ALGEBRAIC COMBINATORICS

Sean Eberhard & Mikhail Muzychuk

Fusions of tensor powers of Johnson schemes

Volume 6, issue 2 (2023), p. 533-538.

https://doi.org/10.5802/alco.271

*Algebraic Combinatorics is published by The Combinatorics Consortium
and is a member of the Centre Mersenne for Open Scientific Publishing*
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

Algebraic Combinatorics
Volume 6, issue 2 (2023), p. 533–538
https://doi.org/10.5802/alco.271

Fusions of tensor powers of Johnson schemes

Sean Eberhard & Mikhail Muzychuk

Abstract This paper is a follow-up to [5], in which the first author studied primitive association schemes lying between a tensor power $\mathcal{T}_m^d$ of the trivial association scheme and the Hamming scheme $\mathcal{H}(d,m)$. A question which arose naturally in that study was whether all primitive fusions of $\mathcal{T}_m^d$ lie between $\mathcal{T}_{m^e}^{d/e}$ and $\mathcal{H}(d/e,m^e)$ for some $e \mid d$. This note answers this question positively provided that m is large enough. We similarly classify primitive fusions of the dth tensor power of a Johnson scheme on $\binom{m}{k}$ points when m is large enough in terms of k and d.

1. Introduction

Association schemes are objects of central importance in algebraic combinatorics. For an introduction to association schemes, the reader could refer to any of [4, 2, 13, 1, 5].

All our association schemes are symmetric. If $\mathfrak{X}, \mathfrak{Y}$ are association schemes on a common vertex set then we write $\mathfrak{X} \leqslant \mathfrak{Y}$ if $\mathfrak{X}$ refines $\mathfrak{Y}$ as a partition. A fusion of an association scheme is a coarsening which is again an association scheme. Notice that some authors, for example [6], use an opposite order on the set of association schemes. Our choice was motivated by keeping notation consistent with [1, 5]. Notice that with this choice of ordering the inclusion $\mathfrak{X} \leqslant \mathfrak{Y}$ implies a similar inclusion between the automorphism groups $\mathsf{Aut}(\mathfrak{X}) \leqslant \mathsf{Aut}(\mathfrak{Y})$.

We denote the Hamming scheme of order m^d and rank $d+1$ by $\mathcal{H}(d,m)$ and the Johnson scheme of order $\binom{m}{k}$ and rank $k+1$ by $\mathcal{J}(m,k)$. The special case $\mathcal{H}(1,m) \cong \mathcal{J}(m,1)$ is the *trivial scheme*, denoted $\mathcal{T}_m$. The dth tensor power of an association scheme $\mathfrak{X}$ is denoted $\mathfrak{X}^d$. The symmetrized dth tensor power of the Johnson scheme $\mathcal{J}(m,k)$ is called the *Cameron scheme* and denoted $\mathcal{C}(m,k,d)$.

In this paper we classify primitive fusions of $\mathcal{J}(m,k)^d$ assuming m is sufficiently large in terms of k and d.

Theorem 1.1. *For any positive integers k, d there exists a constant $m_0(k,d)$ such that any primitive fusion $\mathfrak{X}$ of $\mathcal{J}(m,k)^d$ with $m \geqslant m_0(k,d)$ belongs, up to permuting coordinates, to one of the following intervals:*

(1) $\mathcal{J}(m,k)^d \leqslant \mathfrak{X} \leqslant \mathcal{C}(m,k,d)$,
(2) $\mathcal{T}_{M^e}^{d/e} \leqslant \mathfrak{X} \leqslant \mathcal{H}(d/e, M^e)$ *for some integer $e \mid d$ where $M = \binom{m}{k}$.*

Manuscript received 30th May 2022, revised 21st September 2022 and 27th September 2022, accepted 27th September 2022.

Keywords. association schemes, permutation groups.

Acknowledgements. SE has received funding from the European Research Council (ERC), via the EU's Horizon 2020 grant No. 803711. MM was supported by the Israeli Ministry of Absorption.

The special cases $k = 1$ and $d = 1$ are worth highlighting individually. In [9] it was shown that $\mathcal{H}(d,m)$ has no nontrivial fusions for $m > 4$. The case $k = 1$ of Theorem 1.1 more generally classifies primitive fusions of $\mathcal{T}_m^d$ (for m sufficiently large).

