Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

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Tome 73, n° 4 (2023), p. 1385-1420.

https://doi.org/10.5802/aif.3540

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
73, 4 (2023) 1385-1420

E-SERIES OF CHARACTER VARIETIES OF NON-ORIENTABLE SURFACES

by Emmanuel LETELLIER & Fernando RODRIGUEZ-VILLEGAS

Abstract. — In this paper we are interested in two kinds of stacks associated to a compact non-orientable surface Σ. (A) We consider simply the quotient stack of the space of representations of the fundamental group of Σ to GL_n. (B) We choose a set S of k-punctures of Σ and a generic k-tuple of semisimple conjugacy classes of GL_n, and we consider the stack of anti-invariant local systems on the orientation cover of Σ with local monodromies around the punctures given by the prescribed conjugacy classes. We compute the number of points of these spaces over finite fields from which we get a formula for their E-series (a certain specialization of the mixed Poincaré series). In case (B), unexpectedly (see Remark 1.9), when the Euler characteristic of Σ is even, our formulas turn out to be closely related to those arising from the character varieties of punctured compact orientable surfaces studied in [13] and [14].

Résumé. — Dans cet article nous nous intéressons à deux types de champs associés à une surface compacte non-orientable Σ. (A) On considère simplement le champ quotient de l'espace des représentations du groupe fondamental de Σ dans GL_n. (B) On choisit un ensemble S de k points de Σ et un k-uplet générique de classes de conjugaison semsimples de GL_n et on considère le champ des systèmes locaux anti-invariants sur le revêtement d'orientation de $\Sigma\backslash S$ avec monodromies locales dans les classes de conjugaison choisies. On calcule le nombre de points de ces champs sur un corps fini et on en déduit une formule pour leur E-série (une certaine spécialisation de la série de Poincaré mixte). Dans le cas (B), étonnamment (voir Remarque 1.9), lorsque la caractéristique d'Euler de Σ est paire, nos formules sont très proches de celles provenant des variétés de caractères de surfaces compactes orientables épointées étudiées dans [13] et [14].

1. Introduction

Let K an algebraically closed field of characteristic $\neq 2$.

We let r be a non-negative integer, put $\varrho = r - 2$, and let Σ denote a non-orientable compact surface of Euler characteristic $-\varrho$ (the connected

Keywords: Character varieties, non-orientable surfaces.
2020 *Mathematics Subject Classification:* 00X99.

sum of r real projective planes). Let $S=\{\alpha_1,\dots,\alpha_k\}$ be a set of k points of Σ. Fix a base point $b\in\Sigma\backslash S$. The fundamental group $\Pi=\pi_1(\Sigma\backslash S,b)$ has the following standard presentation

$$\Pi=\left\langle a_1,\dots,a_r,x_1,\dots,x_k\,\middle|\,a_1^2\cdots a_r^2x_1\cdots x_k=1\right\rangle$$

Here x_i represents a small appropriately oriented loop encircling the i-th puncture.

Put $\mathrm{G}=\mathrm{GL}_n(K)$ and let $\sigma:\mathrm{G}\to\mathrm{G}$ be the Cartan involution $g\mapsto {}^tg^{-1}$. We let $\mathrm{G}^+=\mathrm{G}\rtimes\langle\sigma\rangle$ be the corresponding semi-direct product. Fix a k-tuple $\mathcal{C}=(C_1,\dots,C_k)$ of conjugacy classes of G and for $\varepsilon\in\langle\sigma\rangle$ consider the representation variety $\mathrm{Hom}^\varepsilon_{\mathcal{C}}(\Pi,\mathrm{G}^+)$ of $\rho\in\mathrm{Hom}(\Pi,\mathrm{G}^+)$ such that

$$\pi\circ\rho(a_i)=\varepsilon,\qquad\text{and}\qquad\rho(x_j)\in\iota(C_j)$$

for all $i=1,\dots,r$ and $j=1,\dots,k$, where $\pi:\mathrm{G}^+\to\langle\sigma\rangle$ is the quotient map and $\iota:G\to G^+$ the natural inclusion.

The conjugation action of G on G^+ induces an action on $\mathrm{Hom}^\varepsilon_{\mathcal{C}}(\Pi,\mathrm{G}^+)$ and we consider the quotient stack

$$\mathcal{M}^\varepsilon_{\mathcal{C}}:=\left[\mathrm{Hom}^\varepsilon_{\mathcal{C}}(\Pi,\mathrm{G}^+)/\mathrm{G}\right].$$

If $\varepsilon=1$ then $\mathrm{Hom}^\varepsilon_{\mathcal{C}}(\Pi,\mathrm{G}^+)$ is the space

$$\mathrm{Hom}_{\mathcal{C}}(\Pi,\mathrm{G})=\{\rho\in\mathrm{Hom}(\Pi,\mathrm{G})\mid\rho(x_i)\in C_i, i=1,\dots,k\}$$

which has the following explicit description

$$\left\{(A_1,\dots,A_r,X_1,\dots,X_k)\in\mathrm{G}^r\times\prod_{i=1}^kC_i\,\middle|\,A_1^2\cdots A_r^2\,X_1\cdots X_k=1\right\}.$$

On the other hand, if $\varepsilon=\sigma$ then $\mathrm{Hom}^\varepsilon_{\mathcal{C}}(\Pi,\mathrm{G}^+)$ is the set of

$$(A_1,\dots,A_r,X_1,\dots,X_k)\in\mathrm{G}^r\times\prod_{i=1}^kC_i$$

satisfying the equation

$$A_1\sigma(A_1)\cdots A_r\sigma(A_r)X_1\cdots X_k=1.$$

The last space has an interpretation in terms of anti-invariant representations $\widetilde{\Pi}\to\mathrm{G}$ of the fundamental group $\widetilde{\Pi}$ of $\widetilde{\Sigma}$, where $p:\widetilde{\Sigma}\to\Sigma$ is the orientation covering (see Section 4.1 for details). As it will become clear below, this space is much better behaved than the previous one.

In this paper we consider the following two cases :

(A) $\varepsilon=1$ and $S=\emptyset$.

(B) $\varepsilon = \sigma$, the k-tuple $\mathcal{C} = (C_1, \dots, C_k)$ is *generic* and the C_i are semisimple.

We prove that the corresponding stacks $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ are *rational count*. In other words, the number of points over generic finite fields is given by a fixed rational function in the size of the field (see Definition 2.6 for the formal definition). As in the case of polynomial count (see [15, Appendix]) the corresponding counting rational function has a geometric meaning. Indeed, by Theorem 2.9 it coincides with the E-series $E(\mathcal{M}^{\varepsilon}_{\mathcal{C}}; x)$, which is the specialization $t \mapsto -1$ of the mixed Poincaré series $H_c(\mathcal{M}^{\varepsilon}_{\mathcal{C}}; x, t)$. This latter series encodes the dimension of the successive subquotients of the weight filtration on the compactly supported cohomology of $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ (see Section 2.2).

As we will see the genericity assumption on $\mathcal{C}$ will simplify the calculation when $\varepsilon = \sigma$.

It is not clear how to impose genericity in the case $\varepsilon = 1$. For example, if $r = 1 = k$ then the equation

$$A^2 = \xi\, \mathrm{I}_n$$

defining $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ for a given ξ is just equivalent to the case $\xi = 1$ by replacing A by $\sqrt{\xi} A$.

1.1. E-series of $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ in case (A)

Put

$$\mathcal{M}_{\varrho,n} := \left[\left\{(A_1, \dots, A_r) \in \mathrm{G}^r \mid A_1^2 \cdots A_r^2 = \mathrm{I}_n\right\}/\mathrm{G}\right], \qquad \varrho := r - 2,$$

and consider the generating function

$$M_\varrho(q, T) = \sum_{n \geqslant 0} M_{\varrho,n}(q)\, T^n := 1 + \sum_{n \geqslant 1} q^{-\varrho\binom{n}{2}} \left|\mathcal{M}_{\varrho,n}(\mathbb{F}_q)\right| T^n. \tag{1.1}$$

Let $\mathcal{P}$ be the set of all partitions. For $\lambda \in \mathcal{P}$ denote by

$$H_\lambda(q) = \prod_x (q^{h(x)} - 1) \tag{1.2}$$

the *hook polynomial* (where x runs over the set of boxes of the Young diagram of λ and $h(x)$ is the hook length).

For an integer ϱ consider the generating function

$$Z_\varrho(q, T) := \sum_{\lambda \in \mathcal{P}} \left(q^{-n(\lambda)} H_\lambda(q)\right)^{\varrho} T^{|\lambda|}, \tag{1.3}$$

where $n(\lambda) = \sum_{i>0} (i-1)\lambda_i$ if $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}$. Define

$\{V_{\varrho,n}(q)\}_n$ by the formula

$$\sum_{n\geqslant 1} V_{\varrho,n}(q)T^n := \mathrm{Log}\left(Z_\varrho(q,T)\right) \tag{1.4}$$

and, following [12], let for any positive integer k

$$V_{\varrho,n,k}(q) := \sum_{\substack{m|k^\infty \\ m|n}} \frac{1}{m} V_{\varrho,\frac{n}{m}}(q^m), \tag{1.5}$$

where $m \mid k^\infty$ means that m divides a sufficiently high power of k or equivalently that m is only divisible by primes dividing k.

We have the following (see Section 3.4 for the proof).

Theorem 1.1.

(i) *The stack $\mathcal{M}_{\varrho,n}$ has rational count for $r = 1$ ($\varrho < 0$) and polynomial count for $r > 1$ ($\varrho \geqslant 0$).*

(ii) *Let*

$$W_{\varrho,n}(q) := 2V_{\varrho,n}(q) + (q-2)V_{2\varrho,n/2}(q) + \tfrac{1}{2}(q-1)\left(V_{\varrho,n/2,2}(q^2) - V_{2\varrho,n/2,2}(q)\right),$$

where $V_{\varrho,n}$ and $V_{\varrho,n,k}$ are defined in (1.4) *and* (1.5) *for integer n and set to zero if n is not an integer. Then*

$$M_\varrho(q,T) = \mathrm{Exp}\left(\sum_{n\geqslant 1} W_{\varrho,n}(q)T^n\right)$$

(iii) *The E-series of $\mathcal{M}_{\varrho,n}$ (see* (2.1)*) is given by*

$$E(\mathcal{M}_{\varrho,n};q) = q^{\varrho\binom{n}{2}}\mathrm{Coeff}_{T^n}\left(\mathrm{Exp}\left(\sum_{n\geqslant 1} W_{\varrho,n}(q)T^n\right)\right).$$

Remark 1.2. — Let Σ^g be a compact Riemann surface of genus g with fundamental group Π^g and set $\varrho = 2g-2$. It is interesting to point out that the function $Z_\varrho(q,T)$ computes also the number of points of the quotient stack $[\mathrm{Hom}(\Pi^g, \mathrm{GL}_n)/\mathrm{GL}_n]$ over $\mathbb{F}_q$, namely we have [15, Section 3.8]

$$1 + \sum_{n\geqslant 1} q^{-\varrho\binom{n}{2}} \left|[\mathrm{Hom}(\Pi^g, \mathrm{GL}_n)/\mathrm{GL}_n](\mathbb{F}_q)\right| T^n = \mathrm{Exp}\left((q-1)\sum_{n\geqslant 1} V_{\varrho,n}(q)T^n\right).$$

We now give some examples to illustrate the theorem.

