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FIBERED COHOMOLOGY CLASSES IN DIMENSION THREE, TWISTED ALEXANDER POLYNOMIALS AND NOVIKOV HOMOLOGY

by Jean-Claude SIKORAV

Abstract. — We prove that for "most" closed 3-dimensional manifolds M, the existence of a closed non singular one-form in a given cohomology class $u \in H^1(M, \mathbb{R}) = \mathrm{Hom}(\pi_1(M), \mathbb{R})$ is equivalent to the fact that every twisted Alexander polynomial $\Delta^H(M, u) \in \mathbb{Z}[G/\ker u]$ associated to a normal subgroup with finite index $H < \pi_1(M)$ has a unitary u-minimal term.

Résumé. — Nous prouvons que pour « la plupart » des variétés fermées de dimension trois, l'existence d'une forme fermée non singulière dans une classe de cohomologie donnée $u \in H^1(M, \mathbb{R}) = \mathrm{Hom}(\pi_1(M), \mathbb{R})$ équivaut au fait que tout polynôme d'Alexander tordu $\Delta^H(M, u) \in \mathbb{Z}[G/\ker u]$ associé à un sous-groupe distingué d'indice fini $H < \pi_1(M)$ a un terme u-minimal unitaire.

1. Introduction and statement of the main result

We consider M a closed connected 3-manifold. Let $G := \pi_1(M)$ and let u be a nonzero element of $\mathrm{Hom}(G, \mathbb{R})$, which will be identified with $H^1(M, \mathbb{R})$. Denote by $\mathrm{rk}(u)$ the rank of u, i.e. the number of free generators of $G/\ker u$. We are interested in the following

Question. — *Does there exist a nonsingular closed 1-form ω in the class u?*

If such a form exists, we say that u is *fibered*. The reason is that if $\mathrm{rk}(u) = 1$ so that $au(G) \subset \mathbb{Z}$ for a suitable $a \neq 0$, such a form is $a^{-1} f^* \mathrm{d}t$ for f a fibration to $S^1 = \mathbb{R}/\mathbb{Z}$. More generally, by [23], if u fibers then M

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fibers over S^1: perturb ω to $\omega' = \omega + \varepsilon$ such that $\mathrm{rk}([\omega']) = 1$ and ε is C^0-small. Then ω' is still nonsingular, thus M fibers.

An answer to this question was given in rank one by [21]: *if* $\mathrm{rk}(u) = 1$, u *fibers if and only if* $\ker u$ *is finitely generated.* Actually, Stallings required M to be irreducible, but using Perelman it is unnecessary.

In any rank, the paper [22] introducing the Thurston (semi-)norm on $H^1(M;\mathbb{R})$ proved the following results:

(1) the unit ball of the norm is an "integer polyhedron", i.e. it is defined by a finite number of inequalities $u(g) \leqslant n$, $g \in G$, $n \in \mathbb{N}^*$;
(2) the set of fibered $u \in H^1(M;\mathbb{R}) \setminus \{0\}$ is a cone over the union of some maximal open faces of the unit sphere of the Thurston norm.

Note that thanks to Stallings, to know if a given face is "fibered", it suffices to test one element u of rank one and see if $\ker u$ is finitely generated.

In the 2000s and beginning of 2010s, S. Friedl and S. Vidussi studied this question again, mostly in rank one, in connection with what was then a conjecture of Taubes: u *fibers if and only if* $u \wedge [\mathrm{d}t] + a \in H^1(M \times S^1)$ *is represented by a symplectic form, where* $a \in H^2(M;\mathbb{R})$ *satisfies* $a \wedge u \neq 0$. The starting point was the relation of Seiberg–Witten invariants of $M \times S^1$ and *twisted Alexander polynomials*, see below and Section 3. They ultimately solved that conjecture in [7], and obtained as a byproduct a new answer for the characterization of fibered classes in the case of rank 1: *if* $\mathrm{rk}(u) = 1$, u *fibers if and only if all twisted Alexander polynomials* $\Delta^H(G,u)$ *are nonzero.*

Let us describe briefly what are these twisted Alexander polynomials (for a detailed presentation, see [5]). In fact, we do it only for a special case, which is already sufficient: those associated to finite covers, see [6, Section 3.2].

Recall first the definition of the *order* of a finitely generated module $\mathcal{M}$ over a Noetherian UFD R: it is the greatest common divisor of the p-minors of A in a finite presentation

$$R^q \xrightarrow{\times A} R^p \to \mathcal{M},$$

where $\times A$ is the right multiplication by a matrix $A \in \mathrm{M}_{q,p}(R)$. Thus it is an element of R defined up to multiplication by a unit. We denote it by $\mathrm{ord}_R(\mathcal{M})$, usually viewed as an element of R. See Section 3.1.

Since $G/\ker u \approx \mathbb{Z}^r$, the ring $\mathbb{Z}[G/\ker u]$ is isomorphic to $\mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$, thus it is a Noetherian UFD. In particular, this order vanishes if and only if there are no p-minors or they all vanish.

Then let H be a normal subgroup of G with finite index, denoted by $H \triangleleft_{f.i.} G$. We define $H_1(H;\mathbb{Z}[G/\ker u])$ as the homology of H with coefficients in the H-module $\mathbb{Z}[G/\ker u]$. It is naturally a module over $\mathbb{Z}[G/\ker u]$ by action on the coefficients, which is finitely generated since H is finitely generated. By definition, the twisted Alexander polynomial of (G,u) associated to H is

$$\Delta^H(G,u) := \mathrm{ord}_{\mathbb{Z}[G]}(H_1(H;\mathbb{Z}[G/\ker u])).$$

Since the units of $\mathbb{Z}[G/\ker u]$ are $\pm(G/\ker u)$ (i.e. $\pm t_1^{i_1}\dots t_r^{i_r}$), it is an element of $\mathbb{Z}[G/\ker u]/\pm(G/\ker u)$.

It is not too difficult to prove that, if u is fibered, $\Delta^H(G,u)$ is always *u-monic*, i.e. its u-minimal term has a coefficient ± 1: see Proposition 3.4. In the rank one case, this goes back to Alexander.

We can now state our main result.

Theorem 1.1. — *Let M be a closed 3-manifold such that $\widetilde{M}$ is contractible and $G := \pi_1(M)$ is virtually residually torsion-free nilpotent (VRTFN), and let u be a nonzero element of $\mathrm{Hom}(G,\mathbb{R}) = H^1(M,\mathbb{R})$.*

Assume that $\Delta^H(G,u)$ is u-monic for every $H \triangleleft_{f.i.} G$. Then u is fibered, i.e. represented by a nonsingular closed 1-form.

Comments

(1). — Building on [1], [13] proves that $\pi_1(M)$ is VRTFN for all geometric manifolds which are not Sol. In particular, if M is hyperbolic, this follows from the fact that $\pi_1(M)$ is virtually a right-angled Artin group.

If M is Sol, $\pi_1(M)$ is not virtually nilpotent, but M is either a torus bundle over S^1 with hyperbolic monodromy or has a finite cover of this type and $H^1(M,\mathbb{R}) = 0$. Thus in that case the theorem is obvious.

The hypothesis that $\widetilde{M}$ is contractible can be dispensed with: if it does not hold, then (since $b_1(M) > 0$) we are in one of the two following cases: either M is nonprime thus nonfibered and the twisted Alexander polynomials always vanish; or M fibers over S^1 with fiber S^2 or $\mathbb{RP}^2$.

(2). — In rank one, our result is weaker than [7]. However, even in that case we believe that our proof, which is based on different ideas, may be of interest.

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2. Sketch of the proof and content of the paper

The main idea is to express the fibering condition on u by the vanishing of some *Novikov homology* associated to (G,u) and the nonvanishing of $\Delta^H(G,u)$ by the vanishing of some *Abelianized relative Novikov homology* associated to (G,H,u).

In turn, these vanishings are expressed by the invertibility of some matrix in the Novikov ring associated to (G,u) and of its image in the Novikov ring associated to $(G/(H\cap\ker u),\bar{u})$ ($\bar{u}$ being induced by u).

Then the theorem is reduced to a result about "finite detectability of invertible matrices" for a VRTFN group.

We now describe the content of the paper.

In Section 3, we define the twisted Alexander polynomials $\Delta^H(G,u)$.

In Section 4, we define the Novikov ring $\mathbb{Z}[G]_u$, and the Novikov homology $H_*(G,u)$. We quote the result of [2], building upon previous results of Stallings and Thurston: *if $G=\pi_1(M)$ with M a closed 3-manifold, u fibers if and only if $H_1(G,u)=0$.*

In Section 5, we explain the relations between twisted Alexander polynomials and an "Abelian relative" version of Novikov homology.

In Section 6, we specify the computation of $H_1(G,u)$ for $G=\pi_1(M^3)$ with $\widetilde{M}$ contractible, thanks to the form of a presentation of G given by a Heegaard decomposition and Poincaré duality. We deduce that $(H_1(G,u)=0)$ is equivalent to the invertibility in $\mathrm{M}_{p-1}(\mathbb{Z}[G]_u)$ of some matrix $A\in\mathrm{M}_{p-1}(\mathbb{Z}[G])$ where p is the genus of the decomposition.

Similarly, the vanishing of $H_1^{ab}(G/(H\cap\ker u)),\bar{u})$ is equivalent to the invertibility of the image of A in $\mathrm{M}_{p-1}(\mathbb{Z}[G/H\cap\ker u]_{\bar{u}})$. Thus we have reduced Theorem 1.1 to Theorem 6.2:

> *If G is finitely generated and VRTFN, a matrix $A\in\mathrm{M}_n(\mathbb{Z}[G])$ whose image in $\mathrm{M}_n(\mathbb{Z}[G/H]_u)$ is invertible for every $H\triangleleft_{f.i.} G$, is invertible in $\mathrm{M}_n(\mathbb{Z}[G]_u)$.*

The restriction to finitely generated groups is actually not necessary, but simplifies the proof.

Theorem 6.2 is proven in Section 11. There are two main ingredients:

- (Sections 7 to 9) the case when G is nilpotent, which uses three key facts:
  (1) [10] a simple $\mathbb{Z}[G]$-module is finite;
  (2) [8] when G is nilpotent and torsion-free, $\mathbb{Z}[G]$ has a classical ring of quotients on the right.
- (Section 10) the fact that when G is RTFN, it is orderable, thus one can embed $\mathbb{Z}[G]$ in the Mal'cev–Neumann completion $\mathbb{Q}\langle G\rangle$, which is a division ring (or skew field); moreover, by a remark of [12] the order can be chosen so that $\mathbb{Q}\langle G\rangle$ contains $\mathbb{Z}[G]_u$. Actually, we work mostly with a subfield introduced by [4], which contains $\mathbb{Z}[G]$ and whose elements have "controlled" support.