Corollary 1.2. *Let $\mathfrak{X}$ be a primitive fusion of $\mathcal{T}_m^d$, where $m \geqslant m_0(1,d)$. Then, up to permuting coordinates, $\mathcal{T}_{m^e}^{d/e} \leqslant \mathfrak{X} \leqslant \mathcal{H}(d/e, m^e)$ for some integer $e \mid d$.*

In [10] it was shown that $\mathcal{J}(m,k)$ has no nontrivial fusions for $m \geqslant 3k+4$. This result was improved to $m \geqslant 3k-1$ in [11]. The case $k = 1$ of Theorem 1.1 recovers this result, except for the precise lower bound.

Corollary 1.3. *Let $\mathfrak{X}$ be a fusion of $\mathcal{J}(m,k)$, where $m \geqslant m_0(k,1)$. Then either $\mathfrak{X} = \mathcal{J}(m,k)$ or $\mathfrak{X}$ is trivial.*

Association schemes of the type appearing in the conclusion of Theorem 1.1 are studied in [5], where they are called “Cameron sandwiches” and “Hamming sandwiches,” respectively. The main result of [5] is that there are infinite families of nonschurian Hamming sandwiches.

Remark 1.4. Imprimitive fusions of $\mathcal{J}(m,k)^d$ are not so easily classified. Certainly one must allow arbitrary tensor products of the cases appearing in Theorem 1.1, but there are still many others. For example, the imprimitive wreath product $\mathcal{T}_m \wr \mathcal{T}_m$ (see [4, Section 3.4.1]) is an imprimitive fusion of $\mathcal{T}_m^2$ not fitting this description.

As an application we give an elementary classification of primitive groups containing $(A_m^{(k)})^d$. Here $A_m^{(k)}$ denotes the image of the alternating group A_m in its permutation action on k-sets, and below $S_m^{(k)}$ is defined similarly.

The statement below is a special case of Cameron’s theorem [3, 7, 8], which more generally classifies all large primitive permutation groups. However, while the proof of Cameron’s theorem depends on the classification of finite simple groups, our proof does not.

Corollary 1.5. *Let $G \leqslant S_n$ be a primitive permutation group containing $(A_m^{(k)})^d$, where $n = \binom{m}{k}^d$ and $m \geqslant m_0(k,d)$. Then either*

(1) $(A_m^{(k)})^d \leqslant G \leqslant S_m^{(k)} \wr S_d$ *or*
(2) $(A_{M^e})^{d/e} \leqslant G \leqslant S_{M^e} \wr S_{d/e}$ *for some integer $e \mid d$ where $M = \binom{m}{k}$.*

Proof. Let $\mathfrak{X}$ be the orbital scheme of G. It follows from $(A_m^{(k)})^d \leqslant G$ that $\mathfrak{X}$ is a fusion of the orbital scheme of $(A_m^{(k)})^d$, which coincides with $\mathcal{J}(m,k)^d$. Thus $\mathcal{J}(m,k)^d \leqslant \mathfrak{X}$ and, by Theorem 1.1, either $\mathcal{J}(m,k)^d \leqslant \mathfrak{X} \leqslant \mathcal{C}(m,k,d)$ or $\mathcal{T}_{M^e}^{d/e} \leqslant \mathfrak{X} \leqslant \mathcal{H}(d/e, M^e)$. In the first case, $\mathfrak{X}$ has a constituent graph equal to the Cameron graph $C(m,k,d)$. In the second case, $\mathfrak{X}$ has a constituent graph equal to the Hamming graph $H(d/e, M^e)$ (which is a special case of a Cameron graph). Applying [12, Theorem 8.2.1], either the claimed conclusion holds or G is small: $|G| \leqslant \exp(c(\log n)^3)$. Since $|G| \geqslant |(A_m^{(k)})^d| = m!^d$ and $n = \binom{m}{k}^d \leqslant m^{kd}$, we get $d \log m! \leqslant c(\log n)^3 \leqslant c(kd\log m)^3$, in contradiction to the hypothesis $m \geqslant m_0(k,d)$. $\square$

