Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

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Tome 73, nº 2 (2023), p. 447-478.

https://doi.org/10.5802/aif.3529

e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
73, 2 (2023) 447-478

ORDER 5 BRAUER–MANIN OBSTRUCTIONS TO THE INTEGRAL HASSE PRINCIPLE ON LOG K3 SURFACES

by Julian LYCZAK (*)

Abstract. — We construct families of log K3 surfaces and study the arithmetic of their members. We use this to produce explicit surfaces with an order 5 Brauer–Manin obstruction to the integral Hasse principle.

Résumé. — Nous construisons plusieurs familles de surfaces log K3 et en étudions l'arithmétique. Nous en déduisons des exemples explicites de surfaces avec une obstruction de Brauer–Manin d'ordre 5 au principe de Hasse entier.

Introduction

The goal of this paper is to add to the study of integral points on ample *log K3 surfaces* as started by Harpaz [16]. This is done by first giving a geometrically flavoured construction for such equations. One upshot of this construction is that one even gets a family of such surfaces for which the arithmetic properties of the members can be studied simultaneously. We will present two families of log K3 surfaces for which a positive proportion of the fibres fails the *Hasse principle*, i.e. these surfaces are everywhere locally soluble but they do not admit integral points.

For the geometrically similar K3 surfaces it has been conjectured by Skorobogatov that the *Brauer–Manin obstruction* is the only one to the Hasse principle [27]. This is however not the case for log K3 surfaces [16]. The examples in this paper exhibit new arithmetic behaviour and it is hoped

Keywords: Integral points, log K3 surface, integral Hasse principle, Brauer–Manin obstruction.
2020 *Mathematics Subject Classification:* 14G12, 14J26, 14F22, 11G35.
(*) This paper was completed as part of a project which received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 754411.

that the accompanying ideas for studying log K3 surfaces will contribute to a workable conjecture for integral points on log K3 surfaces.

Main results

We will use the *integral Brauer–Manin obstruction* as introduced by Colliot-Thélène and Xu [7] to prove the failure of the integral Hasse principle, which is based on the Brauer–Manin obstruction by Manin [23]. Let $\mathcal{U}/\mathbb{Z}$ be a model of a variety $U/\mathbb{Q}$ for which we want to prove that $\mathcal{U}(\mathbb{Z}) = \emptyset$. The technique uses an element $\mathcal{A} \in \operatorname{Br} U := \mathrm{H}^2(U, \mathbb{G}_m)$ to define an intermediate set

$$\mathcal{U}(\mathbb{Z}) \subseteq \mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})^{\mathcal{A}} \subseteq \mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})$$

in the inclusion of the integral points of $\mathcal{U}$ in its set of integral adelic points, meaning its points over the ring $\mathbb{A}_{\mathbb{Q},\infty} = \mathbb{R} \times \prod_p \mathbb{Z}_p$. Hence if $\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})$ is non-empty but $\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})^{\mathcal{A}}$ is empty, then $\mathcal{A}$ *obstructs the integral Hasse principle* on $\mathcal{U}$. The *order of the obstruction* is the order of $\mathcal{A}$ in $\operatorname{Br} U/\operatorname{Br} \mathbb{Q}$. We will be mainly interested in obstructions coming from elements in the *algebraic Brauer group* $\operatorname{Br}_1 U := \ker\left(\operatorname{Br} U \to \operatorname{Br} \overline{U}\right) \subseteq \operatorname{Br} U$.

In this paper we have restricted to a specific type of log K3 surface to showcase our ideas. In passing we pick up the first Brauer–Manin obstructions to the Hasse principle of order greater than 3. The existence of a high order element in the Brauer group of our log K3 surfaces depends on the splitting field of a related del Pezzo surface. Recall that the splitting field of a del Pezzo surface is the minimal field over which all its -1-curves are defined.

Theorem 0.1 (Theorem 2.5). — *Let X be a quintic del Pezzo surface over $\mathbb{Q}$ with splitting field K. We consider the log K3 surface $U = X \backslash C$ with C a geometrically irreducible anticanonical divisor on X. We have*

$$\operatorname{Br}_1 U/\operatorname{Br}\mathbb{Q} \cong \begin{cases} \mathbb{Z}/5\mathbb{Z} & \text{if } K/\mathbb{Q} \text{ is cyclic of degree } 5; \\ 0 & \text{otherwise.} \end{cases}$$

Also, each cyclic extension $K/\mathbb{Q}$ of degree 5 is the splitting field of a del Pezzo surface over $\mathbb{Q}$. Such a surface is unique up to isomorphism.

We will consider del Pezzo surfaces with a non-trivial algebraic Brauer group and our first explicit example comes from the quintic extension $\mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1})/\mathbb{Q}$. Consider the projective scheme $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ given by the five quadratic forms

$$u_0u_3 + 22u_0u_4 + 121u_0u_5 - u_1^2 - 121u_1u_3 + 2662u_1u_4 - 36355u_2u_4 - 9306u_2u_5 + 10494u_3u_4 - 242u_3u_5 - 215501u_4^2 + 68123u_4u_5 - 13794u_5^2,$$

$$u_0u_4 + 11u_0u_5 - u_1u_2 - 11u_1u_3 + 242u_1u_4 - 3223u_2u_4 - 847u_2u_5 + 902u_3u_4 - 11u_3u_5 - 19272u_4^2 + 6413u_4u_5 - 1331u_5^2,$$

$$u_0u_5 - u_1u_3 + 22u_1u_4 - u_2^2 - 286u_2u_4 - 77u_2u_5 + 77u_3u_4 - 1694u_4^2 + 572u_4u_5 - 121u_5^2,$$

$$u_1u_4 - u_2u_3 - 11u_2u_4 - 77u_4^2 + 55u_4u_5 - 11u_5^2,$$

$$u_1u_5 - u_2u_4 - 11u_2u_5 - u_3^2 + 11u_3u_4 - 44u_4^2.$$

This scheme is constructed and studied in Section 3. In Section 4 we prove the arithmetic properties stated in the following theorem.

Theorem 0.2. — *For each geometrically irreducible hyperplane section $\mathcal{C}_h \coloneqq \{h = 0\} \cap \mathcal{X}$ we define $\mathcal{U}_h = \mathcal{X} \backslash \mathcal{C}_h$.*

(1) *The scheme $\mathcal{X}/\mathbb{Z}$ is a flat proper model of the del Pezzo surface over $\mathbb{Q}$ which splits over the quintic extension $\mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1})$.*
(2) *The existence of an algebraic Brauer–Manin obstruction to the integral Hasse principle on $\mathcal{U}_h$ only depends on the reduction of h modulo 11.*
(3) *There exists an h, and hence even a residue class $h \bmod 11$, for which $\mathcal{U}_h$ has an order 5 obstruction to the integral Hasse principle.*

The same construction can be used to produce many more examples. The arithmetic behaviour is mainly determined by the primes which are ramified in the splitting field K. Any tamely ramified prime can be studied in a similar matter. For completeness we also add an example in Section 5 involving the wildly ramified prime 5.

Theorem 0.3. — *There exists a scheme $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ with the following properties.*

(1) *The scheme $\mathcal{X}$ is a flat model for the del Pezzo surface $X = \mathcal{X}_{\mathbb{Q}}$ over $\mathbb{Q}$ which splits over the unique quintic number field $K \subseteq \mathbb{Q}(\zeta_{25})$.*

(2) *The existence of an algebraic Brauer–Manin obstruction on* $\mathcal{U}_h := \mathcal{X}\backslash\{h = 0\}$ *to the integral Hasse principle only depends on the reduction of* h *modulo* 25.
(3) *There exists an* h*, and hence even a residue class* $h \bmod 25$*, for which* $\mathcal{U}_h$ *has an order* 5 *obstruction to the integral Hasse principle.*

We will also briefly touch upon the cases where the Brauer–Manin obstruction and many other obstructions, for example those introduced by Jahnel and Schindler [19], are inconclusive. In these situations one is sometimes able to find an explicit point, but the question whether $\mathcal{U}_h(\mathbb{Z})$ is non-empty is still open for many linear forms h.

Let us put these results in context.

Integral points on log K3 surfaces

Integral points on log K3 surfaces are believed to behave to a certain degree in a similar way to rational points on K3 surfaces. For those surfaces it has been conjectured by Skorobogatov [27] that the existence of solutions is completely controlled by the Brauer–Manin obstruction. However, results by Ieronymou and Skorobogatov [18] and Skorobogatov and Zarhin [28] say that there cannot be an odd order Brauer–Manin obstruction to the Hasse principle for smooth diagonal quartic surfaces over the rational numbers and for Kummer varieties over any number field. An algebraic obstruction of order 3 on a K3 surface was found in [10] and Berg and Várilly-Alvarado [2] even produced a transcendental cubic obstruction.

For log K3 surfaces the situation is however different; it was proven that the Brauer–Manin obstruction is not the only obstruction to the integral Hasse principle [16] and [19]. On the other hand, Colliot-Thélène and Wittenberg [9] showed that the Brauer group never obstructs the Hasse principle for the equation $x^3 + y^3 + z^3 = n$ which is in line with the conjecture that this equation has an integral solution for $n \not\equiv \pm 4 \mod 9$.

The Hasse principle and the effectivity of the Brauer–Manin obstruction for the equation $x^3 + y^3 + z^3 - xyz = k$ were studied by Ghosh and Sarnak [13], Colliot-Thélène, Wei and Xu [8], and Loughran and Mitankin [21]. Another classical affine cubic equation was studied in this manner by Bright and Loughran [4].

This paper gives the first examples of higher odd order Brauer–Manin obstructions on any type of scheme; all other known examples of the Brauer–Manin obstruction to the (integral) Hasse principle are of either order 2 or

3. This is the highest possible prime order for such an obstruction on log K3 surfaces; for a generic anticanonical divisor C on a del Pezzo surface X the order of algebraic Brauer groups of $X\backslash C$ is only divisible by the primes 2, 3 and 5, see Table 1 in [5]. The results in [5] also show that the quintic algebraic obstructions described in this paper are particular to the degree 5 case; there is an inclusion $\operatorname{Br} X \hookrightarrow \operatorname{Br}_1 U$ and the order of an element in the cokernel divides the degree of the del Pezzo surface X. Hence if $\operatorname{Br}_1 U / \operatorname{Br} X$ has 5-torsion then X is a quintic del Pezzo surface.

A study in families

The novel approach in this paper is to study affine surfaces U in families by fixing the compactification X and letting the complementary divisor C vary. An understanding of the arithmetic and geometry of X will be helpful in studying the open surfaces U.

We propose a general methodology for studying this setup, which we illustrated in the special setting of del Pezzo surfaces of degree 5. An important result for these surfaces is that they always have a point. This is a classical result by Enriques [12] which was also proved by Swinnerton-Dyer [29], Skorobogatov [26] and many others. This proves that X is rational over k [30, Theorem 2.1], $\operatorname{Br} X / \operatorname{Br} k = 0$ and that X satisfies weak approximation. It also allows one to classify and to construct such surfaces over k, which was done in detail in [14].