Notation. — In the following text, G is a finitely generated group and $u : G \to \mathbb{R}$ a nonzero homomorphism. Thus $G/\ker u \approx \mathbb{Z}^r$, $r = \mathrm{rk}(u)$, and $\mathbb{Z}[G/\ker u] \approx \mathbb{Z}[t_1^{\pm}, \dots, t_r^{\pm 1}]$ is a UFD (unique factorization domain).

3. Twisted Alexander polynomials

3.1. Order of a finitely generated module over a Noetherian UFD

Let $\mathcal{M}$ be a finitely generated R-module where R is a Noetherian UFD. One defines (cf. [3])

- the *Fitting ideal (or elementary) of order* 0 $\mathrm{Fitt}_0(\mathcal{M})$ as the ideal of R generated by the p-minors of a matrix $A \in \mathrm{M}_{q,p}(R)$ where
$$R^q \xrightarrow{\times A} R^p \xrightarrow{p} \mathcal{M}$$
is a presentation of $\mathcal{M}$ with $\times A$ the multiplication on the right by A;
- the *order* $\mathrm{ord}_R(\mathcal{M})$ as the greatest common divisor (gcd) of $\mathrm{Fitt}_0(\mathcal{M})$.

Remark 3.1. — We use multiplication on the right rather than on the left since later we will have mostly noncommutative rings, and we prefer to work with left modules. Thus elements of R^q, R^p are interpreted as row vectors.

It is easy to prove that the definition of $\mathrm{Fitt}_0(\mathcal{M})$ and thus of $\mathrm{ord}_R(\mathcal{M})$ does not depend on the presentation: if $R^{q_1} \xrightarrow{\times A_1} R^{p_1} \xrightarrow{\pi_1} \mathcal{M}$ and $R^{q_2} \xrightarrow{\times A_1}$

$R^{p_2} \xrightarrow{\pi_2} \mathcal{M}$ are two presentations, one can lift π_1 to $\times B : R^{m_1} \to R^{m_2}$ and obtain a presentation

$$R^{q_1} \oplus R^{q_2} \xrightarrow{\times A} R^{p_1} \oplus R^{p_2} \xrightarrow{\pi_1+\pi_2} \mathcal{M}, A = \begin{pmatrix} \mathrm{I}_{p_1} & B \\ 0 & A_2 \end{pmatrix}.$$

Similarly, there is a presentation

$$R^{q_1} \oplus R^{q_2} \xrightarrow{\times C} R^{p_1} \oplus R^{p_2} \xrightarrow{\pi_1+\pi_2} \mathcal{M}, C = \begin{pmatrix} A_1 & 0 \\ D & \mathrm{I}_{p_2} \end{pmatrix}.$$

Thus the kernel of $\pi_1 + \pi_2$ is the row space of $\left(\begin{smallmatrix} \mathrm{I}_{p_1} & B \\ 0 & A_2 \end{smallmatrix}\right)$, and also that of $\left(\begin{smallmatrix} A_1 & 0 \\ D & \mathrm{I}_{p_2} \end{smallmatrix}\right)$: this implies (for any ring) that the ideals generated by the p_1-minors of A_1 and by the p_2-minors of A_2 coincide.

The main property of this order is the

Proposition 3.2. — *Let R be a Noetherian UFD, $\mathcal{M}$ an R-module generated by p elements, and $\mathrm{ann}_R(\mathcal{M})$ its annihilator. Then one has the divisions*

$$\gcd(\mathrm{ann}_R(\mathcal{M})) \mid \mathrm{ord}_R(\mathcal{M}) \mid \big(\gcd(\mathrm{ann}_R(\mathcal{M}))\big)^p.$$

More precisely, one has the inclusions of ideals

$$\mathrm{ann}_R(\mathcal{M}) \supset \mathrm{Fitt}_0(\mathcal{M}) \supset (\mathrm{ann}_R(\mathcal{M}))^p = \langle a^p \mid a \in \mathrm{ann}_R(\mathcal{M})\rangle$$

Proof. — Let $R^q \xrightarrow{\times A} R^p \xrightarrow{p} \mathcal{M}$ be a presentation of $\mathcal{M}$, with $A \in \mathrm{M}_{q,p}(R)$. Then

$$a \in \mathrm{ann}_R(\mathcal{M}) \iff aR^p \subset R^q A \iff (\exists\, X \in \mathrm{M}_{p,q}(R)) XA = a\mathrm{I}_p.$$

Let μ be a p-minor of A. Changing the order of the coordinates, we have $A = \left(\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right)$ where $A_1 \in \mathrm{M}_p(R)$ with $\det A_1 = \mu$. Thus there exists $X_1 \in \mathrm{M}_p(R)$ such that $X_1 A_1 = \mu \mathrm{I}_p$, and $X = \left(\begin{smallmatrix} X_1 & 0 \end{smallmatrix}\right) \in \mathrm{M}_{p,q}(R)$ satisfies $XA = \mu \mathrm{I}_p$. Thus $\mathrm{ann}_R(\mathcal{M})$ contains μ, thus it contains $\mathrm{Fitt}_0(\mathcal{M})$.

For the right inclusion: by the Cauchy–Binet formula for $\det(XA)$, the identity $XA = a\mathrm{I}_n$ implies that $a^p = \sum \mu_{X,i}\mu_{A,i}$ where $\mu_{X,i}$ and $\mu_{A,i}$ are p-minors of A and X, thus $a^p \in \mathrm{Fitt}_0(\mathcal{M})$. □

3.2. Twisted Alexander polynomials

For every subgroup $H \triangleleft_{f.i.} G$, consider the left G-module $\mathbb{Z}[G/\ker u]^{G/H}$, where G acts naturally both on $G/\ker u$ and on G/H, thus permuting the factors $\mathbb{Z}[G/\ker u]$. Thus one can define the homology group

$$H_1(G; \mathbb{Z}[G/\ker u]^{G/H}),$$

which is a module over $\mathbb{Z}[G]$ or over $\mathbb{Z}[G/(H \cap \ker u)]$, but not on $\mathbb{Z}[G/\ker u]$.

On the other hand, the action of G on $G/\ker u$ descends to an action of $G/\ker u$ on $\mathbb{Z}[G/\ker u]^{G/H}$ which does not permute the factors $\mathbb{Z}[G/\ker u]$, and $H_1(G;\mathbb{Z}[G/\ker u]^{G/H})$ becomes a module over $\mathbb{Z}[G/\ker u]$.

Viewing $H_1(G;\mathbb{Z}[G/\ker u]^{G/H})$ as a module over $\mathbb{Z}[G/\ker u]$, which is isomorphic to $\mathbb{Z}[\mathbb{Z}^r]$ thus still a UFD, we define the *twisted Alexander polynomial associated to H*:

$$\Delta^H(G,u) := \mathrm{ord}_{\mathbb{Z}[G/\ker u]}(H_1(G/\ker u;\mathbb{Z}[G]^{G/H})),$$

which is an element of $\mathbb{Z}[G/\ker u]$ mod $\pm G/\ker u$.

Remarks 3.3.

(1) When $H = G$, $\Delta^G(G,u)$ can be denoted by $\Delta(G,u)$ and called "multivariate Alexander polynomial", related to the Alexander polynomial of links. If $\mathrm{rk}(u) = 1$, one recovers the classical Alexander polynomial, as generalized by [16].

(2) If X is a finite complex with $\pi_1(X) = G$ and $\widehat{X}_{H,u}$ the covering associated to $\ker u$, H and $H \cap \ker u$, we have an isomorphism of modules over $\mathbb{Z}[H/H \cap \ker u] \approx \mathbb{Z}[u(H)]$:

$$H_1(G;\mathbb{Z}[G/\ker u]^{G/H}) \approx H_1(\widehat{X}_{u,H};\mathbb{Z}).$$

One can deduce that

$$\Delta^H(G,u) = 0 \text{ is } u\text{-monic} \iff \Delta(H,u_{|H}) \text{ is } u\text{-monic}.$$

3.3. Comparison with [6, 3.2.1 to 3.2.4]

(I change their notation from N to M). Friedl and Vidussi start from

- a free Abelian group F together with a morphism $\psi : G = \pi_1(M) \to F$: in our case, $F = G/\ker u$ and ψ is the natural projection.
- a morphism $\gamma : G \to \mathrm{GL}(k,\mathbb{Z}[F])$: in our case, this is the morphism $G \mapsto \mathrm{GL}(\mathbb{Z}[G/\ker u])^{G/H}$ induced by the action of G on G/H but not on $G/\ker u$.

Thus their $\alpha = \gamma \otimes \psi$ is the morphism $G \mapsto \mathrm{GL}(\mathbb{Z}[G/\ker u])^{G/H}$ induced by the actions of G on $G/\ker u$ and on G/H.

Then they define for any $\alpha : G \to \mathrm{GL}(k,\mathbb{Z}[F])$ the *i-twisted Alexander polynomial of* (G,α), denoted by $\Delta^\alpha_{M,i}$, by

$$\Delta^\alpha_{G,i} = \mathrm{ord}_R(H_i(M;\mathbb{Z}[F]^k))$$

where the (hidden) action of G on $\mathbb{Z}[F]^k$ is α. In our notations, we thus have

$$H_i(G;\mathbb{Z}[G/\ker u]^{G/H}) = H_i(M;\mathbb{Z}[F]^k).$$

(for all i if $\widetilde{M}$ is contractible, for $i \leqslant 1$ if not). Thus

$$\Delta^H(G,u) = \Delta^{\gamma\otimes\psi}_{M,1}.$$

3.4. Fibering implies u-monicity of Alexander polynomials

Proposition 3.4. — *If $G = \pi_1(M)$, M a closed manifold of any dimension and $u \in H^1(M,\mathbb{R})$ fibers, then $\Delta^H(G,u)$ is always u-monic.*

Proof. — Let ω be a nonsingular form in the class u, and let X be a vector field on M such that $\omega(X) = 1$. On the universal cover $\widetilde{M}$, ω lifts to $\mathrm{d}f$, X lifts to $\widetilde{X}$ with $\mathrm{d}f(\widetilde{X}) = 1$. Thus the flow (φ^t_X) lifts to $(\varphi^t_{\widetilde{X}})$ with $f \circ \varphi^t_{\widetilde{X}} - f = t$.

Fix a small cell decomposition of M and lift it to $\widetilde{M}$, and choose lifts $\sigma \mapsto \widetilde{\sigma}$ of cells in M. For $t > 0$ large enough, $\varphi^t_{\widetilde{X}}$ is equivariantly homotopic to an equivariant chain map $\widetilde{\varphi}$ such that $\widetilde{\varphi}(\widetilde{\sigma}) = g(\widetilde{\tau})$ with $u(g) > 0$.