Example 1.3 ($\varrho = -1$). — Here $r = 1$ and

$$|\mathcal{M}_{1,n}(\mathbb{F}_q)| = \frac{I_n(q)}{|GL_n(\mathbb{F}_q)|},$$

where $I_n(q)$ denotes the number of involutions in $\mathrm{G}(\mathbb{F}_q)$, a polynomial in q. Concretely,

$$I_n(q) := |\{x \in \mathrm{G}(\mathbb{F}_q) \mid x^2 = 1\}|$$

with first few values

$$\begin{aligned}
I_1(q) &= 2\\
I_2(q) &= q^2 + q + 2\\
I_3(q) &= 2q^4 + 2q^3 + 2q^2 + 2\\
I_4(q) &= q^8 + q^7 + 4q^6 + 3q^5 + 3q^4 + 2q^3 + 2\\
I_5(q) &= 2q^{12} + 2q^{11} + 4q^{10} + 4q^9 + 6q^8 + 4q^7 + 4q^6 + 2q^5 + 2q^4 + 2
\end{aligned}$$

We have (see Corollary 3.7)

$$M_{-1}(q,T) = \sum_{n \geqslant 0} \frac{q^{\binom{n}{2}} I_n(q)}{|\mathrm{GL}_n(\mathbb{F}_q)|} T^n = \operatorname{Exp}\left(\frac{2}{(q-1)}T + \frac{1}{(q+1)}T^2\right). \tag{1.6}$$

This is a q-analogue of the known generating series for the number of involutions t_n in the symmetric group S_n. Namely [16],

$$\sum_{n \geqslant 0} \frac{t_n}{n!} T^n = e^{T + \frac{1}{2}T^2}.$$

Example 1.4 ($\varrho = 0$). — The first few terms of $M_0(q,T)$ are

$$M_0(q,T) = 1 + 2T + (q+3)T^2 + (2q+6)T^3 + (q^2+4q+9)T^4 + \cdots$$

We have

$$Z_0(q,T) = \prod_{n \geqslant 1} (1 - T^n)^{-1}$$

and hence $V_{0,n}(q) = 1$ for all n. It follows from Theorem 1.1 that

$$\operatorname{Log}(M_0(q,T)) = 2T + qT^2 + 2T^3 + qT^4 + \cdots \tag{1.7}$$

or equivalently,

$$M_0(q,T) = \prod_{n \geqslant 1} (1 - T^{2n-1})^{-2}(1 - qT^{2n})^{-1}. \tag{1.8}$$

Remark 1.5. — As pointed out by Frobenius and Schur [8] the change of variables $x \mapsto xz^{-1}$ in an arbitrary group takes the equation $x^2z^2 = 1$ to $z^{-1}xz = x^{-1}$. Hence $M_{0,n}(q)$ equals the number of real conjugacy classes in

$\mathrm{G}(\mathbb{F}_q)$ (i.e. classes of elements conjugate to their inverses). In this form (1.8) was first proved by Gow [11, 27].

Example 1.6 ($\varrho > 0$). — The expression for $M_{\varrho,n}(q)$ in Theorem 1.1 is easy to program; we do not expect a closed formula. We give below the first few values for $r = 3, (\varrho = 1)$.

$$M_{1,1}(q) = 2q - 2$$
$$M_{1,2}(q) = 3q^4 - 2q^3 - 3q^2 + 2$$
$$M_{1,3}(q) = 2q^9 - 2q^8 + 4q^7 - 12q^6 + 10q^5 - 6q^4 + 6q^3 - 2q^2 + 2q - 2$$

Remark 1.7.

(i) Consider the affine variety

$$\mathcal{U}_r := \left\{(A_1, \ldots, A_r) \in \mathrm{G}^r \,\middle|\, A_1^2 \cdots A_r^2 = \mathrm{I}_n\right\}, \qquad r \geqslant 1.$$

By the Weil conjectures the leading coefficient of the polynomial $|\mathcal{U}_r(\mathbb{F}_q)|$ equals the number of irreducible components of largest dimension of $\mathcal{U}_r$ over the complex numbers. It is not difficult to deduce from Theorem 1.1 the value of this leading coefficient. (In general, counting points over $\mathbb{F}_q$ would not yield anything about components of non-maximal dimension.)

On the other hand the number of irreducible components and their dimensions is computed in [3, Thms 2.1, 2.2, 2.3, 3.2 and Prop. 3.4]. It is a pleasant exercise to verify that everything checks out. Here is a sketch on determining the leading coefficients of the counting polynomials. It is enough to compute the leading coefficient of $W_{\rho,n}$. We start by verifying that the highest power of q in $V_{\rho,n}(q)$ is $\rho n(n+1)/2$. Hence the first term in $W_{\varrho,n}(q)$ dominates for n odd and contributes 2 as its leading coefficient. The first term also dominates with coefficient 2 for n even as long as $\rho n(n+1)/2$ is bigger than $1+\rho(n/2+1)n/2$. This happens if $\rho > 1$ or $\rho = 1$ and $n > 2$. In the special case $\rho = 1$ and $n = 2$ all the terms in $W_{\varrho,n}(q)$ actually contribute giving a total leading coefficient of 3. If $\rho = 0$ and n is even on the other hand, the remaining terms in $W_{\varrho,n}(q)$ rather than the first term are the ones that contribute to the highest power of q with coefficient $1 = 1 + (1/2 - 1/2) + (1/4 - 1/4) + \cdots$. The case $\rho = -1$ is easy to check (see also 3.2).

Here is a small table of leading coefficients summarizing the situation.

$r \backslash n$	1	2	3	4	5	$\cdots$
1	2	1	2	1	2	$\cdots$
2	2	1	2	1	2	$\cdots$
3	2	3	2	2	2	$\cdots$
4	2	2	2	2	2	$\cdots$
5	2	2	2	2	2	$\cdots$
$\vdots$			$\vdots$			

Notice the already mentioned special case $r = 3, n = 2$ where the leading coefficient is 3 breaking the pattern; see [3, Section 3] for the explicit description of the irreducible components.

(ii) It seems that $M_{\varrho,n}(q) \equiv 0 \bmod 2$ for all $\varrho > 0$ and n odd. Perhaps this is a consequence of the involution $A \mapsto -A$ in G acting without fixed points on the irreducible components of $\mathcal{U}_r$, as it happens for $r = 1$ (see Remark 3.2), but we have not pursued this further.

1.2. E-series of $\mathcal{M}_{\mathcal{C}}^{\varepsilon}$ in case (B)

We now consider the case (B) where $\epsilon = \sigma$ and to alleviate the notation we write $\mathcal{M}_{\mathcal{C}}$ instead of $\mathcal{M}_{\mathcal{C}}^{\varepsilon}$. For $i = 1, \ldots, k$, let $\mathbf{x}_i$ be an infinite set of commuting variables and consider for a positive integer r, as in [13], the Cauchy function

$$\Omega(z, w) = \Omega_{r,k}(z, w) \coloneqq \sum_{\lambda \in \mathcal{P}} \mathcal{H}_{r,\lambda}(z, w) \left(\prod_{i=1}^{k} \widetilde{H}_\lambda(\mathbf{x}_i; z^2, w^2) \right),$$

where

$$\mathcal{H}_{r,\lambda}(z, w) \coloneqq \prod \frac{(z^{2a+1} - w^{2l+1})^r}{(z^{2a+2} - w^{2l})(z^{2a} - w^{2l+2})}$$

is the (z, w)-deformed hook function with exponent r and $\widetilde{H}_\lambda(\mathbf{x}_i, z, w)$ denotes the modified Macdonald symmetric function in the variables $\mathbf{x}_i$. As in [15, (2.4.11)] it is easily checked that

$$\mathcal{H}_{r,\lambda}(\sqrt{q}, 1/\sqrt{q}) = q^{-\frac{1}{2}\varrho\langle\lambda,\lambda\rangle} H_\lambda(q)^\varrho, \qquad \varrho = r - 2. \tag{1.9}$$

We stress the fact that unlike [13], however, the integer ρ is not necessarily even. In particular, exchanging z and w involves a sign if r is odd. More precisely,

$$\mathcal{H}_{r,\lambda}(w, z) = (-1)^r \mathcal{H}_{r,\lambda'}(z, w), \tag{1.10}$$

where λ' is the dual partition.

For a k-tuple of partitions $\mu = (\mu^1, \dots, \mu^k)$ of n let

$$\mathbb{H}_\mu(z,w) := (z^2-1)(1-w^2)\left\langle \mathrm{Log}\left(\Omega(z,w)\right), h_\mu \right\rangle, \tag{1.11}$$

where $h_\mu = h_{\mu^1}(\mathbf{x}_1)\cdots h_{\mu^k}(\mathbf{x}_k)$ and $h_{\mu^i}(\mathbf{x}_i)$ is the complete symmetric function in the variables $\mathbf{x}_i$.

We obtain the following (see Section 4.1.2 for the proof).

Theorem 1.8. — *We have*

$$\mathrm{E}(\mathcal{M}_{\mathcal{C}}; q) = \frac{q^{d_\mu/2}}{q-1}\mathbb{H}_\mu\left(\sqrt{q}, \frac{1}{\sqrt{q}}\right), \tag{1.12}$$

where μ is defined from $\mathcal{C}$ as in Section 4.1.2 and d_μ is as in Theorem 4.5.

Remark 1.9.

(i) It follows from Theorem 1.8 that if $r = 2h$ is even and Σ' is an orientable compact Riemann surface of Euler characteristic $r-2$ and punctures $S' = \{\alpha'_1, \dots, \alpha'_k\}$, then comparison with [13] shows that the right hand side of (1.12) is also the E-series of the quotient stack

$$\mathcal{M}'_{\mathcal{C}} := \left[\left\{\rho \in \mathrm{Hom}\left(\pi_1(\Sigma' \backslash S')\right), \mathrm{G} \,\middle|\, \rho(z_i) \in C_i\right\}/\mathrm{G}\right].$$

where z_i is a single loop around the puncture α'_i. We thus have

$$E(\mathcal{M}_{\mathcal{C}}; q) = E(\mathcal{M}'_{\mathcal{C}}; q).$$

(ii) We may construct a correspondence between $\mathcal{M}_{\mathcal{C}}$ and $\mathcal{M}'_{\mathcal{C}}$ in (i), which gives some indication of how these stacks are related. We only give a sketch of the construction here for the case $r = 2$. For fixed $h \in \mathrm{GL}_n$ consider the following varieties:

$$\mathcal{A} := \{(x,z) \mid xz\sigma(x)z^{-1} = h\} \subseteq \mathrm{GL}_n \times \mathrm{GL}_n,$$
$$\mathcal{B} := \{(x,z) \mid xzx^{-1}z^{-1} = h\} \subseteq \mathrm{GL}_n \times \mathrm{GL}_n,$$

as well as

$$\mathcal{C} := \{(x,z,u) \,|\, {}^t x = uxu^{-1}, x(zu)z(zu)^{-1} = h\} \subseteq \mathrm{GL}_n \times \mathrm{GL}_n \times \mathrm{GL}_n\,.$$

Note that changing x to xz^{-1} and then z to ${}^t z$ yields an isomorphism between $\mathcal{A}$ and the variety $\{(x,z) \,|\, x\sigma(x)z\sigma(z) = h\}$.

We have natural maps

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "p_{\mathcal{A}}"'] \arrow[dr, "p_{\mathcal{B}}"] & \\ \mathcal{A} & & \mathcal{B} \end{tikzcd}$$

where $p_{\mathcal{A}}(x,z,u) \coloneqq (x,z)$ and $p_{\mathcal{B}}(x,z,u) = (x,zu)$. These maps are equivariant for the natural action of GL_n on the three varieties given by

$$x \mapsto gxg^{-1}, \qquad z \mapsto gz^tg, \qquad u \mapsto {}^tg^{-1}ug^{-1}, \qquad g \in \mathrm{GL}_n\,.$$

Since any $x \in \mathrm{GL}_n$ is conjugate to its transpose the maps $p_{\mathcal{A}}, p_{\mathcal{B}}$ are surjective with fibers isomorphic to the stabilizer of x in GL_n. In particular, for any finite field $\#\mathcal{A}(\mathbb{F}_q) = \#\mathcal{B}(\mathbb{F}_q)$.

(iii) Our surface Σ arises from a pair $(\widetilde{\Sigma},\sigma)$ with $\widetilde{\Sigma}$ an orientable compact Riemann surface and an orientation-reversing involution on $\widetilde{\Sigma}$ whose fixed-point set $\widetilde{\Sigma}^\sigma$ is empty. In [1], the authors compute the E-polynomials of some character varieties coming from pairs of the form $(\widetilde{\Sigma},\sigma)$ with $\widetilde{\Sigma}^\sigma \neq \emptyset$.

Remark 1.10.

(i) As in [13], there is a natural deformation of the formula for $E(\mathcal{M}_{\mathcal{C}};q)$ to two variables. One might be tempted to extend the conjecture on Hodge numbers made there but this may not be that straightforward. We do however check one potential case of the putative conjecture when $r = k = 1$ in Section 4.2 (see Theorem 4.7).

(ii) It was conjectured in [13], in particular, that for r even $\mathbb{H}_\mu(z,w)$ is a polynomial with integer coefficients. This was recently proved by Mellit [21]. Note that in general $\mathbb{H}_\mu(z,w)$ is not in fact polynomial for $r = 1$ (see for example Lemma 4.3). It appears, however, that in general for r odd the denominator is fairly small. For example, for $k = 1$ we seem to just have denominator q^2+1 and then only for $n \equiv 2 \bmod 4$ and when all parts of μ are divisible by 2. On the other hand, note that the specialization $q^{d_\mu/2}\mathbb{H}_\mu\left(\sqrt{q}, \frac{1}{\sqrt{q}}\right)$ for any $r > 1$ *is* a polynomial by (1.9).