We produce models $\mathcal{X}/\mathbb{Z}$ for $X/\mathbb{Q}$ by following this construction over the integers. In this process one has a few more choices along the way which allow one to control the reductions $\mathcal{X}_\mathfrak{p}$ for all primes $\mathfrak{p}$. To finally construct a model of a log K3 surface one considers the complement $\mathcal{U}_h$ of a hyperplane section $\{h = 0\}$ in $\mathcal{X}$.

Using the abundance of points on quintic del Pezzo surfaces we can deduce that for any h the open subscheme $\mathcal{U}_h$ has points over all $\mathbb{Z}_\ell$, except possibly for a very few small primes ℓ. In our cases only local solubility at $\ell = 2$ is not immediate and will depend on h.

We also use the geometric and arithmetic properties of quintic del Pezzo surfaces to compute the Brauer–Manin obstruction on each $\mathcal{U}_h$. We show that the invariant maps are identically 0 for all but an explicit finite list of primes. To effectively deal with a remaining prime p we show that it is enough to only study the closed fibre of $\mathcal{X} \times \mathbb{Z}_p$; a surprising result especially for the wildly ramified prime $p = 5$. We end up with examples of quintic Brauer–Manin obstructions to both the Hasse principle and strong approximation.

There is no reason why this construction only works for affine opens of quintic del Pezzo surfaces; one could use a similar construction to produce models of rational varieties while controlling the arithmetic of the individual fibres.

Outline

We start by recalling some necessary facts on del Pezzo and log K3 surfaces, Brauer groups and the Brauer–Manin obstruction. Then we compute the algebraic Brauer group of log K3 surfaces $U = X\backslash C$ where X is a del Pezzo surface of degree 5 and C is an anticanonical divisor. In the third section we give an example of a construction of a model $\mathcal{X}/\mathbb{Z}$ of a quintic del Pezzo surface $X/\mathbb{Q}$ such that any anticanonical complement $U_h = X\backslash\{h = 0\}$ has an element of order 5 in the Brauer group. The next section is devoted to the arithmetic of each $\mathcal{U}_h$. In particular we compute the Brauer–Manin obstruction coming from the element of order 5. In the last section we use the same construction to produce a different family of log K3 surfaces $\mathcal{U}_h$ for which the arithmetic behaves differently, but there still is an element of order 5 in the Brauer group.

Notation and conventions

Let k be a field. We will write $\overline{k}$ for a fixed algebraic closure and k^{sep} for the separable closure of k in $\overline{k}$. The absolute Galois group of a field k is denoted by $G_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. A *variety* over a field k is a separated scheme of finite type over $\operatorname{Spec} k$. A *curve* over a field k is a variety over k of pure dimension 1, it need not be irreducible, reduced or smooth. A *surface* over a field k is a geometrically integral variety of dimension 2 over k. A *curve on a surface* over a field k is a closed subscheme of the surface which is a curve over k. For a scheme X over a field k we will write X_K for the base change $X \times_k K$ for any field extension K of k. The notation $\overline{X}$ will be synonymous with $X_{\overline{k}}$.

Acknowledgements

Most results in this article were obtained during my Ph.D. studies at Leiden University. I would like to thank my supervisor Martin Bright for his help and our many discussions on the subject. I also thank the anonymous referee for their careful reading and their many insightful comments and suggestions.

1. Preliminaries

We will consider the existence of integral solutions to polynomial equations defining surfaces. We will first collect the key notions and results on the surfaces we will encounter. Then we will recall the necessary results on Brauer groups and the Brauer–Manin obstruction to integral points.

1.1. Del Pezzo surface of degree 5

We will review some facts about del Pezzo surfaces. The main references will be [24] and [11]. For an overview of the arithmetic of such surfaces one is referred to [30].

DEFINITION 1.1. — *Let k be a field. A* del Pezzo surface *is a smooth projective surface X over k such that the anticanonical line bundle ω_X^{-1} is ample. The degree of a del Pezzo surface is the anticanonical self-intersection K_X^2.*

We will only need del Pezzo surfaces of degree 5. In which case ω_X^{-1} is even very ample over k and we see that every del Pezzo surface of degree 5 can be embedded as a degree 5 surface in $\mathbb{P}_k^5$. Let us collect some facts on the geometry of these surfaces.

LEMMA 1.2. — *Let X be a del Pezzo surface of degree 5 over a separably closed field k.*

(1) *The Picard group $\operatorname{Pic} X$ is free of rank 5 and it has an orthogonal basis $L_0, L_1, \ldots, L_4$ with respect to intersection pairing which satisfies $L_0^2 = 1$ and $L_i^2 = -1$ for $i \neq 0$.*
(2) *In any such basis the canonical class is given by $K_X = -3L_0 + L_1 + L_2 + L_3 + L_4$.*
(3) *There are precisely ten classes $D \in \operatorname{Pic} X$ which satisfy $D^2 = D \cdot K_X = -1$, namely L_i for $i \neq 0$ and $L_{ij} := L_0 - L_i - L_j$ for $0 < i < j \leqslant 4$. Each of these classes contains a unique curve, and these curves are smooth, irreducible and have genus 0.*
(4) *The intersection graph of these ten* -1-curves *is the so-called Petersen graph shown in Figure 1.1.*

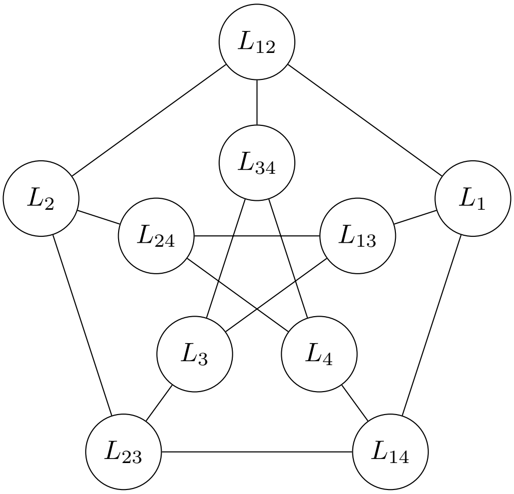

Figure 1.1. The intersection graph of -1*-curves on a del Pezzo surface of degree* 5.

(5) *Let* $A(X)$ *be the group of automorphisms of* $\operatorname{Pic} X$ *which preserve the intersection pairing and the canonical class. Then* $A(X)$ *is isomorphic to* S_5 *and this isomorphism is unique up to conjugation.*

Proof. — The first result follows from the fact that a del Pezzo surfaces of degree 5 is geometrically the blowup of the projective plane in 4 points, no three of which lie on a line, see for example [11, III, Proposition 3]. From here one can deduce the remaining statements.

We do draw attention to a particularly nice proof of the last statement. Note that $A(X)$ permutes the -1-classes and that these classes generate the Picard group. So $A(X)$ is a subgroup of the automorphism group of the Petersen graph. To compute this automorphism group we identify the vertices with $\binom{I}{2}$ for $I = \{1,2,3,4,5\}$ such that $\{\{i,j\},\{k,l\}\}$ is an edge precisely if i, j, k and l are distinct. This makes it straightforward to show that the automorphism group of the Petersen graph is isomorphic to S_5. Next one checks that each automorphism of the graph preserves the relations between the -1-classes and hence extends to an automorphism of the whole Picard group.

It follows that $A(X)$ is non-canonically isomorphic to S_5. However this identification is unique up to conjugation since S_5 is an inner group. □

Let us now switch to del Pezzo surfaces of degree 5 over general fields. The following proposition shows that the geometric Picard group as a Galois module is a principal invariant.

Proposition 1.3. — *Let k be a field. There is a bijection between isomorphism classes of quintic del Pezzo surfaces over k and S_5-conjugacy classes of group homomorphisms $G_k \to S_5$.*

Proof. — This is Lemma 14 in [14]. □

We will describe how to construct a del Pezzo surface from such a group homomorphism as was done in [14].

Proposition 1.4. — *Let k be a field with absolute Galois group G_k and let Λ be the effective generator of $\operatorname{Pic}\mathbb{P}^2_k$. Consider a group homomorphism $\psi\colon G_k \to S_5$. Fix five points $P_i \in \mathbb{P}^2_k(k^{\mathrm{sep}})$ such that no three lie on a line and that G_k acts on these points as S_5 acts on its indices.*

(1) *The linear system $\mathcal{L} = \left|\mathcal{O}_{\mathbb{P}^2_k}(5\Lambda - 2P_1 - 2P_2 - 2P_3 - 2P_4 - 2P_5)\right|$ has dimension 5.*
(2) *The image of the associated map is a del Pezzo X surface of degree 5.*
(3) *The composition $G_k \to A(X^{\mathrm{sep}}) \xrightarrow{\cong} S_5$ recovers ψ up to conjugacy.*
(4) *The isomorphism class of X only depends on the conjugacy class of ψ and is independent of the choice of P_i.*

Proof. — The first two statements are Theorem 5 in [14]. This theorem also shows that the -1-curves on X correspond to the lines on $\mathbb{P}^2_{k^{\mathrm{sep}}}$ passing through two of the points P_i. This shows that the Galois action on the -1-curves on X and hence on $\operatorname{Pic} X^{\mathrm{sep}}$ equals ψ up to conjugacy. This proves the third statement. To conclude the proof we use Proposition 1.3. □

1.2. Log K3 surfaces of dP_5 type

For our interest in integral points we move to surfaces which are not necessarily projective. The following class will be important.

Definition 1.5. — *Let U be a smooth surface over a field k. A* log K3 structure *on U is a triple (X, C, i) consisting of a proper smooth surface X over k, an effective anticanonical divisor C on X with simple normal crossings and an open embedding $i\colon U \to X$, such that i induces an isomorphism between U and $X\backslash C$. A* log K3 surface *is a simply connected,* *smooth surface U over k together with a choice of log K3 structure (X, C, i) on U.*

Let X be a del Pezzo surface of degree 5 and let C be an effective anticanonical divisor on X. The affine surface $U = X\backslash C$ is called a log K3 surface of dP_5 type*.*

Whenever we consider such a surface U without explicitly specifying X we will assume the choice of compactification to be understood from context.

1.3. Brauer groups

Let U be a scheme over a field k. We will need the concept of the *Brauer group* $\mathrm{Br}\, U$ of U. Two common definitions are the étale cohomology group $\mathrm{Br}\, U := \mathrm{H}^2(U, \mathbb{G}_m)$ and the group $\mathrm{Br}_{\mathrm{Az}}\, U$ of equivalence classes of Azumaya algebras on U. There is a natural morphism $\mathrm{Br}_{\mathrm{Az}}\, U \to \mathrm{Br}\, U$ which induces an isomorphism between $\mathrm{Br}_{\mathrm{Az}}\, U$ and $(\mathrm{Br}\, U)_{\mathrm{tors}}$ if U is a quasi-projective scheme over k by an unpublished result by Gabber. Another proof by De Jong can be found in [20]. In Theorem 6.6.7 in [25] we find conditions for $\mathrm{Br}\, U$ to be a torsion group and we conclude that we can identify both types of Brauer groups for regular integral schemes which are quasi-projective over a field. All the varieties for which we will consider the Brauer group will satisfy these conditions and we will pass freely between the two notions.

