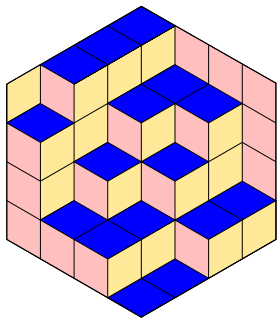

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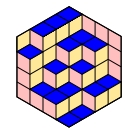

Interval groups related to finite Coxeter groups I

Barbara Baumeister, Georges Neaime & Sarah Rees

Abstract We derive presentations of the interval groups related to all quasi-Coxeter elements in the Coxeter group of type D_n. Type D_n is the only infinite family of finite Coxeter groups that admits proper quasi-Coxeter elements. The presentations we obtain are over a set of generators in bijection with what we call a Carter generating set, and the relations are those defined by the related Carter diagram together with a twisted cycle or a cycle commutator relator, depending on whether the quasi-Coxeter element is a Coxeter element or not. The proof is based on the description of two combinatorial techniques related to the intervals of quasi-Coxeter elements.

In a subsequent work [4], we complete our analysis to cover all the exceptional cases of finite Coxeter groups, and establish that almost all the interval groups related to proper quasi-Coxeter elements are not isomorphic to the related Artin groups, hence establishing a new family of interval groups with nice presentations [4, 5]. Alongside the proof of the main results, we establish important properties related to the dual approach to Coxeter and Artin groups.

Contents

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Keywords. Coxeter groups, Quasi-Coxeter elements, Carter diagrams, Artin(–Tits) groups, dual approach to Coxeter and Artin groups, generalised non-crossing partitions, Garside structures, Interval (Garside) structures.

1. Introduction

The philosophy of interval Garside theory is that starting from suitable intervals in a given group, we construct an interval Garside monoid and group, along with a complex whose fundamental group is the interval Garside group, such that the divisibility relations of the interval provide relevant information about the interval Garside group. Part of the information we obtain are efficient solutions to the word and conjugacy problems, as well as important group-theoretical properties [15]. Interval Garside groups also enjoy important homological, and homotopical properties [14].

Garside theory is relevant in the context of Coxeter and Artin groups. Actually, Garside structures first arose out of observations of properties of Artin's braid group that were made in Garside's Oxford thesis [19] and his article [20]. It was then realised that Garside's approach extends to all Artin groups of spherical type, independently by Brieskorn–Saito and Deligne in two adjacent articles in the Inventiones [11] and [16]. This approach is called the standard approach to Coxeter and Artin groups.

The dual approach consists of analysing the Coxeter group as a group generated by all its reflections. Spherical Artin groups are constructed from intervals called the generalised non-crossing partitions. The dual approach was first considered by Birman–Ko–Lee [9] for the usual braid group, and then generalised by Bessis in [7]. The relevant intervals consist of elements lying below the so-called Coxeter elements that play a prominent role within the dual approach. The Coxeter elements are all conjugate to one another and some of them can be found by taking the product of the elements in the standard generating set in any order.

Coxeter elements are of maximal length over the set of reflections, but they do not exhaust all the elements of maximal length. Quasi-Coxeter elements [3] are of maximal length such that the reflections in a certain reduced decomposition generate the Coxeter group. Among them are Coxeter elements. We call a proper quasi-Coxeter element a quasi-Coxeter element that is not a Coxeter element. Amongst the infinite families of finite Coxeter groups, proper quasi-Coxeter elements exist only in type D_n. Carter [13] classified the conjugacy classes in Weyl groups. Among them are the conjugacy classes of quasi-Coxeter elements. He also defined diagrams related to the conjugacy classes that we call Carter diagrams. Cameron-Seidel-Tsaranov [12] defined presentations of Weyl groups defined on Carter diagrams by adding cycle commutator relators.

We establish presentations of the interval groups related to all quasi-Coxeter elements. Our presentations are compatible with the analysis of Carter [13]. Actually, they are always nicely defined on Carter diagrams by adding either cycle commutator relators or twisted cycle commutator relators depending on whether the quasi-Coxeter element is a Coxeter element or not. Twisted cycle and cycle commutator relators can be written as relations between positive words. For Coxeter elements, where the interval group is the Artin group, some of our group presentations also arise from cluster algebras (see [1, 21] and also [23]). For almost all the other proper quasi-Coxeter

elements, we can establish that the interval group related to each of them is not isomorphic to the corresponding Artin group. Although we obtain nice presentations of these groups, the intervals of proper quasi-Coxeter elements are not lattices in almost all the cases, hence not giving rise to Garside structures. This classifies the interval Garside structures one obtains for quasi-Coxeter elements within the dual approach. Along with the description of the presentations of interval groups, we describe important properties for quasi-Coxeter elements, their divisors, and their lifts to the interval groups.

We divide our analysis into two parts. This paper is the first part of the series. It concerns the only infinite family of Coxeter groups (that of type D_n), where proper quasi-Coxeter elements exist. This family needs a special treatment. The second part [4] deals with the exceptional cases, and (together with [5]) establishes the non-isomorphism results.

The main theorem of this paper is the following. We refer to Sections 2 and 3 for the definitions of a quasi-Coxeter element, its associated Carter diagram Δ, and the group $A(\Delta)$.

THEOREM A. *Let w be a quasi-Coxeter element of the Coxeter group W of type D_n and Δ its associated Carter diagram, as shown in Figure* 1 *of Section* 3.3. *Then the interval group $G([1,w])$ admits a presentation over the generators $x_1,\dots,x_n$ corresponding to the vertices of Δ together with the relations described by Δ and the twisted cycle commutator relator $\mathrm{TC}(x_i,x_j,x_k,x_l)$, associated with the 4-cycle (x_i,x_j,x_k,x_l) within Δ, that is,*

$$G([1,w]) \cong A(\Delta)/\langle\langle \left[x_i, x_j^{-1}x_k x_l x_k^{-1} x_j\right]\rangle\rangle.$$

We shall reformulate Theorem A as Theorem 5.1, and prove that theorem in Section 6.

Our use of the word 'twisted' comes from the fact that when following the cycle (x_i,x_j,x_k,x_l) in the cycle commutator relator, we invert the element x_j. We call the set $\{x_1,\dots,x_n\}$ of generators that appears in Theorem A a Carter generating set.

The case where w is a Coxeter element is a particular case of Theorem A, where there is no 4-cycle. Therefore, in this case, we get a new proof of the fact that $G([1,w])$ is the Artin group of type D_n, that was covered before in [7].

Within the proof of Theorem A, we describe an important combinatorial technique that derives reduced expressions over the set of reflections for the divisors of length $n-1$ of quasi-Coxeter elements. This reveals important information on the poset of quasi-Coxeter elements, on parabolic subgroups, and enables us to establish nice presentations of the interval groups in accordance with Carter diagrams. The algorithms we define use the description of the elements in the Coxeter group of type D_n as monomial matrices. This is relevant to the dual approach for complex reflection groups. We suspect that our algorithms generalise to the context of the infinite families of complex reflection groups.

This first paper is structured as follows. After some preparations and after introducing the notation, Section 2 contains our strategy for the proof of Theorem A. The section also contains a good summary of our results (see Section 2.5). In Section 3, we recall the dual approach to the Coxeter group of type D_n. Next, we describe the combinatorial technique that defines reduced decompositions and introduce diagrams for these decompositions in Section 4. Within our proof, parabolic subgroups play an important role. In Section 5, we decompose the reflections over the Carter generating set and define the lift of these decompositions to the interval groups. Finally, Section 6 finishes our proof by induction.

2. Definitions and Preliminaries

2.1. Coxeter groups and Artin groups.

Definition 2.1. *Suppose that W is a group and S is a subset of W. For s and t in S, let m_{st} be the order of st if this order is finite, and be ∞ otherwise. We say that (W, S) is a Coxeter system, and that W is a Coxeter group with Coxeter system S, if W admits the presentation with generating set S together with the quadratic relations: $s^2 = 1$ for all $s \in S$, and the braid relations: $\underbrace{sts\dots}_{m_{st}} = \underbrace{tst\dots}_{m_{st}}$ for $s, t \in S$, $s \neq t$ and $m_{st} \neq \infty$. We define an element t of W to be a reflection if it is a conjugate of an element of S.*

We define the Artin group $A(W)$ associated with a Coxeter system (W, S) as follows.