Thus the identity of the cell complex $C_*(\widetilde{M})$ is homotopic over $\mathbb{Z}[G]$ to a chain map A whose support in G lies in $\{u > 0\}$. Thus A induces the identity on $H_1(M,\mathbb{Z}[u(G)]^{G/H})$. View A as a matrix in some $\mathrm{M}_N(\mathbb{Z}[G])$, and denote by $\bar{A}$ its image in $\mathrm{M}_{N[G:H]}(\mathbb{Z}[u(G)])$ which acts on $C_*(\widetilde{M}) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[u(G)]^{G/H})$.

Then the support of $\bar{A}$ in $u(G)$ lies in $]0,+\infty[$. On the other hand, since A induces the identity on $H_1(M,\mathbb{Z}[u(G)]^{G/H})$ $\det(\mathrm{Id} - \bar{A})$ annihilates $H_*(M,\mathbb{Z}[u(G)]^{G/H})$ and in particular $H_1(M,\mathbb{Z}[u(G)]^{G/H})$. Since $\det(\mathrm{Id} - A)$ is u-monic, we are done. □

4. Novikov homology

4.1. Novikov ring

We define the *Novikov ring* $\mathbb{Z}[G]_u$ as the following group of formal series over G with coefficients in $\mathbb{Z}$:

$$\mathbb{Z}[G]_u := \{\lambda \in \mathbb{Z}[\![G]\!] \mid (\forall\, C \in \mathbb{R})\ \mathrm{supp}(\lambda) \cap \{u \leqslant C\} \text{ is finite}\},$$

where $\{u \leqslant C\} = \{g \in G \mid u(g) \leqslant C\}$. It is easy to see that the multiplication

$$\sum_{g_1 \in G} a_{g_1} g_1 \cdot \sum_{g_2 \in G} b_{g_2} g_2 = \sum_{g \in G} \left(\sum_{g_1 g_2 = g} a_{g_1} b_{g_2} \right) g$$

is well defined and makes $\mathbb{Z}[G]_u$ a ring containing $\mathbb{Z}[G]$ as a subring.

Units of $\mathbb{Z}[G]_u$. — If $\lambda = 1 + a \in \mathbb{Z}[G]_u$ and $\mathrm{supp}(a) \subset \{u > 0\}$, λ is invertible, with inverse $\lambda^{-1} = \sum_{n=0}^{\infty} (-a)^n$. Thus every element of $\mathbb{Z}[G]_u$ whose u-minimal part is of the form $\pm g$ with $g \in G$, is a unit. We call such elements *u-monic*. More generally, if $A \in \mathrm{M}_n(\mathbb{Z}[G]_u)$ with $\mathrm{supp}(A) \subset \{u > 0\}$, $\mathrm{I}_n + A$ is invertible with $(\mathrm{I}_n + A)^{-1} = \sum_{n=0}^{\infty} (-A)^n$.

In the case when $\mathbb{Z}[G]$ has no zero divisors, in particular for $G = \mathbb{Z}^r$, every unit of $\mathbb{Z}[G]_u$ is u-monic, thus the units of $\mathbb{Z}[G]_u$ coincide with u-monic elements.

4.2. Novikov homology, relation with fibering

The Novikov homology $H_*(G, u)$ is defined as the homology of G with coefficients in the left $\mathbb{Z}[G]$-module $\mathbb{Z}[G]_u$:

$$H_*(G, u) := H_*(G, \mathbb{Z}[G]_u).$$

Although we shall not use it explicitly, let us quote the following easy result, which was a great inspiration for our work. Note the relation with [21].

Theorem 4.1 ([2, 19]). — *If* $\mathrm{rk}(u) = 1$, *the kernel of u is finitely generated if and only if*

$$H_1(G, u) = 0 = H_1(G, -u).$$

For this paper, the interest of Novikov homology lies in the following

Theorem 4.2 ([2]). — *Let $G = \pi_1(M)$, where M is a closed and connected three-manifold. The following are equivalent:*

- *u is fibered.*
- *$H_1(G, u) = 0$.*

Remarks 4.3.

(1) This is their Theorem E, reinterpreted in terms of Novikov homology, cf. p. 456 of the paper.
(2) At the time, one needed M to contain no fake cells, and also the hypothesis $\pi_1(M) \neq \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (to avoid a possible fake $\mathbb{RP}^2 \times S^1$), restrictions removed later thanks to Perelman.

(3) In [19], the equivalence between (i) and $(H_1(G,u)=0=H_1(G,-u))$ was proved as an immediate consequence of [21, 22] and Theorem 4.1. This would suffice to prove our main result with almost no change.

4.3. Computation of $H_1(G,u)$

To simplify the notations, we assume that G is finitely presented (anyhow, we only need this case). Let $\langle x_1,\dots,x_p \mid r_1,\dots,r_q\rangle$ be a presentation of G, and let D_1, D_2 be defined as in Section 3. Then, denoting by $(D_i)_u \in \mathrm{M}_{q,p}(\mathbb{Z}[G]_u)$ the matrix obtained by the base change $\mathbb{Z}[G] \to \mathbb{Z}[G]_u$, we have

$$H_1(G,u) = \ker(\times(D_1)_u)/\operatorname{im}(\times D(_2)_u).$$

Since $u \neq 0$, there exists $i \in \{1,\dots,p\}$, such that $u(x_i) \neq 0$, thus $x_i - 1$ is invertible in $\mathbb{Z}[G]_u$. We can assume that $i = p$. Denoting by $D_2^{(i)}$ the matrix obtained by deleting the i-eth column of D_2, we have $H_1(G,u) \approx \operatorname{coker}(\times D_2^{(i)})_u$.

Corollary 4.4. — *We have*

$$H(G,u) = 0 \iff \times(D_2^{(i)})_u : \mathbb{Z}[G]_u^q \to \mathbb{Z}[G]_u^{p-1} \textit{is onto.}$$

Equivalently, there exists $\widetilde{X} \in \mathrm{M}_{p-1,q}(\mathbb{Z}[G]_u)$ such that $\widetilde{X}D_2^{(i)} = \mathrm{I}_{m-1}$. By truncating $\widetilde{X} = \sum_{g\in G} X_g g$, $X_g \in \mathrm{M}_{p-1,I}(\mathbb{Z})$ below a sufficiently high level of u, i.e. by defining the finite sum $X = \sum_{u(g)\leqslant C} X_g g$ with C sufficiently large, we obtain $X \in \mathrm{M}_{p-1,I}(\mathbb{Z}[G])$ such that $XD_2^{(i)} = \mathrm{I}_{p-1} + A$ with $u > 0$ on $\operatorname{supp}(A)$. Since such a matrix is invertible over $\mathbb{Z}[G]_u$, we obtain the following

Corollary 4.5. — *$(H_1(G,u;R) = 0)$ is equivalent to the existence of a matrix $X \in \mathrm{M}_{m-1,I}(R[G])$ such that $XD_2^{(i)} = \mathrm{I}_{m-1} + A$ with $u > 0$ on $\operatorname{supp}(A)$.*

Corollary 4.6. — *We have*

$$H_1(G,u) = 0 \iff (\exists\, X \in \mathrm{M}_{p-1,q}(\mathbb{Z}[G]))\ u > 0 \textit{ on } \operatorname{supp}(XD_2^{(i)} - \mathrm{I}_{p-1}).$$

Proof. — By Proposition 4.4, the left hand side is equivalent to the existence of $\widetilde{X} \in \mathrm{M}_{p-1,q}(\mathbb{Z}[G]_u)$ such that $\widetilde{X}(D_2^{(i)}) = \mathrm{I}_{p-1}$. By truncating $\widetilde{X} = \sum_{g\in G} X_g g$, $X_g \in \mathrm{M}_{p-1,q}(\mathbb{Z}[G])$ below a sufficiently high level of u, i.e. by defining the finite sum $X = \sum_{u(g)\leqslant C} X_g g$ with C sufficiently large, we obtain $X \in \mathrm{M}_{p-1,q}(\mathbb{Z}[G])$ such that $u > 0$ on $\operatorname{supp}(XD_2^{(i)} - \mathrm{I}_{p-1})$.

Conversely, since such a matrix X is invertible over $\mathbb{Z}[G]_u$, this proves the corollary. □

5. Abelianized relative Novikov homology and twisted Alexander polynomials

Consider the induced morphism $\bar{u} : G/\ker u \to \mathbb{R}$ and the associated Novikov ring $\mathbb{Z}[G/\ker u]_{\bar{u}}$, which is a left $\mathbb{Z}[G]$-module, and define the *Abelianized Novikov homology*

$$H_1^{ab}(G,u) := H_1(G;\mathbb{Z}[G/\ker u]_{\bar{u}}).$$

It is in fact the original homology defined in [Novikov 1981]. Since $G/\ker u$ is free Abelian of rank $r = \mathrm{rk}(u)$, $\mathbb{Z}[G/\ker u]_{\bar{u}})$ is Abelian.

If $H \triangleleft_{f.i} G$ is a normal subgroup with finite index, we generalize the above definition. Consider the induced morphism $\bar{u} : G/(H \cap \ker u) \to \mathbb{R}$ and the associated Novikov ring $\mathbb{Z}/[G/(H \cap \ker u)]_{\bar{u}}$, and define the *Abelianized relative Novikov homology*

$$H_1^{ab}(G,H,u) := H_1(G;\mathbb{Z}[G/(H \cap \ker u)]_{\bar{u}}).$$

Now we can state and prove the relation between Alexander polynomials and Abelianized relative Novikov homology.