Using the functoriality of associating the Brauer group to the scheme we can define the following filtration: $\mathrm{Br}_0\, U \subseteq \mathrm{Br}_1\, U \subseteq \mathrm{Br}\, U$, where the *constant Brauer group* $\mathrm{Br}_0\, U$ is defined as $\mathrm{Im}(\mathrm{Br}\, k \to \mathrm{Br}\, U)$ and the *algebraic Brauer group* $\mathrm{Br}_1\, U$ is $\ker(\mathrm{Br}\, U \to \mathrm{Br}\, U^{\mathrm{sep}})$. We will denote the quotient $\mathrm{Br}_1\, U / \mathrm{Br}_0\, U$ by $\mathrm{Br}_1\, U / \mathrm{Br}\, k$ although the map $\mathrm{Br}\, k \to \mathrm{Br}_1\, U$ need not be injective.

If k is either a local or a global field, it follows from the Hochschild–Serre spectral sequence that $\mathrm{Br}_1\, U / \mathrm{Br}\, k$ is isomorphic to $\mathrm{H}^1(G_k, \mathrm{Pic}\, U^{\mathrm{sep}})$ in certain cases. This is well-known if U is proper, see for example [25, Corollary 6.7.8], but the proof actually works under the weaker condition $\mathbb{G}_m(U^{\mathrm{sep}}) = k^{\mathrm{sep},\times}$.

If U is an integral noetherian regular scheme over a field of characteristic 0 the natural map $\mathrm{Br}\, U \to \mathrm{Br}\, \kappa(U)$ is an inclusion [15, Section II.1]. So in this case we can represent elements of the Brauer group by classes of central simple algebras over the field $\kappa(U)$. We will construct Azumaya algebras on U as cyclic algebras over the function field.

Definition 1.6. — *Let κ be a field and n an integer not dividing the characteristic of κ. An Azumaya algebra in the image of the cup product*

$$\mathrm{H}^1(\kappa,\mu_n)\times\mathrm{H}^1(\kappa,\mathbb{Z}/n\mathbb{Z})\to\mathrm{H}^2(\kappa,\mu_n)\cong\operatorname{Br}k[n]$$

is called a cyclic algebra *over κ.*

A cyclic extension κ'/κ of degree n with a fixed generator $\sigma\in\mathrm{Gal}(\kappa'/\kappa)$ determines an element of $\mathrm{Hom}(G_\kappa,\mathbb{Z}/n\mathbb{Z})\cong\mathrm{H}^1(\kappa,\mathbb{Z}/n\mathbb{Z})$ by sending σ to 1. Any element $a\in\kappa^\times$ gives an element in $\mathrm{H}^1(\kappa,\mu_n)\cong\kappa^\times/(\kappa^\times)^n$. The cyclic algebra $a\cup(\kappa'/\kappa,\sigma)$ is denoted by (a,κ',σ).

For more details, see [25, Section 1.5.7]. Here one also finds the following important result.

Lemma 1.7. — *A cyclic algebra (a,κ',σ) is trivial in $\operatorname{Br}\kappa$ precisely when $a\in N_{\kappa'/\kappa}(\kappa'^\times)$.*

To see if a cyclic algebra in $\operatorname{Br}\kappa(U)$ comes from $\operatorname{Br}U$ we have the following lemma.

Lemma 1.8. — *Consider a smooth and geometrically integral variety U over a field k satisfying $\mathbb{G}_m(U^{\mathrm{sep}})=k^{\mathrm{sep},\times}$. Fix a finite cyclic extension K/k, a generator $\sigma\in\mathrm{Gal}(K/k)$, and an element $g\in\kappa(U)^\times$.*

The cyclic algebra $\mathcal{A}=(g,\kappa(U_K)/\kappa(U),\sigma)$ lies in the image of $\operatorname{Br}U\to\operatorname{Br}\kappa(U)$ precisely if $\operatorname{div}g=\mathrm{Nm}_{K/k}(D)$ for some divisor D on U_K. If k, and hence K, is a number field, and U is everywhere locally soluble then $\mathcal{A}$ is constant exactly when D can be taken to be principal.

Proof. — This lemma is similar to Proposition 4.17 from [6]. The difference is that the projectivity assumption is replaced by the weaker condition $\mathbb{G}_m(U^{\mathrm{sep}})=k^{\mathrm{sep},\times}$. One can check that under this assumption the proof presented in [6] is still valid. □

1.4. Brauer–Manin obstruction

In some cases elements of the Brauer group allow us to prove that there are no integral points on a scheme. Let $\mathcal{U}/\mathbb{Z}$ be a model of $U=\mathcal{U}_\mathbb{Q}$. The *Brauer–Manin set* of $\mathcal{A}$ is the subset of the *integral adelic points* $\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})=U(\mathbb{R})\times\prod_\ell\mathcal{U}(\mathbb{Z}_\ell)$ defined by

$$\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})^{\mathcal{A}}=\left\{(P_\ell)\in\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})\;\middle|\;\sum_\ell\operatorname{inv}_\ell\mathcal{A}(P_\ell)=0\right\}.$$

Here the *invariant maps* inv_ℓ are those defined in [25, Theorem 1.5.34]. Note that the infinite sum is well-defined by [25, Proposition 8.2.1]. The Brauer–Manin set is of particular interest because of the property described in the following theorem from [7].

LEMMA 1.9. — *Let $\mathcal{U}$ be a scheme over the integers and let U be the generic fibre over $\mathbb{Q}$. For any element $\mathcal{A} \in \operatorname{Br} U$ we have the following chain of inclusions*

$$\mathcal{U}(\mathbb{Z}) \subseteq \mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})^{\mathcal{A}} \subseteq \mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty}).$$

When $\mathcal{A}$ is a cyclic algebra we can use Lemma 1.7 to compute the images of the invariant maps and we might gain some information on the set of integral points.

DEFINITION 1.10. — *We say that an element $\mathcal{A} \in \operatorname{Br} U$* obstructs the integral Hasse principle *if $\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})$ is non-empty, but $\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})^{\mathcal{A}}$ is empty. The* order *of the obstruction is the order of $\mathcal{A}$ in $\operatorname{Br} U/\operatorname{Br}\mathbb{Q}$.*

2. The interesting Galois action

The main goal will be to construct affine schemes $\mathcal{U} \subseteq \mathbb{A}^5_{\mathbb{Z}}$ which have a Brauer–Manin obstruction to the integral Hasse principle. In all our examples we will construct $\mathcal{U}$ in such a way that $U = \mathcal{U}_{\mathbb{Q}}$ is a log K3 surface of dP_5 type. This means that we will be interested in the Brauer group of such surfaces. The following terminology will turn out to be helpful in that regard.

DEFINITION 2.1. — *Let X be a del Pezzo surface of degree 5 over a field k. Let K be the minimal Galois extension of k over which all -1-curves on X are defined. We say that X is* interesting *if $[K : k] = 5$. A log K3 surface of dP_5 type $U = X\backslash C$ is called* interesting *if X is an interesting del Pezzo surface and C is geometrically irreducible.*

The field K is called the splitting field *of the interesting surfaces X and U.*

Consider an interesting log K3 surface $U = X\backslash C$. By definition of a log K3 surface we see that C is smooth. The curve C is also geometrically irreducible since U is interesting. The results in this paper are also true for the complement of a geometrically irreducible anticanonical curve C on a del Pezzo surface X of degree 5. To be able to use the language of log K3 surfaces we do keep the superfluous condition that C is smooth.

The following lemma shows that two interesting del Pezzo surfaces over a field k are isomorphic precisely if their splitting fields are isomorphic.

Lemma 2.2. — *Let K/k be a Galois extension of degree 5 in a fixed separable closure k^{sep} of k. There exists an interesting del Pezzo surface X over k with splitting field K. The surface X is unique up to isomorphism.*

Proof. — Fix a generator $\sigma \in \mathrm{Gal}(K/k)$ of the Galois group of the cyclic extension K/k. Consider a non-trivial group homomorphism $\mathrm{Gal}(K/k) \to S_5$ by mapping σ to any element of order 5. This gives a group homomorphism $G_k \to \mathrm{Gal}(K/k) \to S_5$. By Proposition 1.3 it corresponds to an isomorphism class of quintic del Pezzo surfaces X. Since G_K is the kernel of $G_k \to S_5$ we conclude that X is interesting.

The uniqueness follows from the fact that any two non-trivial group homomorphisms $G_K \to S_5$ are S_5-conjugate, hence any two interesting del Pezzo surfaces with splitting field $K \subseteq k^{\mathrm{sep}}$ give conjugate group homomorphisms $G_k \to S_5$. By Proposition 1.3 we see that the surfaces are isomorphic over k. □

Definition 2.3. — *Let K/k be a Galois extension of degree 5. The isomorphism class of interesting del Pezzo surfaces of degree 5 over k which are split by K is denoted by $\mathrm{dP}_5(K)$.*

Note that the proof of Lemma 2.2 can be used to identify the action of the Galois group on $\mathrm{Pic}\, X^{\mathrm{sep}}$. In particular we have the following result.

Lemma 2.4. — *On an interesting del Pezzo surface there are two Galois orbits of geometric -1-curves, each of size 5. The sum of the -1-curves in one such orbit is an anticanonical divisor.*

Proof. — Let K be the splitting field of X. Since X is interesting the extension K/k is by definition of degree 5. It follows from the minimality of K that $\mathrm{Gal}(K/k)$ does not fix any of the ten -1-curves, hence there must be two orbits of size 5. After choosing a possibly different basis of $\mathrm{Pic}\, X^{\mathrm{sep}}$ we see that these two orbits are the two regular pentagons in Figure 1.1 and that there is a $\sigma \in \mathrm{Gal}(K/k)$ which acts on the outer pentagon by rotating counter-clockwise. Since σ preserves the intersection pairing it will rotate the inner pentagon counter-clockwise. This determines the action of σ on the -1-classes:

$$L_1 \mapsto L_{12} \mapsto L_2 \mapsto L_{23} \mapsto L_{14} \mapsto L_1,$$

$$L_3 \mapsto L_4 \mapsto L_{13} \mapsto L_{34} \mapsto L_{24} \mapsto L_3.$$

The last statement is now easily checked. □

If we consider the complement U of a geometrically irreducible anticanonical divisor C on a del Pezzo surface of degree 5 over a number field k we can compute its algebraic Brauer group modulo constants as $\mathrm{H}^1(G_k, \operatorname{Pic} U^{\mathrm{sep}})$. The following proposition shows that the action of G_k on $\operatorname{Pic} X^{\mathrm{sep}}$ is interesting precisely when $\mathrm{Br}_1 U/\operatorname{Br} k$ is non-trivial.