2. Notation

Let $[0,k]^d = \{0,\dots,k\}^d$. We use the following notation for vectors $a,b,c \in [0,k]^d$:

$$
\begin{aligned}
& a \leqslant b \iff a_i \leqslant b_i \text{ for all } i \text{ (in this case we say that } b \textit{ dominates } a\text{)},\\
& |a-b| = (|a_1-b_1|,\dots,|a_d-b_d|),\\
& \min(a,b) = (\min(a_1,b_1),\dots,\min(a_d,b_d)),\\
& \max(a,b) = (\max(a_1,b_1),\dots,\max(a_d,b_d)),\\
& a! = a_1!\cdots a_d!,\\
& \binom{a}{b} = \binom{a_1}{b_1}\cdots\binom{a_d}{b_d},\\
& \mathrm{wt}(a) = a_1+\cdots+a_d,\\
& \mathrm{supp}(a) = \{i : a_i > 0\},\\
& [a] = \{b \in [0,k]^d : b \leqslant a\},\\
& (x)^d = (x,\dots,x),\\
& e_i = (0,\dots,0,1,0,\dots,0).
\end{aligned}
$$

We understand $\binom{a}{b}$ to be zero unless $(0)^d \leqslant b \leqslant a$. We call $\mathrm{wt}(a)$ the *weight* of a and $\mathrm{supp}(a)$ the *support* of a.

If p is a polynomial in one variable we write $\deg(p)$ for its degree and $\lambda(p)$ for its leading term.

3. Proof

The structure constants of $\mathcal{J}(m,k)$ are given by

$$p^a_{b,c}(m) = \sum_i \binom{k-a}{i}\binom{a}{k-b-i}\binom{a}{k-c-i}\binom{m-k-a}{b+c+i-k}$$

(as in [5, 6]). Here $0 \leqslant a,b,c \leqslant k$, and i can be restricted to the range

$$\max(0, k-a-b, k-b-c, k-a-c) \leqslant i \leqslant \min(k-a, k-b, k-c, m-a-b-c).$$

In the following we always assume $m \geqslant 3k$. Under this assumption we have $p^a_{b,c}(m) > 0$ if and only if a,b,c satisfy the triangle inequalities [5, Lemma 4.1]. More precisely we have the following.

Lemma 3.1. *We have $p^a_{b,c}(m) > 0$ if and only if $|b-c| \leqslant a \leqslant b+c$. Assuming that $0 \leqslant b-c \leqslant a \leqslant b+c$, the leading term of $p^a_{b,c}(m)$ is*

$$\lambda(p^a_{b,c}(m)) = \begin{cases} \binom{a}{b}\binom{a}{c}\frac{1}{(b+c-a)!}m^{b+c-a} & : a \geqslant b,\\ \binom{k-a}{k-b}\binom{a}{b-c}\frac{1}{c!}m^c & : a \leqslant b. \end{cases}$$

In particular, $\deg(p^a_{b,c}(m)) \leqslant \min(b,c)$*, and equality holds if and only if* $a \leqslant \max(b,c)$.

Now consider $\mathcal{J}(m,k)^d$. For $a,b,c \in [0,k]^d$, let

$$p^a_{b,c}(m) = \prod_{i=1}^d p^{a_i}_{b_i,c_i}(m).$$

These are the structure constants of $\mathcal{J}(m,k)^d$. The following lemma generalizes the previous one.

Lemma 3.2. *Let $a,b,c \in [0,k]^d$. Then $p^a_{b,c}(m) > 0$ if and only if $|b-c| \leqslant a \leqslant b+c$ Moreover $\deg(p^a_{b,c}(m)) \leqslant \mathrm{wt}(\min(b,c))$, with equality if and only if $a \leqslant \max(b,c)$. If $\deg(p^a_{b,c}(m)) = \mathrm{wt}(b) = \mathrm{wt}(c)$ then $a \leqslant b = c$ and the leading term of $p^a_{b,c}(m)$ is*

$$\lambda(p^a_{b,c}(m)) = \binom{(k)^d - a}{(k)^d - b} \frac{1}{b!} m^{\mathrm{wt}(b)}.$$

Proof. It follows from $p^a_{b,c}(m) = \prod_{i=1}^d p^{a_i}_{b_i c_i}(m)$ that $p^a_{b,c}(m) > 0$ iff each triple a_i, b_i, c_i satisfies the triangle condition $|b_i - c_i| \leqslant a_i \leqslant b_i + c_i$. In this case

$$\deg(p^a_{b,c}(m)) = \sum_{i=1}^d \deg(p^{a_i}_{b_i,c_i}(m)) \leqslant \sum_{i=1}^d \min(b_i, c_i) = \mathrm{wt}(\min(b,c)).$$

Equality holds if and only if $a_i \leqslant \max(b_i, c_i)$ for all i.