Definition 2.2. *The Artin group $A(W)$ associated with a Coxeter system (W, S) is defined by a presentation with generating set $\boldsymbol{\mathcal{S}}$ in bijection with S and the braid relations: $\underbrace{\boldsymbol{sts}\dots}_{m_{st}} = \underbrace{\boldsymbol{tst}\dots}_{m_{st}}$ for $\boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{\mathcal{S}}$ and $\boldsymbol{s} \neq \boldsymbol{t}$, where $m_{st} \in \mathbb{Z}_{\geqslant 2}$ is the order of st in W.*

These presentations are often represented graphically using a Coxeter diagram Γ. This is a graph with vertex set S, in which the edge $\{s, t\}$ exists if $m_{st} \geqslant 3$, and is labelled with m_{st} when $m_{st} \geqslant 4$. Let Γ be such a diagram. We denote by $W(\Gamma)$ and $A(\Gamma)$ the related Coxeter and Artin groups W and $A(W)$. The finite Coxeter groups are precisely the real reflection groups, and the spherical Artin groups are the Artin groups related to the finite Coxeter groups. The corresponding Coxeter diagrams are the three infinite families of types A, B, and D, and the exceptional cases of types E_6, E_7, E_8, F_4, H_3, H_4, and $I_2(e)$. In the remainder of the article, W will always be a finite Coxeter group.

Recall that the Coxeter group of type A_n ($n \geqslant 1$) is the symmetric group $\mathrm{Sym}(n+1)$ and the related Artin group is the usual braid group

$$\mathcal{B}_{n+1} = \langle \boldsymbol{s}_1, \dots, \boldsymbol{s}_n \mid \boldsymbol{s}_i\boldsymbol{s}_{i+1}\boldsymbol{s}_i = \boldsymbol{s}_{i+1}\boldsymbol{s}_i\boldsymbol{s}_{i+1} \text{ for } 1 \leqslant i \leqslant n-1 \text{ and}$$
$$\boldsymbol{s}_i\boldsymbol{s}_j = \boldsymbol{s}_j\boldsymbol{s}_i \text{ for } |i-j| > 1\rangle.$$

2.2. Quasi-Coxeter elements.

Let (W, S) be a finite Coxeter system, and let the set of all its reflections be $T := \cup_{w \in W} S^w$. As each $w \in W$ is a product of reflections in T, we can define

$$\ell_T(w) := \min\{k \in \mathbb{Z}_{\geqslant 0} \mid w = t_1 t_2 \dots t_k;\ t_i \in T\},$$

the reflection length of w. If $w = t_1 t_2 \dots t_k$ with $t_i \in T$ and $k = \ell_T(w)$, we call $(t_1, t_2, \dots, t_k)$ (or $t_1 t_2 \dots t_k$ by abuse of notation) a reduced decomposition of w.

Now we define the notion of quasi-Coxeter elements.

Definition 2.3. *An element w of a finite Coxeter group W is called a quasi-Coxeter element if there exists a reduced decomposition $t_1 t_2 \dots t_n$ of w where n is the cardinality of S such that $\langle t_1, t_2, \dots, t_n\rangle = W$.*

A Coxeter element is a conjugate of any element that is written as the product of the simple generators of W in any order. Note that every Coxeter element is a quasi-Coxeter element. A quasi-Coxeter element is called proper if it is not a Coxeter element.

It is shown in [6] that the quasi-Coxeter elements in simply laced Coxeter groups are precisely those elements that admit a reduced decomposition into reflections such

that the roots related to these reflections form a basis of the related root lattice. In the non simply laced case, it is also required that the system of coroots generates the coroot lattice.

Recall that a parabolic subgroup of W is a subgroup generated by a conjugate of a subset of S. Note that a more general definition of parabolic subgroups, which is in fact equivalent to our definition for finite Coxeter systems, is used in [2, 3]. We call an element in W a parabolic quasi-Coxeter element if it is a quasi-Coxeter element in a parabolic subgroup of W.

Since the set T of reflections is closed under conjugation, there is a natural way to obtain new reflection decompositions from a given one. The braid group $\mathcal{B}_n$ acts on the set T^n of n-tuples of reflections via

$$\begin{aligned}
\boldsymbol{s}_i(t_1,\dots,t_n) &= (t_1,\dots,t_{i-1},\ t_it_{i+1}t_i,\quad t_i\quad,\ t_{i+2},\dots,t_n),\\
\boldsymbol{s}_i^{-1}(t_1,\dots,t_n) &= (t_1,\dots,t_{i-1},\quad t_{i+1}\quad,\ t_{i+1}t_it_{i+1},\ t_{i+2},\dots,t_n),\quad i=1,\dots,n-1,
\end{aligned}$$

the so-called Hurwitz action of $\mathcal{B}_n$ on T^n. It is readily observed that this action restricts to the set of all reduced reflection decompositions of a given element $w \in W$. If the latter action is transitive, then we say that the dual Matsumoto property holds for w.

The dual Matsumoto property characterises the parabolic quasi-Coxeter elements (see [3, Theorem 1.1]).

THEOREM 2.4. *An element $w \in W$ is a parabolic quasi-Coxeter element if and only if the dual Matsumoto property holds for w.*

We recall the following fact, and thereby introduce the notation P_w for parabolic quasi-Coxeter elements $w \in W$. The result is a consequence of [2, Theorem 1.4] and [3, Theorem 1.2].

LEMMA 2.5. *Let w be a parabolic quasi-Coxeter element in a Coxeter group W and $w = t_1t_2\dots t_k$ be a reduced decomposition into reflections. Then $P_w \coloneqq \langle t_1,\dots,t_k\rangle$ is a parabolic subgroup and the definition of P_w is independent of the choice of the reduced reflection decomposition of w.*

2.3. DECOMPOSITION DIAGRAMS. We introduce diagrams related to reduced decompositions.

DEFINITION 2.6. *Let $t_1t_2\dots t_k$ be a reduced decomposition of $w \in W$. We define a decomposition diagram related to $t_1t_2\dots t_k$ as follows. The vertices of the diagram correspond to the reflections $t_1, t_2, \dots, t_k$. If two reflections commute, we put no edge between the related vertices. Otherwise, we put an edge, and we label it by the order of the product of the two reflections when this order is strictly bigger than 3.*

In Carter's classification of the conjugacy classes in the Weyl groups [13], it is shown that every element w in W is the product $w = w_1w_2$ of two involutions, and that each involution is the product of commuting reflections, which then provides a bipartite decomposition of w. Carter exhibited the list of conjugacy classes of proper quasi-Coxeter elements by describing for each class a diagram related to a bipartite decomposition for a representative of the class (see [13, Table 2]) which we call a Carter diagram. Note that Definition 2.6 generalises the notion of Carter diagrams.

2.4. INTERVAL GROUPS OF QUASI-COXETER ELEMENTS. We start by defining left and right division.

DEFINITION 2.7. *Let $v, w \in W$. We say that v is a (left) divisor of w, and write $v \preceq w$, if $w = vu$ with $u \in W$ and $\ell_T(w) = \ell_T(v) + \ell_T(u)$, where $\ell_T(w)$ is the length over T of $w \in W$. The order relation $\preceq$ is called the absolute order relation on W.*

The interval $[1,w]$ *related to an element* $w \in W$ *is defined to be the set of divisors of* w *for* $\preceq$*, that is* $[1,w] = \{v \in W \mid v \preceq w\}$*.*

Similarly, we define divisibility from the right. We say that v *is a right divisor of* w*, and write* $v \preceq_r w$*, if* $w = uv$ *with* $u \in W$ *and* $\ell_T(w) = \ell_T(v) + \ell_T(u)$*. Similarly, we also define the interval* $[1,w]_r$ *of right divisors of an element* $w \in W$*.*

Remark 2.8. A quasi-Coxeter element has the inductive property that every left divisor of it is a parabolic quasi-Coxeter element (see [3, Corollary 6.11]). Therefore, if w is a quasi-Coxeter element, then every element in the interval $[1,w]$ is a parabolic quasi-Coxeter element.