Proposition 5.1. — *We have the equivalence*

$$\Delta^H(G,u) \text{ is } u\text{- monic} \iff H_1^{ab}(G,H,u) = 0.$$

Proof (inspired by the referee). — Set $R = \mathbb{Z}[G/\ker u]$, $S = \mathbb{Z}[G/\ker u]_{\bar{u}}$, which are UFDs with $R \subset S$. Set

$$M_R = \mathbb{Z}[G/\ker u]^{G/H} \subset M_S = \mathbb{Z}[G/\ker u]_{\bar{u}}^{G/H}.$$

Then M_R and M_S are G-modules for the natural actions on $G/\ker u$ and on G/H, and $H_1(G;M_R)$ and $H_1(G;M_S)$ can be viewed as modules over R and S respectively, with $G/\ker u$ acting without permuting the factors. Moreover, we have

$$\Delta^H(G,u) = \mathrm{ord}_R(H_1(G;M_R)), H_1^{ab}(G,H,u) = H_1(G;M_S).$$

Recall that for an element of S, in particular of R, to be u-monic means to be a unit in S. Denote by S^* the units of S. Using Proposition 3.2, it suffices to prove that

$$(\spadesuit) \qquad \mathrm{ann}_R(H_1(G;M_R)) \cap S^* \neq \emptyset \iff H_1(G;M_S) = 0.$$

Using a presentation $\langle x_1,\dots,x_p \mid r_1,\dots,r_q\rangle$ be a presentation of G, we have an exact complex

$$C_2=\mathbb{Z}[G]^q \xrightarrow{\times D_2} C_1=\mathbb{Z}[G]^p \xrightarrow{\times D_1} C_0=\mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z}\to 0,$$

where $D_1=\left(\begin{smallmatrix} x_1-1\\ \cdots\\ x_p-1\end{smallmatrix}\right)$ and ε is the augmentation. Thus $H_1(G;M_R)$ and $H_1(G;M_S)$ are the H_1 of the induced sequences with $\mathbb{Z}[G]$ replaced by M_R and M_S. Thus we have (up to isomorphisms)

$$H_1(G;M_R)=\frac{\ker(\times B)}{\operatorname{im}(\times A)},\quad H_1(G;M_S)=\frac{\ker(\times B_S)}{\operatorname{im}(\times A_S)},$$

where $A\in \mathrm{M}_{a,b}(R)$, $B\in \mathrm{M}_{b,c}(R)$, and $A_S=A$, $B_S=B$ viewed as matrices with coefficients in S. Also, $a=q[G:H]$, $b=p[G:H]$, $c=[G:H]$ and $B=\left(\begin{smallmatrix} (\bar{x}_1-1)\mathrm{I}_c\\ \cdots\\ (\bar{x}_p-1)\mathrm{I}_c\end{smallmatrix}\right)$.

The key point is that, since $u\neq 0$, there exists i such that $u(x_i)\neq 0$, thus the image $\bar{x}_i-1\in R$ belongs to S^*. We can assume that $i=p$, thus

$$B=\begin{pmatrix} B_1 (\bar{x}_p-1)\mathrm{I}_c\end{pmatrix}\quad \text{with } B_1\in \mathrm{M}_{b-c,c}(R).$$

Write $A=\left(\begin{smallmatrix} A_1 & A_2\end{smallmatrix}\right)$, where $A_1\in \mathrm{M}_{a,b-c}(R)$ and $A_2\in \mathrm{M}_{a,c}(R)$. Similarly, we have $A_S=\left(\begin{smallmatrix} (A_1)_S & (A_2)_S\end{smallmatrix}\right)$. Since $AB=0$, we have

$$(\heartsuit)\qquad A_1B_1+(\bar{x}_p-1)A_2=0.$$

Proposition 5.1 will result from the following lemma, a variant of 4.4.

Lemma 5.2. — *Up to isomorphisms, we have*

$$(\bar{x}_p-1)\operatorname{coker}(\times A_1)\subset H_1(G;M_R)\subset \operatorname{coker}(\times A_1)$$
$$H_1(G;M_S)=\operatorname{coker}(\times (A_1)_S).$$

Proof of Lemma 5.2. — The secund line follows from the first by replacing R by S and using the fact that $\bar{x}_p-1$ is invertible in S. It can also be proved directly as Corollary 4.4. Thus it suffices to prove the first line.

Step 1. — Consider the map

$$i: w=(\bar{x}_p-1)v\in(\bar{x}_p-1)R^{b-c}\longmapsto ((\bar{x}_p-1)v, vB_1)\in\ker(\times B).$$

If $z\in R^a$, we have

$$i(w)=zA \iff (\bar{x}_p-1)v=zA_1 \text{ and } vB_1=zA_2.$$

In view of $(\heartsuit)$, the right hand side is equivalent to $((\bar{x}_p-1)v=zA_1)$. Thus $i^{-1}(\operatorname{im}(\times A))\subset \operatorname{im}(\times A_1)$, thus i induces an injection

$$(\bar{x}_p-1)\operatorname{coker}(\times A_1)\to \frac{\ker(\times B)}{\operatorname{im}(\times A)}=H_1(G;M_R).$$

Step 2. — Denoting an element of R^b by (v,w) with $v \in R^{b-c}$ and $w \in R^c$, consider the map

$$\pi : (v,w) \in \ker(\times B) \mapsto v \in R^{a-c}.$$

If $\pi(v,w) \in \mathrm{im}(\times A_1)$, i.e. there exists $z \in A^a$ such that $zA_1 = v$, we have

$$zA - (v,w) = (0, zA_2 - w).$$

Since $AB = 0$ and $(v,w)B = 0$, this implies

$$(0, zA_2 - w)B = (\bar{x}_p - 1)(zA_2 - w) = 0,$$

thus $w = zA_2$. Thus

$$(v,w) = (zA_1, zA_2) = zA.$$

Thus $\pi^{-1}(\mathrm{im}(\times A_1)) \subset \mathrm{im}(\times A)$, thus π induces an injection

$$H_1(G; M_R) = \frac{\ker(\times B)}{\mathrm{im}(\times A)} \to \mathrm{coker}(\times A_1). \qquad \square$$

End of the proof of Proposition 5.1. — Since $\bar{x}_p - 1 \in S^*$, the lemma implies

$$\begin{aligned} (5.1) \quad \mathrm{ann}_R(H_1(G;M_R)) \cap S^* \neq \emptyset \\ \iff (\exists\, \lambda \in R \cap S^*) \lambda R^b \subset M_R^{b-c} A_1 \\ \iff (\exists\, \lambda \in R \cap S^*, X \in \mathrm{M}_{b-c,a}(R)) X A_1 = \lambda \mathrm{I}_{b-c} \end{aligned}$$

$$(5.2) \qquad H_1(G; M_S) = 0(1) \iff (\exists\, \widehat{X} \in \mathrm{M}_{b-c,a}(S)) \widehat{X}(A_1)_S = \mathrm{I}_{b-c}.$$

Clearly, (5.1) $\Rightarrow$ (5.2). Conversely, if $\widehat{X}(A_1)_S = \mathrm{I}_{b-c}$, truncating $\widehat{X}$ below a sufficiently high level of u gives an identity with coefficents in R:

$$YA_1 = \mathrm{I}_{b-c} + C \text{ with } u > 0 \text{ on } \mathrm{supp}(C).$$

Thus $\det(YA_1) \in R \cap S^*$, which implies that $X = YA_1\widetilde{(YA_1)}^T$ (transpose of the cofactor matrix) satisfies (5.1). This finishes the proof of Proposition 5.1. $\square$

6. Computations in dimension three and reduction of the main result

In this section we consider the case where $G = \pi_1(M)$ where M is a closed and connected three-manifold with a contractible universal covering $\widetilde{M}$.

6.1. A convenient complex for computing the homology of G

Using a handle decomposition of genus p [or a self-indexing Morse function with one minimum and one maximum], one can obtain $H_*(\widetilde{M};\mathbb{Z})$ by a complex of left modules over $\mathbb{Z}[G]$, of the form

$$C_* = (\mathbb{Z}[G] \overset{\times D_3}{\longrightarrow} \mathbb{Z}[G]^p \overset{\times D_2}{\longrightarrow} Z[G]^p \overset{\times D_1}{\longrightarrow} \mathbb{Z}[G])$$

with

$$D_1 = \begin{pmatrix} x_1 - 1 \\ \dots \\ x_p - 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} y_1 - 1 & \cdots & y_p - 1 \end{pmatrix}.$$

$(x_1, \dots, x_p)$ and $(y_1, \dots, y_p)$ are generating systems for G. Since $u \neq 0$, we can reorder them so that $u(x_p)$ and $u(y_p)$ are nonzero, thus $x_p - 1$ and $y_p - 1$ are invertible in $\mathbb{Z}[G]_u$. Then we denote by c the column $(x_i - 1)_{i<m}$ and ℓ the row $(y_j - 1)_{j<m}$, so that

$$D_1 = \begin{pmatrix} c \\ x_p - 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} \ell & y_p - 1 \end{pmatrix}.$$

Without using the contractibility of $\widetilde{M}$, this complex gives a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$ up to degree 2, thus

$$H_1(G, u) \approx \ker(\partial_1)_{\mathbb{Z}[G]_u} / \operatorname{im}(\partial_2)_{\mathbb{Z}[G]_u},$$

where we have changed the coefficients from $\mathbb{Z}[G]$ to $\mathbb{Z}[G]_u$.

By the contractibility of $\widetilde{M}$, it is a complete free resolution, and G is a group with 3-dimensional Poincaré duality, which can be expressed as follows. Denote by w the orientation morphism $G \to \{1, -1\}$, and define modified adjoint isomorphisms

$$\lambda = \sum_g a_g g \mapsto \lambda^* = \sum_g a_g \varepsilon(g) g^{-1}, A = (a_{i,j})^* = (a^*_{j,i}).$$

Then Poincaré duality can be expressed by the fact that C_* is quasi-isomorphic to the complex $(C^*_i = C_{3-i}, \times D^*_{4-i})$.

Let us write $D_2 = \left(\begin{smallmatrix} A & C \\ L & a \end{smallmatrix}\right)$ where $\mathrm{A} \in \mathrm{M}_{p-1}(\mathbb{Z}[G])$, $L \in M_{1,p-1}(\mathbb{Z}[G])$, $C \in \mathrm{M}_{p-1,1}(\mathbb{Z}[G])$ and $a \in \mathbb{Z}[G]$. Note that $D_2^{(p)} = \left(\begin{smallmatrix} A & C \end{smallmatrix}\right)$.

Since $\partial_1 \circ \partial_2 = 0$ and $\partial_2 \circ \partial_3 = 0$, we have $D_2 D_1 = 0$ and $D_3 D_2 = 0$. Working over $\mathbb{Z}[G]_u$, we obtain

$$\begin{aligned} &C = Ac(1 - x_p)^{-1}, \quad a = Lc(1 - x_p)^{-1} \\ &L = (1 - y_p)^{-1} \ell A \\ &a = (1 - y_p)^{-1} \ell C = (1 - y_p)^{-1} \ell A c (1 - x_p)^{-1}. \end{aligned}$$

Thus (D_3^*, D_2^*, D_1^*) has "the same shape" as (D_1, D_2, D_3) in the following sense: it is obtained from (D_1, D_2, D_3) by replacing (x_i, y_i, A, C, L, a) by $(y_i^{-1}, x_i^{-1}, A^*, L^*, C^*, a^*)$, and one has $u(y_p) \neq 0$.

These computations have the following consequence.