Theorem 2.5. — *Let $U = X\backslash C$ be a log K3 surface of* dP_5 *type over a number field k with C geometrically irreducible. We have*

$$\mathrm{Br}_1 U/\operatorname{Br} k \cong \begin{cases} \mathbb{Z}/5\mathbb{Z} & \text{if } U \text{ is interesting;} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — It was mentioned in [5, Remark at the end of Section 2.1] that the algebraic Brauer group modulo constants of log K3 surfaces of dP_5 type with a geometrically irreducible anticanonical divisor C is trivial except for one specific action of the Galois group on the geometric Picard group. So it suffices to verify the statement for interesting del Pezzo surfaces over k. We will fix a basis $(L_0, L_1, L_2, L_3, L_4)$ of $\operatorname{Pic} \overline{X}$ as in the proof of Lemma 2.4.

Since $C \subseteq X$ is geometrically irreducible we find the following exact sequence of Galois modules

$$0 \to \mathbb{Z} \xrightarrow{j} \operatorname{Pic} \overline{X} \to \operatorname{Pic} \overline{U} \to 0,$$

where j maps n to $-nK_X$. This shows that $\operatorname{Pic} \overline{U} \cong \operatorname{Pic} \overline{X}/\mathbb{Z}C \cong \mathbb{Z}^4$, since the anticanonical divisor class $-K_X = 3L_0 - L_1 - L_2 - L_3 - L_4$ is primitive. So $\operatorname{Pic} \overline{U}$ is torsion free and from the inflation–restriction sequence we conclude that the inflation homomorphism induces an isomorphism

$$\mathrm{H}^1(\mathrm{Gal}(K/k), \operatorname{Pic} U_K) \xrightarrow{\inf} \mathrm{H}^1(G_k, \operatorname{Pic} \overline{U}).$$

We will compute the action of σ on the quotient $\operatorname{Pic} \overline{U}$ of $\operatorname{Pic} \overline{X}$ using the specific action of σ on $\operatorname{Pic} \overline{X}$ in the proof of Lemma 2.4. We first determine that σ maps $L_0 = L_{12} + L_1 + L_2$ to $2L_0 - L_1 - L_2 - L_3$.

The classes $[L_0]$, $[L_1]$, $[L_2]$ and $[L_3]$ in $\operatorname{Pic} U_K$ form a basis and in this basis the class of L_4 becomes $[L_4] = 3[L_0] - [L_1] - [L_2] - [L_3]$. So σ acts on $\operatorname{Pic} \overline{U}$ as

$$\sigma = \begin{pmatrix} 2 & 1 & 1 & 3 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{pmatrix}.$$

By results on group cohomology of cyclic groups [31, Theorem 6.2.2] we get

$$\mathrm{H}^1(G, \operatorname{Pic} \overline{U}) \cong \ker(1 + \sigma + \sigma^2 + \sigma^3 + \sigma^4)/\mathrm{Im}(1 - \sigma).$$

Since $1+\sigma+\sigma^2+\sigma^3+\sigma^4=0$ and the image of $1-\sigma$ is generated by $(1,0,0,2)$, $(0,1,0,4)$, $(0,0,1,4)$ and $(0,0,0,5)$ we find

$$\mathrm{Br}_1 U/\mathrm{Br}\, k \cong \mathbb{Z}/5\mathbb{Z}. \qquad \square$$

Consider an interesting log K3 surface U. On the compactification X of U we have three important effective anticanonical divisors. First of all $C = X\backslash U$, but also the two divisors supported on -1-curves as described in Lemma 2.4. These anticanonical sections are important enough to introduce some notation.

Definition 2.6. — *Let X be an interesting del Pezzo surface of degree 5 over a field k. Let $l_1, l_2 \in \mathrm{H}^0(X, \omega_X^\vee)$ be the anticanonical sections supported on -1-curves from Lemma 2.4.*

We will use these elements to construct explicit generators of $\mathrm{Br}_1 U/\mathrm{Br}\, k$ for an interesting log K3 surface $U = X\backslash C$.

Lemma 2.7. — *Let K be the splitting field of an interesting log K3 surface $U = X\backslash C$ over a number field k. Fix a generator σ of $\mathrm{Gal}(K/k) \cong \mathbb{Z}/5\mathbb{Z}$. Let $h \in \mathrm{H}^0(X, \omega_X^\vee)$ be a global section whose divisor of zeroes is C.*

The cyclic $\kappa(X)$-algebras

$$\left(\frac{l_1}{h}, \sigma\right) \quad \text{and} \quad \left(\frac{l_2}{h}, \sigma\right)$$

are similar over $\kappa(X)$, their class $\mathcal{A}$ lies in the subgroup $\mathrm{Br}\, U \subseteq \mathrm{Br}\, \kappa(X)$ and generates $\mathrm{Br}_1 U/\mathrm{Br}\, k$.

Proof. — As $\mathrm{div}_U(\frac{l_1}{h})$ and $\mathrm{div}_U(\frac{l_2}{h})$ are orbits of -1-curves defined over K it follows from Lemma 1.8 that the cyclic algebras lie in the subgroup $\mathrm{Br}\, U$. The algebras $(\frac{l_1}{h}, \sigma) \otimes (\frac{l_2}{h}, \sigma)^{\mathrm{opp}}$ and $\left(\frac{l_1}{l_2}, \sigma\right)$ are similar and $\mathrm{div}_U(\frac{l_1}{l_2})$ is the norm of a principal divisor on U since this is even the case on X. Indeed, the divisors $L_{14} + L_1 - L_2$ and L_{24} are linearly equivalent on X, and their norms $\mathrm{Nm}_{K/k}(L_{14} + L_1 - L_2)$ and $\mathrm{Nm}_{K/k}(L_{24})$ are the divisors of zeroes of l_1 and l_2. It follows again from Lemma 1.8 that $\left(\frac{l_1}{l_2}, \sigma\right)$ is trivial in $\mathrm{Br}\, U$.

The algebra $\mathcal{A}$ is split by the degree 5 extension K and this implies that $\mathcal{A}$ is either trivial or of order 5. Suppose that the class of $\mathcal{A}$ is trivial, then by Lemma 1.8 there is a principal divisor D on U_K such that $\mathrm{Nm}_{K/k} D = \mathrm{div}_U l_1$. This implies that there is a $g \in \kappa(U_K)$ such that $\mathrm{div}_{U_K} g = D$. Consider g as a function on X_K and D as a divisor on X_K. Then $\mathrm{div}_{X_K} g = D + nC$ for an integer n, since C is geometrically irreducible. From $\mathrm{Nm}_{K/k} D =$

$\mathrm{div}_U l_1$ we find $K_{X_K} \cdot D = -1$ and we conclude that

$$0 = K_{X_K} \cdot \mathrm{div}_{X_K} g = K_{X_K} \cdot D + nK_{X_K} \cdot C = -1 - 5n,$$

which is a contradiction. □

Note that l_1 and l_2 are only defined up to multiplication by an element in $k^\times$. From now on we will denote the class in Lemma 2.7 by $\mathcal{A} \in \mathrm{Br}_1(U)$ which is uniquely defined up to an element in $\mathrm{Br}\, k$. Fix for the moment an interesting del Pezzo surface X. We will consider the class $\mathcal{A}_h$ on U_h as h varies over all linear forms. We have seen that $\mathcal{A}_h$ is of order 5 if h cuts out a geometrically irreducible curve. The next lemma shows that this only fails for specific choices of h.

Lemma 2.8. — *Let $X \subseteq \mathbb{P}^5_k$ be an interesting del Pezzo surface over a field k. A hyperplane section given by the vanishing of an $h \in \mathrm{H}^0(X, \mathcal{O}(1))$ fails to be geometrically irreducible if and only if h is a scalar multiple of either l_1 or l_2.*

Proof. — Consider a hyperplane section $C \subseteq X$. Let D be a k-irreducible component of C and consider a -1-curve L on X^{sep}. It follows that $L \cdot D^{\mathrm{sep}} = \sigma(L) \cdot D^{\mathrm{sep}}$ and as the Galois orbit of L is an anticanonical divisor, we find

$$5 \geqslant -K_X \cdot D = 5L \cdot D^{\mathrm{sep}} > 0,$$

since the degree of $D \subseteq \mathbb{P}^5_k$ is positive and at most the degree of C, which equals 5. This proves that $L \cdot D^{\mathrm{sep}} = 1$ for all -1-curves L and hence $C - D$ is an effective divisor of degree 0. We conclude that $C = D$ and this proves that any anticanonical section C is irreducible over k.

If C is not geometrically irreducible, then it must have at least two geometrically irreducible components of the same degree d since the Galois group acts on the set of geometrically irreducible components of C. Since C is of degree 5 we find $2d \leqslant 5$ and hence d is either 1 or 2. But in both cases we see that C contains a geometrically irreducible curve of degree 1, which must be a -1-curve L. Then C also contains all conjugates of L and hence C is the Galois orbit of a -1-curve. This proves that C is defined by the vanishing of either l_1 or l_2. □

In the next section we will use Proposition 1.4 to produce explicit equations for an interesting del Pezzo surface.

3. A model of $\mathrm{dP}_5(\mathbb{Q}(\zeta_{11})^+)$ over the integers

We have seen that all interesting del Pezzo surfaces split by a given quintic extension K of the base field k are isomorphic. We can use Proposition 1.4 to construct such a surface as the image of a rational map $\mathbb{P}^2_k \dashrightarrow \mathbb{P}^5_k$. We will also use this proposition to recover the anticanonical sections l_1 and l_2; in the notation of Proposition 1.4, let $\Lambda_{i,j}$ be the line through the points P_i and P_j in $\mathbb{P}^2_k(k^{\mathrm{sep}})$. Without loss of generality we can assume that there is a generator $\sigma \in \mathrm{Gal}(K/k)$ which maps P_i to P_{i+1}, where we consider the indices modulo 5. The divisors $\sum \Lambda_{i,i+1}$ and $\sum \Lambda_{i,i+2}$ over K are Galois invariant and hence their classes lie in the linear system $\mathcal{L}$ over k. These are the only divisors in $\mathcal{L}$ supported on lines and correspond to l_1 and l_2 on X. We will now give an explicit first example of how one can construct models of this surface.

We will use the quintic extension $K = \mathbb{Q}(\alpha)$ of $k = \mathbb{Q}$ where $\alpha = \zeta_{11} + \zeta_{11}^{-1}$. We will write m_α for the minimal polynomial of α over $\mathbb{Q}$. Let α_i be the conjugates of α.

Definition 3.1. — *Let $\mathcal{Q} \subseteq \mathbb{Z}[x,y,z]_{(5)}$ be the sub-$\mathbb{Z}$-module consisting of all quintic polynomials which vanish at least twice at the points $P_i = (\alpha_i^2 \colon \alpha_i \colon 1) \in \mathbb{P}^2_{\mathbb{Q}}$.*

Lemma 3.2. — *The $\mathbb{Z}$-module $\mathcal{Q}$ is free of rank 6 and $\mathbb{Z}[x,y,z]_{(5)}/\mathcal{Q}$ is torsion free.*

Proof. — Clearly $\mathcal{Q}$ is a free $\mathbb{Z}$-module. To compute its rank we use the result in the proof of Theorem 5 in [14] which says that $\mathcal{Q} \otimes \mathbb{Q}[x,y,z]$ has dimension 6. The last statement follows from the fact that for $\lambda \in \mathbb{Z}\backslash\{0\}$ and $q \in \mathbb{Z}[x,y,z]_{(5)}$ we have $\lambda q \in \mathcal{Q}$ precisely if $q \in \mathcal{Q}$. □

Let us fix a basis $q_i \in \mathcal{Q}$.