(iii) In any case, we infer a purely combinatorial identity involving Macdonald polynomials. Namely, we conjecture (see Conjecture 4.4) that

$$(z^2-1)(1-w^2)\,\mathrm{Log}\left(\sum_{\lambda\in\mathcal{P}}\prod \frac{(z^{2a+1}-w^{2l+1})}{(z^{2a+2}-w^{2l})(z^{2a}-w^{2l+2})}\widetilde{H}_\lambda(\mathbf{x};z^2,w^2)\right) = (z-w)m_{(1)}(\mathbf{x}) + \frac{1}{z^2+1}m_{(2)}(\mathbf{x}) + m_{(1^2)}(\mathbf{x})$$

where for a partition λ, we denote by $m_\lambda(\mathbf{x})$ the corresponding monomial symmetric function.

Acknowledgements

It is a pleasure to thank T. Scognamiglio for very useful remarks on this paper and M. Olsson for his help with the cohomology of stacks. We are grateful for the hospitality of the institutions were this work was carried out: the first author would like to thank the University of Texas at Austin and the ICTP, and the second author would like to thank Oxford University and the Université de Caen were the work was started several years ago and IHP and the Université de Paris were it was completed.

2. Preliminaries

2.1. Log and Exp

We let Λ be the field $\mathbb{Q}(x_1,\dots,x_r)$ where $x_1,\dots,x_r$ are indeterminate which commute and $\Lambda[\![T]\!]$ the ring of series with coefficients in Λ.

Consider

$$\psi_n:\Lambda[\![T]\!]\to\Lambda[\![T]\!],\ f(q,T)\mapsto f(x_1^n,\dots,x_r^n,T^n).$$

The ψ_n are called the *Adams operations*.

Define $\Psi:T\Lambda[\![T]\!]\to T\Lambda[\![T]\!]$ by

$$\Psi(f)=\sum_{n\geqslant 1}\frac{\psi_n(f)}{n}.$$

Its inverse is given by

$$\Psi^{-1}(f)=\sum_{n\geqslant 1}\mu(n)\frac{\psi_n(f)}{n}$$

where μ is the ordinary Möbius function.

Define $\mathrm{Log}:1+T\Lambda[\![T]\!]\to T\Lambda[\![T]\!]$ and its inverse $\mathrm{Exp}:T\Lambda[\![T]\!]\to 1+T\Lambda[\![T]\!]$ as

$$\mathrm{Log}(f)=\Psi^{-1}\left(\log(f)\right)$$

and

$$\mathrm{Exp}(f)=\exp\left(\Psi(f)\right).$$

They satisfy the following obvious properties.

$$\mathrm{Log}(f\cdot g)=\mathrm{Log}(f)+\mathrm{Log}(g),\qquad \mathrm{Exp}(h+l)=\mathrm{Exp}(h)\cdot\mathrm{Exp}(l).$$

They also commute with the Adams operations, namely for any integer $r>0$, we have

$$\mathrm{Log}\circ\psi_r=\psi_r\circ\mathrm{Log},\qquad \mathrm{Exp}\circ\psi_r=\psi_r\circ\mathrm{Exp}\,.$$

Remark 2.1. — Note that the map $T \mapsto -T$ is not preserved under Log and Exp as $1+q^iT^j = (1-q^{2i}T^{2j})/(1-q^iT^j)$.

Remark 2.2. — Let $f \in 1 + T\Lambda[\![T]\!]$. If we write

$$\mathrm{Log}(f) = \sum_{n \geqslant 1} \frac{1}{n} U_n T^n, \qquad \mathrm{Log}(f) = \sum_{n \geqslant 1} V_n T^n,$$

then

$$V_r(q) = \frac{1}{r} \sum_{d|r} \mu(d) \psi_d(U_{r/d}).$$

We have the following results (details may be found for instance in Mozgovoy [22]).

For $g \in \Lambda$ and $n \geqslant 1$ we put

$$g_n := \frac{1}{n} \sum_{d|n} \mu(d) \psi_{\frac{n}{d}}(g).$$

This is the Möbius inversion formula of $\psi_n(g) = \sum_{d|n} d \cdot g_d$.

Lemma 2.3. — *Let $g \in \Lambda$ and $f_1, f_2 \in 1 + T\Lambda[\![T]\!]$ such that*

$$\mathrm{Log}(f_1) = \sum_{d=1}^{\infty} g_d \cdot \mathrm{Log}(\psi_d(f_2)).$$

Then

$$\mathrm{Log}(f_1) = g \cdot \mathrm{Log}(f_2).$$

2.2. Mixed Poincaré series

Let K be $\overline{\mathbb{F}}_q$ and choose a prime ℓ which does not divide q. Let $\mathfrak{X}_o$ be an algebraic stack of finite type defined over $\mathbb{F}_q$, whose lift to K is denoted by $\mathfrak{X}$. We denote by $H_c^i(\mathfrak{X}, \overline{\mathbb{Q}}_\ell)$ the compactly supported i-th ℓ-adic cohomology group of $\mathfrak{X}$ as defined in [17, 18].

We denote by $F : \mathfrak{X} \to \mathfrak{X}$ the geometric Frobenius and by F^* the induced Frobenius on ℓ-adic cohomology. Let $W_\bullet^k$ be the weight filtration on $H_c^k(\mathfrak{X}, \overline{\mathbb{Q}}_\ell)$, i.e. the F^*-stable increasing filtration such that for all integer $n > 0$, the eigenvalues of $(F^*)^n$ on the subquotient W_m^k / W_{m-1}^k are pure of weight nm.

We define then the mixed Poincaré series of $\mathfrak{X}$, a power formal series in $x^{1/2}$, as

$$\mathrm{H}_c(\mathfrak{X};x,t) \coloneqq \sum_{k,m} \dim\left(W_m^k/W_{m-1}^k\right) x^{m/2} t^k.$$

When it is well defined (i.e. when the sum $\sum_k (-1)^k \dim\left(W_m^k/W_{m-1}^k\right)$ is finite) we let the E-series of $\mathfrak{X}$ be

$$\mathrm{E}(\mathfrak{X};x) \coloneqq \mathrm{H}_c(\mathfrak{X};x,-1) = \sum_m \sum_k (-1)^k \dim\left(W_m^k/W_{m-1}^k\right) x^{m/2}. \tag{2.1}$$

Remark 2.4. — Let $X/\mathbb{C}$ be a separated scheme of finite type over $\mathbb{C}$. The compactly supported cohomology groups $H_c^i((X/\mathbb{C})^{\mathrm{an}}, \mathbb{Q})$ carry a mixed Hodge structure defined by Deligne and so one can define the corresponding E-polynomial

$$\mathrm{E}(X/\mathbb{C};a,b) = \sum_{i,j}\sum_k (-1)^k h_c^{i,j;k}(X/\mathbb{C}) a^i b^j,$$

where $\{h_c^{i,j;k}(X/\mathbb{C})\}_{i,j}$ are the mixed Hodge numbers of $H_c^k((X/\mathbb{C})^{\mathrm{an}}, \mathbb{C})$.

We then consider

$$\mathrm{E}(X/\mathbb{C};x) \coloneqq E(X/\mathbb{C};\sqrt{x},\sqrt{x}) = \sum_r \sum_k (-1)^k \sum_{i+j=r} h_c^{i,j;k}(X/\mathbb{C}) x^{r/2}.$$

If $X/\mathbb{C}$ is projective and smooth then the cohomology is pure of weight k, i.e. $h_c^{i,j;k}$ are zero unless $i+j=k$, and so

$$\mathrm{E}(X/\mathbb{C};x) = \sum_k (-1)^k \dim H_c^k(X/\mathbb{C},\mathbb{C})\, x^{k/2}.$$

Let R be a subring of $\mathbb{C}$ which is finitely generated as a $\mathbb{Z}$-algebra and let X/R be an R-scheme of finite type such that $X/\mathbb{C}$ is obtained from X/R by scalar extension. Then there is an open subset U of $\mathrm{Spec}(R)$ for which the following is true : for any ring homomorphism $\varphi : R \to \mathbb{F}_q$ such that the image of $\mathrm{Spec}(\mathbb{F}_q) \to \mathrm{Spec}(R)$ is in U we have

$$\mathrm{E}(X/\mathbb{C};x) = \mathrm{E}(X/_\varphi\overline{\mathbb{F}}_q;x).$$

If $X/\mathbb{C}$ is smooth projective then this is true because $H_c^k(X/\mathbb{C},\mathbb{C})$ and $H_c^k(X/_\varphi\overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_\ell)$ are pure of weight k [4, Théorème I.6].

The general case reduces to the smooth projective case using that we always have a decomposition

$$[X/\mathbb{C}] = [S/\mathbb{C}] - [T/\mathbb{C}]$$

in the Grothendieck group of the category of separated $\mathbb{C}$-schemes of finite type with $S/\mathbb{C}$ and $T/\mathbb{C}$ both projective and smooth [15, Appendix, Lemma 6.1.1], and that E-polynomials are additive.

Theorem 2.5. — *Let G be a connected linear algebraic group over K acting on a separated scheme X of finite type over K. Assume that X, G and the action are all defined over $\mathbb{F}_q$. The E-series of the quotient stack $[X/G]$ is well-defined and*

$$\mathrm{E}([X/G];x) = \mathrm{E}(X;x)\,\mathrm{E}(\mathrm{B}(G);x),$$

where $\mathrm{B}(G) \coloneqq [\mathrm{Spec}(K)/G]$ *is the classifying stack of* G.

Proof. — We consider the cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & {[X/G]} \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] & \mathrm{B}(G) \end{tikzcd}$$

Following the same lines as in [2, Section 2.5] with compactly supported cohomology instead, we show that we have an E_2 spectral sequence of finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces

$$H^i_c(\mathrm{B}(G), \overline{\mathbb{Q}}_\ell) \otimes H^j_c(X, \overline{\mathbb{Q}}_\ell) \Rightarrow H^{i+j}_c([X/G], \overline{\mathbb{Q}}_\ell)$$

which is compatible with the action of F^*. The theorem now follows from the above spectral sequence as in the proof of the Lefschetz trace formula in [2, Section 2.5]. $\square$

Definition 2.6. — *An algebraic stack of finite type $\mathfrak{X}$ defined over $\mathbb{F}_q$ has* rational count *if there exists a rational function $Q(t) \in \mathbb{Q}(t)$ such that for all integer $n > 0$, we have*

$$|\mathfrak{X}(\mathbb{F}_{q^n})| = Q(q^n).$$

Remark 2.7. — Note that if X and G are as in Theorem 2.5 then

$$|[X/G](\mathbb{F}_q)| = \frac{|X(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|}$$

and so $[X/G]$ has rational count if and only if X has polynomial count. Indeed, if $[X/G]$ has rational count, it follows that X has rational count and therefore by Lemma 2.8 it has polynomial count.

Lemma 2.8. — *Let $S \in \mathbb{Q}(t)$ be a rational function. If $S(x_r)$ is an integer for infinitely many integers $x_1 < x_2 < \cdots$ then $S \in \mathbb{Q}[t]$.*

Proof. — Write $S = P/T$ with $P, T \in \mathbb{Q}[t]$ and let $Q, R \in \mathbb{Q}[t]$ be such that

$$P = QT + R, \qquad \deg(R) < \deg(T).$$

Let $Q' \in \mathbb{Z}[X]$ and $N \in \mathbb{Z}$ be such that $Q = Q'/N$.

We thus have

$$NS = Q' + \frac{NR}{T}.$$

There exists r' such that for all $x > x_{r'}$

$$\left|\frac{NR(x)}{T(x)}\right| < 1$$

Hence for all $r > r'$, we have

$$NS(x_r) = Q'(x_r)$$

since $S(x_r) \in \mathbb{Z}$, and so S is a polynomial. □

Theorem 2.9. — *Let X and G be as in Theorem 2.5. If $[X/G]$ has rational count with counting rational function $Q(t)$, then*

$$\mathrm{E}([X/G]; x) = Q(x).$$

Proof. — By Theorem 2.5 we are reduced to the prove the theorem in the two following cases :

(i) $G = 1$.
(ii) X is a point.

The case (ii) follows from an explicit computation of the dimension of $H_c^i(\mathrm{B}(G), \overline{\mathbb{Q}}_\ell)$ which is pure of weight i (see [5, 2]).

Let us prove (i). We have for all $r > 0$

$$|X(\mathbb{F}_{q^r})| = \sum_k (-1)^k \mathrm{Tr}\left((F^*)^r, H_c^k(X, \overline{\mathbb{Q}}_\ell)\right). \tag{2.2}$$

For a complex number $\xi \in \mathbb{C}$ and a finite dimensional $\mathbb{Q}_\ell$ vector space V with an action of F^* let $m_\xi(V)$ be the multiplicity of ξ as a root of $\det(1 - F^*|_V T)$.

