RD with respect to the word length associated with any finite symmetric generating set.*
(2) *If a discrete group Λ has RD with respect to some proper, discrete length function L, then each subgroup $H \leqslant \Lambda$ has RD with respect to the induced length function $L_{|_H}$.*

(3) *If an amenable finitely-generated group has RRD with respect to some proper, discrete length function L, then it has polynomial growth with respect to L.*

These three lemmas are combined in the following criterion, useful to prove that some discrete groups do not have RD:

Proposition 6.2. — *Let Λ be a discrete group endowed with a discrete, proper length function L. Let H be an amenable finitely-generated subgroup of Λ. Then if H has exponential growth with respect to $L_{|H}$ (and this is the case if H has exponential growth with respect to a word length), then Λ does not have RD with respect to L.*

6.2. The lamplighter subgroup

Let us consider the subgroup

$$H := \left\{ \begin{pmatrix} X^n & P \\ 0 & X^{-n} \end{pmatrix} \,\middle|\, n \in \mathbb{Z}, \quad P \in A \right\}$$

of $\mathrm{SL}_2(A)$ and let S denote the finite subset of H

$$\left\{ \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix}, \begin{pmatrix} X^{-1} & 0 \\ 0 & X \end{pmatrix}, \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pm X \\ 0 & 1 \end{pmatrix} \right\}.$$

The following proposition is a routine exercise for people working in geometric group theory. We give the proof for readers with a different background.

Proposition 6.3. — *The subgroup H is amenable, S is a symmetric generating set of H, and H has exponential growth with respect to the word-length associated to S.*

Proof. — If $P \in \mathbb{F}_q[X, X^{-1}]$, define

$$\gamma(P) := \begin{pmatrix} 1 & P \\ 0 & 1 \end{pmatrix}$$

so that $\gamma : \mathbb{F}_q[X, X^{-1}] \to H$ is a morphism. Define also

$$\psi\left(\begin{pmatrix} X^n & P \\ 0 & X^{-n} \end{pmatrix} \right) := n$$

so that $\psi : H \to \mathbb{Z}$ is a morphism. Then

$$0 \longrightarrow \mathbb{F}_q[X, X^{-1}] \xrightarrow{\gamma} H \xrightarrow{\psi} \mathbb{Z} \longrightarrow 0$$

is a short exact sequence so H is solvable, hence amenable.

Now let us prove that H has exponential growth with respect to the word-length associated to S. Let $n \in \mathbb{N}$, and $P := \sum_{i=0}^{n} a_i X^{2i}$, where $a_i \in \{0,1\}$. There are 2^{n+1} such P, and we will prove that every $\gamma(P)$ can be written as a product of $3n+1$ (or less) elements of S.

To do so, define

$$A_0 := \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix}$$

$$A_{j+1} := \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} A_j \begin{pmatrix} X^{-1} & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} 1 & a_{n-(j+1)} \\ 0 & 1 \end{pmatrix}$$

It is straightforward to see that $A_n = \gamma(P)$, and by definition, A_n is the product of (at most) $3n+1$ elements of S. $\square$

Remark 6.4. — Such a subgroup H is a variant of the usual lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$.

We now prove Corollary 2.11.

Proof. — Proposition 6.2 shows that if (Γ, L) has the property RD, then it cannot contain an amenable finitely-generated exponential growth subgroup, and Proposition 6.3 shows that H is such a subgroup. $\square$

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Adrien BOYER
Université Paris 7 (France)
aadrien.boyer@gmail.com

Antoine PINOCHET LOBOS
I2M, CNRS UMR7373, Université d’Aix-Marseille (France)
antoine.pinochet-lobos@univ-amu.fr

Christophe PITTET
I2M, CNRS UMR7373, Université d'Aix-Marseille (France)
Section de Mathématiques, Faculté des Sciences, Université de Genève (Switzerland)
pittet@math.cnrs.fr