Proposition 6.1. — *Let $u \in H^1(M;\mathbb{R}) \setminus \{0\}$. The two following properties are equivalent:*

(1) *u is fibered.*
(2) *The matrix $A \in \mathrm{M}_{p-1}(\mathbb{Z}[G])$ becomes invertible in $\mathrm{M}_{p-1}(\mathbb{Z}[G]_u)$.*

Proof. — By [2], (1) is equivalent to $(H_1(G,u) = 0)$ thus to the exactness of $C_* \otimes_{\mathbb{Z}G} \mathbb{Z}[G]_u$ in degree 1. Since $u(x_p) \neq 0$, $\bar{x}_p - 1$ is a unit of $\mathbb{Z}[G]_u$. By 4.4, this is equivalent to the left invertibility of $\begin{pmatrix} A L \end{pmatrix}$ with $\mathbb{Z}[G]_u$-coefficients. Since L is of the form λA, this left invertibilty is equivalent to that of A in $\mathrm{M}_{p-1}(\mathbb{Z}[G]_u)$.

Since C^* is quasi-isomorphic to C_*, (1) is equivalent to the exactness of $C^* \otimes_{\mathbb{Z}G} \mathbb{Z}[G]_u$ in degree 1. Since (D_3^*, D_2^*, D_1^*) has the same shape as (D_1, D_2, D_3) in the sense explained above, we can apply the same argument to prove that (1) is equivalent to the left invertibility of A^* in $\mathrm{M}_{p-1}(\mathbb{Z}[G]_{-u})$, i.e. the right-invertibility of A in $\mathrm{M}_{p-1}(\mathbb{Z}[G]_u)$. This proves Proposition 6.1. □

Remark. — In [20] it was established that $\mathbb{Z}[G]_u$ is always *stably finite*: a matrix $A \in \mathrm{M}_n(\mathbb{Z}[G]_u)$ is invertible if and only if it is left invertible. This is a well-known result of Kaplansky for $\mathbb{Z}[G]$, which was proved by [14] for $\mathbb{Z}[G]_u$ when $\mathrm{rk}(u) = 1$. With Poincaré duality, this allows to prove $(H_1(G,u) = 0 \Rightarrow H_1(G,-u) = 0)$ without using the results of Stallings and Thurston.

Proposition 6.1 reduces the proof of the main result to the following result.

Theorem 6.2. — *Assume that G is finitely generated and VRTFN. Let p be a prime and $n \in \mathbb{N}$. Let $A \in \mathrm{M}_n(\mathbb{Z}[G])$ be such that for every $H \triangleleft_{f.i.} G$ its image in $\mathrm{M}_n(\mathbb{Z}[G/H \cap \ker u]_{\bar{u}})$ is invertible.*

Then A is invertible in $\mathrm{M}_n(\mathbb{Z}[G]_u)$.

Remarks 6.3.

(1) Note that we state the result for $A \in \mathrm{M}_n(\mathbb{Z}[G])$, not in $\mathrm{M}_n(\mathbb{Z}[G]_u)$. Presumably, the result would remain true, but it is not needed and I have not been able to prove it.

(2) The validity of Theorem 6.2 for a finite index subgroup $G_0 \subset G$ implies its validity for G: this follows from the fact that an n-matrix A over $\mathbb{Z}[G]_u$ can be represented by a $(n.[G:G_0])$-matrix $\widetilde{A}$ over $\mathbb{Z}[G_0]_{u_{|G_0}}$, and that the invertibility of A is equivalent to the bijectivity of the left and right multiplications by A, thus to the invertibility of $\widetilde{A}$ (and similarly for the finite invertibility).

Thus it suffices to prove Theorem 6.2 when G is finitely generated and RTFN.

7. Finitely detectable units and full left ideals in group rings

Definition 7.1.

(1) *A matrix $A \in \mathrm{M}_n(\mathbb{Z}[G])$ is* finitely invertible *if its image in every quotient $\mathrm{M}_n(\mathbb{Z}[G/H])$ for $H \triangleleft_{f.i.} G$ is invertible. The ring $\mathrm{M}_n(\mathbb{Z}[G])$ has* finitely detectable units *if every finitely invertible matrix is invertible.*

(2) *A left ideal $I \subset \mathbb{Z}[G]$ is* finitely full *if the natural projection $I_H \subset \mathbb{Z}[G/H]$ is equal to $\mathbb{Z}[G/H]$ for every $H \triangleleft_{f.i.} G$. The ring $\mathbb{Z}[G]$ has* finitely detectable full left ideals *if every left ideal which is finitely full is equal to $\mathbb{Z}[G]$.*

Remark 7.2. — Since $\mathbb{Z}[G]$ is anti-isomorphic to itself via $\sum a_g g \mapsto \sum a_g g^{-1}$, if $\mathbb{Z}[G]$ has finitely detectable full left ideals, it also has detectable full right ideals.

Proposition 7.3. — *Assume that $\mathbb{Z}[G]$ has finitely detectable full left ideals.*

(1) *Every left $\mathbb{Z}[G]$-submodule $\mathcal{M} \subset \mathbb{Z}[G]^n$ which projects onto $\mathbb{Z}[G/H]^n$ for every $H \triangleleft_{f.i.} G$ is equal to $\mathbb{Z}[G]^n$.*

(2) *For every $n \in \mathbb{N}^*$, $\mathrm{M}_n(\mathbb{Z}[G])$ has finitely detectable units.*

Proof.

(1). — For $n = 1$, it is the hypothesis. Assume that $n > 1$ and the result is true for $n-1$. Consider the set I of $\lambda \in \mathbb{Z}[G]$ such that there exists $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{Z}[G]$ with $(\lambda_1, \ldots, \lambda_{n-1}, \lambda) \in \mathcal{M}$. It is a left ideal, which projects onto every quotient $\mathbb{Z}[G/H]$ with $H \triangleleft_{f.i.} G$. Thus $I = \mathbb{Z}[G]$, i.e. $\mathcal{M}$ contains an element $x = (\lambda_1, \ldots, \lambda_{n-1}, 1)$.

One has a direct sum decomposition

$$\mathbb{Z}[G]^n = \mathbb{Z}[G]^{n-1} \oplus \mathbb{Z}[G]x.$$

Subtracting $\mu_n x$ from every element $(\mu_1, \dots, \mu_n) \in \mathcal{M}$, one sees that

$$\mathcal{M} = (\mathcal{M} \cap \mathbb{Z}[G]^{n-1}) \oplus \mathbb{Z}[G]x.$$

It suffices to prove that $\mathcal{M} \cap \mathbb{Z}[G]^{n-1} = \mathbb{Z}[G]^{n-1}$. Clearly, $\mathcal{M} \cap \mathbb{Z}[G]^{n-1}$ is a left submodule which projects onto every $(\mathbb{Z}[G/H])^{n-1}$, $H \triangleleft_{f.i.} G$. Thus the proposition follows from the induction hypothesis.

(2). — If $A \in \mathrm{M}_n(\mathbb{Z}[G])$ is finitely invertible, the left submodule $\mathbb{Z}[G]^n A \subset \mathbb{Z}[G]^n$ is finitely full. Thus $\mathbb{Z}[G]^n A = \mathbb{Z}[G]^n$, i.e. A is left invertible. Similarly, $A\mathbb{Z}[G]^n = \mathbb{Z}[G]^n$, thus A is right invertible. Thus A is a unit. □

8. Facts on nilpotent groups and their group rings

We collect here a few facts that we will use about (mostly finitely generated) nilpotent groups and their group rings.

(1) If G is nilpotent and finitely generated, G is polycyclic ([11, Theorem 17.2.2 p. 119]; [18, 5.2.17 p. 13]).

(2) If G is nilpotent and finitely generated, it has a torsion-free subgroup of finite index ([11, Theorem 17.2.2 p. 119]).

(3) If G is nilpotent and finitely generated, it is residually finite ([11, Exercise 17.2.8 p. 124, follows from (2) and the Mal'cev embedding of a finitely generated torsion-free nilpotent group in some $\mathrm{SL}(n, \mathbb{Z})$, Theorem 17.2.8 p. 120]).

(4) If G is nilpotent and finitely generated, every subgroup of G is finitely generated ([18, 5.2.18 p. 137]).

(5) If G is any group and $(\gamma_n(G))$ its lower central series ($\gamma_1(G) = G$, $\gamma_{n+1}(G) = [G, \gamma_n(G)]$), the set

$$G_n := \sqrt{\gamma_n(G)} := \{g \in G \mid (\exists\, k \in \mathbb{N}^*)\ g^k \in \gamma_n(G)\}$$

is a normal subgroup, moreover G/G_n is torsion-free and $[G_n, G_m] \subset G_{n+m}$([17, Lemma 1.8 p. 473]). Note that the sequence (G_n) is finite iff G is nilpotent and torsion-free.

(6) If G is nilpotent and torsion-free, it is *orderable*, i.e. it has a total order such that $x \leqslant y \Rightarrow xz \leqslant yz$ and $zx \leqslant zy$. (See also Section 10) ([17, Lemma 1.6 p. 587]).

(7) If G is nilpotent and torsion-free, $\mathbb{Z}[G]$ is a domain (easy consequence of (6)).

(8) If G is polycyclic (in particular, nilpotent and finitely generated), $\mathbb{Z}[G]$ is left (and right) Noetherian ([9]; [17, Corollary 2.8 p. 425]).

(9) Let G be a polycyclic group, $\mathcal{M}$ a finitely generated left or right module over$\mathbb{Q}[G]$, and z a central element of G, making $\mathcal{M}$ into a $\mathbb{Q}[t,t^{-1}]$-module. Then $\mathcal{M}$ does not contain any $\mathbb{Q}[t,t^{-1}]$-submodule $\mathcal{N}$ having a free submodule $\mathcal{N}_0$ with $\mathcal{N}/\mathcal{N}_0$ torsion and having an infinite family of elements with distinct annihilators ([10, Lemma 6]).

(10) If G is nilpotent and finitely generated, a left $\mathbb{Z}[G]$-module which is simple (or irreducible) is finite. Equivalently, if I is a maximal left ideal of $\mathbb{Z}[G]$, $\mathbb{Z}[G]/I$ is finite ([10, Lemma 2 and Theorem 3.1]; [17, Corollaries 2.9 and 2.10 p. 544].).

(11) If G is nilpotent and torsion-free, $\mathbb{Z}[G]$ is contained in a division ring D which is a *classical ring of quotients on the right and on the left*:

$$D = \{xy^{-1} \mid x \in \mathbb{Z}[G], y \in \mathbb{Z}[G] \setminus \{0\}\} = \{y^{-1}x \mid x \in \mathbb{Z}[G], y \in \mathbb{Z}[G] \setminus \{0\}\}.$$

([8, Theorem 1] and [15, Corollary 10.23 p. 304]: A right Noetherian domain has a classical ring of quotients).