Definition 3.3. — *Let $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ be the image of the rational map $\mathbb{P}^2_{\mathbb{Z}} \dashrightarrow \mathbb{P}^5_{\mathbb{Z}}$ defined by the q_i.*

There are two primitive elements of $\mathcal{Q}$ which factor into linear polynomials over $\overline{\mathbb{Q}}$. These correspond to the two primitive linear forms $l_1, l_2 \in \mathbb{Z}[u_0, u_1, \ldots, u_5]$.

Note that the scheme $\mathcal{X}$ does not depend on the choice of basis of $\mathcal{Q}$. It does however depend on the choice of α. The statements are easier, but not by much, since we have chosen an integral α; we could have picked any generator of K over $\mathbb{Q}$.

Proposition 3.4. — *The scheme $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ is given by the quadratic polynomials*

$$u_0u_3 + 22u_0u_4 + 121u_0u_5 - u_1^2 - 121u_1u_3 + 2662u_1u_4 - 36355u_2u_4 \\ - 9306u_2u_5 + 10494u_3u_4 - 242u_3u_5 - 215501u_4^2 + 68123u_4u_5 - 13794u_5^2,$$

$$u_0u_4 + 11u_0u_5 - u_1u_2 - 11u_1u_3 + 242u_1u_4 - 3223u_2u_4 - 847u_2u_5 \\ + 902u_3u_4 - 11u_3u_5 - 19272u_4^2 + 6413u_4u_5 - 1331u_5^2,$$

$$u_0u_5 - u_1u_3 + 22u_1u_4 - u_2^2 - 286u_2u_4 - 77u_2u_5 + 77u_3u_4 \\ - 1694u_4^2 + 572u_4u_5 - 121u_5^2,$$

$$u_1u_4 - u_2u_3 - 11u_2u_4 - 77u_4^2 + 55u_4u_5 - 11u_5^2,$$

$$u_1u_5 - u_2u_4 - 11u_2u_5 - u_3^2 + 11u_3u_4 - 44u_4^2.$$

In this example, the two relevant hyperplane sections are given by

$$l_1 = u_0 + 22u_1 - 363u_2 + 165u_3 - 1859u_4 + 484u_5,$$

$$l_2 = u_0 + 22u_1 - 352u_2 + 143u_3 - 1595u_4 + 363u_5.$$

Also,

(1) *the generic fibre $X = \mathcal{X}_{\mathbb{Q}}$ is isomorphic to* $\mathrm{dP}_5(K)$*, and*
(2) *$\mathcal{X}$ is the flat closure of X in $\mathbb{P}^5_{\mathbb{Z}}$.*

Proof. — The MAGMA code for these computations can be found on the journal's website under `https://doi.org/10.5802/aif.3529`. We use those computations also for the proofs of some of the following statements.

(1) This follows from the fact that $\mathrm{dP}_5(K)$ is the image of the rational map $\mathbb{P}^2_{\mathbb{Q}} \dashrightarrow \mathbb{P}^5_{\mathbb{Q}}$ using a basis of $\mathcal{Q} \otimes \mathbb{Q}$.
(2) We used a Gröbner basis computation to compute the image of the rational map $\mathbb{P}^2_{\mathbb{Z}} \dashrightarrow \mathbb{P}^5_{\mathbb{Z}}$. The upshot of this that the equations above also define a Gröbner basis of the ideal $I \subseteq \mathbb{Z}[u_0, u_1, \dots, u_5]$ of $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$. Since the leading coefficients of the basis elements are units we conclude from [1, Proposition 4.4.4] that $I\mathbb{Q}[u_0, u_1, \dots, u_5] \cap \mathbb{Z}[u_0, u_1, \dots, u_5]$ is equal to I and hence that $\mathcal{X}$ is the flat closure of its generic fibre. □

This last proof also shows that $\mathcal{X}$ itself is integral, since X is integral. From this or the fact that $\mathcal{X}$ is flat over $\mathbb{Z}$ we deduce the important fact that all fibres $\mathcal{X}_\ell$ are equidimensional of dimension 2.

3.1. Fibres of the model

We can now study almost all fibres of $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ using the reduction of the minimal polynomial m_α modulo primes.

Lemma 3.5. — *Let $\ell \in \mathbb{Z}$ be a prime for which the reduction $\overline{m}_\alpha \in \mathbb{F}_\ell[s]$ is separable. The fibre $\mathcal{X}_\ell$ is a del Pezzo surface of degree 5. The hyperplane sections given by the vanishing of l_1 and l_2 each cut out five -1-curves on $\mathcal{X}_\ell$.*

Proof. — Consider the rational map $\mathbb{P}^2_{\mathbb{F}_\ell} \dashrightarrow \mathbb{P}^5_{\mathbb{F}_\ell}$ using the basis $q_i \otimes 1$ of $\mathcal{Q} \otimes \mathbb{F}_\ell$. By construction this morphism lands in $\mathcal{X}$.

If m_α is separable modulo ℓ then the reductions of the points P_i modulo ℓ are distinct and $\mathcal{Q} \otimes \mathbb{F}_\ell$ consists of all quintics over $\mathbb{F}_\ell$ vanishing at least twice at these points. By Proposition 1.4 we see that the image Y is a del Pezzo surface of degree 5. Hence we have $Y \subseteq \mathcal{X}_\ell \subseteq \mathbb{P}^5_{\mathbb{F}_\ell}$.

From Corollary III.9.6 in [17] we see that all irreducible components of $\mathcal{X}_\ell$ are of dimension 2. Hence Y is one such component of $\mathcal{X}_\ell$. By flatness $\mathcal{X} \to \mathrm{Spec}\,\mathbb{Z}$ we see that $\mathcal{X}_\ell$ has degree 5 in $\mathbb{P}^5_{\mathbb{F}_\ell}$, just like Y. Hence $\mathcal{X}_\ell$ has no other irreducible components.

The statement about l_1 and l_2 also follows from the flatness of $\mathcal{X}$ over $\mathbb{Z}$. □

There are actually two possibilities if the reduction of m_α modulo ℓ is separable.

Corollary 3.6.

(1) *If m_α is irreducible modulo ℓ then $\mathcal{X}_\ell$ is interesting.*
(2) *If m_α splits completely in $\mathbb{F}_\ell$ with distinct roots, then $\mathcal{X}_\ell$ is split, i.e. all -1-curves are defined over $\mathbb{F}_\ell$.*

Proof. — In the proof of the previous lemma we have seen that $\mathcal{X}_\ell$ is the image of $\mathbb{P}^2_{\mathbb{F}_\ell}$ of all quintics vanishing at least twice at the five points $P_i = (\alpha_i^2 : \alpha_i : 1)$ modulo ℓ. The action of Galois on the -1-curves on $\mathcal{X}_\ell$ is determined by the action of Galois on the points P_i.

We have seen before that an interesting del Pezzo surface is obtained precisely if the five points are defined over a quintic extension, but are conjugate over the base field. A split del Pezzo surface of degree 5 corresponds to the case that all points are defined over the base field. □

Remark 3.7. — It is possible to determine the fibres of $\mathcal{X}/\mathbb{Z}$ directly from the splitting of m_α in $\mathbb{F}_\ell$ even for primes which divide the discriminant of m_α. But this requires a long geometric treatise of singular del Pezzo surfaces, see the Ph.D. thesis of the author [22]. For our explicit examples it is much shorter to just study the remaining finitely many fibres separately.

For this example we are only left with the fibre over $\ell = 11$, since $\Delta(m_\alpha) = 11^4$.

Lemma 3.8. — *The fibre $\mathcal{X}_{11}$ is an integral surface in $\mathbb{P}^5_{\mathbb{F}_{11}}$ with precisely one singular point.*

The divisor on $\mathcal{X}_{11}$ defined by l_1 is supported on a line L. The singular point lies on this line. Also, a hyperplane section given by the vanishing of $h \in \mathbb{F}_{11}[u_0, u_1, \ldots, u_5]$ contains L precisely if h lies in the ideal $(u_0, u_1, u_2, u_3) \subseteq \mathbb{F}_{11}[u_0, \ldots, u_5]$.

Proof. — We have explicit equations for $\mathcal{X}$ and hence for $\mathcal{X}_{11}$ and all statements can be checked explicitly. □

The surface $\mathcal{X}_{11}$ is actually well-understood. It is a singular del Pezzo surface and the unique singular point which is of type A_4 lies on a unique line L on $\mathcal{X}_{11} \subseteq \mathbb{P}^2_{\mathbb{F}_{11}}$, i.e. a -1-curve (on its minimal desingularisation).

We even have a birational morphism $\mathbb{P}^2 \dashrightarrow \mathcal{X}_{11}$ which restricts to an isomorphism $\mathbb{A}^2 \xrightarrow{\cong} \mathcal{X}_{11} \backslash L$,

$$(1\colon y\colon z) \mapsto (1\colon y\colon z\colon y^2\colon yz\colon y^3 + z^2).$$

This will allow one to transfer many problems on $\mathcal{X}_{11}$ to a problem on the affine or the projective plane.

Remark 3.9. — This is not at all particular to this one example; for any choice $\alpha \in \overline{\mathbb{Q}}$ of degree 5 we can construct a relative surface $\mathcal{X}$ over $\mathbb{Z}$. If the minimal polynomial m_α reduces to the fifth power of a linear polynomial modulo ℓ then $\mathcal{X}_\ell$ always has these properties.

We will forgo this general approach and stick to our explicit examples.

4. A family of log K3 surfaces

Consider the model $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ of the interesting del Pezzo surface of the previous section. We will use it to construct a family of log K3 surfaces of dP_5 type together with their models.

Definition 4.1. — *Let $h \in \mathbb{Z}[u_0, u_1, u_2, u_3, u_4, u_5]_{(1)}$ be a primitive linear form. Let $\mathcal{U}_h$ be the complement of $\mathcal{C}_h = \{h = 0\} \cap \mathcal{X}$ in $\mathcal{X}$.*

We will consider when $\mathcal{U}_h$ does not have integral points. First of all this happens when $\mathcal{U}_h$ is not everywhere locally soluble. We can make precise when this happens.

Lemma 4.2. — *The affine surface* $\mathcal{U}_h$ *is everywhere locally soluble precisely when*

$$h \not\equiv u_2 + u_5 \mod 2.$$

Proof. — One can check that the points

$$\begin{array}{cc} (1\colon 0\colon 0\colon 0\colon 0\colon 0) & (-693\colon -88\colon -11\colon 0\colon 1\colon 1) \\ (-725\colon -120\colon -11\colon 1\colon 0\colon 1) & (967\colon 122\colon 11\colon -1\colon 0\colon 1) \\ (-3345\colon -328\colon -46\colon -4\colon 4\colon 4) & (-3497\colon -331\colon -34\colon 1\colon 1\colon 0) \\ (-6138\colon -407\colon -44\colon 0\colon 1\colon 0) & \end{array}$$

lie on $\mathcal{X}$. Also, their coordinates as vectors in $\mathbb{Z}^6$ define a lattice of dimension 6 of index 2. This proves that for any prime $\ell \neq 2$ and any linear form h at least one of these points P satisfies $h(P) \not\equiv 0 \mod \ell$. This shows that such a point determines an element in $\mathcal{U}_h(\mathbb{Z}_\ell)$.