If X has polynomial count with counting polynomial $P(T) = \sum_n c_n T^n$ then by definition $P(q^r) = |X(\mathbb{F}_{q^r})|$ for all $r > 0$. Considering the generating series summing over all $r > 0$ and taking logarithmic derivative we conclude that

$$\sum_{\xi\in\mathbb{C}} \sum_k (-1)^k \sum_i m_\xi(W_i^k/W_{i-1}^k) \frac{\xi T}{1-\xi T} = \sum_n c_n \frac{q^n T}{1-q^n T}.$$

Comparing poles on both sides of this equality and considering that

$$m_\xi(W_i^k/W_{i-1}^k) > 0$$

implies that $|\xi| = q^{i/2}$ we deduce that $i = 2n$ must be even and

$$\sum_{\xi\in\mathbb{C}} \sum_k (-1)^k m_\xi(W_{2n}^k/W_{2n-1}^k) \frac{\xi T}{1-\xi T} = \sum_k (-1)^k m_{q^n}(W_{2n}^k/W_{2n-1}^k) = c_n.$$

We also have $\dim(W_i^k/W_{i-1}^k) = \sum_{\xi\in\mathbb{C}} m_\xi(W_i^k/W_{i-1}^k)$ and therefore

$$\sum_k (-1)^k \dim(W_{2n}^k/W_{2n-1}^k) = c_n$$

proving our claim. $\square$

2.3. Mass formula

Consider the group Π with presentation

$$\Pi = \left\langle a_1,\dots,a_r,x_1,\dots,x_k \;\middle|\; a_1^2\cdots a_r^2 x_1\cdots x_k = 1\right\rangle .$$

Let $\mathrm{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$, let $\sigma : \mathrm{G} \to \mathrm{G}$ be the involution $g \mapsto {}^t g^{-1}$ and consider the semi-direct product $\mathrm{G}^+ = \mathrm{G} \rtimes \langle\sigma\rangle$. Consider on G the $\mathbb{F}_q$-structure induced by the Frobenius that raises coefficients of matrices to their q-th power and fix a k-tuple $\mathcal{C} = (C_1,\dots,C_k)$ of conjugacy classes of $\mathrm{G}(\mathbb{F}_q)$. For $\varepsilon = (\varepsilon_1,\dots,\varepsilon_r) \in \langle\sigma\rangle^r$ consider the representation variety

$$\begin{aligned}&\mathrm{Hom}^{\varepsilon}_{\mathcal{C}}(\Pi,\mathrm{G}^+(\mathbb{F}_q))\\ &:= \left\{\rho\in\mathrm{Hom}(\Pi,\mathrm{G}^+(\mathbb{F}_q)) \;\middle|\; \begin{matrix}\epsilon(\rho(a_i)) = \varepsilon,\\ \rho(x_j)\in\iota(C_j)\end{matrix} \text{ for all } 1\leqslant i\leqslant r \text{ and } 1\leqslant j\leqslant k\right\},\end{aligned}$$

where $\epsilon : \mathrm{G}^+ \to \langle\sigma\rangle$ is the quotient map and $\iota : G \to G^+$ the natural inclusion.

Recall that for an irreducible complex character χ of some finite group U, the Schur indicator $c_\chi \in \{-1,0,1\}$ is defined as

$$c_\chi := \frac{1}{|U|}\sum_{u\in U}\chi(u^2).$$

An irreducible character of U is afforded by a *real* representation (we call its character *real*) if and only if $c_\chi = 1$. We denote by $\widehat{U}$ the set of irreducible complex characters of U and by $\widehat{U}_{\mathrm{real}}$ the subset of real irreducible characters. It is known [11] that for $U = \mathrm{G}(\mathbb{F}_q)$ the Schur indicator c_χ is either 0 or 1. Frobenius and Schur [8, (9), p. 197] proved the mass formula

$$\frac{1}{|U|}\left|\{a_1^2\cdots a_r^2 = 1\}\right| = \sum_\chi c_\chi\left(\frac{|U|}{\chi(1)}\right)^{r-2}, \tag{2.3}$$

where the sum is over all irreducible characters of U.

We need the following generalization of (2.3) for our setting.

Theorem 2.10 (Mass formula). — *We have*

$$\frac{|\mathrm{Hom}^{\varepsilon}_{\mathcal{C}}(\Pi,\mathrm{G}^{+}(\mathbb{F}_q))|}{|\mathrm{G}(\mathbb{F}_q)|} = \begin{cases} \sum_{\chi\in\widehat{G(\mathbb{F}_q)}} \left(\frac{|\mathrm{G}^F|}{\chi(1)}\right)^{r-2} \prod_{i=1}^k \frac{\chi(C_i^F)\,|C_i^F|}{\chi(1)}, & \text{if } \varepsilon_i=\sigma \text{ for all } i,\\ \sum_{\chi\in\widehat{G(\mathbb{F}_q)}_{\mathrm{real}}} \left(\frac{|\mathrm{G}^F|}{\chi(1)}\right)^{r-2} \prod_{i=1}^k \frac{\chi(C_i^F)\,|C_i^F|}{\chi(1)}, & \text{otherwise.}\end{cases} \tag{2.4}$$

Proof. — Let $\mathcal{C}(\mathrm{G}(\mathbb{F}_q))$ denote the $\mathbb{C}$-vector space of central functions $\mathrm{G}(\mathbb{F}_q)\to\mathbb{C}$. It is equipped with a convolution product $*$ defined as

$$(f_1 * f_2)(g) = \sum_{xy=g} f_1(x)f_2(y),$$

for $f_1,f_2\in\mathcal{C}(\mathrm{G}(\mathbb{F}_q))$ and $g\in\mathrm{G}(\mathbb{F}_q)$.

Define the function $\eta^\varepsilon : \mathrm{G}(\mathbb{F}_q)\to\mathbb{C}$ by

$$\eta^\varepsilon(y) = \#\{(a_1,\dots,a_r)\in(\mathrm{G}(\mathbb{F}_q))^r \mid E_{\varepsilon_1}(a_1)\cdots E_{\varepsilon_r}(a_r)=y\}$$

where $E_1(a) \coloneqq a^2$ and $E_\sigma(a)=a\sigma(a)$.Then

$$\left|\mathrm{Hom}^{\varepsilon}_{\mathcal{C}}(\Pi,\mathrm{G}^{+}(\mathbb{F}_q))\right| = (\eta^\varepsilon * 1_{C_1} * \cdots * 1_{C_k})(1), \tag{2.5}$$

where 1_{C_i} denotes the function on $\mathrm{G}(\mathbb{F}_q)$ that takes the value 1 on elements of C_i and 0 elsewhere.

Denote by $\mathrm{F}(\widehat{\mathrm{G}(\mathbb{F}_q)})$ be the $\mathbb{C}$-vector space of complex valued functions on $\widehat{\mathrm{G}(\mathbb{F}_q)}$ and let $\mathcal{F} : \mathrm{F}(\mathrm{G}(\mathbb{F}_q))\to\mathrm{F}(\widehat{\mathrm{G}(\mathbb{F}_q)})$ be defined by

$$\mathcal{F}(f)(\chi) \coloneqq \sum_{g\in\mathrm{G}(\mathbb{F}_q)} \frac{f(g)\chi(g)}{\chi(1)}.$$

It satisfies $\mathcal{F}(f*g)=\mathcal{F}(f)\cdot\mathcal{F}(g)$ where $\cdot$ denotes the pointwise multiplication on $\mathrm{F}(\widehat{\mathrm{G}(\mathbb{F}_q)})$.

By [13, Proposition 3.1.1], for all $f\in\mathcal{C}(\mathrm{G}(\mathbb{F}_q))$ we have

$$f(1) = \frac{1}{|\mathrm{G}(\mathbb{F}_q)|}\sum_{\chi\in\widehat{\mathrm{G}(\mathbb{F}_q)}} \chi(1)^2\mathcal{F}(f)(\chi).$$

We thus deduce from (2.5) that

$$\frac{|\mathrm{Hom}^{\varepsilon}_{\mathcal{C}}(\Pi,\mathrm{G}^{+}(\mathbb{F}_q))|}{|\mathrm{G}(\mathbb{F}_q)|} = \sum_{\chi\in\widehat{\mathrm{G}(\mathbb{F}_q)}} \frac{\chi(1)^2}{|\mathrm{G}(\mathbb{F}_q)|^2}\,\mathcal{F}(\eta^\varepsilon)(\chi)\prod_{i=1}^k \frac{\chi(C_i)\,|C_i|}{\chi(1)}.$$

On the other hand we have $\eta^{\varepsilon}=\eta^{\varepsilon_1}*\cdots*\eta^{\varepsilon_r}$. Hence

$$\frac{|\mathrm{Hom}^{\varepsilon}_{\mathcal{C}}(\Pi,\mathrm{G}^{+}(\mathbb{F}_q))|}{|\mathrm{G}(\mathbb{F}_q)|}=\sum_{\chi\in\widehat{\mathrm{G}(\mathbb{F}_q)}}\frac{\chi(1)^2}{|\mathrm{G}(\mathbb{F}_q)|^2}\prod_{i=1}^{r}\mathcal{F}(\eta^{\varepsilon_i})(\chi)\prod_{i=1}^{k}\frac{\chi(C_i)\,|C_i|}{\chi(1)}.$$

By [9], [11, Theorem 3] we have

$$\eta^{\sigma}(y)=|\{x\in\mathrm{GL}_n(\mathbb{F}_q)\,|,xx^{\sigma}=y\}|\sum_{\chi\in\widehat{\mathrm{G}(\mathbb{F}_q)}}\chi(y)$$

for all $y\in\mathrm{G}(\mathbb{F}_q)$ and so

$$\mathcal{F}(\eta^{\sigma})(\chi)=\sum_{g\in G(\mathbb{F}_q)}\sum_{\psi\in\widehat{\mathrm{G}(\mathbb{F}_q)}}\frac{\psi(g)}{\psi(1)}\chi(g)=\frac{|\mathrm{G}(\mathbb{F}_q)|}{\chi(1)}.$$

The claim now follows from

$$\mathcal{F}(\eta^{1})(\chi)=\frac{|\mathrm{G}(\mathbb{F}_q)|}{\chi(1)}\,c_{\chi},$$

since, as mentioned, $c_{\chi}^2=c_{\chi}$ [11]. □

3. Character varieties of non-orientable surfaces

Consider

$$\Pi=\left\langle a_1,\dots,a_r\,\middle|\,a_1^2\cdots a_r^2=1\right\rangle.$$

As before, we consider on G the $\mathbb{F}_q$-structure induced by the Frobenius that raises coefficients of matrices to their q-th power.

In this section, we prove that the quotient stack

$$\mathcal{M}=[\mathrm{Hom}(\Pi,\mathrm{G})/\mathrm{G}]$$

has rational count and we compute its E-series by counting points over finite fields thanks to Theorem 2.9.

3.1. Case $r=1$: involutions

For $n\in\mathbb{Z}_{\geqslant 0}$ let $I_n(q)$ be the number of involutions in $\mathrm{G}(\mathbb{F}_q)$, i.e.,

$$I_n(q)\coloneqq|\{x\in\mathrm{G}(\mathbb{F}_q)\mid x^2=1\}|.$$

We have, using standard notation

$$\begin{bmatrix}n\\r\end{bmatrix}\coloneqq\frac{(q)_n}{(q)_r(q)_{n-r}}$$

with $(a)_n\coloneqq\prod_{k=0}^{n-1}(1-aq^k)$, the following.

Proposition 3.1.

$$I_n(q) = \sum_{r=0}^{n} q^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix} \tag{3.1}$$

Proof. — If $x \in \mathrm{G}(\mathbb{F}_q)$ is such that $x^2 = 1$ then it must be conjugate to a diagonal matrix with, say, r eigenvalues equal to 1 and $n-r$ eigenvalues equal to -1. A calculation shows that the size of this conjugacy class is $q^{r(n-r)} \left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$. □

In particular, I_n is a polynomial in q with non-negative integer coefficients (see Section 3.1 for the first few values).

Remark 3.2.

(i) Note that the map $x \mapsto -x$ induces an involution of the affine variety $\mathcal{U} := \{x \in \mathrm{G} \,|\, x^2 = 1\}$ permuting the conjugacy classes of semisimple elements with eigenvalues in $\{-1, 1\}$. When n is odd this involution does not fix any of these conjugacy classes. This explains why the polynomials $I_n(q)$, with n odd, are divisible by 2.