Now we introduce the definition of an interval group related to quasi-Coxeter elements in W. Let w be a quasi-Coxeter element in W. Consider the interval $[1,w]$ of divisors of w.

Definition 2.9. *We define the group* $G([1,w])$ *by a presentation with set of generators* $\mathbf{[1,w]}$ *in bijection with the interval* $[1,w]$*, and relations corresponding to the relations in* $[1,w]$*, meaning that* $\boldsymbol{uv} = \boldsymbol{r}$ *if* $u,v,r \in [1,w]$*,* $uv = r$*, and* $u \preceq r$*, i.e.,* $\ell_T(r) = \ell_T(u) + \ell_T(v)$*.*

By transitivity of the Hurwitz action on the set of reduced decompositions of w (see Lemma 2.4), the next result follows immediately.

Proposition 2.10. *Let* $w \in W$ *be a quasi-Coxeter element, and let* $\boldsymbol{T} \subset \mathbf{[1,w]}$ *be the copy of the set of reflections* T *in* W*. Then*

$$G([1,w]) = \langle \boldsymbol{T} \mid \boldsymbol{tt'} = \boldsymbol{t't''} \text{ for } \boldsymbol{t},\boldsymbol{t'},\boldsymbol{t''} \in \boldsymbol{T} \text{ if } t \neq t', t'' \in T \text{ and } tt' = t't'' \preceq w \rangle$$

is a presentation of the interval group with respect to w*.*

Notice that the relations described in Proposition 2.10 are the relations that are visible in the poset $([1,w], \preceq)$ in heights one and two. They are called the dual braid relations (see [7]).

The following result is due to Michel as stated by Bessis in [7, Theorem 0.5.2] and explained in [15, Chapter VI, pg. 318] (see also [8]). It is the main theorem in interval Garside theory.

Theorem 2.11. *If the two intervals* $[1,w]$ *and* $[1,w]_r$ *are equal (we say that* w *is balanced) and if both posets* $([1,w], \preceq)$ *and* $([1,w]_r, \preceq_r)$ *are lattices, then the interval group* $G([1,w])$ *is an interval Garside group.*

Since T is stable under conjugation, quasi-Coxeter elements are always balanced. The only obstruction to obtaining interval Garside groups is the lattice property. In the case when the quasi-Coxeter element is a Coxeter element, Bessis [7] showed the following.

Theorem 2.12. *Let* W *be a finite Coxeter group. Then the interval group* $G([1,w])$ *for* $w \in W$ *a Coxeter element is an interval Garside group isomorphic to the corresponding Artin group* $A(W)$*.*

The main purpose of our work is to continue the analysis of the interval groups related to all quasi-Coxeter elements.

We introduce the following notation, which we shall use in the remainder of the article.

Notation 2.13. We denote by $\mathrm{B}(x,y)$ the braid relator $xyx(yxy)^{-1}$ or $xy(yx)^{-1}$, by $\mathrm{TC}(x,y,z,t)$ the twisted cycle commutator relator $\left[x, y^{-1}ztz^{-1}y\right]$, and by $\mathrm{CC}(x,y,z,t)$ the cycle commutator relator $\left[x, yztz^{-1}y^{-1}\right]$.

2.5. Strategy of the proof. We describe here our general strategy for the proof of Theorem A (see also Theorem 5.1). We are also going to mention some important results that we established within the proof, as they are interesting in themselves.

Let W be the Coxeter group of type D_n. We employ the description of W as a group of monomial matrices as will be explained in Section 3.1. Let w be a quasi-Coxeter element of the Coxeter group W of type D_n. Actually, there exists a reduced decomposition of w whose reflections $s_1, s_2, \ldots, s_n$ satisfy the relations that can be described by the Carter diagram Δ (see Figure 1 in Section 3.3). The quasi-Coxeter elements in type D_n are characterised in Proposition 3.8. We let $S \coloneqq \{s_1, s_2, \ldots, s_n\}$ from now on.

By [12], the Coxeter group W admits a presentation on the set S of generators whose relations are the quadratic relations ($s_i^2 = 1$ for $1 \leqslant i \leqslant n$) along with the relations of the diagram Δ and the cycle commutator relator $\mathrm{CC}(s_i, s_j, s_k, s_l)$ corresponding to the unique 4-cycle (s_i, s_j, s_k, s_l) of Δ. Note that in W, the cycle commutator and the twisted cycle commutator relators associated with the 4-cycle are the same. All this is described in Section 3.

Consider the interval group $G([1, w])$. By Proposition 2.10, the group $G([1, w])$ is generated by a copy $\boldsymbol{T}$ of T along with the dual braid relations $\boldsymbol{t}\boldsymbol{t}' = \boldsymbol{t}'\boldsymbol{t}''$ ($\boldsymbol{t} \in \boldsymbol{T}$ corresponds to $t \in T$), whenever $tt' = t't'' \preceq w$, $t \neq t'$ and $t, t', t'' \in T$.

We want to prove that $G([1, w])$ is isomorphic to the group $\boldsymbol{G}$ that is defined by a presentation on the set of generators $\boldsymbol{\mathcal{S}} = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_n\} \subset \boldsymbol{T}$ corresponding to the subset S of T with the corresponding relations described by Δ, along with the twisted cycle commutator relator $\mathrm{TC}(\boldsymbol{s}_i, \boldsymbol{s}_j, \boldsymbol{s}_k, \boldsymbol{s}_l)$ corresponding to the 4-cycle $(\boldsymbol{s}_i, \boldsymbol{s}_j, \boldsymbol{s}_k, \boldsymbol{s}_l)$, where $\boldsymbol{s}_i, \boldsymbol{s}_j, \boldsymbol{s}_k, \boldsymbol{s}_l$ correspond to s_i, s_j, s_k, s_l, respectively.

Step 1: Definition of f: We define a map f from $\boldsymbol{G}$ to $G([1, w])$ by setting $f(\boldsymbol{s}_i) = \boldsymbol{s}_i$ for each i. It will follow from Proposition 3.12 and Lemma 5.12 that the braid relators $\mathrm{B}(\boldsymbol{s}_i, \boldsymbol{s}_j)$ and the twisted cycle commutator relator $\mathrm{TC}(\boldsymbol{s}_i, \boldsymbol{s}_j, \boldsymbol{s}_k, \boldsymbol{s}_l)$ specified by the presentation given for $\boldsymbol{G}$ hold in $G([1, w])$ as well. Hence f extends to a homomorphism from $\boldsymbol{G}$ to $G([1, w])$.

Step 2: Reduced decompositions and their diagrams: Let w_0 be a divisor of length $n - 1$ of w. We will describe a particular reduced decomposition of $w_0 = t_1 t_2 \ldots t_{n-1}$ in Sections 4.2 and 4.3, and characterise whether w_0 is a Coxeter element or a proper quasi-Coxeter element in the corresponding subgroup $P_{w_0} \coloneqq \langle t_1, \ldots, t_{n-1} \rangle \subseteq W$ (see for instance Propositions 4.5, 4.15, and 4.16). In order to describe these reduced decompositions, we describe a combinatorial technique (in Section 4.1) by using the description of the elements of W as monomial matrices.

The reduced decomposition $t_1 t_2 \ldots t_{n-1}$ of w_0 corresponds to a decomposition diagram (see Definition 2.6) that we denote by Δ_0. We show that Δ_0 is a disjoint union of Coxeter diagrams of types A or D or of the same type as Δ (but with fewer generators) with a (single) 4-cycle (see Propositions 4.5, 4.15, and 4.16). Thereby we are able to determine the type of w_0.