We have seen in Lemma 3.5 that $\mathcal{X}_2$ is smooth. One can check that $\#\mathcal{X}(\mathbb{F}_2) = 5$ and that these points lie on the indicated hyperplane section over $\mathbb{F}_2$. □

4.1. Obstructions coming from $\mathcal{A}_h$

Note that if $\mathcal{C}_\mathbb{Q}$ is geometrically irreducible, i.e. h is not a multiple of l_1 or l_2 by Lemma 2.8, then we see that $\operatorname{Br} U_h / \operatorname{Br} \mathbb{Q}$ contains an element of order 5. Let us compute the invariant maps for this element.

Lemma 4.3. — *Consider a geometrically irreducible hyperplane section given by a primitive* h. *Let* ℓ *be a prime and let* $\mathcal{A}$ *be a generator for* $\operatorname{Br}_1 U_h / \operatorname{Br} \mathbb{Q}$. *We consider the invariant map*

$$\operatorname{inv}_\ell \mathcal{A} \colon \mathcal{U}_h(\mathbb{Z}_\ell) \to \mathbb{Q}/\mathbb{Z}.$$

If $\ell \neq 11$ *then* $\operatorname{inv}_\ell \mathcal{A}$ *is identically zero.*

Proof. — The statement is immediate for the infinite place and primes ℓ which split completely in K, since in those cases $\mathcal{A}_{\mathbb{Q}_\ell} \cong \mathcal{A}_{K_\mathfrak{l}}$ is trivial in $\operatorname{Br} U_{\mathbb{Q}_\ell}$ for any prime $\mathfrak{l}$ of K above ℓ.

Now suppose that m_α is irreducible modulo ℓ. The Kummer–Dedekind theorem implies that ℓ is inert in $\mathbb{Z}[\alpha]$. This also proves that ℓ is inert in $\mathcal{O}_K$ and there is a unique prime $\mathfrak{l}$ above ℓ. Also, $X_{\mathbb{Q}_\ell}$ is an interesting del Pezzo surface since m_α is irreducible over $\mathbb{Q}_\ell$. Hence the hyperplane section given by the vanishing of l_1 modulo ℓ is a cycle of five conjugate lines and does not contain $\mathbb{F}_\ell$-points. Hence l_1 is invertible on all points in $U(\mathbb{Z}_\ell)$. This shows that $\frac{l_1}{h}(P) \in \mathbb{Z}_\ell^\times$ for all $P \in \mathcal{U}(\mathbb{Z}_\ell)$. Since the extension $K_\mathfrak{l}/\mathbb{Q}_\ell$

of local fields is unramified we see that any unit is a norm. Hence $\mathrm{inv}_\ell \mathcal{A}$ is also in this case constantly 0. ☐

Lemma 4.4. — *Let L be the unique line on $\mathcal{X}_{11} \subseteq \mathbb{P}^5_{\mathbb{F}_{11}}$. If L does not lie in the zero locus of h then* $\mathrm{inv}_{11} \mathcal{A} \colon \mathcal{U}_h(\mathbb{Z}_{11}) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ *is surjective.*

Proof. — We have seen in Lemma 3.8 that the condition is equivalent to $h \mod 11$ not lying in $\mathbb{F}_{11}[u_0, \dots, u_3]$. Let $\overline{h}$ be this reduction modulo 11.

On points P for which $\frac{l_1}{h}(P) \in \mathbb{Z}_{11}$ is invertible we can use Lemma 1.7 to compute the $\mathrm{inv}_{11} \mathcal{A}(P)$. To be precise, the invariant at such a point P only depends on $\frac{l_1}{h}(P) \in \mathbb{F}_{11}$ up to fifth powers and there is an isomorphism $\psi \colon \mathbb{F}_{11}^\times/\{\pm 1\} \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ such that $\mathrm{inv}_{11} \mathcal{A}(P) = \psi\left(\frac{l_1}{h}(P)\right)$.

Hence it will suffice to prove the following stronger statement: the map $\frac{l_1}{h} \colon (\mathcal{U}_h \backslash L)(\mathbb{F}_{11}) \to \mathbb{F}_{11}^\times$ is surjective for $\overline{h} \notin (u_0, \dots, u_3)$. Note that both the domain and the map depend on our choice of h. For this statement we only have the finitely many $\overline{h}$ which we need to evaluate on a subset of the finitely many points $\mathcal{X}_{11}(\mathbb{F}_{11})$. The code for this computation can be found on the journal's website under `https://doi.org/10.5802/aif.3529`. ☐

Proposition 4.5. — *Define $f = h(1, y, z, y^2, yz, y^3 + z^2) \in \mathbb{F}_{11}[y, z]$. The value $0 \in \mathbb{Q}/\mathbb{Z}$ lies in the image of* $\mathrm{inv}_{11} \mathcal{A}$ *precisely when the polynomial f assumes at least one of the values ± 1 modulo 11 for $y, z \in \mathbb{F}_{11}$.*

The image of $\mathrm{inv}_{11} \mathcal{A} \colon \mathcal{U}_h(\mathbb{Z}_{11}) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ *has size*

- 1 *precisely when f is a constant;*
- 4 *precisely when f is a separable quadratic polynomial in y;*
- 5 *in all other cases.*

Note in the second case that f is in particular independent of z.

Proof. — Using the last lemma we will only need to consider the $\overline{h} \in (u_0, \dots, u_3)$. In this case we have that

$$(\mathcal{U}_h \backslash L)(\mathbb{Z}_{11}) = \mathcal{U}_h(\mathbb{Z}_{11})$$

since L lies in the zero locus of h. Furthermore, the value of $\mathrm{inv}_{11} \mathcal{A}$ at a point P only depends on $\frac{l_1}{h}(P)$ modulo 11 or equivalently the reduction $\overline{P} \in \mathcal{U}_h(\mathbb{F}_{11})$ of P. The statement can now be checked completely by a computer.

We would however like to provide a little more insight. Using the isomorphism $\mathcal{X}_{11} \backslash L \xrightarrow{\cong} \mathbb{A}^2_{\mathbb{F}_{11}}$ from Lemma 3.8 we see that $\mathcal{U}_{h,11} \backslash L \xrightarrow{\cong} \mathbb{A}^2_{\mathbb{F}_{11}} \backslash \{f = 0\}$. Hence we are interested in the image of $f \colon \mathbb{A}^2_{\mathbb{F}_{11}} \backslash \{f = 0\} \to \mathbb{F}_{11}^\times/\{\pm 1\}$. If $\overline{h}$ depends on u_5, then f is a cubic polynomial and $f = c$ for any $c \in \mathbb{F}_{11}$ is likely to have a solution, as made precise in the previous lemma. If $\overline{h}$ is independent of u_5 but does depend on either u_2 or u_4, then f is linear in z with the leading coefficient being linear in y. Fixing y to be a suitable y_0 shows that $f(y_0, z) = c$ always has a solution in $\mathbb{F}_{11}$. Hence in these cases $(\mathcal{U}_h \backslash L)(\mathbb{Z}_{11}) \to \mathbb{F}_{11}^\times/\{\pm 1\}$ is surjective.

For the remaining cases we have that $\overline{h}$ is independent of u_2, u_4 and u_5. This implies that f is a polynomial independent of z of degree at most 2. We also see from Lemma 3.8 that $\mathcal{U}_{h,11} = \mathcal{U}_{h,11} \backslash L \xrightarrow{\cong} \mathbb{A}^2_{\mathbb{F}_{11}} \backslash \{f = 0\}$. So it suffices to study the image of $f\colon \mathbb{A}^2_{\mathbb{F}_{11}} \backslash \{f = 0\} \to \mathbb{F}_{11}^\times/\{\pm 1\}$. This is further simplified since f does not depend on the variable z.

When f is constant we immediately get the first case. Whenever f is linear or an inseparable quadratic polynomial with root $\rho \in \mathbb{F}_{11}$ the surjectivity of $f\colon \mathbb{F}_{11} \backslash \{\rho\} \to \mathbb{F}_{11}^\times/\{\pm 1\}$ is immediate.

For the last case it is easily checked that for a quadratic separable polynomial $f = c(y-\rho_1)(y-\rho_2)$ the image of $f\colon \mathbb{F}_{11} \backslash \{\rho_1, \rho_2\} \to \mathbb{F}_{11}^\times/\{\pm 1\}$ has size four. This is independent of whether f splits over $\mathbb{F}_{11}$ or over $\mathbb{F}_{11^2}$. ☐

We can now apply the above results to compute the Brauer–Manin obstruction for a fixed h and find actual algebraic obstructions of order 5 to the integral Hasse principle.

Theorem 4.6. — *Let $\mathcal{H}$ be the hyperplane in $\mathbb{P}^5_{\mathbb{Z}}$ given by the vanishing of $u_1 - 7u_3$. The complement $\mathcal{U} = \mathcal{X} \backslash \mathcal{H}$ has points over $\mathbb{Q}$ and every $\mathbb{Z}_\ell$, but there is an order 5 Brauer–Manin obstruction to the existence of integral points.*

Remark 4.7. — Let S be a set of rational primes which split completely in K. The proof of the above statement can easily be adapted to show that there are no S-integral points on $\mathcal{X}$.

On the other hand if ℓ is an inert prime then $\mathrm{inv}_\ell\, \mathcal{A}$ need not be constant on $\mathbb{Q}_\ell$-points even if it is so on $\mathbb{Z}_\ell$-points. Although our model $\mathcal{U}$ is regular this does not contradict Theorem 1 in [3]. Hence the concept of a regular model is not as useful for S-integral points as it is for rational points.

A careful analysis of the proof of Proposition 4.5 yields the following result.

Theorem 4.8. — *Let $\mathcal{U}_h$ be the complement in $\mathcal{X}$ of a geometrically irreducible hyperplane section given by a primitive linear form h. The class of h modulo 2 determines whether the affine surface $\mathcal{U}_h$ is locally soluble. The existence of an algebraic obstruction to the Hasse principle for integral points depends only on the reduction of h modulo 11. Out of the $11^6 - 1 = 1771560$ possible reductions $\overline{h}$ of h modulo 11 precisely 228 give an obstruction.*

Note that this does not mean that the reduction of h modulo 2 and 11 is the only condition; the proof still uses the assumption that h is primitive. It follows from Lemma 2.8 that the condition that the section is geometrically irreducible is immediately satisfied if h does not reduce to $\pm u_0$ modulo 11. For hyperplanes h reducing to either of these two forms it is easily shown that $\mathrm{inv}_{11} \mathcal{A}$ is identically equal to 0 on $\mathcal{U}_h(\mathbb{Z}_{11})$.