(ii) These conjugacy classes are the irreducible components of

$$\mathcal{U} = \mathrm{Hom}(\pi_1(\Sigma), GL_n(\mathbb{C})),$$

where Σ is the real projective plane ([3, Thm 2.1]).

Comparison to the untwisted, orientable case discussed in [15, Section 3.8] suggests considering the generating series

$$\sum_{n \geqslant 0} \frac{q^{n^2/2} I_n(q)}{|\mathrm{G}(\mathbb{F}_q)|} T^n.$$

To avoid dealing with powers of $\sqrt{q}$ we consider instead

$$I(q,T) := \sum_{n \geqslant 0} \frac{(-1)^n q^{\binom{n}{2}} I_n(q)}{|\mathrm{G}(\mathbb{F}_q)|} T^n, \tag{3.2}$$

which amounts to shifting T by a factor of $\sqrt{q}$; the factor of $(-1)^n$ simplifies later formulas. Alternatively,

$$I(q,T) = \sum_{n \geqslant 0} \frac{I_n(q)}{(q)_n} T^n, \tag{3.3}$$

since

$$|\mathrm{G}(\mathbb{F}_q)| = (-1)^n q^{\binom{n}{2}} (q)_n. \tag{3.4}$$

It follows from the Mass formula (2.10) that

$$I_n(q) = \sum_{\chi} \chi(1), \tag{3.5}$$

where the sum is over the real irreducible characters of $\mathrm{G}(\mathbb{F}_q)$. Hence we also have

$$I(q,T) = \sum_{n\geqslant 0} \sum_{\chi \text{ real}} \frac{\chi(1)}{|\mathrm{G}(\mathbb{F}_q)|} (-1)^n q^{\binom{n}{2}} T^n. \tag{3.6}$$

Proposition 3.3. — *The following identity holds*

$$(q-1)\operatorname{Log}\left(I(q,T)\right) = -2T + T^2 \tag{3.7}$$

or equivalently

$$I(q,T) = \prod_{n\geqslant 0} \frac{(1-q^nT^2)}{(1-q^nT)^2}. \tag{3.8}$$

The identity (3.7) will follow from a more general formula that we now describe. Using (3.1) we have

$$I(q,T) = \sum_{n\geqslant 0} \sum_{r=0}^{n} \frac{q^{r(n-r)}}{(q)_r (q)_{n-r}} T^n.$$

This suggests that we introduce another variable and consider the series

$$I^*(q,X,Y) \coloneqq \sum_{r\geqslant 0} \sum_{s\geqslant 0} \frac{q^{rs}}{(q)_r (q)_s} X^r Y^s, \tag{3.9}$$

with

$$I(q,T) = I^*(q,T,T).$$

The following generalization of Proposition 3.3 holds.

Proposition 3.4.

$$(q-1)\operatorname{Log}\left(I^*(q,X,Y)\right) = -X - Y + XY \tag{3.10}$$

or equivalently

$$I^*(q,X,Y) = \prod_{n\geqslant 0} \frac{(1-q^nXY)}{(1-q^nX)(1-q^nY)}. \tag{3.11}$$

Proof. — Following Fadeev–Kasahev [7], start with the q-binomial theorem

$$\sum_{n\geqslant 0} \frac{(X)_n}{(q)_n} Y^n = \frac{(XY)_\infty}{(Y)_\infty}$$

and replace $(X)_n$ by $(X)_\infty/(Xq^n)_\infty$. Now use Euler's formula

$$(u)_\infty^{-1} = \sum_{n\geqslant 0} \frac{u^n}{(q)_n}$$

with $u = Xq^n$ to finish the proof. □

Remark 3.5. — As the characteristic is different from 2, involutions are in bijection with projections

$$\{x^2 = 1\} \longleftrightarrow \{e^2 = e\}$$
$$x \longleftrightarrow \tfrac{1}{2}(1-x).$$

Hence $I_n(q)$ equals the q-Stirling number $S_{n,2}$ that counts the number of non-trivial splittings of a vector space of dimension n over $\mathbb{F}_q$ into two direct summands [26, Example 5.5.2(b), pp. 45–6], [6].

Proposition 3.6. — *The following identity holds*

$$\mathrm{Log}\,(Z_{-1}(q,T)) = \frac{T}{(q-1)} + \frac{T^2}{(q^2-1)(q-1)}, \tag{3.12}$$

Proof. — The identity is a specialization of a corresponding identity for the Schur symmetric functions. Indeed from [19, p. 45] we know that

$$s_\lambda(1,q,q^2,\ldots) = (-1)^{|\lambda|}\mathcal{H}_{\lambda'}(q)^{-1}.$$

On the other hand [19, p. 76]

$$\sum_\lambda s_\lambda(x_1,x_2,\ldots) = \prod_i (1-x_i)^{-1} \prod_{i<j}(1-x_ix_j)^{-1}$$

and (Cauchy's formula)

$$\sum_\lambda s_\lambda(x_1,x_2,\ldots)s_\lambda(y_1,y_2,\ldots) = \prod_{i,j}(1-x_iy_j)^{-1}.$$

It follows that

$$\begin{aligned}
Z_{-1}(q,T) &= \sum_\lambda \mathcal{H}_\lambda(q)\,(-T)^{|\lambda|} \\
&= \prod_{i\geqslant 0}(1+q^iT)^{-1} \prod_{0\leqslant i<j}(1-q^{i+j}T^2)^{-1} \\
&= \prod_{i\geqslant 0}(1-q^iT)\prod_{i\geqslant 0}(1-q^{2i}T^2)^{-1} \prod_{0\leqslant i<j}(1-q^{i+j}T^2)^{-1}
\end{aligned}$$

and hence

$$\mathrm{Log}\,(Z_{-1}(q,T)) = T\sum_{i\geqslant 0} q^i + T^2 \sum_{0\leqslant i\leqslant j} q^{i+j}.$$

Summing the series finishes the proof. □

We leave to the reader to deduce the following corollary from the identity (3.12).

Corollary 3.7. — *We have*

$$\operatorname{Log}\left(M_{-1}(q,T)\right) = \frac{2}{(q-1)}T + \frac{1}{(q+1)}T^2. \tag{3.13}$$

Comparing (3.2) with (1.1) we see that $M_{-1}(q,T) = I(q,-T)$ hence Corollary 3.7 also follows from (3.7) (keeping in mind Remark 2.1).

Remark 3.8. — Proposition 3.4 is essentially the quantum version of the 5-term relation of the dilogarithm of Fadeev–Kasahev [7]. Indeed, if we let u, v satisfy the relation $vu = quv$ then

$$I^*(q,u,v) = E(q,v)E(q,u),$$

where

$$E(q,T) \coloneqq \sum_{n\geqslant 0} \frac{T^n}{(q)_n} = \prod_{n\geqslant 0} (1-q^nT)^{-1}.$$

We also have

$$E(q,T)^{-1} = \sum_{n\geqslant 0} \frac{(-1)^n q^{n(n-1)/2}}{(q)_n} T^n.$$

It is easily checked by induction that

$$v^r u^s = q^{rs} u^s v^r, \qquad (vu)^n = q^{n(n-1)/2} u^n v^n.$$

A calculation now shows that if

$$E(q,X)E(q,Y)E(q,XY)^{-1} = \sum_{m,n\geqslant 0} c_{m,n}(q)\, X^n Y^m$$

then

$$E(q,u)E(q,-vu)E(q,v) = \sum_{m,n\geqslant 0} c_{m,n}(q)\, u^n v^m$$

and hence Proposition 3.4 is equivalent to

$$E(q,v)E(q,u) = E(q,u)E(q,-vu)E(q,v).$$

3.2. $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-orbits on a set

Fix an infinite set X. Let $F : X \to X$ be an automorphism of infinite order such that for all $x \in X$, the set $\{F^i(x) \mid i \in \mathbb{Z}\}$ is finite, and let $\sigma \in \operatorname{Aut}(X)$ be an involution that commutes with F. Consider the action of $\Gamma \coloneqq \mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ on X where the first factor acts via F and the second factor via σ.

For $x \in X$, let $r = r(x)$ be the smallest non-negative integer such that

$$F^r(x) = \sigma(x),$$

if one exists, otherwise set $r = \infty$. Let also $d = d(x)$ be the *degree* of x, i.e. the size of its F-orbit. We will call $\nu = (r, d)$ the Γ-degree of the Γ-orbit of x; these are of the following three kinds.

	ν	$\|\nu\|$
(i)	$(0, d)$	d
(ii)	$(r, 2r)$	$2r$
(iii)	(∞, d)	$2d$

where $|\nu|$ denotes the size of the corresponding Γ-orbit and $r > 0$ in case (ii). Orbits of the first kind are of the form $\{x, F(x), \ldots, F^{d-1}(x)\}$ with $x \in X^\sigma$. Orbits of the second kind are of the form $\{x, F(x), \ldots, F^{2r-1}(x)\}$ with x of degree $2r$ satisfying $F^r(x) = \sigma(x)$. Finally, orbits of the third kind are of the form

$$\{x, F(x), \ldots, F^{d-1}(x), \sigma(x), \ldots, F^{d-1}\sigma(x)\},$$

where x has degree d and does not satisfy any equation of the form $F^r(x) = \sigma(x)$.

For ν a given Γ-degree let $\widetilde{N}_\nu(q)$ be the number of Γ-orbits of Γ-degree ν. For integers $r, d > 0$, define

$$N_d := |\{x \in X^\sigma \mid F^d(x) = x\}|, \qquad N'_r := |\{x \in X - X^\sigma \mid F^r(x) = \sigma(x)\}|,$$
$$N_d^\# := |\{x \in X - X^\sigma \mid F^d(x) = x\}|.$$

We denote by μ the ordinary Möbius function.

Proposition 3.9. — *We have*

(i)
$$\widetilde{N}_{(0,d)} = \frac{1}{d}\sum_{r|d} \mu\left(\frac{d}{r}\right) N_d.$$

Let $\widetilde{N}_d^\#$ be the number of F-orbit of $X - X^\sigma$ of size d. Then

$$\widetilde{N}_d^\# = \frac{1}{d}\sum_{e|d} \mu\left(\frac{d}{e}\right) N_e^\#.$$

(ii)
$$\widetilde{N}_{(r,2r)} = \frac{1}{2r}\sum_{s|r,\ r/s \text{ odd}} \mu\left(\frac{r}{s}\right) N'_s.$$

(iii)
$$\widetilde{N}_{(\infty,d)} = \frac{1}{2}\widetilde{N}_d^\# - \begin{cases} \frac{1}{2}\widetilde{N}_{(d/2,d)} & \text{if } d \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — We only prove (ii). Put

$$X'_r := \{x \in X - X^\sigma \mid F^r(x) = \sigma(x)\},$$

and let $X_{(s,2s)}$ be the subset of elements x of X'_s such that $r(x) = s$. Since σ is an involution we have:

$$X'_r = \bigcup_{s|r,\, r/s\,\mathrm{odd}} X_{(s,2s)}.$$

Hence

$$N'_r = \sum_{s|r,\, r/s\,\mathrm{odd}} |X_{(s,2s)}|$$

From the Möbius inversion formula we find that

$$|X_{(r,2r)}| = \sum_{s|r,\, r/s\,\mathrm{odd}} \mu\left(\frac{r}{s}\right) N'_s.$$

We thus deduce (ii) by noticing that $\widetilde{N}_{(r,2r)} = \frac{1}{2r}\,|X_{(r,2r)}|$. □

3.3. Colorings on varieties, infinite products formulas

The first part of this section is a minor extension of [23] which we recall for the convenience of the reader.

We keep the notation of Section 3.2 but here we assume that X is an algebraic variety over $\overline{\mathbb{F}}_q$ which is defined over $\mathbb{F}_q$ and that $F : X(\overline{\mathbb{F}}_q) \to X(\overline{\mathbb{F}}_q)$ is the corresponding Frobenius endomorphism (for any integer $r \geqslant 1$, we have $X^{F^r} = X(\mathbb{F}_{q^r})$). Here we use the notation $\widetilde{N}_{(0,d)}(q)$, $\widetilde{N}_{(d,2d)}(q)$ and $\widetilde{N}_{(\infty,d)}(q)$ instead of $\widetilde{N}_{(0,d)}$, $\widetilde{N}_{(d,2d)}$ and $\widetilde{N}_{(\infty,d)}$.