If w_0 is a Coxeter element in P_{w_0}, then by [7], the dual braid relation $\boldsymbol{t}\boldsymbol{t}' = \boldsymbol{t}'\boldsymbol{t}''$ satisfied in $G([1, w_0])$ where $tt' = t't'' \preceq w$ is a consequence of the relators $\mathrm{B}(\boldsymbol{t}_i, \boldsymbol{t}_j)$ for $t_i, t_j \in \{t_1, t_2, \ldots, t_{n-1}\}$ $(i \neq j)$. If w_0 is a proper quasi-Coxeter element, then induction on n implies that the dual braid relation $\boldsymbol{t}\boldsymbol{t}' = \boldsymbol{t}'\boldsymbol{t}''$ satisfied in $G([1, w_0])$ is a consequence of the relations $\mathrm{B}(\boldsymbol{t}_i, \boldsymbol{t}_j)$ and $\mathrm{TC}(\boldsymbol{t}_i, \boldsymbol{t}_j, \boldsymbol{t}_k, \boldsymbol{t}_l)$ for $t_i, t_j \in \{t_1, t_2, \ldots, t_{n-1}\}$ $(i \neq j)$ and (t_i, t_j, t_k, t_l) the 4-cycle of Δ_0. In this way, we have shown that all the dual braid relations are consequences of the relations between $\boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_{n-1}$ corresponding to the

relations between $t_1, t_2, \ldots, t_{n-1}$ implied by each decomposition diagram Δ_0 corresponding to a divisor w_0 of length $n-1$ of w.

Step 3: Decomposition of elements in T and $\boldsymbol{T}$: In order to find a homomorphism $g : G([1,w]) \to \boldsymbol{G}$, we describe a decomposition of each element $\boldsymbol{t}$ in $\boldsymbol{T}$ in terms of elements in $\boldsymbol{\mathcal{S}}$. This can be done because of the dual Matsumoto property for w, i.e. the transitive Hurwitz action on the set of reduced decompositions over T of w. This is based on particular decompositions over S of the reflections in T. This is done in Propositions 5.7 and 5.10.

Let g be the map that sends $\boldsymbol{t_i} \in G([1,w])$ to its decomposition over $\boldsymbol{S}$ as given in Proposition 5.10. We show that the map g is a homomorphism. Suppose that $\boldsymbol{tt'} = \boldsymbol{t't''}$ is a dual braid relation of $G([1,w])$. We need to check that the image of this relation under g holds within $\boldsymbol{G}$. There exists a divisor w_0 of length $n-1$ of w such that tt' is a prefix of w_0. Therefore, $\boldsymbol{tt'} = \boldsymbol{t't''}$ holds in $G([1,w_0])$. So rather than checking that the images by g of all the dual braid relations hold in $\boldsymbol{G}$, our strategy is to shift the analysis to the groups $G([1,w_0])$, from which we can establish the desired homomorphism. This analysis was done in Step 2.

Step 4: Lift of the relations: In order to conclude homomorphism for g, we finally need to show that the image by g of all the defining relations between the elements $\boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_{n-1}$ of $G([1,w_0])$ corresponding to the relations between $t_1, t_2, \ldots, t_{n-1}$ implied by each decomposition diagram Δ_0 can be derived from the relations between the elements of $\boldsymbol{\mathcal{S}}$ that are implied by the diagram Δ. This is proved in Section 6.3 by induction on n. The base of our induction is the cases $n = 4$ and $n = 5$ (see Sections 6.1 and 6.2). Note that we also separate $n = 5$ as a base of induction so that we do not need anymore to show twisted cycle commutator relators in Section 6.3. This is possible since, apart from one special case (Equation (11) with $i = n-1$), in the reduced decomposition $t_1 t_2 \ldots t_{n-1}$ of each w_0, the only reflection t_i such that $g(\boldsymbol{t}_i)$ contains $\boldsymbol{s}_n$ in its decomposition over $\boldsymbol{\mathcal{S}}$ is precisely the last one, that is t_{n-1}.

Section 6.4 concludes the proof that g is a homomorphism. Isomorphism between $G([1,w])$ and $\boldsymbol{G}$ is proved once the composites $f \circ g$ and $g \circ f$ have been shown to be identity maps.

Note that it might be possible to apply this strategy more generally, but this article only deals with the case where W is of type D_n.

3. Dual approach to the Coxeter group of type D_n

3.1. The Coxeter group of type D_n. We employ the description of the Coxeter group W of type D_n ($n \geqslant 4$) as the group of $n \times n$ monomial matrices such that the nonzero coefficients are equal to 1 or -1 and their product is equal to 1. This description will help us to describe our combinatorial technique and to easily explain our arguments.

Note that this description of W corresponds to the case $d = 1, e = 2$ of the infinite series $G(de, e, n)$ of complex reflection groups (see [27]).

Notation 3.1. A monomial matrix $w \in W$ is associated with a permutation in $\mathrm{Sym}(n)$ that has been marked by overlining some elements within its cycles. We call the result, σ_w, a marked permutation, where an entry i ($1 \leqslant i \leqslant n$) of a cycle indicates that the coefficient in row i of w is equal to 1, while an entry $\overline{i}$ indicates that this coefficient is equal to -1.

The monomial matrix w is denoted by σ_w, so we have $w = \sigma_w$. When there is no confusion, we remove the cycles (i), for $1 \leqslant i \leqslant n$, of length 1 from σ_w.

We note that, for $w \in W$, the marked permutation σ_w must always have an even number of overlined entries. We also note that the notation σ_w is not the cycle decomposition of the permutation π_w of the unit vectors $\pm e_i, 1 \leqslant i \leqslant n$, of $\mathbb{R}^n$ that is also naturally associated with w. In fact, each cycle of length k in σ_w corresponds to either two cycles of length k or a single cycle of length $2k$ in π_w.

Example 3.2. Let

$$w = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

be an element of W of type D_6. Using Notation 3.1, we have $w = (\overline{1}, 3, 4)(2, \overline{5}, 6)$, while we have $\pi_w = (1, -3, -4, -1, 3, 4)(2, 5, -6, -2, -5, 6)$ where, for brevity, we label the unit vectors $\pm i$ with $1 \leqslant i \leqslant 6$.

We set the following convention for the remainder of the article.

Convention 3.3. *When $i = i'$, we interpret each cycle*

$$(i', i'+1, \ldots, \overline{i}), (i', i'-1, \ldots, \overline{i}), (\overline{i}, i-1, \ldots, i'), \text{ or } (\overline{i}, i+1, \ldots, i')$$

as the 1-cycle $(\overline{i})$. By convention, we also set $\overline{\overline{i}} = i$ for any positive integer i. We also suppose that a decreasing-index cycle of the form $(x_i, x_{i-1}, \ldots, x_{i'})$ is the identity element when $i \leqslant i'$ and an increasing-index expression of the form $(x_i, x_{i+1}, \ldots, x_{i'})$ is the identity element when $i \geqslant i'$. Finally, we also assume that a cycle of length $\geqslant 3$ that contains n should start by n.

Lemma 3.4. *The set T of reflections in W is represented by the set of elements*

$$\{(i, j),\ (\overline{i}, \overline{j}) \mid 1 \leqslant i \neq j \leqslant n\}.$$

Note that the reflection (i, j) represents a transposition matrix whose entries are all 1, while $(\overline{i}, \overline{j})$ represents a matrix derived from the previous one by changing the signs in rows i and j. For $2 \leqslant i \leqslant n$, we denote by s_i the reflection $(i-1, i)$.

The following lemma is straightforward to prove.

Lemma 3.5. *Let t and t' be two reflections in W, with $t = (i, j)$ or $(\overline{i}, \overline{j})$ and $t' = (k, l)$ or $(\overline{k}, \overline{l})$. If $\{i, j\}$ does not intersect $\{k, l\}$, then the reflections t and t' commute. If the cardinality of the intersection is 1, then we get $(i, j)(k, l)(i, j) = (k, l)(i, j)(k, l)$.*

3.2. Length function over the set of reflections. Shi computed in [28] the length function over the set of reflections in the infinite series of complex reflection groups. The Coxeter group W of type D_n corresponds to the group $G(2, 2, n)$. The length function over the set of reflections in $G(2, 2, n)$ appears in [28, Corollary 3.2]. Let us recall this result.