The last statement has the following consequence.

Corollary 8.1 ([15, p. 301]). — *If G is nilpotent and torsion-free and $E \subset \mathbb{Z}[G]$ is finite, its elements can be reduced to a common denominator: there exists $x \in \mathbb{Z}[G] \setminus \{0\}$ such that $E \subset \mathbb{Z}[G]x^{-1}$.*

9. Theorem 6.2 for nilpotent groups

In this section, G is a finitely generated nilpotent group. We first prove the finite detectability of full ideals, then the result in the title.

Proposition 9.1. — *Let G be a finitely generated nilpotent group. Then $\mathbb{Z}[G]$ has finitely detectable full left ideals.*

Proof. — We argue by contradiction, thus we assume that I is a finitely full left ideal in $\mathbb{Z}[G]$ which is not full. Then I is contained in a maximal left ideal I_1, without the axiom of choice since $\mathbb{Z}[G]$ is Noetherian. Then I_1 is again a finitely full left ideal in $\mathbb{Z}[G]$ which is not full, thus we can assume that I is maximal.

Thus $\mathcal{M} := \mathbb{Z}[G]/I$ is a simple $\mathbb{Z}[G]$-module, and by [10], $\mathcal{M}$ is finite.Thus

$$H := \ker(G \to \mathrm{Aut}(\mathcal{M}))$$

has finite index. Thus $\mathcal{M}$ is isomorphic to a quotient of $\mathbb{Z}[G/H]/I_H$ with $H \triangleleft_{f.i.} G$ and I_H the image of I. By hypothesis, $I_H = \mathbb{Z}[G/H]$, thus $\mathcal{M} = 0$, contradiction. □

Now we prove Theorem 6.2 for nilpotent finitely generated groups.

Proposition 9.2. — *Let G be a nilpotent and finitely generated group, and let $A \in \mathrm{M}_m(\mathbb{Z}[G])$ be such that, for every $H \triangleleft_{f.i.} G$, the image $A_{H,u}$ of A in $\mathrm{M}_m(\mathbb{Z}[G/H \cap \ker u]_{\overline{u}})$ is invertible. Then A is invertible in $\mathrm{M}_m(\mathbb{Z}[G]_u)$.*

Proof. — By Remark 6.3 (2), we can assume that G is torsion-free, thus $\mathbb{Z}[G]$ is contained in a division ring D which is a classical ring of fractions on the right and on the left.

If u is injective, the result is obvious. In general, we make an induction over the polycyclic length of G, i.e. in the torsion-free case the length n of any subnormal sequence of subgroups

$$G = G_0 > G_1 > \cdots > G_n > G_{n+1} = \{1\}, \quad G_i/G_{i+1} \approx \mathbb{Z}.$$

By the Schreier refinement theorem ([11, 4.4.4]), this length is independent of the sequence.

We can assume that $\ker u \neq \{1\}$ and that the Proposition is already known when the Hirsch length is smaller than that of G. Then u is not injective on the center $C(G)$, otherwise we would have $C(G) \cap [G,G] = \{1\}$ thus $[G,G] = \{1\}$ and G would be Abelian, giving a contradiction.

Moreover, every element of G which has a nontrivial power in $C(G)$ is already in $C(G)$ ([11, Exercise 16.2.9]). Thus we can find $z \in C(G) \cap \ker u$ such that $\Gamma := G/\langle z \rangle$ has no torsion. Then the Hirsch length of Γ is smaller than that of G. □

Lemma 9.3. — *There is an identity $AB = x\mathrm{I}_m$ in $\mathrm{M}_m(\mathbb{Z}[G]_u)$, with $x \in \mathbb{Z}[G]_u \setminus \{0\}$.*

Proof. — The right multiplication by A is injective on $(\mathbb{Z}[G]_u)^n$: if $LA = 0$, we obtain $L_{H,u}A_{H,u} = 0$ for every $H \triangleleft_{f.i.} G$, where $L_{H,u}$ is the image of L in $(\mathbb{Z}[G/(H \cap \ker u)]_{\overline{u}})^m$. Since by hypothesis $A_{H,u}$ is invertible, $L_{H,u} = 0$. Since this is true for all H and G is residually finite, $L = 0$.

Thus A has an inverse in $\mathrm{M}_m(D)$, which by Corollary 8.1 is of the form $A^{-1} = Bx^{-1}$ with $B \in \mathrm{M}_m(\mathbb{Z}[G])$, $x \in \mathbb{Z}[G] \setminus \{0\}$. This proves Lemma 9.3. □

To prove Proposition 9.2, it suffices to prove that x divides B on the right, i.e. $B = B_1 x$ in $\mathrm{M}_n(\mathbb{Z}[G]_u)$, thus $(AB_1)x = x\mathrm{I}_n$, and since $x \neq 0$ and $\mathbb{Z}[G]_u$ is a domain, this implies $AB_1 = \mathrm{I}_n$. A similar argument with "left" and "right" exchanged proves that A is left-invertible, thus invertible.

Lemma 9.4. — *For $n \in \mathbb{N}^*$, let x_n, B_n be the images of x, B in $\mathbb{Z}[G/\langle z^n \rangle]_{\overline{u}})$ and $\mathrm{M}(m, \mathbb{Z}[G/\langle z^n \rangle]_{\overline{u}}))$. Then x_n divides B_n on the right.*

Proof. — The image A_n of A in $\mathrm{M}_m(\mathbb{Z}[G/\langle z^n\rangle])$ gives rise to a matrix $\widetilde{A}_n \in \mathrm{M}_{mn}(\mathbb{Z}[\Gamma])$ whose images in every $\mathrm{M}_{mn}(\mathbb{Z}[\Gamma/H \cap \ker \overline{u}]_{\overline{u}})$ is invertible, and $\widetilde{A}_n$ is invertible in $\mathrm{M}_{mn}(\mathbb{Z}[\Gamma]_{\overline{u}})$ if and only if A_n is invertible in $\mathrm{M}_m(\mathbb{Z}[G/\langle z^n\rangle]_{\overline{u}})$.

By the induction hypothesis, $\widetilde{A}_n$ is invertible in $\mathrm{M}_{mn}(\mathbb{Z}[\Gamma]_{\overline{u}})$ thus A_n is invertible in $\mathrm{M}_m(\mathbb{Z}[G/\langle z^n\rangle]_{\overline{u}})$. Denote its inverse by A_n^{-1} and multiply the identity $A_nB_n = x_n\mathrm{I}_m$ on the left by A_n^{-1}, we obtain $B_n = A_n^{-1}x_n$, which proves Lemma 9.4. □

We shall need the two following objects.

(1) For $\lambda = \sum_{g\in G} a_g g \in \mathbb{Z}[G]_u \setminus \{0\}$, define $\mu = \min(u_{|\mathrm{supp}(z)})$ and

$$\widetilde{m}_u(\lambda) = \sum_{g | u(g)=\min(u_{|\mathrm{supp}(\lambda)})} a_g g \in \mathbb{Z}[G].$$

We have $\widetilde{m}_u(\lambda) = g m_u(\lambda)$ with $g \in G$ and $m_u(\lambda) \in \mathbb{Z}[G/\ker u]$, where $m_u(\lambda)$ is defined up to multiplication by an element of $\ker u$. We call $m_u(\lambda)$ the *u-minimal part of* λ.

(2) Let $\zeta_n \in \mathbb{C}$ be a primitive n-root of unity, and let $\Phi_n \in \mathbb{Z}[t]$ be its minimal polynomial (the n-th cyclotomic polynomial). The rings $\mathbb{Z}[G]$ and $\mathbb{Z}[G]_{\overline{u}}$ can be factored by the ideal generated by $\Phi_n(z)$, to give quotients of $\mathbb{Z}[G/\langle z^n\rangle]$, and $\mathbb{Z}[G/\langle z^n\rangle]_{\overline{u}}$. The quotients may be expressed as twisted rings

$$\mathbb{Z}[G]/(\Phi_n(z)) = \mathbb{Z}[\zeta_n][\Gamma]$$
$$\mathbb{Z}[G]_{\overline{u}}/(\Phi_n(z)) = \mathbb{Z}[\zeta_n][\Gamma]_{\overline{u}}.$$

Since $\mathbb{Z}[\zeta_n]$ is a domain, these rings are also domains.

Let y be a coefficient of B. Denote by $\bar{x}_n$ and $\bar{y}_n$ the images of x and y in $\mathbb{Z}[\zeta_n][\Gamma]_{\overline{u}}$. By Lemma 9.4, $\bar{x}_n$ divides $\bar{y}_n$ since they are also images of $x_n, y_n \in \mathbb{Z}[G/\langle z^n\rangle]_{\overline{u}}$. Moreover, $\bar{x}_n$ and $\bar{y}_n$ have $\bar{u}$-minimal parts

$$m_{\overline{u}}(\bar{x}_n), m_{\overline{u}}(\bar{y}_n) \in \mathbb{Z}[\zeta_n][\ker \overline{u}],$$

defined up to multiplication by an element of $\pm \ker \overline{u}$.

Since $m_u(x) \neq 0$, for $n \gg 1$ its image $\overline{m_u(x)}_n$ is nonzero, thus equal to $m_{\overline{u}}(\bar{x}_n)$. And since $\bar{x}_n$ divides $\bar{y}_n$ and $\mathbb{Z}[\ker \overline{u}]$ is a domain, this implies that $\overline{m_u(x)}_n$ divides $\overline{m_u(y)}_n$ for $n \gg 1$, equivalently that $m_u(x)$ divides $m_u(y)$ modulo $\Phi_n(z)$ for $n \gg 1$.

To finish the proof of Proposition 9.2, it suffices to prove the following lemma.

Lemma 9.5.

(1) *If* $P, Q \in \mathbb{Z}[t]$ *and* $P(\zeta_n)$ *divides* $Q(\zeta_n)$ *in* $\mathbb{Z}[\zeta_n]$ *for* $n \gg 1$*, then* P *divides* Q *in* $\mathbb{Z}[t]$*.*
(2) *If* $\lambda, \mu \in \mathbb{Z}[\ker u]$ *and* λ *divides* μ *modulo* $\Phi_n(z)$ *for* $n \gg 1$*, then* λ *divides* μ *in* $\mathbb{Z}[\ker u]$*.*
(3) *If* $x, y \in \mathbb{Z}[G]_u$ *and* $m_u(x)$ *divides* $m_u(y)$ *modulo* $\Phi_n(z)$ *for* $n \gg 1$*, then* x *divides* y *in* $\mathbb{Z}[G]_u$*.*

Indeed, modulo the lemma we have $B = B_1 x$ with $B_1 \in \mathrm{M}_n(\mathbb{Z}[G]_u)$, thus $(AB_1)x = x\mathrm{I}_n$, and since $x \neq 0$ and $\mathbb{Z}[G]_u$ is a domain, this implies $AB_1 = \mathrm{I}_n$, thus A is right invertible. A similar argument as above with "right" and "left" exchanged proves that A is left-invertible, thus A is invertible.