We also make the assumption that there exists polynomials in $\mathbb{Q}[T]$

$$\widetilde{N}_{(0,1)}(T), \qquad \widetilde{N}_{(1,2)}(T), \qquad \widetilde{N}_{(\infty,1)}(T)$$

such that for any finite field extension $\mathbb{F}_{q^d}$ of $\mathbb{F}_q$, we have

$$\widetilde{N}_{(0,1)}(q^d) = \widetilde{N}_{(0,d)}(q)$$
$$\widetilde{N}_{(1,2)}(q^d) = \widetilde{N}_{(d,2d)}(q)$$
$$\widetilde{N}_{(\infty,1)}(q^d) = \widetilde{N}_{(\infty,d)}(q)$$

Then there exist also polynomials in $\mathbb{Q}[T]$

$$N_1(T), \qquad N'_1(T), \qquad N_1^{\#}(T),$$

such that for any finite field extension $\mathbb{F}_{q^d}$ we have

$$N_1(q^d) = N_d, \quad N'_1(q^d) = N'_d, \quad N_1^{\#}(q^d) = N_d^{\#}.$$

Denote by $\mathcal{P}$ the set of all partitions and denote by 0 the unique partition of 0. For $\lambda\in\mathcal{P}$, we denote by $|\lambda|$ the size of λ. Assume given a *weight function* $W_\lambda:\mathcal{P}\to\mathbb{Q}(T)$, $\lambda\mapsto W_\lambda(T)$ with $W_0(T)=1$. Let X/Γ denotes the set of Γ-orbits of $X(\overline{\mathbb{F}}_q)$. For a map $f:X/\Gamma\to\mathcal{P}$ we let $|f|:=\sum_{\gamma\in X/\Gamma}|\gamma|\cdot|f(\gamma)|\in\mathbb{Z}_{\geqslant 0}\cup\{\infty\}$ be the size of the support of f. We also denote by O_0,O_1,O_∞ the union of the orbits of the first kind, the second kind and the third kind respectively (so that $O_0\cup O_1\cup O_\infty=X/\Gamma$).

Then for $f:X/\Gamma\to\mathcal{P}$ with finite support, we put

$$W_f(q):=\prod_{\gamma\in O_0}W_{f(\gamma)}\left(q^{|\gamma|}\right)\prod_{\gamma\in O_1}W_{f(\gamma)}\left(q^{|\gamma|}\right)\prod_{\gamma\in 0_\infty}W_{f(\gamma)}\left(q^{|\gamma|/2}\right)^2.$$

Consider

$$Z(q,T):=\sum_\lambda W_\lambda(q)T^{|\lambda|},\qquad Z_2(q,T):=\sum_\lambda W_\lambda(q)^2T^{|\lambda|}.$$

Proposition 3.10. — *We have*

$$\sum_{\{f:X/\Gamma\to\mathcal{P},\,|f|<\infty\}}W_f(q)T^{|f|}$$
$$=\prod_{d\geqslant 1}Z(q^d,T^d)^{\widetilde{N}_{(0,d)}(q)}\prod_{r\geqslant 1}Z(q^{2r},T^{2r})^{\widetilde{N}_{(r,2r)}(q)}\prod_{d\geqslant 1}Z_2(q^d,T^{2d})^{\widetilde{N}_{(\infty,d)}(q)}.\tag{3.14}$$

Proof. — We have

$$\begin{aligned}\sum_{\{f:X/\Gamma\to\mathcal{P},\,|f|<\infty\}}&W_f(q)T^{|f|}\\
&=\left(\sum_{f:O_0\to\mathcal{P}}\left(\prod_{\gamma\in O_0}W_{f(\gamma)}\left(q^{|\gamma|}\right)\right)T^{|f|}\right)\left(\sum_{f:O_1\to\mathcal{P}}\left(\prod_{\gamma\in O_1}W_{f(\gamma)}\left(q^{|\gamma|}\right)\right)T^{|f|}\right)\\
&\qquad\times\left(\sum_{f:O_\infty\to\mathcal{P}}\left(\prod_{\gamma\in O_\infty}W_{f(\gamma)}\left(q^{|\gamma|/2}\right)^2\right)T^{|f|}\right)\\
&=\prod_{\gamma\in O_0}\left(\sum_\lambda W_\lambda\left(q^{|\gamma|}\right)T^{|\gamma|\cdot|\lambda|}\right)\cdot\prod_{\gamma\in O_1}\left(\sum_\lambda W_\lambda\left(q^{|\gamma|}\right)T^{|\gamma|\cdot|\lambda|}\right)\\
&\qquad\cdot\prod_{\gamma\in O_\infty}\left(\sum_\lambda W_\lambda\left(q^{|\gamma|/2}\right)^2T^{|\gamma|\cdot|\lambda|}\right)\end{aligned}$$

□

In order to express the left hand side of 3.14 as an infinite product in the variables q and T, we need to compute the Log of the right hand side of (3.14) (see [23] for more details). It is convenient to work formally and consider the following general case.

Put

$$F_0 \coloneqq \prod_{d\geqslant 1}\left(\Omega_0(q^d,T^d)\right)^{\widetilde{N}_{(0,d)}(q)}, \qquad F_1 \coloneqq \prod_{r\geqslant 1}\left(\Omega_1(q^{2r},T^{2r})\right)^{\widetilde{N}_{(r,2r)}(q)},$$
$$F_\infty \coloneqq \prod_{d\geqslant 1}\left(\Omega_\infty(q^d,T^{2d})\right)^{\widetilde{N}_{(\infty,d)}(q)}$$

for some $\Omega_i(q,T)\in 1+T\Lambda[\![T]\!]$ with $i\in\{0,1,\infty\}$.

For $i=1,\infty$, define $\{H_{i,n}(q)\}_n$ by

$$\sum_{n\geqslant 1} H_{i,n}(q)T^n = \operatorname{Log}\left(\Omega_i(q,T)\right).$$

Theorem 3.11. — *We have*

(i) $$\operatorname{Log}(F_0) = N_1(q)\operatorname{Log}\left(\Omega_0(q,T)\right).$$

(ii) $$\operatorname{Log}(F_1) = \frac{1}{2}N_1'(q)\sum_{m\geqslant 1}\left(\sum_{j=0}^{v_2(m)}\frac{1}{2^j}H_{1,m/2^j}(q^{2^{j+1}})\right)T^{2m}.$$

(iii) $$\operatorname{Log}(F_\infty) = \frac{1}{2}N_1^{\#}(q)\sum_{m\geqslant 1}H_{\infty,m}(q)T^{2m} - \frac{1}{2}N_1'(q)\sum_{m\geqslant 1}\sum_{j=1}^{v_2(m)}\frac{1}{2^j}H_{\infty,m/2^j}(q^{2^j})T^{2m}.$$

where v_2 denotes the valuation at 2 and $H_{1,x}=H_{\infty,x}=0$ if x is not an integer.

Proof. — The first identity follows from Lemma 2.3 and Proposition 3.9(i). For the second identity, we compute $\operatorname{Log}(F_1)$ using the two steps procedure mentioned in Remark 2.2. We thus define $\{R_n(q)\}_{n\geqslant 1}$ by

$$\operatorname{Log}(\Omega_1(q,T)) = \sum_{n\geqslant 1} R_n(q)\frac{T^n}{n}.$$

We have

$$\begin{aligned}
\mathrm{Log}(F_1) &= \sum_{r\geqslant 1}\widetilde{N}_{(r,2r)}(q)\,\mathrm{Log}(\Omega_1(q^{2r},T^{2r}))\\
&= \sum_{r\geqslant 1}\widetilde{N}_{(r,2r)}(q)\sum_{n\geqslant 1}R_n(q^{2r})\frac{T^{2rn}}{n}\\
&= \sum_{m\geqslant 1}\left(\sum_{r|m}2r\widetilde{N}_{(r,2r)}(q)R_{\frac{m}{r}}(q^{2r})\right)\frac{T^{2m}}{2m}\\
&= \sum_{N\geqslant 1}C_N(q)\frac{T^N}{N}
\end{aligned}$$

where

$$C_N(q) := \begin{cases}\sum_{r|\frac{N}{2}}2r\widetilde{N}_{(r,2r)}(q)R_{\frac{N}{2r}}(q^{2r}) & \text{if } N \text{ is even}\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\mathrm{Log}(F_1) = \sum_{n\geqslant 1}V_n(q)T^n$$

where $V_n(q) := \frac{1}{n}\sum_{d|n}\mu(d)C_{n/d}(q^d)$.

Hence $V_n(q) = 0$ if n is odd and

$$\begin{aligned}
V_{2m}(q) &= \frac{1}{2m}\sum_{\{d|2m\,,\,\frac{2m}{d}\text{ even}\}}\mu(d)C_{2m/d}(q^d)\\
&= \frac{1}{2m}\sum_{d|m}\mu(d)C_{2m/d}(q^d)\\
&= \frac{1}{2m}\sum_{d|m}\mu(d)\sum_{r|\frac{m}{d}}2r\widetilde{N}_{(r,2r)}(q^d)R_{\frac{m}{rd}}(q^{2rd})\\
&= \frac{1}{m}\sum_{k|m}\sum_{d|k}\mu(d)\frac{k}{d}\widetilde{N}_{(\frac{k}{d},2\frac{k}{d})}(q^d)R_{\frac{m}{k}}(q^{2k})\\
&= \frac{1}{m}\sum_{k|m}R_{\frac{m}{k}}(q^{2k})\sum_{d|k}\mu(d)\frac{k}{d}\widetilde{N}_{(\frac{k}{d},2\frac{k}{d})}(q^d)\\
&= \frac{1}{m}\sum_{k|m}R_{\frac{m}{k}}(q^{2k})\sum_{d|k}\mu\left(\frac{k}{d}\right)d\widetilde{N}_{(d,2d)}(q^{k/d})
\end{aligned}$$

By Proposition 3.9 we have

$$\begin{aligned} r\widetilde{N}_{(r,2r)}(q) &= \frac{1}{2}\sum_{s|r,\, r/s\,\text{odd}} \mu\left(\frac{r}{s}\right) N_1'(q^s) \\ &= \frac{1}{2}\sum_{s|r'} \mu(s) N_1'(q^{r/s}), \end{aligned}$$

where $r' \coloneqq r/2^{v_2(r)}$.

Put $g_s(q) \coloneqq \mu(s)N_1'(q^{1/s})$ and $f_r(q) \coloneqq \sum_{s|r} g_s(q)$. Note that $r\widetilde{N}_{(r,2r)}(q) = \frac{1}{2} f_{r'}(q^r)$. We thus have

$$\begin{aligned} V_{2m}(q) &= \frac{1}{2m}\sum_{k|m} R_{\frac{m}{k}}(q^{2k}) \sum_{d|k} \mu\left(\frac{k}{d}\right) f_{d'}(q^k) \\ &= \frac{1}{2m}\sum_{k|m} R_{\frac{m}{k}}(q^{2k}) \sum_{d|k'} \sum_{j=0}^{v_2(k)} \mu\left(2^{v_2(k)-j}\frac{k'}{d}\right) f_d(q^k) \\ &= \frac{1}{2m}\sum_{k|m} R_{\frac{m}{k}}(q^{2k}) \sum_{d|k'} f_d(q^k) \left(\sum_{j=0}^{v_2(k)} \mu\left(2^{v_2(k)-j}\frac{k'}{d}\right)\right) \end{aligned}$$

Since in the above sum k'/d is odd, the sum on the right hand side equals 0 unless $v_2(k) = 0$. Hence

$$V_{2m}(q) = \frac{1}{2m}\sum_{\{k|m,\, k \text{ odd}\}} R_{\frac{m}{k}}(q^{2k}) \sum_{d|k} \mu\left(\frac{k}{d}\right) f_d(q^k)$$

By the Möbius inversion formula we have

$$\sum_{d|k} \mu\left(\frac{k}{d}\right) f_d(q) = g_k(q).$$

Hence

$$\begin{aligned} V_{2m}(q) &= \frac{1}{2m} N_1'(q) \sum_{\{k|m,\, k \text{ odd}\}} \mu(k) R_{\frac{m}{k}}(q^{2k}) \\ &= \frac{1}{2m} N_1'(q) \sum_{k|m'} \mu(k) R_{\frac{m}{k}}(q^{2k}) \\ &= \frac{1}{2} N_1'(q) \left(\sum_{j=0}^{v_2(m)} \frac{1}{2^j} H_{1,m/2^j}(q^{2^{j+1}})\right), \end{aligned}$$

from which we deduce the second identity.