Proposition 3.6. *Let $w \in G(2, 2, n)$, and suppose that w is represented by a marked permutation σ_w as described in Notation* 3.1 *Suppose that σ_w is written as a product of r cycles, and define e to be the number of these cycles that have an even number of overlined entries. Then, the length $\ell_T(w)$ over T of w is equal to $n - e$.*

Example 3.7. Let $w \in G(2, 2, 6)$ be as in Example 3.2. Then $w = (\overline{1}, 3, 4)(2, \overline{5}, 6)$. Here we have $C_1 = (\overline{1}, 3, 4)$ and $C_2 = (2, \overline{5}, 6)$. Both cycles have an odd number (equal to 1) of overlined entries. Hence $e = 0$ and we get $\ell_T(w) = 6$.

Note also that the $n \times n$ identity matrix corresponds to the marked permutation with n cycles each containing a single entry i. Each cycle then contains 0 overlined entries. So $e = n$ and we see that the $\ell_T(Id) = 0$.

3.3. Quasi-Coxeter elements in type D_n. Let W be a Coxeter group of type D_n. It is a consequence of Carter (see [13] and [3, Remark 8.3 (b)]) that W contains $\lfloor \frac{n}{2} \rfloor$ conjugacy classes of quasi-Coxeter elements. We fix an integer m with $1 \leqslant m \leqslant \lfloor n/2 \rfloor$; this fixes a conjugacy class of quasi-Coxeter elements in W. The m-th conjugacy class is associated by Carter [13] with the diagram $\Delta_{m,n}$ displayed in Figure 1. The elements s_i where $2 \leqslant i \leqslant n$ are defined after Lemma 3.4 and we set $s_1 \coloneqq (\overline{m}, \overline{m+1})$. When there is no confusion, we denote $\Delta_{m,n}$ by Δ.

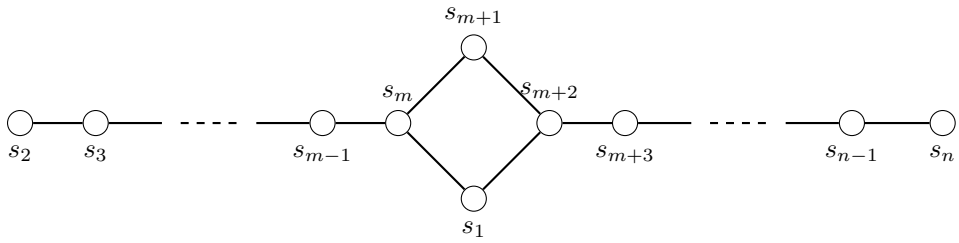


Figure 1. Carter diagram $\Delta_{m,n}$ of type D_n.

In Δ an edge between two nodes s_i and s_j describes the relation $s_i s_j s_i = s_j s_i s_j$, and when there is no edge between s_i and s_i, this means that the two reflections commute. In the next proposition, we choose a particular representative of each conjugacy class of quasi-Coxeter elements that will be helpful in the description of our main result.

Proposition 3.8. *The m-th conjugacy class of quasi-Coxeter elements contains a representative*

$$w = (m, m-1, \ldots, 2, \overline{1})(n, n-1, \ldots, \overline{m+1}).$$

The element w can be written as the product $s_2 s_3 \ldots s_m s_1 s_{m+1} s_{m+2} s_{m+3} s_{m+4} \ldots s_n$.

Proof. See [13, Proposition 25] for representatives of the conjugacy classes, where Carter defines the notion of signed cycle-type. The second sentence of the proposition is readily checked. □

We call the set $\{s_1, s_2, \ldots, s_n\}$ a Carter generating set. As w is a quasi-Coxeter element, every Carter generating set generates the Coxeter group.

Note that the Carter diagram Δ contains $m-2$ and $n-m-2$ vertices on the left- and right-hand sides of the single 4-cycle within Δ, respectively.

For $m = 1$, the element w is a Coxeter element and $\Delta_{1,n}$ is the Coxeter diagram of type D_n, and w becomes $(\overline{1})(n, n-1, \ldots, \overline{2})$. We call a proper Carter diagram of type D_n a diagram $\Delta_{m,n}$ that is not the Coxeter diagram of type D_n, that is with $m \geqslant 2$.

The Carter diagram Δ is the decomposition diagram (see Definition 2.6) related to the reduced decomposition $s_2 s_3 \ldots s_m s_1 s_{m+1} s_{m+2} s_{m+3} s_{m+4} \ldots s_n$ of the quasi-Coxeter element w. We are using this particular decomposition of the quasi-Coxeter element since it will be helpful to describe the divisors of length $n-1$ of w in Section 4 and to describe necessary combinatorial techniques for our analysis in Sections 4 and 5.

Example 3.9. The element $w = (3, 2, \overline{1})(6, 5, \overline{4})$ is a representative of a conjugacy class of quasi-Coxeter elements in type D_6. In this case $m = 3$.

LEMMA 3.10. *If $h \in W$ has a reduced reflection decomposition $h = t_1 \dots t_r$, $t_i \in T$, whose decomposition diagram is a Coxeter diagram, then this diagram is the Carter diagram of the conjugacy class that contains h, and h is a Coxeter element in $\langle t_1, \dots, t_r \rangle$.*

Proof. The reflection subgroup $W_h \coloneqq \langle t_1, \dots, t_r \rangle$ of W is a Coxeter group itself, according to Dyer [17, Theorem (3.3)]. If the decomposition diagram of h is a Coxeter diagram, we can choose the signs of the roots $\alpha_1, \dots, \alpha_r$ related to the reflections $t_1, \dots, t_r$ such that their dihedral angles are obtuse, as the diagram does not contain any cycles. Therefore $\{\alpha_1, \dots, \alpha_r\}$ is a simple system for W_h (see Dyer [17, Theorem 4.4] or [6, Lemma 4.1]). This yields that h is a Coxeter element in W_h. Therefore the decomposition diagram is the Carter diagram of the conjugacy class which contains h. □

The next result is a consequence of Cameron–Seidel–Tsaranov [12, Theorem 3.10].

PROPOSITION 3.11. *The Coxeter group has a presentation with set of generators the Carter generators. The relations are $s_i^2 = 1$ for $1 \leqslant i \leqslant n$ and the relations described by $\Delta_{m,n}$ together with the cycle commutator relation*

$$[s_m, s_{m+1}s_{m+2}s_1s_{m+2}s_{m+1}] = (s_m s_{m+1}s_{m+2}s_1s_{m+2}s_{m+1})^2 = 1.$$

We end this section by the following statement that will be used to construct the homomorphism f introduced in Step 1 of the strategy of our proof in Section 2.5.

PROPOSITION 3.12.

(1) *We have either $s_is_j \preceq w$ or $s_js_i \preceq w$ or $s_is_j = s_js_i \preceq w$, and s_is_j is of order 2, for $|i-j| > 1$.*
(2) *We have that $s_is_{i+1} \preceq w$, and s_is_{i+1} is of order 3, for $2 \leqslant i \leqslant n-1$.*
(3) *Let $2 \leqslant i \leqslant n$. We have that $s_1s_m \preceq w$, $s_{m+2}s_1 \preceq w$, and $s_1s_i \preceq w$. Further the elements s_1s_m and $s_{m+2}s_1$ are of order 3, and else s_1s_i is of order 2.*
(4) *Let $t \coloneqq s_1^{s_m s_{m+1}} = (\overline{m-1}, \overline{m})$. We have that $ts_{m+2} \preceq w$ and ts_{m+2} is of order 2.*

Proof. The result is an immediate consequence of the Hurwitz action and of the choice of the elements s_i. □

4. MAXIMAL DIVISORS OF QUASI-COXETER ELEMENTS

As we pointed out in the strategy of our proof (Step 2), our method depends on an analysis of maximal divisors of a quasi-Coxeter element w, and in particular of the decomposition of each such as a product of $n-1$ reflections. In Section 4.1 we identify 11 different cases for such maximal divisors w_0, which fall into three types, I, II and III, and then in the following sections, we find reduced decompositions for elements w_0 of type I (in Section 4.2), and of types II and III (in Section 4.3), as well as their decomposition diagrams (see Definition 2.6).