Proof.

(1). — Since $\mathbb{Z}[t]$ is a UFD, one can reduce to the case when P is irreducible. The resultants $\mathrm{res}(P, \Phi_n)$ and $\mathrm{res}(P, Q)$ satisfy

$$\mathrm{res}(P, \Phi_n) = \pm \prod_{\zeta \in \Phi_n^{-1}(0)} P(\zeta_n), \quad \mathrm{res}(P, Q) = \pm \prod_{\zeta \in \Phi_n^{-1}(0)} P(\zeta_n).$$

Since $P(\zeta_n)$ divides $Q(\zeta_n)$ in $\mathbb{Z}[\zeta_n]$ for $n \gg 1$, this implies

$$(\forall\, n \gg 1) \quad \mathrm{res}(P, \Phi_n) \text{ divides } \mathrm{res}(P, Q) \text{ in } \mathbb{Z}.$$

We want to prove that $\mathrm{res}(P, Q) = 0$. It suffices to prove that $\mathrm{res}(P, \Phi_n)$ takes infinitely many values.

If p is a prime number, we have

$$\mathrm{res}(P, \Phi_p) = \mathrm{res}(P, t^{p-1} + \cdots + 1) = \prod_{\alpha \in P^{-1}(0)} (\alpha^{p-1} + \cdots + 1).$$

We distinguish three cases:

- The zeros $\alpha_1, \ldots, \alpha_d$ of P are not algebraic units. Then for some non-Archimedean absolute value $|\cdot|_v$ on $\mathbb{Q}(\alpha_1, \ldots, \alpha_d)$ we have $|\alpha_1|_v = \cdots = |\alpha_d|_v \neq 1$. Replacing P by $t^d P(t^{-1})$, we can assume that $|\alpha_1|_v > 1$, thus as $p \to \infty$ the formula for $\mathrm{res}(P, \Phi_p)$ implies that when $p \to \infty$ we have

$$|\mathrm{res}(P, \Phi_p)|_v \sim C|\alpha_1|^{dp}, C > 0.$$

  Thus $\mathrm{res}(P, \Phi_p)$ takes infinitely many values.
- The zeros of P are algebraic units but not roots of unity. Then at least one has modulus 1. Say that $|\alpha_1|, \ldots, |\alpha_k| > 1 \geqslant |\alpha_{k+1}, \ldots, \alpha_d|$.

Then the formula for $\mathrm{res}(P, \Phi_p)$ implies

$$|\mathrm{res}(P, \Phi_p)| \sim C|\alpha_1|^{kp}, C > 0.$$

Again, $\mathrm{res}(P, \Phi_p)$ takes infinitely many values.

- The zeros of P are roots of unity, i.e. $P = \pm\Phi_k$ for some k. Then if p is a large prime, it does not divides k, thus

$$\Phi_{kp} = \sum_{i=0}^{p-1} t^{ki},$$

which implies

$$\mathrm{res}(\Phi_k, \Phi_{kp}) = \prod_{\zeta \in \Phi_k^{-1}(\{0\})} \Phi_{kp}(\zeta) = p^{\varphi(k)}.$$

Thus $\mathrm{res}(P, \Phi_{kp})$ takes infinitely many values.

(2). — Let $\sigma : G/\langle z\rangle \to G$ be a section, then λ, μ can be written uniquely

$$\lambda = \sum_{\gamma \in G\langle z\rangle} P_\gamma(z)\sigma(\gamma), \mu = \sum_{\gamma \in G\langle z\rangle} Q_\gamma(z)\sigma(\gamma).$$

Let $P(z)$ (resp. $Q(z)$) be the gcd of the $P_\gamma(z)$ (resp. the $Q_\gamma(z)$) in $\mathbb{Z}[z, z^{-1}]$, defined up to multiplication by $\pm z^k$. The hypothesis implies that for n large enough $P(z)$ divides each coefficient $Q_\gamma(z)$ modulo $\Phi_n(z)$. By (1), $P(z)$ divides $Q_\gamma(z)$, thus $P(z)$ divides μ.

Thus we are reduced to the case where $P(z) = 1$, i.e. λ is not divisible by any $R(z) \neq \pm z^k$. This means that the right $\mathbb{Z}$-module $M = \mathbb{Z}[G]/\lambda\mathbb{Z}[G]$ is torsion-free over $\mathbb{Z}[z, z^{-1}]$. The image $\bar{\mu}$ of μ in M is divisible by $\Phi_n(z)$ for n large enough, and we want to prove that $\bar{\mu} = 0$. Assume by contradiction that $\bar{\mu} \neq 0$, then we shall obtain a contradiction to Lemma 6 of [10], whose statement we recall:

(♣) *Let G be a polycyclic group, $\mathcal{M}$ a finitely generated left or right module over $\mathbb{Q}[G]$, and z a central element of G, making $\mathcal{M}$ into a $\mathbb{Q}[t, t^{-1}]$-module. Then $\mathcal{M}$ does not contain any $\mathbb{Q}[t, t^{-1}]$-submodule $\mathcal{N}$ having a free submodule $\mathcal{N}_0$ with $\mathcal{N}/\mathcal{N}_0$ torsion and having an infinite family of elements with distinct annihilators.*

The $\mathbb{Z}[z, z^{-1}]$-submodule

$$N = \{x \in M \mid (\exists\, P(z) \in \mathbb{Z}[z, z^{-1}] \setminus \{0\})\ P(z)x \in \mathbb{Z}[z, z^{-1}]\bar{\mu}\}$$

contrains the free submodule $N_0 = \mathbb{Z}[z, z^{-1}]\bar{\mu}$. Moreover, N/N_0 is torsion, and for n large enough N/N_0 admits an element with annihilator the principal ideal

$$I_n = \Phi_n(z)\mathbb{Z}[z, z^{-1}].$$

Tensoring everything over $\mathbb{Q}$, we obtain

$$\mathcal{M}, \mathcal{N}, \mathcal{N}_0, (I_n)_{\mathbb{Q}} = \Phi_n(z)\mathbb{Q}[z, z^{-1}]$$

which contradict (♣). Thus $\bar{\mu} = 0$, which proves (2).

(3). — We define a "division algorithm by increasing value of u" (analogous to the division algorithm in $\mathbb{Q}[\![t]\!]$). Since in (2) we can replace μ by $\mu - \alpha\lambda$, we have that $m_u(x)$ divides $m_u(y - \alpha x)$ for every $\alpha \in \mathbb{Z}[\ker u]$. This allows to define by induction a sequence (x_n) in $\mathbb{Z}[G]_u$ by $x_0 = x$, $x_1 = y$ and

$$(\forall\, n \geqslant 2) \quad x_n = x_{n-1} - \widetilde{m}_u(x_{n-1})\widetilde{m}_u(x)^{-1}.$$

By construction, we have

$$(\diamondsuit) \qquad y = (\widetilde{m}_u(x_1)\widetilde{m}_u(x)^{-1} + \cdots + \widetilde{m}_u(x_{n-1})\widetilde{m}_u(x)^{-1})x + x_n$$

Define $S = \operatorname{supp}(y)$ and

$$T = \{g^{-1}h \mid h \in \operatorname{supp}(\widetilde{m}_u(x)),\ g \in \operatorname{supp}(t_u(x))\},$$

which is a subset of $[a, +\infty[$ for some $a > 0$. For $n \geqslant 2$, we have

$$\operatorname{supp}(x_n) \subset \operatorname{supp}(x_{n-1}) \cup T \operatorname{supp}(x_{n-1}).$$

Since $\operatorname{supp}(x_1) = S$, this implies

$$\operatorname{supp}(x_n) \subset E = \bigcup_{i=0}^{\infty} ST^i, T^i = \{t_1 \ldots t_i \mid t_1, \ldots, t_i \in T\}.$$

Let $\mu_n = \min(u_{|\operatorname{supp}(x_n)})$. By construction, μ_n is increasing and belongs to $u(E)$. Since $u \geqslant a > 0$ on T, E has only a finite number of elements in $\{u \leqslant C\}$ for every $C \in \mathbb{R}$, thus $\mu_n \to +\infty$. Thus $\sum_{n=1}^{\infty} \widetilde{m}_u(x_n)\widetilde{m}_u(x)^{-1}$ is a well-defined element $\alpha \in \mathbb{Z}[G]_u$, and by $(\diamondsuit)$ we have $y = \alpha x$. This finishes the proof of Lemma 9.5 and thus of Proposition 9.2. □

10. Mal'cev–Neumann completion of $\mathbb{Z}[G]$

Here we assume that G is residually torsion-free nilpotent (RTFN), i.e. there exists a series of normal subgroups of $G = G_0 > G_1 > \cdots > G_n$, such that G/G_n is torsion-free nilpotent and $\bigcap_{n\in\mathbb{N}} G_n = \{1\}$. We also require that G be finitely generated (countable would suffice).

10.1. Order on G

Following [4], we define an order on G as follows. First, one defines

$$G_n := \sqrt{\gamma_n(G)} = \{x \in G \mid (\exists\, m > 0) x^m \in \gamma_n(G)\}.$$

As we recalled, since G is nilpotent they are subgroups. Clearly, they are normal in G, and G/G_n is torsion-free. Moreover, one has $[G_n, G_n] \subset G_{2n+1} \subset G_{n+1}$ by [17, Lemma 1.8 p. 473]. Thus G_n/G_{n+1} is torsion-free Abelian. It is also finitely generated since it is contained in G/G_n which is nilpotent and finitely generated.

One orders arbitrarily each G_n/G_{n+1}. Then one defines $x \in G_0$ to be positive if and only if, for the unique n such that $x \in G_n \setminus G_{n+1}$, one has $xG_{n+1} > 1$ in G_n/G_{n+1}.

In other words, an element $x \in G$ is > 1 if and only if its first nontrivial image in a subquotient G_{n-1}/G_n is > 1. It is clear that G^+ is indeed the positive cone of an order on G.

10.2. Mal'cev–Neumann completion, comparison with Novikov

We recall a celebrated result of A.I. Mal'cev and B.H. Neumann: if G is a bi-invariantly ordered group, the formal series

$$\mathbb{Q}\langle G\rangle := \{\lambda \in \mathbb{Q}[\![G]\!] \mid \operatorname{supp}(\lambda) \text{ is well-ordered}\}$$

form a division ring (or skew field) for the natural operations, containing $\mathbb{Q}[G]$ as a subring. (Actually, one can replace $\mathbb{Q}$ by any field, or even any division ring).