By Proposition 3.9(iii) we have

$$F_\infty = \left(\prod_{d\geqslant 1} \Omega_\infty(q^d, T^{2d})^{\frac{1}{2}\widetilde{N}_d^{\#}(q)}\right)\cdot\left(\prod_{n\geqslant 1}\Omega_\infty(q^{2n},T^{4n})^{-\frac{1}{2}\widetilde{N}_{(n,2n)}(q)}\right).$$

where $\widetilde{N}_d^{\#}(q)$ is the number of F-orbits in $X(\overline{\mathbb{F}}_q)-X^\sigma(\overline{\mathbb{F}}_q)$ of size d. Hence

$$\operatorname{Log}(F_\infty) = \frac{1}{2}N_1^{\#}(q)\operatorname{Log}\left(\Omega_\infty(q,T^2)\right) + \operatorname{Log}\left(\prod_{n\geqslant 1}\Omega_\infty(q^{2n},T^{4n})^{-\frac{1}{2}\widetilde{N}_{(n,2n)}(q)}\right).$$

The result is then a consequence of the fact that

$$\operatorname{Log}\Omega_\infty(q,T^2) = \sum_{m\geqslant 1} H_{\infty,m}(q)\,T^{2m}.$$ □

3.4. Proof of Theorem 1.1

By Theorem 2.10 we have

$$\frac{\left|\left\{(x_1,\dots,x_r)\in \mathrm{G}(\mathbb{F}_q)^r \mid x_1^2\cdots x_r^2=1\right\}\right|}{|\mathrm{G}(\mathbb{F}_q)|} = \sum_{\chi\in\widehat{\mathrm{G}(\mathbb{F}_q)}_{\mathrm{real}}}\left(\frac{|\mathrm{G}(\mathbb{F}_q)|}{\chi(1)}\right)^{r-2}. \tag{3.15}$$

To describe the real characters of $\mathrm{G}(\mathbb{F}_q)$ we will use the idea of colorings on varieties [23]. Let $\sigma\in\operatorname{Aut}(\mathbb{G}_m)$ be the involution $\sigma(x):=x^{-1}$ We have an action of $\Gamma:=\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{G}_m(\overline{\mathbb{F}}_q)$, where the first factor acts via the Frobenius F and the second via σ. These orbits are described in Section 3.2. However note that here we have $\mathbb{G}_m^\sigma=\{1,-1\}$, and so in the case i) we have only two orbits of the kind $(0,1)$ which are $\{1\}$ and $\{-1\}$. For ν a given Γ-degree let $\widetilde{N}_\nu(q)$ be the number of Γ-orbits of Γ-degree ν. The following formulae for these quantities hold (see Proposition 3.9).

Proposition 3.12.

(i)
$$\widetilde{N}_{(0,1)} = 2$$

(ii)
$$\widetilde{N}_{(r,2r)} = \frac{1}{2r}\sum_{s|r,\ r/s\text{ odd}}\mu\left(\frac{r}{s}\right)(q^s-1)$$

(iii)
$$\widetilde{N}_{(\infty,d)} = \frac{1}{2}\widetilde{N}_d^{\#}(q) - \begin{cases}\frac{1}{2}\widetilde{N}_{(d/2,d)} & \text{if } d \text{ is even}\\ 0 & \text{otherwise.}\end{cases}$$

with

$$\widetilde{N}_d^{\#}(q) = \frac{1}{d}\sum_{e|d}\mu\left(\frac{d}{e}\right)(q^e-3).$$

A real character of $\mathrm{G}(\mathbb{F}_q)$ is uniquely described by a map Λ from Γ-orbits (technically on the dual of $\mathbb{G}_m(\overline{\mathbb{F}}_q)$ but for convenience we will still think of $\mathbb{G}_m(\overline{\mathbb{F}}_q)$) to the set of partitions such that

$$|\Lambda| := \sum_{\gamma\in\mathbb{G}_m/\Gamma} |\gamma|\cdot|\Lambda(\gamma)| = n.$$

Recall that for an integer ϱ we defined (see (1.3))

$$Z_\varrho(q,T) = \sum_\lambda \mathcal{H}_\lambda(q)^\varrho\, T^{|\lambda|},$$

where $\mathcal{H}_\lambda(q)$ is the hook polynomial (1.2), and $\{V_{\varrho,n}(q)\}_n$ by the formula (see (1.4))

$$\sum_{n\geqslant 1} V_{\varrho,n}(q)T^n := \operatorname{Log}\left(Z_\varrho(q,T)\right).$$

Note that $V_{\varrho,n}(q)$ is a rational function of q for $\varrho < 0$ and a Laurent polynomial for $\varrho \geqslant 0$.

By the Mass Formula (3.15) we have

$$M_{\varrho,n}(q) := \frac{\left|\left\{(x_1,\dots,x_r)\in\mathrm{G}(\mathbb{F}_q)^r \mid x_1^2\cdots x_r^2 = 1\right\}\right|}{|\mathrm{G}(\mathbb{F}_q)|} \tag{3.16}$$

$$= \sum_{\{\Lambda:\mathbb{G}_m/\Gamma\to\mathcal{P},\,|\Lambda|=n\}} \left(\frac{|\mathrm{G}(\mathbb{F}_q)|}{\chi_\Lambda(1)}\right)^\varrho, \tag{3.17}$$

with $\varrho = r-2$. Using the formula for $\chi(1)$ in terms of hook polynomials (see for instance [20, IV, 6.7]) we obtain the following.

$$q^{-\varrho\binom{n}{2}} M_{\varrho,n}(q) = \sum_{\{\Lambda:\mathbb{G}_m/\Gamma\to\mathcal{P},\,|\Lambda|=n\}} \mathcal{H}_\Lambda(q)^\varrho,$$

By Formula (3.14) we obtain the following

Proposition 3.13.

$$\begin{aligned} M_\varrho(q,T) &= 1 + \sum_{n\geqslant 1} q^{-\varrho\binom{n}{2}} M_{\varrho,n}(q)\, T^n \\ &= \prod_{d\geqslant 1} Z_\varrho(q^d,T^d)^{\widetilde{N}_{(0,1)}(q)} \prod_{r\geqslant 1} Z_\varrho(q^{2r},T^{2r})^{\widetilde{N}_{(r,2r)}(q)} \\ &\qquad\qquad \times \prod_{d\geqslant 1} Z_{2\varrho}(q^d,T^{2d})^{\widetilde{N}_{(\infty,d)}(q)}. \end{aligned}$$

Now we can complete the proof of Theorem 1.1: (ii) follows by combining Proposition 3.13 with Theorem 3.11; (i) is a direct consequence of (ii) given the above observation on the nature of $V_{\varrho,n}(q)$; (iii) is just a restatement of (ii).

Remark 3.14. — The statement (Theorem 1.1(i)) that $M_{\varrho,n}(q)$ is a polynomial in q for $r > 1$ also follows from the main result of [10] as the abelianization of the fundamental group

$$\Pi = \left\langle a_1, \ldots, a_r \,\middle|\, a_1^2 \cdots a_r^2 = 1 \right\rangle .$$

is infinite for $r > 1$ (but finite for $r = 1$).

4. Orientation cover of non-orientable surfaces

4.1. Character varieties

Let $r \geqslant 1$ be an integer, put $\varrho = r - 2$ and denote by Σ a compact non-orientable surface of Euler characteristic $-\varrho$ (the connected sum of r real projective planes). Consider a set $S = \{\alpha_1, \ldots, \alpha_k\}$ of k-points of Σ ($k \geqslant 1$). Fix a base point $b \in \Sigma \backslash S$. The fundamental group $\Pi = \pi_1(\Sigma\backslash S, b)$ has the presentation

$$\Pi = \left\langle a_1, \ldots, a_r, x_1, \ldots, x_k \,\middle|\, a_1^2 \cdots a_r^2 x_1 \cdots x_k = 1 \right\rangle .$$

Let K be an algebraically closed field and consider a *generic* k-tuple $\mathcal{C} = (C_1, \ldots, C_k)$ of semisimple conjugacy classes of $\mathrm{G} = \mathrm{GL}_n(K)$ such that

$$\prod_{i=1}^{k} \det(C_i) = 1.$$

The genericity condition means that for any $1 \leqslant s < n$, if we select s eigenvalues of C_i for each $i = 1, \ldots, k$ (possibly with multiplicities), then the product of the sk selected eigenvalues is not equal to 1.

We denote by σ the automorphism $\mathrm{G} \to \mathrm{G}$, $g \mapsto {}^t g^{-1}$ and G^+ the semidirect product $\mathrm{G} \rtimes \langle \sigma \rangle$. We consider the representation varieties $\mathcal{U}_{\mathcal{C}}$ of the $\rho \in \mathrm{Hom}(\Pi, \mathrm{G}^+)$ such that

$$\forall j = 1, \ldots, r \quad \forall i = 1, \ldots, k, \quad \rho(a_j) \in \mathrm{G}\sigma, \rho(x_i) \in \iota(C_i)$$

It can be explicitly described as the space of $k + r$-tuples

$$(A_1, \ldots, A_r, X_1, \ldots, X_k) \in \mathrm{G}^r \times C_1 \times \cdots \times C_k$$

satisfying the equation

$$A_1\sigma(A_1)\cdots A_r\sigma(A_r)X_1\cdots X_k = 1.$$

Let G act on $\mathcal{U}_\mathcal{C}$ by

$$A_j \mapsto gA_j\sigma(g^{-1}), \qquad x_i \mapsto gx_ig^{-1}, \qquad g \in G$$

and we consider the quotient stack

$$\mathcal{M}_\mathcal{C} := [\mathcal{U}_\mathcal{C}/\mathrm{G}].$$

Remark 4.1. — Note that unlike in the situation of [13], even though the k-tuple $\mathcal{C}$ is assumed to be generic, the action of G is far from being free. For instance, if $r = 1$, $n = 2$ and $k = 1$ with the central matrix $X_1 = \mathrm{diag}(-1,-1)$ at the puncture, then $A = \mathrm{diag}(1,-1)$ is a solution to the equation

$$A\sigma(A)X_1 = I_2$$

stabilized by the torus of diagonal matrices.

Remark 4.2. — The character stack $\mathcal{M}_\mathcal{C}$ has an alternative description that can be found in [25, 24] and we now describe briefly. Denote by $p : \widetilde{\Sigma} \to \Sigma$ the orientation covering of Σ. This is an unramified double covering with $\widetilde{\Sigma}$ a compact Riemann surface of genus $g = r - 1$. Considering the homomorphism $\mathrm{Gal}(p) \to \mathrm{Aut}(\mathrm{G})$ that maps the non-trivial element of the Galois group of p to σ; let us identify $\mathrm{Gal}(p)$ with $\langle\sigma\rangle$. The orientation character $\chi : \Pi \to \langle\sigma\rangle$ maps a loop that preserves the orientation to 1 and a loop that reverses the orientation to σ. It thus maps a_j to σ and x_i to 1. Denote by $\widetilde{\Pi}$ the fundamental group of $\widetilde{\Sigma}\backslash p^{-1}(S)$ with some base point $\widetilde{b}$ above b.

A representation $\rho \in \mathcal{U}_{\overline{\mathcal{C}}}$ defines (by restriction) a representation $\widetilde{\rho} : \widetilde{\Pi} \to \mathrm{G}$ making the following diagram commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widetilde{\Pi} & \xrightarrow{p_*} & \Pi & \xrightarrow{\chi} & \langle\sigma\rangle & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \widetilde{\rho}} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 1 & \longrightarrow & \mathrm{G} & \xrightarrow{\iota} & \mathrm{G}^+ & \xrightarrow{\pi} & \langle\sigma\rangle & \longrightarrow & 1 \end{array} \tag{4.1}$$

In order to understand local monodromies we need to choose the generators of Π precisely. For $i = 1, \ldots, k$, we let D_i be a small open neighbourhood (homeomorphic to an open disc in $\mathbb{C}$) of α_i in Σ, such that p is trivial over D_i. Let β_i be a point in D_i and let λ_i be a path from x to β_i. We choose a single loop ℓ_i in D_i based at β_i around α_i and we take the generator x_i of Π to be $\lambda_i^{-1}\ell_i\lambda_i$. The path λ_i lifts to a unique path $\widetilde{\lambda}_i$ from $\widetilde{b}$ to some point $\widetilde{\beta}_{i1}$ above β_i. We let $\widetilde{D}_{i1}$ be the connected component of $p^{-1}(D_i)$ containing $\widetilde{\beta}_{i1}$, $\widetilde{\alpha}_{i1}$ the unique point above α_i in $\widetilde{D}_{i1}$ and $\widetilde{\ell}_{i1}$ the loop in $\widetilde{D}_{i1}$ based at $\widetilde{\beta}_{i1}$ around $\widetilde{\alpha}_{i1}$ which lifts ℓ_i. Then we let x_{i1} be the generator $\widetilde{\lambda}_i^{-1}\widetilde{\ell}_i\widetilde{\lambda}_i$ of $\widetilde{\Pi}$. Let $\gamma_\sigma \in \Pi$ be a loop reversing the orientation. If instead of the path λ_i from x to β_i we choose the path $\lambda_i\gamma_\sigma$, then we get into the other connected component of $p^{-1}(D_i)$ with punctures α_{i2} and we obtain an other generator x_{i2} of $\widetilde{\Pi}$ which is mapped to $\gamma_\sigma^{-1}x_i\gamma_\sigma$ via p.