4.1. DIVISORS OF LENGTH $n-1$. Let $w = (m, m-1, \dots, 2, \overline{1})(n, n-1, \dots, \overline{m+1})$ be a quasi-Coxeter element. Since the maximum possible length of an element in W is n, the elements of length $n-1$ that divide w consist of all the products $w(i,j)$ and $w(\overline{i}, \overline{j})$ for which $1 \leqslant i < j \leqslant n$. We denote by w_0 a divisor of length $n-1$ of w. We compute these divisors in Equations (1) to (11) below. We distinguish 3 types that we denote by I, II, and III and that are displayed in the following Tables 1, 2, and 3. The first column of each table represents the cases for i and j. The second column is the divisor w_0. Notice that we get from type II to type III by applying symmetry.

$1 \leqslant i \leqslant m$, $m+1 \leqslant j \leqslant n$	(1) $w(i,j) = (n, n-1, \ldots, j+1, i, i-1, \ldots, \overline{1}, m, m-1, \ldots, i+1, j, j-1, \ldots, \overline{m+1})$.
$i \neq m$, $j \neq n$	(2) $w(\overline{i}, \overline{j}) = (n, n-1, \ldots, \overline{j+1}, i, i-1, \ldots, \overline{1}, m, m-1, \ldots, \overline{i+1}, j, j-1, \ldots, \overline{m+1})$.
$i = m$, $j \neq n$	(3) $w(\overline{m}, \overline{j}) = (n, n-1, \ldots, \overline{j+1}, m, m-1, \ldots, 1, j, j-1, \ldots, \overline{m+1})$.
$i \neq m$, $j = n$	(4) $w(\overline{i}, \overline{n}) = (n, n-1, \ldots, m+1, i, i-1, \ldots, \overline{1}, m, m-1, \ldots, \overline{i+1})$.
$i = m$, $j = n$	(5) $w(\overline{m}, \overline{n}) = (n, n-1, \ldots, m+1, m, m-1, \ldots, 1) = (n, n-1, \ldots, 1)$.

Table 1. Type I: $1 \leqslant i \leqslant m$ and $(m+1) \leqslant j \leqslant n$.

$1 \leqslant i < j \leqslant m$	(6) $w(i,j) = (m, m-1, \ldots, j+1, i, i-1, \ldots, \overline{1})(i+1, j, j-1, \ldots, i+2)(n, n-1, \ldots, \overline{m+1})$.
$j \neq m$	(7) $w(\overline{i}, \overline{j}) = (m, m-1, \ldots, \overline{j+1}, i, i-1, \ldots, \overline{1})(\overline{i+1}, j, j-1, \ldots, i+2)(n, n-1, \ldots, \overline{m+1})$.
$j = m$	(8) $w(\overline{i}, \overline{m}) = (i, i-1, \ldots, 1)(\overline{i+1}, m, m-1, \ldots, i+2)(n, n-1, \ldots, \overline{m+1})$.

Table 2. Type II: $1 \leqslant i < j \leqslant m$.

$m+1 \leqslant i < j \leqslant n$	(9) $w(i,j) = (m, m-1, \ldots, \overline{1})(n, n-1, \ldots, j+1, i, i-1, \ldots, \overline{m+1})(j, j-1, \ldots, i+1)$.
$j \neq n$	(10) $w(\overline{i}, \overline{j}) = (m, m-1, \ldots, \overline{1})(n, n-1, \ldots, \overline{j+1}, i, i-1, \ldots, \overline{m+1})(j, j-1, \ldots, \overline{i+1})$.
$j = n$	(11) $w(\overline{i}, \overline{n}) = (m, m-1, \ldots, \overline{1})(i, i-1, \ldots, m+1)(n, n-1, \ldots, \overline{i+1})$.

Table 3. Type III: $(m+1) \leqslant i < j \leqslant n$.

Remark also that Equation (2) is similar to Equation (1); the difference is that two entries are further overlined in Equation (2). We see the same similarities between Equations (6) and (7), and Equations (9) and (10).

Notice that each element w_0 of the 11 Equations admits exactly one cycle with an even number of overlined elements (we assume that 0 is even). Hence each element is of length $n-1$ by Proposition 3.6. In Sections 4.2 and 4.3, we describe a reduced

decomposition over the set T of reflections for each divisor w_0 of w of type I and of types II and III, respectively.

We provide an example where we explicitly write the monomial matrices.

EXAMPLE 4.1. Let W be a Coxeter group of type D_5. Let $m = 2$ and let $w = (2,\overline{1})(5,4,\overline{3})$ be a proper quasi-Coxeter element. As a monomial matrix,

$$w = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let us multiply w from the right by the transposition $(1,4)$ as described in Equation (1). So we get

$$w(1,4) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the marked permutation notation introduced in Notation 3.1, we have that $w(1,4) = (5,\overline{1},2,4,\overline{3})$ (the coefficient is equal to -1 on row numbers 1 and 3 of the matrix w_1) which is compatible with the result of Equation (1).

4.2. REDUCED DECOMPOSITIONS AND DIAGRAMS FOR TYPE I. Suppose that w_0 has type I (see Table 1). As a marked permutation, it is a cycle of the form $(x_1, x_2, x_3, \ldots, x_n)$, where each x_k is equal to p or $\overline{p}$ $(1 \leqslant p \leqslant n)$, with $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$, and with an even number of overlined entries (see Equations (1) to (5)). We will describe how to produce a reduced reflection decomposition of length $n-1$ for this element.

We continue the study of Example 4.1 that will help the understanding of a procedure that describes the reduced decompositions. The general idea is to multiply the marked permutation $w_0 = (x_1, x_2, x_3, \ldots, x_n)$ from the right by a sequence of reflections in order to obtain the identity matrix. A decomposition of w_0 is given by the product in reverse order of all the reflections used in the procedure. It turns out that this decomposition is reduced.

EXAMPLE 4.2. Let $w = (2,\overline{1})(5,4,\overline{3})$ and $w_0 = w(1,4) = (5,\overline{1},2,4,\overline{3})$ be as in Example 4.1. We follow the cycle $(5,\overline{1},2,4,\overline{3})$. The first two entries are 5 and $\overline{1}$. We multiply w_0 from the right by the transposition $(1,5)$ and get

$$w_1 = w_0(1,5) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (\overline{1},2,4,\overline{3})(5),$$

and 5 becomes a fixed point.

We continue by following the cycle $(\overline{1},2,4,\overline{3})$. The first two entries are $\overline{1}$ and 2. We multiply w_1 from the right by the reflection $(\overline{1},\overline{2})$ and get

$$w_2 = w_1(\overline{1},\overline{2}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (2,4,3)(5)(1).$$

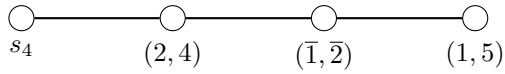
This operation permutes the columns 1 and 2 and multiplies their entries by -1. This then yields that 1 becomes a fixed point. Remark that by this operation, the third entry in $(2,4,3)(5)(1)$ is not overlined anymore.

We continue by following the cycle $(2,4,3)$. Here, the first two entries are 2 and 4. Then, we multiply w_2 from the right by the transposition $(2,4)$. We therefore obtain a coefficient equal to 1 in diagonal position $[2,2]$. We get

$$w_3 = w_2(2,4) = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&0&0&0&1 \end{pmatrix} = (4,3)(5)(1)(2).$$

Now, we arrive at the last step. We multiply w_3 from the right by the transposition $(3,4)$ and finally obtain the identity matrix denoted by $I_5 = (5)(1)(2)(4)(3)$.

This implies that $w_0 = (3,4)(2,4)(\overline{1},\overline{2})(1,5) = s_4(2,4)(\overline{1},\overline{2})(1,5)$. These are the reflections that we used before in reverse order. By Proposition 3.6, the length of w_0 is equal to 4. Hence $(3,4)(2,4)(\overline{1},\overline{2})(1,5)$ is a reduced decomposition of w_0 as it consists of 4 reflections. By Lemma 3.5, the related decomposition diagram is described by the following standard type A_4 diagram.