If $\deg(p^a_{b,c}(m)) = \mathrm{wt}(b) = \mathrm{wt}(c)$ then $\mathrm{wt}(\min(b,c)) = \mathrm{wt}(b) = \mathrm{wt}(c)$, which implies $b = c$, and moreover we have seen that we must have $a \leqslant b$. Multiplying the leading terms of $p^{a_i}_{b_i,c_i}(m)$ given by the previous lemma, we get the claimed formula. ☐

Let $\mathfrak{X}$ be a fusion of $\mathcal{J}(m,k)^d$. Since $\mathfrak{X}$ is a coarsening of $\mathcal{J}(m,k)^d$ there is a partition $\mathfrak{S}$ of $[0,k]^d$ such that $\mathfrak{X} = \{R_\alpha : \alpha \in \mathfrak{S}\}$, where

$$(u,v) \in R_\alpha \iff (|u_1 \smallsetminus v_1|, \ldots, |u_d \smallsetminus v_d|) \in \alpha.$$

In this situation we write $\mathfrak{X} = \mathcal{J}(m,k)^{\mathfrak{S}}$ [5, Section 4]. For $\beta, \gamma \in \mathfrak{S}$ and $a \in [0,k]^d$ define

$$p^a_{\beta,\gamma}(m) = \sum_{b \in \beta, c \in \gamma} p^a_{b,c}(m).$$

For $\mathfrak{X} = \mathcal{J}(m,k)^{\mathfrak{S}}$ to be an association scheme, $\mathfrak{S}$ must satisfy two conditions:

(1) $\{(0)^d\} \in \mathfrak{S}$,
(2) $p^a_{\beta,\gamma}(m) = p^{a'}_{\beta,\gamma}(m)$ for all $\alpha, \beta, \gamma \in \mathfrak{S}$ and $a, a' \in \alpha$.

We may denote the common value of $p^a_{\beta,\gamma}(m)$ $(a \in \alpha)$ by $p^\alpha_{\beta,\gamma}(m)$; these are the structure constants of $\mathfrak{X}$.

We call the sets $\alpha \in \mathfrak{S}$ the *basic* $\mathfrak{S}$*-sets*; their unions are called $\mathfrak{S}$*-sets*. For nonempty $S \subseteq [0,k]^d$ let $\mathrm{wt}(S) = \max\{\mathrm{wt}(b) \mid b \in S\}$. For $\alpha \in \mathfrak{S}$ let α^* be the set of $a \in \alpha$ of maximal weight. Let $D_\alpha = \bigcup_{a \in \alpha^*} [a]$. Note that $\alpha^* \subseteq D_\alpha$.

For any $\alpha, \beta \subseteq [0,k]^d$ and $a \in [0,k]^d$ the structure constant $p^a_{\alpha,\beta}(m)$ is a real polynomial in m the coefficients of which depend on α, β and a. For every pair of distinct real polynomials $f, g \in \mathbb{R}[x]$ there exists a real number $c_{f,g} \in \mathbb{R}$ such that $f(x) \neq g(x)$ holds for all $x > c_{f,g}$. Therefore, there exists a constant $m_0(k,d)$ such that, provided $m \geqslant m_0(k,d)$, the following condition holds for all $\alpha, \beta \subseteq [0,k]^d$ and $a, b \in [0,k]^d$:

$$p^a_{\alpha,\beta}(m) = p^b_{\alpha,\beta}(m) \implies p^a_{\alpha,\beta}(m) = p^b_{\alpha,\beta}(m) \text{ as polynomials in } m \implies \lambda(p^a_{\alpha,\beta}(m)) = \lambda(p^b_{\alpha,\beta}(m)). \tag{1}$$

In what follows we assume that $m \geqslant m_0(k,d)$ and hence $p^a_{\beta,\gamma}(m) = p^{a'}_{\beta,\gamma}(m)$ only if $p^a_{\beta,\gamma}(m)$ and $p^{a'}_{\beta,\gamma}(m)$ are equal as polynomials in m.

Proposition 3.3. *Let $\beta \in \mathfrak{S}$ be a basic $\mathfrak{S}$-set.*

(1) *D_β is an $\mathfrak{S}$-set, and $\mathrm{wt}(D_\beta \smallsetminus \beta) < \mathrm{wt}(\beta)$.*
(2) *For every basic set $\alpha \in \mathfrak{S}$ there is a constant N^β_α such that*

$$\sum_{b : a \leqslant b \in \beta^*} \binom{(k)^d - a}{(k)^d - b} = N^\beta_\alpha \qquad (a \in \alpha).$$

(3) *Every element of β is dominated by a unique element of β^*.*

(4) *Either* $\mathrm{wt}(\beta) = \mathrm{wt}(D_\beta \smallsetminus \beta) + 1$ *or* $\beta^* \subseteq \{0,k\}^d$.