Proof of Theorem 4.8. — Let us count the non-zero linear forms $\overline{h}$ over $\mathbb{F}_{11}$ for which such an obstruction exists. In Proposition 4.5 we saw that we get no obstruction unless f is either constant or a separable quadratic polynomial in y.

If f is constant then we see that $\mathrm{inv}_{11} \mathcal{A}$ is constant and we get an obstruction if f is one of the 8 non-fifth powers modulo 11.

For an $\overline{h}$ such that f is a quadratic inseparable polynomial we have seen that $f\colon \mathbb{F}_{11}\backslash\{\rho_1, \rho_2\} \to \mathbb{F}_{11}^\times/\{\pm 1\}, x \mapsto f(x)$ misses exactly one value. If f misses the value $q \in \mathbb{F}_{11}^\times/\{\pm 1\}$ we see that λf for $\lambda \in \mathbb{F}_{11}^\times$ misses the class of λq. There are $10 \cdot 11^2$ quadratic polynomials over $\mathbb{F}_{11}$ and $10 \cdot 11$ of these are inseparable. The group $\mathbb{F}_{11}^\times$ acts on the remaining $10^2 \cdot 11$ quadratic polynomials by multiplication. All orbits have size 10 and in such an orbit exactly 2 miss the unit element in $\mathbb{F}_{11}^\times/(\mathbb{F}_{11}^\times)^5$. This proves that for an h for which the invariant map at 11 assumes precisely 4 values there is an obstruction if the associated polynomial f is one of these $2 \cdot 10 \cdot 11 = 220$ separable quadratic polynomials. □

Remark 4.9. — We can also consider the affine surfaces $\mathcal{U}_h$ for which $\mathcal{U}_h(\mathbb{A}_\mathbb{Q})^\mathcal{A}$ is non-empty. Of course rational points on X are Zariski dense since it is a del Pezzo surface of degree 5. The question now is whether some of those points $x \in X(\mathbb{Q})$ lie on $\mathcal{U}_h$, i.e. $h(x_0, \ldots, x_5) = \pm 1$ for the primitive coordinates x_i of the point x. For some hyperplanes such a point is easily found, but not for the linear form $h = u_2 - u_3 - u_4$.

For this h we see from Proposition 4.5 that there is no algebraic obstruction. Let us consider other types of obstructions. Let $C \subseteq X$ be the curve cut out by $h = u_2 - u_3 - u_4 = 0$. This curve is smooth, has a point and hence is an elliptic curve. We now apply Remark 4.10(iii) in [19] which holds for a smooth anticanonical section on any del Pezzo surface. In our situation we see that an n-torsion element of $\mathrm{Br}\, U_h/\mathrm{Br}_1 U_h$ gives a non-zero $\mathbb{Q}$-isogeny $C \to D$ of degree n where $D(\mathbb{Q})$ has a proper n-torsion point. Since one can prove that C is only $\mathbb{Q}$-isogenous to itself and $C(\mathbb{Q})$ is torsion free, we conclude that $\mathrm{Br}\, U_h = \mathrm{Br}_1 U_h$. Hence there is also no transcendental Brauer–Manin obstruction.

Next we consider the set of real points and the notions of an obstruction at infinity coming from [19]. First we prove that $U(\mathbb{R})$ is connected. Consider the morphism $U \to \mathbb{A}^1_{\mathbb{Q}}, \vec{u} \mapsto \frac{u_4}{h}$ which induces a continuous map $\pi\colon U(\mathbb{R}) \to \mathbb{R}$. Every -1-curve on X is defined over the real numbers and can be checked to induce a section of this map. One can even show that, for any hyperplane section $h = 0$ defined over the rationals, the set of points in $U(\mathbb{R})$ which lie on a -1-curve is connected. Now consider the fibres of π. Since the hyperplane sections $u_4 = 0$ and $h = 0$ intersect in precisely one real point, each fibre of π is an affine real curve which is the complement of a single point on a genus 1 curve. A computation shows that the associated proper elliptic fibration has four singular fibres which lie over the real numbers $t_1 < t_2 < t_3 < t_4 = 0$. The curves above the intervals (t_1, t_2) and (t_3, t_4) have two components, but in either case each component meets at least one of the -1-curves. This proves that $U(\mathbb{R})$ is connected and there is no strong obstruction at infinity. By Theorem 2.7 of [19] we even conclude that $\mathcal{U}_h$ is not weakly obstructed at infinity.

So, many obstructions to the existence of integral points are inconclusive. We can however use the above map π to search for points more effectively; for a real point $\vec{u}$ on X with $h(\vec{u}) = 1$ and a fixed value of u_4 the value of u_5 lies in a bounded interval, for example $(\frac{u_4}{6}, 4u_4)$ if $u_4 \geqslant 1$. Fixing an integral value for u_4 and then trying all possible values of u_5 yields a single integral point

$$x = (-78849858 \colon 1180075 \colon -473168 \colon -513818 \colon 40649 \colon 85706).$$

with $|u_4| \leqslant 10^5$.

For surfaces $\mathcal{U}_h$ for some other linear form h with small coefficients, such as $h = u_2 - u_3 - u_4 + u_5$, it is still unclear if there are no integral points, or if there are only none of small height.

Remark 4.10. — We can use the same construction to produce models $\mathcal{X}$ with a different splitting field K. However if K is ramified at a prime $p > 11$ then one can show that the image of $\mathrm{inv}_p \mathcal{U}_h(\mathbb{Z}_p)$ has either size 1 or 5. Furthermore, the first case happens precisely when $\frac{l_1}{h}$ is constant modulo p similar to above. The interesting thing to note is that the intermediate case in which the invariant map assumes 4 invariants does not occur any more.

We are left with the case of quintic fields $K/\mathbb{Q}$ which are ramified at 5.

5. Explicit examples with splitting field $K \subseteq \mathbb{Q}(\zeta_{25})$

It is also possible to find obstructions of order 5 to the integral Hasse principle when $\mathcal{X}$ is a model of the interesting del Pezzo surface X split by the unique quintic extension K contained in $\mathbb{Q}(\zeta_{25})$. In that case K has 5 as a wildly ramified prime. For example, define the field $K \subseteq \mathbb{Q}(\zeta_{25})$ as the splitting field of the polynomial

$$m_\alpha = s^5 - 20s^4 + 100s^3 - 125s^2 + 50s - 5.$$

This produces the projective surface $\mathcal{X}$ over the integers given by the five equations

$$u_0u_3 + 40u_0u_4 + 400u_0u_5 - u_1^2 - 400u_1u_3 + 16000u_1u_4 - 365050u_2u_4 \\ -49995u_2u_5 + 51985u_3u_4 - 200u_3u_5 - 2029975u_4^2 + 392250u_4u_5 - 39375u_5^2,$$

$$u_0u_4 + 20u_0u_5 - u_1u_2 - 20u_1u_3 + 800u_1u_4 - 18125u_2u_4 - 2500u_2u_5 \\ + 2550u_3u_4 - 5u_3u_5 - 101015u_4^2 + 19800u_4u_5 - 2000u_5^2,$$

$$u_0u_5 - u_1u_3 + 40u_1u_4 - u_2^2 - 900u_2u_4 - 125u_2u_5 + 125u_3u_4 \\ - 5000u_4^2 + 985u_4u_5 - 100u_5^2,$$

$$u_1u_4 - u_2u_3 - 20u_2u_4 - 125u_4^2 + 50u_4u_5 - 5u_5^2,$$

$$u_1u_5 - u_2u_4 - 20u_2u_5 - u_3^2 + 20u_3u_4 - 100u_4^2.$$

The two linear forms over $\mathbb{Z}$ cutting out the two quintuples of -1-curves are

$$l_1 = u_0 + 25u_1 - 700u_2 + 200u_3 - 3425u_4 + 575u_5,$$

$$l_2 = u_0 + 75u_1 - 1675u_2 + 375u_3 - 5175u_4 + 575u_5.$$

By construction this scheme shares many properties with the previous example.

Proposition 5.1.

(1) *The scheme $\mathcal{X}$ is integral, with integral fibres.*
(2) *If m_α is irreducible modulo ℓ then $\mathcal{X}_\ell$ is an interesting del Pezzo surface.*
(3) *If m_α has five distinct roots in $\mathbb{F}_\ell$ then $\mathcal{X}_\ell$ is a split del Pezzo surface of degree 5.*

(4) *For* $\ell = 5$ *the surface* $\mathcal{X}_5$ *contains a unique line* L *for which* $\mathrm{div}_{\mathcal{X}_5}(l_1) = 5L$, *and* $\mathcal{X}_5$ *has a unique singular point of type* A_4 *which lies on* L. *There is a birational map* $\mathcal{X}_5 \dashrightarrow \mathbb{P}^2_{\mathbb{F}_5}$ *which restricts to an isomorphism* $\mathcal{X}_5\backslash L \xrightarrow{\cong} \mathbb{A}^2_{\mathbb{F}_5}$.

The only fibre not discussed in this lemma is the one over 7, since $\Delta(K) = 5^8 7^6$. Although $\mathcal{X}_7$ is again a singular del Pezzo surface, we will not need any information about this fibre this since 7 splits completely in K.

Proof. — One can follow the proofs in Section 3 for this different choice of α and corresponding equations for $\mathcal{X}$. □

We will consider $\mathcal{U}_h = \mathcal{X}\backslash\{h = 0\}$ like in the previous sections. As before, local solubility is immediate at most primes.

LEMMA 5.2. — *The surface* $\mathcal{U}_h$ *is everywhere locally soluble precisely when*

$$h \not\equiv u_2 + u_3 \mod 2.$$

Proof. — As for the proof of Lemma 4.2 it is easy enough to find enough points on $\mathcal{X}$ whose reductions do not lie on a hyperplane modulo $\ell > 2$. For $\ell = 2$ the fibre $\mathcal{X}$ is again smooth, $\#\mathcal{X}(\mathbb{F}_2) = 5$ and all $\mathbb{F}_2$-points lie on the unique hyperplane given by $u_2 + u_3 \equiv 0 \mod 2$. □

The computation of the invariant maps at the unramified primes is the same computation as in Lemma 4.3 for the previous example.

LEMMA 5.3. — *Consider the invariant map*

$$\mathrm{inv}_\ell \mathcal{A} \colon \mathcal{U}_h(\mathbb{Z}_\ell) \to \mathbb{Q}/\mathbb{Z}.$$

If $\ell \neq 5$, *then the invariant map is identically zero.*

Let us consider the remaining prime.

THEOREM 5.4. — *Then* $\mathrm{inv}_5 \mathcal{A} \colon \mathcal{U}(\mathbb{Z}_5) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ *is not surjective precisely when there exist integers* λ, c_1 *and* c_3 *satisfying* $5 \nmid \lambda$, *and* $5 \mid c_1, c_3$ *or* $5 \nmid c_3$ *such that*

$$h \equiv \lambda u_0 + 5(c_1 u_1 + c_3 u_3) \mod 25.$$

The invariant map is constant when $5 \mid c_1, c_3$ *and otherwise the size of its image is* 3.