The general procedure is as follows.

Procedure 4.3. *Let $x = (x_1, x_2, \ldots, x_r)$. For k from 1 to $r-1$:*

- *If $x_k = p$, then whether $x_{k+1} = q$ or $\overline{q}$, we multiply $(x_k, x_{k+1}, \ldots, x_r)$ from the right by the reflection (p,q) and get $(x_k)(x_{k+1}, x_{k+2}, \ldots, x_r)$, whose length is one less than the length of $(x_k, x_{k+1}, \ldots, x_r)$.*
- *If $x_k = \overline{p}$, then whether $x_{k+1} = q$ or $\overline{q}$, we multiply $(x_k, x_{k+1}, \ldots, x_r)$ from the right by the reflection $(\overline{p},\overline{q})$ and get $(x_k)(x_{k+1}, x_{k+2}, \ldots, \overline{x}_r)$, whose length is one less than the length of $(x_k, x_{k+1}, \ldots, x_r)$.*

Remark that the entry $\overline{x}_r$ can be equal to $\overline{\overline{u}}$ for $1 \leqslant u \leqslant n$ in which case it is equal to u, by the convention that $\overline{\overline{i}} = i$ for any positive integer i (see Convention 3.3).

Proposition 4.4. *Let w_0 be a divisor of w of type I. A reduced decomposition of w_0 is obtained as the product in reverse order of the reflections that are applied in Procedure* 4.3.

Proof. Let w_0 be a divisor of w of type I. It is of the form $(x_1, x_2, x_3, \ldots, x_n)$. Applying Procedure 4.3 for all k from 1 to $n-1$, the element $(x_1, x_2, x_3, \ldots, x_n)$ is transformed into the identity matrix $(x_1)(x_2)\ldots(x_n)$ in $n-1$ steps. Since all reflections are of order two, a decomposition of the element $(x_1, x_2, x_3, \ldots, x_n)$ is given by the product in reverse order of all the reflections used in this procedure.

Since $\ell(w_0) = n-1$ by Proposition 3.6, the decomposition we obtain is reduced (as it consists of $n-1$ reflections). □

Proposition 4.5. *The decomposition diagram of each reduced decomposition corresponding to Type I is represented by a Coxeter diagram of type A_{n-1}, and w_0 is a Coxeter element in P_{w_0}. In particular, w_0 is a parabolic Coxeter element in W.*

Proof. It is an immediate consequence of Proposition 4.4 that the decomposition diagram of the reduced decomposition of w_0 produced in Procedure 4.3 is a string. Therefore, Lemmas 3.10 and 2.5 yield the assertion. □

We apply Proposition 4.4 to establish decompositions of type I divisors w_0 of w, as described in the following lemmas.

LEMMA 4.6. *Let $w_0 = w(i,j)$ be a divisor of w of type I, with $1 \leqslant i \leqslant m$, $m{+}1 \leqslant j \leqslant n$, so that we are in the situation of Equation* (1)*. Then w_0 has one of the following decompositions of length $n-1$ over T.*

(1) Suppose $i \neq m, j \neq n-2, n-1, n$:
$w_0 = s_{m+2}s_{m+3}\dots s_{j-1}s_j(i+1,j)s_{i+2}s_{i+3}\dots s_{m-1}s_m(\overline{1},\overline{m})s_2s_3\dots$
$\dots s_{i-1}s_i(i,j+1)s_{j+2}\dots s_{n-1}s_n.$

(2) Suppose $i \neq m, j = n-2$:
$w_0 = s_{m+2}\dots s_{n-2}s_{n-1}(i+1,n-2)s_{i+2}\dots s_{m-1}s_m(\overline{1},\overline{m})s_2\dots s_{i-1}s_i(i,n-1)s_n.$

(3) Suppose $i = m, j = n-2$:
$w_0 = s_{m+2}\dots s_{n-3}s_{n-2}(\overline{1},\overline{n-2})s_2\dots s_{m-1}s_m(m,n-1)s_n.$

(4) Suppose $i \neq m, j = n-1$:
$w_0 = s_{m+2}\dots s_{n-2}s_{n-1}(i+1,n-1)s_{i+2}\dots s_{m-1}s_m(\overline{1},\overline{m})s_2\dots s_{i-1}s_i(i,n).$

(5) Suppose $i = m, j = n-1$:
$w_0 = s_{m+2}\dots s_{n-2}s_{n-1}(\overline{1},\overline{n-1})s_2\dots s_{m-1}s_m(m,n).$

(6) Suppose $i \neq m, j = n$:
$w_0 = s_{i+2}\dots s_{m-1}s_m(\overline{1},\overline{m})s_2\dots s_{i-1}s_i(\overline{i},\overline{m+1})s_{m+2}\dots s_{n-1}s_n.$

(7) Suppose $i = m, j = n$:
$w_0 = s_2\dots s_{m-1}s_ms_1s_{m+2}\dots s_{n-1}s_n.$

Proof. We apply Proposition 4.4 to Equation (1). □

LEMMA 4.7. *Let $w_0 = w(\overline{i},\overline{j})$ be a divisor of w of type I, with $1 \leqslant i < m$, $m{+}1 \leqslant j < n$, so that we are in the situation of Equation* (2)*. Then w_0 has one of the following decompositions of length $n-1$ over T.*

(1) Suppose $i \neq m$, $j < n-2$:
$w_0 = s_{m+2}\dots s_{j-1}s_j(\overline{i+1},\overline{j})s_{i+2}\dots s_{m-1}s_m(\overline{1},\overline{m})s_2\dots s_{i-1}s_i(\overline{i},\overline{j+1})s_{j+2}\dots s_n.$

(2) Suppose $i \neq m$, $j = n-2$:
$w_0 = s_{m+2}\dots s_{n-3}s_{n-2}(\overline{i+1},\overline{n-2})s_{i+2}\dots s_{m-1}s_m(\overline{1},\overline{m})s_2\dots s_{i-1}s_i(\overline{i},\overline{n-1})s_n.$

(3) Suppose $i \neq m$, $j = n-1$:
$w_0 = s_{m+2}\dots s_{n-2}s_{n-1}(\overline{i+1},\overline{n-1})s_{i+2}\dots s_{m-1}s_m(\overline{1},\overline{m})s_2\dots s_{i-1}s_i(\overline{i},\overline{n}).$

Proof. We apply Proposition 4.4 to Equation (2). □

LEMMA 4.8. *Let $w_0 = w(\overline{m},\overline{j})$ be a divisor of w of type I, where $m+1 \leqslant j < n$, so that we are in the situation of Equation* (3)*. Then w_0 has one of the following decompositions of length $n-1$ over T.*

(1) Suppose $j < n-2$:
$w_0 = s_{m+2}\dots s_{j-1}s_j(1,j)\dots s_{m-1}s_m(\overline{m},\overline{j+1})s_{j+2}\dots s_{n-1}s_n.$

(2) Suppose $j = n-2$:
$w_0 = s_{m+2}\dots s_{n-3}s_{n-2}(1,n-2)s_2\dots s_{m-1}s_m(\overline{m},\overline{n-1})s_n.$

(3) Suppose $j = n-1$:
$w_0 = s_{m+2}\dots s_{n-2}s_{n-1}(1,n-1)s_2\dots s_{m-1}s_m(\overline{m},\overline{n}).$

Proof. We apply Proposition 4.4 to Equation (3). □

Similarly, we get the next result.

Lemma 4.9. *Let w_0 be as in the situation of Equations* (4) *and* (5)*. Then w_0 has the following decompositions of length $n-1$ over T.*

(1) Suppose $i \neq m$ and $j = n$, so that we are in the situation of Equation (4):
$w_0 = s_{i+2} \dots s_{m-1} s_m (\overline{1}, \overline{m}) s_2 \dots s_{i-1} s_i (i, m+1) s_{m+2} \dots s_{n-1} s_n$

(2) Suppose $i = m$ and $j = n$, so that we are in the situation of Equation (5):
$w_0 = s_2 s_3 \dots s_n$.

4.3. Reduced decompositions and diagrams for types II and III. In this section, we find reduced decompositions for the maximal divisors w_0 of w that are of types II and III. They are listed in Equations (6) to (11).