Proof. Let $w = \mathrm{wt}(\beta)$ and $D = D_\beta$. Then, since the polynomials $p^a_{b,c}(m)$ have positive leading coefficient whenever they are nonzero, the degree of $p^a_{\beta,\beta}(m)$ is, by the previous lemma,

$$\deg(p^a_{\beta,\beta}(m)) = \max_{b,c\in\beta} \deg(p^a_{b,c}(m)) \leqslant \max_{b,c\in\beta} \mathrm{wt}(\min(b,c)) \leqslant w,$$

with equality holding if and only if $a \leqslant b = c \in \beta^*$, i.e., if and only if $a \in D$. Since $p^a_{\beta,\beta}(m)$ should depend only on the cell of $\mathfrak{S}$ containing a, it follows that D is an $\mathfrak{S}$-set. Since β is basic, $\beta \subseteq D$, and since $\beta \supseteq \beta^*$ we have $\mathrm{wt}(D \smallsetminus \beta) < \mathrm{wt}(\beta)$.

Moreover if $a \in D$ then the leading term of $p^a_{\beta,\beta}$ is

$$\lambda(p^a_{\beta,\beta}(m)) = \sum_{b: a\leqslant b\in\beta^*} \binom{(k)^d - a}{(k)^d - b} \frac{1}{b!} m^w,$$

and again this should depend only on the cell α containing a. Taking $\alpha = \beta$ and $a \in \beta^*$, it follows that $a!$ is a constant for $a \in \beta^*$. Hence (2) holds (if α is not contained in D then $N^\beta_\alpha = 0$).

Next apply (2) with $\alpha = \beta$. Taking $a \in \beta^*$ shows $N^\beta_\beta = 1$, so (3) holds.

Finally let $b \in \beta^*$ and suppose $a = b - e_i \geqslant 0$. If $a \in \beta$ then we get

$$\binom{k - b_i + 1}{k - b_i} = N^\beta_\beta = 1,$$

so $b_i = k$. Hence either $\mathrm{wt}(b) = \mathrm{wt}(\beta) + 1$ or $b_i \in \{0,k\}$ for all i, which implies (4). ☐

We can define a partial ordering on $\mathfrak{S}$ by saying $\alpha \preceq \beta$ if every $a \in \alpha$ is dominated by some $b \in \beta$. By part (3) of the proposition, this is equivalent to $N^\beta_\alpha > 0$. It is obvious that $\preceq$ is reflexive and transitive on $\mathfrak{S}$. To verify antisymmetry, note that if $\alpha \preceq \beta \preceq \alpha$ and $a \in \alpha^*$ then there are $b \in \beta$ and $a' \in \alpha$ such that $a \leqslant b \leqslant a'$, which by maximality of a implies $a = b = a'$ and hence $\alpha = \beta$ since $\mathfrak{S}$ is a partition. We say $\alpha \in \mathfrak{S}$ is *minimal* if it is minimal in $(\mathfrak{S} \smallsetminus \{(0)^d\}, \preceq)$.

Corollary 3.4. *Let $\alpha \in \mathfrak{S}$ be a minimal basic set. Then $D_\alpha = \alpha \cup \{(0)^d\}$. Moreover, the elements of α^* have disjoint equal-sized supports, and we either have* $\mathrm{wt}(\alpha) = 1$ *or $\alpha^* \subseteq \{0,k\}^d$.*

Proof. Apply the proposition. Note that if β is a basic subset of $D_\alpha \smallsetminus \alpha$ then $\beta \prec \alpha$. By minimality of α this implies $\beta = \{(0)^d\}$. Hence $D_\alpha = \alpha \cup \{(0)^d\}$. Next, for any $a \in \alpha^*$ and $i \in \mathrm{supp}(a)$ we have $e_i \in D_\alpha$. By part (3) of Proposition 3.3, a is the unique element of α^* dominating e_i. Therefore the elements of α^* have disjoint supports. By part (4), either $\mathrm{wt}(\alpha) = 1$ or $\alpha^* \subseteq \{0,k\}^d$. If $\mathrm{wt}(\alpha) = 1$ then all $a \in \alpha^*$ have singleton support, and otherwise $|\mathrm{supp}(a)| = \mathrm{wt}(a)/k = \mathrm{wt}(\alpha)/k$ for all $a \in \alpha^*$. ☐