The value $0 \in \mathbb{Q}/\mathbb{Z}$ *lies in the image of* $\mathrm{inv}_{11} \mathcal{A}$ *precisely when* $\lambda + 5(c_1 y + c_3 y^2)$ *assumes one of values* $\pm 1, \pm 7$ *modulo* 25 *for* $y \in \mathbb{Z}$.

To prove this result one can use the fact that the model $\mathcal{X}$ is regular. However, if 5 is ramified in K one has a similar statement for any model

$\mathcal{X}_\alpha$, which does not need to be regular. The following chain of results also implies a similar result in the more general case.

Lemma 5.5. — *Consider a point $\overline{P} \in (\mathcal{U}_h \backslash L)(\mathbb{F}_5)$. Let $\mathcal{P}$ be the set of the 25 lifts of $\overline{P}$ in $\mathcal{X}(\mathbb{Z}/25\mathbb{Z})$. The image of*

$$\frac{l_1}{h} \colon \mathcal{P} \to (\mathbb{Z}/25\mathbb{Z})^\times$$

is either of size 1 or 5.

Proof. — Define $\mathcal{V} := \mathcal{U}_{h,5} \backslash L \subseteq \mathbb{A}^5_{\mathbb{F}_5}$ on which $\frac{h}{l_1}$ is given by $h_{\mathrm{aff}} = a_0 + a_1 u_1 + \cdots + a_5 u_5$. Now let $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$ be a 5-tuple of integers reducing to $\overline{P} \in \mathcal{V}(\mathbb{F}_5)$. We will first show that the lifts of $\overline{P}$ to points in $\mathcal{X}(\mathbb{Z}/25\mathbb{Z})$ are $\vec{x} + 5\vec{w}$ where $\vec{w}$ is any vector in a translation of the tangent space of $\mathcal{V}$ at $\overline{P}$.

Indeed, suppose that $\mathcal{X}$ is given by polynomials g_j in the variables u_i. The tangent space at $\overline{P}$ is by definition

$$T_{\overline{P}}\mathcal{V} = \left\{ \vec{v} \in \mathbb{F}_5^5 \; : \; \sum_{i=1}^{5} \frac{\partial g_j}{\partial u_i}(\overline{P}) v_i \equiv 0 \bmod 5 \;\text{ for all } j \right\}$$

and if $\vec{x} + 5\vec{w} \in \mathcal{X}(\mathbb{Z}/25\mathbb{Z})$ then for all j

$$0 \equiv g_j(\vec{x} + 5\vec{w}) \equiv g_j(\vec{x}) + 5 \sum_{i=1}^{5} \frac{\partial g_j}{\partial u_i}(\vec{x}) w_i \mod 25$$

which proves that the set of possible $\vec{w}$ is an affine translation of $T_{\overline{P}}\mathcal{V}$.

To compute h_{aff} at these lifts, let us write $\vec{a} = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{Z}^5$. Then we find

$$h_{\mathrm{aff}}(\vec{x} + 5\vec{w}) \equiv h_{\mathrm{aff}}(\vec{x}) + 5\vec{a} \cdot \vec{w} \mod 25.$$

This concludes the proof since the $\vec{w}$ live in a linear space over $\mathbb{F}_5$. ☐

This result is very powerful when combined with the following fact.

Lemma 5.6. — *An element $a \in \mathbb{Z}_5^\times$ is a fifth power precisely if it is so modulo 25, i.e. if its reduction $\hat{a} \in \mathbb{Z}/25\mathbb{Z}$ lies in $\{\pm 1, \pm 7\}$.*

Hence, the five lifts of any $\bar{a} \in (\mathbb{Z}/5\mathbb{Z})^\times$ in $(\mathbb{Z}/25\mathbb{Z})^\times$ lie in different classes modulo fifth powers.

To prove Theorem 5.4 using brute computational force one would need to list all possible linear forms h and points in $\mathcal{X}(\mathbb{Z}_5)$ modulo 25. We will use these last two results to show we can do this computation while only using the points and linear forms modulo 5; drastically improving on the time needed for the computations.

Let us first show we can ignore the singular point on $\mathcal{X}_5$, or even the unique line $L \subseteq \mathcal{X}_5$ containing this point.

Proposition 5.7. — *If h is not a multiple of l_1 modulo 5 then*

$$\mathrm{inv}_5\,\mathcal{A}\colon \mathcal{U}_h(\mathbb{Z}_5) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$$

is surjective.

Proof. — We will prove a stronger statement. Define $\mathcal{V} := \mathcal{U}_5 \backslash L \subseteq \mathbb{A}^5_{\mathbb{F}_5}$ on which the linear form is given by $h = a_0 + a_1u_1 + \cdots + a_5u_5$. We will show that there is an $\mathbb{F}_5$-point $\overline{P}$ on $\mathcal{V}$ such that $h\colon T_{\overline{P}}\mathcal{V} \to \mathbb{F}_5$ is surjective; note that one can consider the tangent space as a linear or affine subspace of $\mathbb{F}_5^5$, since this does not change the size of the image of this functional.

One can prove this stronger statement by translating it back to a study of plane curves using the isomorphism $\mathcal{V} \cong \mathbb{A}^2_{\mathbb{F}_5} \backslash \{f = 0\}$ for

$$f = h(1, y, z, y^2, yz, y^3 + z^2)$$

and conclude that for every h there are at least 10 such points.

Now let $\overline{P}$ be such an $\mathbb{F}_5$-point in $\mathcal{V} \subseteq \mathcal{X}$. By the defining property of $\overline{P}$ we see that $\mathrm{inv}\,\mathcal{A}_h$ assumes five values on the points in $\mathcal{U}_h(\mathbb{Z}_5)$ reducing to $\overline{P}$. □

Corollary 5.8. — *If h does not vanish on the line $L \subseteq \mathcal{X}_5$ then* $\mathrm{inv}_5\,\mathcal{A}$ *is surjective.*

We can now efficiently prove Theorem 5.4.

Proof of Theorem 5.4. — By Proposition 5.7 we only need to consider the case that h is a multiple of u_0 modulo 5 hence we can write $h = \lambda u_0 + 5(c_1u_1 + \cdots + c_5u_5) \in \mathbb{Z}[u_0, u_1, \dots, u_5]$. Let us write $k = c_1u_1 + \cdots + c_5u_5$. Since $l_1 \equiv u_0 \mod 25$ we see that the value of $\frac{h}{l_1} = \lambda + 5k(\frac{u_1}{u_0}, \dots, \frac{u_5}{u_0})$ mod 25 at any $P \in \mathcal{U}(\mathbb{Z}_5)$ only depends on its reduction $\overline{P} \in \mathcal{U}(\mathbb{F}_5)$.

We are interested in the values k takes on $\mathcal{U}(\mathbb{Z}_5)$ modulo 5. A computer check available on the journal's website under `https://doi.org/10.5802/aif.3529` shows that for the listed cases k assumes the indicated number of values in $\mathbb{F}_5$. Hence $\frac{h}{l_1}$ assumes the same number of values in $(\mathbb{Z}/25\mathbb{Z})^\times$ each of which is a different lift of $\lambda \in \mathbb{F}_5^\times$. This shows that $\frac{l_1}{h}$ assumes exactly 1, 3 or 5 values in $\mathbb{Z}_5^\times$ modulo fifth powers. Hence we see that $\mathrm{inv}_5\,\mathcal{A}$ assumes these many values on $\mathcal{U}_h(\mathbb{Z}_5)$.

To provide a little more insight we can again use the isomorphism $\mathcal{U}_5 \cong \mathbb{A}^2_{\mathbb{F}_5} \backslash \{f = 0\}$ now using $f = k(1, y, z, y^2, yz, y^3 + z^2)$. One can check that f is surjective to $\mathbb{F}_5^\times$ if it describes a line, a conic with two distinct rational points at infinity, a geometrically integral conic with a single point at infinity, or a cubic curve. The remaining cases are the constant functions and the quadratics which are independent of z. In these cases we have $k \equiv c_1u_1 + c_3u_3 \mod 5$.

The hyperplane section of $\mathbb{P}^5_{\mathbb{F}_5}$ defined by the vanishing of $k \equiv c_1u_1+c_3u_3 \mod 5$ corresponds to the polynomial $c_1y + c_3y^2$ on $\mathbb{A}^2_{\mathbb{F}_5}$ which is quadratic if $c_3 \neq 0$ and constant if $c_1 = c_3 = 0$. By symmetry we see that a quadratic in one variable over $\mathbb{F}_5$ assumes exactly 3 values. And obviously $h \equiv \lambda l_1 \mod 25$ precisely when c_1 and c_3 are zero in $\mathbb{F}_5$. □

For completeness we will give an example of a hyperplane section for which the associated affine scheme over the integers does not have integral solutions.

Theorem 5.9. — *Consider an h which cuts out a geometrically irreducible hyperplane section such that 0 does not lie in the image of $\operatorname{inv}_5 \mathcal{A}$. The reduction of h modulo 25 is one of 176 out of the $(5^2)^6-5^6 = 244125000$ possible linear forms over $\mathbb{Z}/25\mathbb{Z}$. For example, the surface $\mathcal{U}_h/\mathbb{Z}$ for $h = 2u_0 - 15u_1 + 10u_3$ admits a Brauer–Manin obstruction of order 5 to the existence of integral points.*

Proof. — Let $(\mathbb{Z}/25\mathbb{Z})^\times$ act by multiplication on the linear forms modulo 25 for which $\operatorname{inv}_5 \mathcal{A}$ is not surjective. Multiplication by λ translates the image of the invariant map by an element of $\frac{1}{5}\mathbb{Z}/\mathbb{Z}$ corresponding to the class of λ in $(\mathbb{Z}/25\mathbb{Z})^\times$ modulo fifth powers. So if the size of the image of an invariant map corresponding to a hyperplane $\{h = 0\}$ has one element, then $\frac{4}{5}$ of the scalar multiples of h do not have 0 in the image. For invariant maps whose image is of size 3 precisely $\frac{2}{5}$ of the scalar multiples have this property. Using Theorem 5.4 we see that the number of linear forms modulo 25 for which 0 does not lie in the image of the invariant map is $\frac{4}{5} \cdot 20 + \frac{2}{5} \cdot 20 \cdot 4 \cdot 5 = 176$.

Now consider the linear form $h = 2u_0-15u_1+10u_3$. The affine surface $\mathcal{U}_h$ is locally soluble by Lemma 5.2. The result follows from the Theorem 5.4; take $\lambda = 2$, $c_1 = -3$ and $c_3 = 2$ and note that $2 - 15x + 10x^2$ only assumes the values $2, 12, 22 \mod 25$. So 0 does not lie in the image of the invariant map at 5 and the invariant maps at the other primes are all constant zero by Lemma 5.3. □

We can also show that in the two explicit examples with splitting fields with conductor 11 and 25 the absence of integral points is not explained by the principle of obstructions at infinity as introduced by Jahnel and Schindler in [19].

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Manuscrit reçu le 30 septembre 2020,
révisé le 9 avril 2021,
accepté le 23 juin 2021.

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