Therefore if $\rho(x_i) \in \iota(C_i)$ then $\widetilde{\rho}(x_{i1}) \in C_i$ and $\widetilde{\rho}(x_{i2}) \in \sigma(C_i)$. Put $C_{ij} = C_i$ if $j = 1$ and $\sigma(C_i)$ if $j = 2$, and consider the representation variety

$$\widetilde{\mathcal{U}}_{\mathcal{C}} := \left\{\widetilde{\rho} \in \mathrm{Hom}(\widetilde{\Pi}, \mathrm{G}) \,\middle|\, \widetilde{\rho}(x_{ij}) \in \iota(C_{ij})\right\}$$

Say that a pair $(\widetilde{\rho}, h_\sigma) \in \widetilde{\mathcal{U}}_{\mathcal{C}} \times G$ is σ-invariant if we have

$$\begin{cases} h_\sigma\widetilde{\rho}(z)h_\sigma^{-1} = \sigma\left(\widetilde{\rho}(\gamma_\sigma^{-1}z\gamma_\sigma)\right) & \text{for all } z \in \widetilde{\Pi} \\ \widetilde{\rho}(\gamma_\sigma^2) = h_\sigma^{-1}\sigma(h_\sigma^{-1}) \end{cases}.$$

If the two above conditions are satisfied then the representation $\widetilde{\rho}$ can be extended to an homomorphism $\rho : \Pi \to \mathrm{G}^+$ making the diagram (4.1) commutative. The group G acts on the space $\widetilde{\mathcal{U}}_{\mathcal{C},\sigma}$ of σ-invariant pairs as follows

$$g \cdot (\widetilde{\rho}, h_\sigma) = (g \cdot \widetilde{\rho}, \sigma(g)h_\sigma g^{-1}),$$

where $g \cdot \widetilde{\rho}$ is the representation obtained by composing $\widetilde{\rho}$ with the conjugation by g.

Then [25, Theorem 2.2.1] we have an isomorphism

$$\mathcal{M}_{\mathcal{C}} \simeq [\widetilde{\mathcal{U}}_{\mathcal{C},\sigma}/\mathrm{G}].$$

The stack $[\widetilde{\mathcal{U}}_{\mathcal{C},\sigma}/\mathrm{G}]$ can be described in terms of local systems $\mathcal{L}$ on $\widetilde{\Sigma}\backslash p^{-1}(S)$ satisfying $\mathcal{L} \simeq \sigma^*(\mathcal{L}^\vee)$ (where $\mathcal{L}^\vee$ is the dual local system of $\mathcal{L}$) with local monodromy in C_{ij} at the puncture α_{ij}, see [25, Section 6.1.2] for more details.

4.1.1. The Cauchy function

Recall that in Section 1.2 we defined the Cauchy function $\Omega(z, w)$. Let $\mathbb{H}_n(z, w)$ be the degree n component of $(z^2 - 1)(1 - w^2)\,\mathrm{Log}\,(\Omega(z, w))$ and for a k-tuple of partitions $\mu = (\mu^1, \dots, \mu^k)$ of n let

$$\mathbb{H}_n(z, w) := (z^2 - 1)(1 - w^2)\left\langle \mathrm{Log}\,(\Omega(z, w))\,, h_\mu\right\rangle, \tag{4.2}$$

where $h_\mu = h_{\mu^1}(\mathbf{x}_1)\cdots h_{\mu^k}(\mathbf{x}_k)$ and $h_{\mu^i}(\mathbf{x}_i)$ is the complete symmetric function in the variables $\mathbf{x}_i$.

For a partition λ denotes by $m_\lambda = m_\lambda(\mathbf{x})$ the corresponding monomial symmetric function in the variables $\mathbf{x} = \{x_1, x_2, \dots\}$. A computation shows that we have

Lemma 4.3. — *If $r = k = 1$, we have*

$$\mathbb{H}_n(z,w) = \begin{cases} (z-w)m_{(1)} & n=1 \\ \frac{1}{z^2+1}m_{(2)} + m_{(1^2)} & n=2. \end{cases}$$

Conjecture 4.4. — *If $r = 1$ and $k = 1$. We have $\mathbb{H}_n(z,w) = 0$ for $n \geqslant 2$. Equivalently,*

$$(z^2-1)(1-w^2)\operatorname{Log}\left(\sum_{\lambda\in\mathcal{P}}\prod \frac{(z^{2a+1}-w^{2l+1})}{(z^{2a+2}-w^{2l})(z^{2a}-w^{2l+2})}\widetilde{H}_\lambda(\mathbf{x};z^2,w^2)\right)$$
$$= (z-w)m_{(1)}(\mathbf{x}) + \frac{1}{z^2+1}m_{(2)}(\mathbf{x}) + m_{(1^2)}(\mathbf{x}).$$

(The equivalence follows from Lemma 4.3.)

4.1.2. Proof of Theorem 1.8

Assume that $K = \overline{\mathbb{F}}_q$ and we consider on G the $\mathbb{F}_q$-structure induced by the Frobenius that raises coefficients of matrices to their q-th power. We also assume that the eigenvalues of the conjugacy classes $C_1, \dots, C_k$ are in $\mathbb{F}_q$.

As G is connected we have

$$\begin{aligned}|\mathcal{M}_\mathcal{C}(\mathbb{F}_q)| &= \frac{|\mathcal{U}_\mathcal{C}(\mathbb{F}_q)|}{|\mathrm{G}(\mathbb{F}_q)|} \\ &= \frac{\left|\left\{((A_i)_i,(X_i)_i)\in(\mathrm{G}(\mathbb{F}_q))^r\times\prod_{i=1}^k C_i(\mathbb{F}_q)\,\middle|\,\prod_{i=1}^r A_i\sigma(A_i)\prod_{i=1}^k X_i = 1\right\}\right|}{|\mathrm{G}(\mathbb{F}_q)|}.\end{aligned}$$

For each $i = 1, \dots, k$, let $\mu^i = (\mu^i_1, \mu^i_2, \dots)$ be the partition of n given by the multiplicities of the eigenvalues of the conjugacy class $\mathcal{C}_i$ and let μ be the k-tuple $(\mu^1, \dots, \mu^k)$.

Theorem 4.5. — *The stack $\mathcal{M}_\mathcal{C}$ has rational count. More precisely,*

$$|\mathcal{M}_\mathcal{C}(\mathbb{F}_q)| = \frac{q^{d_\mu/2}}{q-1}\mathbb{H}_\mu\left(\sqrt{q}, \frac{1}{\sqrt{q}}\right).$$

with $d_\mu = n^2(r-2+k) + 2 - \sum_{i,j}(\mu^i_j)^2$.

Proof. — Using Theorem 2.10 the proof is completely similar to that of [13, Theorem 5.2.1]. □

Now Theorem 1.8 follows from Theorem 2.9.

Remark 4.6. — If $r = 2h$ is even, let Σ' be a compact Riemann surface of Euler characteristic $r-2$ and a subset $S' = \{\alpha'_1, \dots, \alpha'_k\} \subset \Sigma'$. Consider the stacky character variety

$$\mathcal{M}'_{\mathcal{C}} = \left[\left\{\rho \in \mathrm{Hom}\left(\pi_1(\Sigma' \backslash S')\right), \mathrm{G} \,\middle|\, \rho(z_i) \in C_i\right\}/\mathrm{G}\right].$$

where z_i is a single loop around the puncture α'_i. Then by Theorem 4.5 and [13, Theorem 1.2.3] we have

$$\mathrm{E}(\mathcal{M}'_{\mathcal{C}}; x) = \mathrm{E}(\mathcal{M}_{\mathcal{C}}; x).$$

4.2. Mixed Poincaré series in the case $r = k = 1$

Theorem 4.7. — *Assume that $r = k = 1$. Then $\mathcal{M}_{\mathcal{C}}$ is empty unless $n = 1, 2$ in which case we have*

$$\mathrm{H}_c(\mathcal{M}_{\mathcal{C}}; q, t) = \frac{\left(t\sqrt{q}\right)^{d_\mu}}{qt^2 - 1} \mathbb{H}_\mu\left(t\sqrt{q}, -\frac{1}{\sqrt{q}}\right). \tag{4.3}$$

Proof. — Consider the solutions to the equation

$$A\sigma(A) = X, \qquad A \in \mathrm{G}, \tag{4.4}$$

where $X := \mathrm{diag}(\xi_1, \dots, \xi_n)$. (For a calculation of the number of solutions for arbitrary X see [9].)

For $n = 1$, we always have $A\sigma(A) = 1$ and so the space $\mathcal{M}_{\mathcal{C}}$ is nothing but the quotient stack $[\mathbb{G}_m/\mathbb{G}_m]$ for the trivial action of $\mathbb{G}_m$ on itself. The later is isomorphic to $\mathbb{G}_m \times \mathrm{B}\mathbb{G}_m$. The mixed Poincaré polynomial of $\mathbb{G}_m$ is $t + qt^2$ and the mixed Poincaré series of the classifying stack $\mathrm{B}\mathbb{G}_m$ is $1/(qt^2 - 1)$ [5, 9.1.1, 9.1.4], therefore by Künneth formula

$$\mathrm{H}_c(\mathcal{M}_{\mathcal{C}}; q, t) = \frac{t + qt^2}{qt^2 - 1},$$

and so (4.3) is true when $r = k = n = 1$ by Lemma 4.3.

Going back to Equation (4.4), if $A = (a_{i,j})$ this amounts to solving the equations $\xi_i a_{j,i} = a_{i,j}$ for $i, j = 1, \dots, n$. Hence if $\xi_i \neq 1$ then $a_{i,i} = 0$. The equations imply $\xi_i \xi_j a_{i,j} = a_{i,j}$. If x is generic and $n > 2$ we do not have $\xi_i \xi_j = 1$ for $i \neq j$ or $\xi_i = 1$. Therefore the only solution is identically zero and (4.4) has no solution in G.

For $n = 2$ we have two possibilities for generic X:

(i) $\xi_1 = \xi_2 = -1$ and $A = \left(\begin{smallmatrix} 0 & \alpha \\ -\alpha & 0 \end{smallmatrix}\right)$, with α non-zero or
(ii) $\xi_1 = \xi, \xi_2 = \xi^{-1}$ for some $\xi \neq \pm 1$ and $A = \left(\begin{smallmatrix} 0 & \xi\alpha \\ \alpha & 0 \end{smallmatrix}\right)$, with α non-zero.

In case (i), we see from the above calculation that $\mathcal{M}_{\mathcal{C}}$ is the quotient stack $[\mathbb{G}_m / \mathrm{GL}_2]$ where GL_2 acts on $\mathbb{G}_m$ by the determinant. Writing $\mathbb{G}_m$ as the quotient $\mathrm{GL}_2 / \mathrm{SL}_2$, we find that $[\mathbb{G}_m / \mathrm{GL}_2]$ is the classifying stack $\mathrm{B}(\mathrm{SL}_2)$ whose mixed Poincaré series is $\frac{1}{qt^2(qt^2-1)(qt^2+1)}$ by [5, 9.1.1, 9.1.4]. Now (4.3) follows from Lemma 4.3.

In case (ii), the stack $\mathcal{M}_{\mathcal{C}}$ is isomorphic to the stack $[\mathbb{G}_m/\mathrm{T}_2]$, where the group $\mathrm{T}_2 \subset \mathrm{GL}_2$ of diagonal matrices acts by the determinant. Writing $\mathbb{G}_m$ as $\mathrm{T}_2/\mathrm{T}_2'$, where $\mathrm{T}_2' = \mathrm{T}_2 \cap \mathrm{SL}_2 \simeq \mathbb{G}_m$, we find that

$$\mathrm{H}_c(\mathcal{M}_{\mathcal{C}}; q, t) = \frac{1}{qt^2 - 1},$$

from which, together with Lemma 4.3, we deduce (4.3) in this case. □

Remark 4.8. — Note that the combinatorial Conjecture (4.4) implies that the right-hand side of (4.3) is zero for $n > 2$. In particular, the above theorem together with Conjecture (4.4) imply that (4.3) is true for all n.

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Manuscrit reçu le 14 septembre 2020,
révisé le 25 juin 2021,
accepté le 4 novembre 2021.

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