Until now $\mathfrak{X}$ could be imprimitive. Now we specialize to the primitive case to complete the proof of Theorem 1.1. Let $\alpha \in \mathfrak{S}$ be minimal. If $\mathfrak{X}$ is primitive then R_α must be connected, which implies that α^* covers $\{1,\dots,d\}$. Hence α^* is an equipartition of $\{1,\dots,d\}$. Let $w = \mathrm{wt}(\alpha)$. If $w = 1$ then α^* must be the set of elements of weight 1, so R_α is the Cameron graph. Since the Weisfeiler–Leman stabilization of the Cameron graph is the Cameron scheme, we find $\mathfrak{X} \leqslant \mathcal{C}(m,k,d)$. If $w > 1$ then $\alpha^* \subseteq \{0,k\}^d$. If the elements of α^* have support size e then R_α is the Hamming graph $H(M^e, d/e)$, so $\mathfrak{X} \leqslant \mathcal{H}(d/e, M^e)$. To finish we must show $\mathcal{T}^{d/e}_{M^e} \leqslant \mathfrak{X}$. For this it suffices to prove that for every $\beta \in \mathfrak{S}$, the elements of β^* are sums of elements of α^* (and in particular $\beta^* \subseteq \{0,k\}^d$).

We apply Proposition 3.3(2) to α and β. Let $i \in \{1,\dots,d\}$. Taking $a = e_i$, we find that N^β_α is at least the number of $b \in \beta^*$ such that $b_i > 0$. On the other hand, taking a

to be the unique element of α^* dominating e_i, since $a \in \{0,k\}^d$ we find that N_α^β is equal to the number of $b \in \beta^*$ such that $a \leqslant b$. Hence $b_i > 0$ implies $a \leqslant b$. This implies that b is the sum of those $a \in \alpha^*$ such that $a \leqslant b$, as required.

References

[1] R. A. Bailey, *Association schemes*, Cambridge Studies in Advanced Mathematics, vol. 84, Cambridge University Press, Cambridge, 2004, Designed experiments, algebra and combinatorics.

[2] Eiichi Bannai and Tatsuro Ito, *Algebraic combinatorics. I: Association schemes*, The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA, 1984.

[3] Peter J. Cameron, *Finite permutation groups and finite simple groups*, Bull. London Math. Soc. **13** (1981), no. 1, 1–22.

[4] Gang Chen and Ilia Ponomarenko, *Lectures on coherent configurations*, online lecture notes, 2019, `http://www.pdmi.ras.ru/~inp/ccNOTES.pdf`.

[5] Sean Eberhard, *Hamming sandwiches*, preprint, 2022, `https://arxiv.org/abs/2203.03687`, to appear in *Combinatorica*.

[6] L. A. Kalužnin and M. H. Klin, *Certain maximal subgroups of symmetric and alternating groups*, Mat. Sb. (N.S.) **87(129)** (1972), 91–121.

[7] Martin W. Liebeck, *On minimal degrees and base sizes of primitive permutation groups*, Arch. Math. (Basel) **43** (1984), no. 1, 11–15.

[8] Attila Maróti, *On the orders of primitive groups*, J. Algebra **258** (2002), no. 2, 631–640.

[9] M. E. Muzychuk, *Subschemes of Hamming association schemes $H(n,q)$, $q \geqslant 4$*, in Interactions between algebra and combinatorics (A. A. Ivanov and M. H. Klin, eds.), Springer, Dordrecht, 1992, Acta Appl. Math. **29** (1992), no. 1-2, pp. 119–128.

[10] ———, *Subschemes of the Johnson scheme*, European J. Combin. **13** (1992), no. 3, 187–193.

[11] Daiyu Uchida, *On the subschemes of the Johnson scheme*, Mem. Fac. Sci. Kyushu Univ. Ser. A **46** (1992), no. 1, 85–92.

[12] John Wilmes, *Structure, automorphisms, and isomorphisms of regular combinatorial objects*, ProQuest LLC, Ann Arbor, MI, 2016, Ph.D. thesis – The University of Chicago.

[13] Paul-Hermann Zieschang, *Theory of association schemes*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005.

Sean Eberhard, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB (U.K.)
E-mail : `eberhard@maths.cam.ac.uk`

Mikhail Muzychuk, Department of Mathematics, Ben Gurion University, P.O.B. 653, Beer Sheva 8410501 (Israel)
E-mail : `muzychuk@bgu.ac.il`