In presence of a nonzero morphism $u : G \to \mathbb{R}$, following [12], we shall require the order to be compatible with u in the sense that $(u(x) > 0 \Rightarrow x > 1)$. This is possible by changing the definition of (G_n), setting

- $G_1^{new} = \ker u$
- $G_n^{new} = G_{n-1}$ if $n \geqslant 2$.

and defining the order on G/G_1 by embedding it in $\mathbb{R}$ via $\bar{u}$ induced by u. The interest of this is that we then have

$$\mathbb{Z}[G]_u \subset \mathbb{Q}\langle G\rangle.$$

10.3. Subfield with controlled coefficients

We shall work mostly in a subfield of $\mathbb{Z}\langle G\rangle$, introduced by [4], which contains $\mathbb{Z}[G]$: by definition,

$$S(G) := \bigcup_{n=1}^{\infty} \mathbb{Q}[G_n]\langle G/G_n\rangle,$$

where

$$\mathbb{Q}[G_n]\langle G/G_n\rangle := \left\{\lambda \in \mathbb{Q}[\![G]\!] \;\middle|\; \lambda = \sum_{t\in T} \lambda_t t, \lambda_t \in \mathbb{Q}[G_n], t \in T\right\},$$

where T is a well-ordered subset of some transversal T_n of G_n in G (clearly, this does not depend on the choice of T_n). This is clearly a subring of $\mathbb{Q}\langle G\rangle$, which contains $\mathbb{Q}[G]$.

Proposition 10.1 ([4, Proposition 4.3]). — *$S(G)$ is a subfield of $\mathbb{Q}\langle G\rangle$.*

We provide a proof of this proposition since that of Eizenbud–Lichtman is incorrect. I thank Andrei Jaikin (personal communication, January 2020) for alerting me about this.

Proof. — We can choose the transversals so that $T_n \subset T_{n+1}$. We have to prove that every nonzero $\lambda \in \mathbb{Q}[G_n]\langle G/G_n\rangle$ is invertible in $\mathbb{Q}[G_m]\langle G/G_m\rangle$ for m large enough. We can assume that $\lambda = \sum_{t\in T} \lambda_t t$, with T a well-ordered subset of T_n, $\min T = 1$, $\lambda_t \in \mathbb{Q}[G_n]$ and $\lambda_1 \neq 0$. And also that

$$\lambda_1 = 1 + a_1 g_1 + \cdots + a_k g_k, a_i \in \mathbb{Q}, g_i > 1.$$

Let $m \geqslant n$ be large enough that none of the g_i is in G_m. Thus $g_i = \gamma_i t_i$ with $\gamma_i \in G_m$ and $t_i \in T_m \cap G^+$. Thus

$$\lambda = 1 + \sum_{i=1}^{k} a_i \gamma_i t_i + \sum_{t\in T\setminus\{t_0\}} \lambda_t t.$$

Moreover, $\lambda_t t \in \mathbb{Q}[G_m]F_t$ where F_t is a finite subset of T_m, with $t < t' \Rightarrow F_t < F_{t'}$ (every element of F_t is less than every element of $F_{t'}$). Since T is well-ordered, $\widehat{T} := \{t_1, \ldots, t_m\} \cup \bigcup_{t\in T} F_t$ is a well-ordered subset of $T_m \cap G^+$, and we have

$$\lambda = 1 + \sum_{t\in\widehat{T}} \mu_t t, \mu_t \in \mathbb{Q}[G_m].$$

Thus in $\mathbb{Q}\langle G\rangle$ we have

$$\lambda^{-1} = 1 - \sum_{t\in\widehat{T}} \mu_t t + \cdots + (-1)^r \sum_{t_1,\ldots,t_r\in\widehat{T}} \mu_{t_1} t_1 \ldots \mu_{t_r} t_r + \ldots.$$

Each product $\mu_{t_1}t_1\dots\mu_{t_r}t_r$ can be rewritten $\nu_t t$ with $\nu_t \in \mathbb{Q}[G_m]$ and $t \in (\widehat{T})^+$ (a positive word in $\widehat{T}$). By the proof of Mal'cev–Neumann [17, Lemma 2.10 p. 599-601], $(\widehat{T})^+$ is well-ordered and every element belongs to at most finitely many sets $(\widehat{T})^n$. Thus we obtain

$$\lambda^{-1} = 1 + \sum_{t\in(\widehat{T})^+} \alpha_t t, \alpha_t \in \mathbb{Q}[G_m],$$

thus $\lambda^{-1} \in \mathbb{Q}[G_m]\langle G_m\rangle$. □

The interest of $S(G)$ lies in the following

Proposition 10.2. — *The projection $\pi_\lambda : \mathrm{supp}(\lambda) \to G/G_n$ has finite fibers, thus there is a well-defined morphism*

$$\lambda \in \mathbb{Q}[G_n]\langle G/G_n\rangle \mapsto \bar{\lambda} \in \mathbb{Q}\langle G/G_n\rangle$$

which extends the natural morphism $\mathbb{Q}[G] \to \mathbb{Q}[G/G_n]$.

Proof. — Writing $\lambda = \sum_{t\in T}\lambda_t t$, we have $\mathrm{im}(\pi_\lambda) = p(T)$ where p is the projection $T_n \to G/G_n$, which is bijective and increasing since T_n is a transversal. Then $\pi_\lambda^{-1}(\{p(t)\}) = \mathrm{supp}(\lambda_t)$, which is finite.

To prove "thus", define

$$\bar{\lambda} = \sum_{t\in T}\lambda_t p(t) = \sum_{g\in p(T)} \lambda_{p^{-1}(g)} g \in \mathbb{Q}[\![G/G_n]\!].$$

Its support is $\bigcup_{g\in p(T)} \mathrm{supp}(\lambda_{p^{-1}(g)})$: it is well-ordered since $p(T)$ is ordered as T and the $\mathrm{supp}(\lambda_t)$ are finite. □

Corollary 10.3. — *If $\lambda \in \mathbb{Z}[G_n]\langle G/G_n\rangle$ and $\bar{\lambda} \in \mathbb{Z}[G/G_n]_{\bar{u}}$, then $\lambda \in \mathbb{Z}[G]_u$.*

Proof. — Let $c \in \mathbb{R}$. By hypothesis, $\mathrm{supp}(\bar{\lambda}) \cap \{u < c\}$ is finite. Since $\mathrm{supp}(\lambda) \to G/G_m$ has finite fibers, $\mathrm{supp}(\lambda) \cap \{u < c\}$ is finite, thus $\lambda \in \mathbb{Z}[G]_u$. □

11. Proof of Theorem 6.2

By Remark 6.3 (2), it suffices to treat the case when G is RTFN. We define the order on G, $\mathbb{Q}\langle G\rangle$, $S(G)$ and $\mathbb{Q}[G_m]\langle G/G_m\rangle$ as in the previous section. We assume $(u(x) > 0 \Rightarrow x > 1)$, thus $\mathbb{Z}[G]_u \subset \mathbb{Q}\langle G\rangle$.

Let $A \in \mathrm{M}_n(\mathbb{Z}[G])$ such that every image in $\mathrm{M}_n(\mathbb{Z}[G/(H\cap\ker u]_{\bar{u}})$ (for $H \triangleleft_{f.i.} G$) is invertible. We want to prove that A is invertible in $\mathrm{M}_n(\mathbb{Z}[G]_u)$.

Step 1. — We first prove that A is invertible in $\mathrm{M}_n(S(G))$. Assume the contrary, then since $S(G)$ is a division ring, there exists $L \in (S(G))^n \setminus \{0\}$ such that $LA = 0$. By Proposition 10.2, for N large enough the image $\overline{L} \in (\mathbb{Q}\langle G/G_N\rangle)^n$ is well-defined and nonzero, and we have $\overline{L}\,\overline{A} = 0$, where $\overline{A}$ is the image of A in $\mathrm{M}_n(\mathbb{Q}\langle G/G_N\rangle)$.

Since $\mathrm{supp}(\overline{L}) \subset G/G_N$ is finite, we find $k \in \mathbb{N}^*$ such that $m\overline{L} \in (\mathbb{Z}\langle G/G_N\rangle)^n$. Thus $\overline{A}$ is not invertible in $\mathrm{M}_n(\mathbb{Z}[G/G_N])$ and a fortiori in $\mathrm{M}_n(\mathbb{Z}[G/G_N]_{\overline{u}})$. Since G/G_N is nilpotent, Proposition 9.2 implies that for some subgroup $K \lhd_{f.i.} G/G_N$, the image of $\overline{A}$ is not invertible in

$$\mathrm{M}_n(\mathbb{Z}[(G/G_N)/(K \cap \ker u)]_{\overline{u}}).$$

We have $K = H/H_N$ with $H \lhd_{f.i} G$, and there is a natural isomorphism

$$(G/G_N)/((H/H_N) \cap \ker u) \approx G/(H \cap \ker u).$$

Thus the image of A is not invertible in $\mathrm{M}_n(\mathbb{Z}[(G/(G \cap \ker u)]_{\overline{u}})$, contradiction.

Step 2. — Let B be the inverse of A in $\mathrm{M}_n(S(G))$, and let N_0 be such that $B \in \mathrm{M}_n(\mathbb{Q}[G_{N_0}]\langle G/G_{N_0}\rangle)$. By Proposition 10.2, for every $N \geqslant N_0$, B has a well-defined image $\overline{B} \in \mathrm{M}_N(\mathbb{Q}\langle G/G_N\rangle)$, and $\overline{A}\,\overline{B} = \mathrm{I}_n = \overline{B}\,\overline{A}$. Thus $\overline{B}$ is the inverse of $\overline{A} \in \mathrm{M}_n(\mathbb{Q}\langle G/G_N\rangle)$. Since $\overline{A}$ is invertible already in $\mathrm{M}_n(\mathbb{Z}[G/G_N]_{\overline{u}})$, we have $\overline{B} \in \mathrm{M}_n(\mathbb{Z}[G/G_N]_{\overline{u}})$.

Since this is true for all $N \geqslant N_0$, B has integer coefficients, ie $B \in \mathrm{M}_n(\mathbb{Z}[G_{N_0}]\langle G/G_{N_0}\rangle)$. Finally, its image in $\mathrm{M}_n(\mathbb{Z}\langle G/G_{N_0}\rangle)$ belongs to $\mathrm{M}_n(\mathbb{Z}[G/G_{N_0}]_{\overline{u}})$, thus by Corollary 10.3, we have $B \in \mathrm{M}_n(\mathbb{Z}[G]_u)$. □

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