We define a combinatorial technique that enables us to obtain a reduced decomposition, whose decomposition diagram Δ_0 is the union of Coxeter diagrams of type A or D, or a proper Carter diagram of type D.

First, observe that each element w_0 defined in one of the Equations (6)– (11) is the product of three cycles. Since $\ell(w_0) = n-1$, by Proposition 3.6 each w_0 admits exactly one cycle with an even number of overlined elements. The other two cycles contain an odd number of overlined elements. Observing these equations, we recognise that these cycles contain exactly one overlined element at the end of the cycle. Assume that the cycles are $x \coloneqq (x_1, x_2, \dots, x_p), y \coloneqq (y_1, y_2, \dots, y_q), z \coloneqq (z_1, z_2, \dots, z_r)$ such that an even number of entries of $(z_1, z_2, \dots, z_r)$ are overlined, $x_p = \overline{u}$, and $y_q = \overline{v}$. Also, observe that we always have $p + q + r = n$.

The combinatorial technique is based on Procedure 4.3. We formulate it in the following procedure.

Procedure 4.10.

Step 1: *If $r \geqslant 2$, then apply Procedure* 4.3 *to the cycle $(z_1, z_2, \dots, z_r)$. In this way, we obtain $(z_1)(z_2)\dots(z_r)$.*

Step 2: *If $p \geqslant 2$, then apply Procedure* 4.3 *to the cycle $(x_1, x_2, \dots, x_p)$. In this way, we obtain $(x_1)(x_2)\dots(x_{p-1})(\overline{u})$.*

Step 3: *If $q \geqslant 2$, then apply Procedure* 4.3 *to the cycle $(y_1, y_2, \dots, y_q)$. In this way, we obtain $(y_1)(y_2)\dots(y_{q-1})(\overline{v})$.*

Furthermore, we impose an additional condition: if n does not appear in the cycle $(z_1, z_2, \dots, z_r)$, then we choose $(x_1, x_2, \dots, x_p)$ to be the cycle that contains n in Step 2.

Proposition 4.11. *Let w_0 be a divisor of w of type II or III. We continue with the notations introduced at the beginning of the section. A reduced decomposition of w_0 is obtained as the product $(u, v)(\overline{u}, \overline{v})$ followed by the reflections used in Procedure* 4.10 *in reverse order.*

Proof. After application of Procedure 4.10, the monomial matrix w_0 is transformed into the diagonal matrix with diagonal coefficients equal to 1 everywhere apart from diagonal positions $[u, u]$ and $[v, v]$, where the two coefficients are equal to -1. Multiplying this diagonal matrix by $(u, v)(\overline{u}, \overline{v})$, it is transformed into the identity matrix. A decomposition of w_0 is therefore the product of (u, v) by $(\overline{u}, \overline{v})$ followed by the reflections used in Procedure 4.10 in reverse order.

The decomposition is reduced if its length is equal to $n-1$. In the first step of Procedure 4.10, the number of reflections that have been used is equal to $r-1$, while $p-1$ and $q-1$ reflections are used in each of Steps 2 and 3. In addition, we multiplied at the end by two reflections: (u, v) and $(\overline{u}, \overline{v})$. Therefore, the number of reflections used in this decomposition is $(r-1)+(p-1)+(q-1)+2 = (r+p+q)-1 = n-1$. □

We explain Procedure 4.10 and Proposition 4.11 in the following two examples. The first example corresponds to type II and the second to type III.

EXAMPLE 4.12. We continue with our running Example 4.1, so $n = 5$, $m = 2$ and $w = (2, \overline{1})(5, 4, \overline{3})$. Let $w_0 = w(1, 2) = (\overline{1})(2)(5, 4, \overline{3})$, whose cycle decomposition is given in Equation (6).

We apply Procedure 4.10 to w_0.

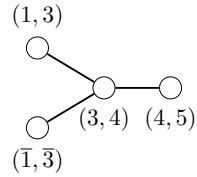
Step 1: The even cycle is (2) and corresponds to $r = 1$ in Procedure 4.10. Step 1 does not apply and we move to Step 2.

Step 2: The cycle containing a unique overlined entry and containing $n = 5$ is $(5, 4, \overline{3})$. Here, we have $p = 3$. Then, this step applies, and we execute Procedure 4.3 to the cycle $(5, 4, \overline{3})$. Therefore, we first multiply $(5, 4, \overline{3})$ by $(4, 5)$ and then the result by $(3, 4)$ from the right, and get $(5, 4, \overline{3})(4, 5)(3, 4) = (5)(4)(\overline{3})$. We set $w_2 \coloneqq w_0(4, 5)(3, 4)$.

Step 3: The second cycle contains a unique overlined entry $(\overline{1})$. In this case, we have $q = 1$. Therefore, we do not modify $w_2 = (\overline{1})(2)(5)(4)(\overline{3})$ in Step 3.

By Proposition 4.11, we multiply w_2 from the right by $(1, 3)(\overline{1}, \overline{3})$ so that it is transformed into the identity matrix. A reduced decomposition of w_0 is then obtained by adding to $(1, 3)(\overline{1}, \overline{3})$ the reflections used in Procedure 4.10 in reverse order. Hence we obtain $w_0 = (1, 3)(\overline{1}, \overline{3})(3, 4)(4, 5)$.

The decomposition diagram associated to this reduced decomposition is a Coxeter diagram of type D_4. This is readily checked by Lemma 3.5. The diagram is then the following.



The element w_0 is therefore a Coxeter element in the subgroup generated by the reflections $(1, 3), (\overline{1}, \overline{3}), (3, 4), (4, 5)$ that compose the reduced decomposition.

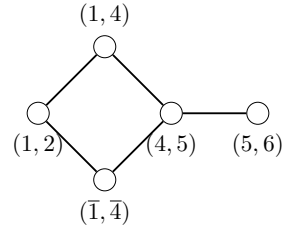
EXAMPLE 4.13. Let $n = 6$ and $m = 2$. Consider the proper quasi-Coxeter element $w = (2, \overline{1})(6, 5, 4, \overline{3})$ and $w_0 = w(\overline{3}, \overline{6})$. By Equation (11), it is $w_0 = (2, \overline{1})(3)(6, 5, \overline{4})$.

We apply now Procedure 4.10. The cycle (3) contains only one element. So, we move to Step 2.

Step 2: We apply Procedure 4.3 to the cycle $(6, 5, \overline{4})$ that contains $n = 6$. We obtain $(6, 5, \overline{4})(5, 6)(4, 5) = (6)(5)(\overline{4})$. We set $w_2 \coloneqq w_0(5, 6)(4, 5)$.

Step 3: We apply Procedure 4.3 to the cycle $(2, \overline{1})$, and get $w_3 \coloneqq w_2(1, 2) = (2)(\overline{1})(3)(6)(5)(\overline{4})$. By Proposition 4.11, a reduced decomposition of w_0 is obtained by adding to $(1, 4)(\overline{1}, \overline{4})$ the reflections used in Procedure 4.10 in reverse order. Therefore, we obtain $w_0 = (1, 4)(\overline{1}, \overline{4})(1, 2)(4, 5)(5, 6)$.

By Lemma 3.5, the decomposition diagram associated with this reduced decomposition is a proper Carter diagram of type D_5. The diagram is the following.



We will show later that the element $w_0 = (1,4)(\overline{1},\overline{4})(1,2)(4,5)(5,6)$ is a proper quasi-Coxeter element in the subgroup generated by the reflections $(1,4)$, $(\overline{1},\overline{4})$, $(1,2)$, $(4,5)$, $(5,6)$ that compose the reduced decomposition.

Now, we characterise the diagrams Δ_0 of the reduced decompositions, and whether the elements w_0 are Coxeter or proper quasi-Coxeter elements in the subgroup P_{w_0} generated by the reflections that compose the reduced decomposition. In fact, Procedures 4.3 and 4.10 are tailored in order to obtain a Coxeter diagram of type A or D, or a proper Carter diagram of type D. Observe first the following. We continue to use the notation of x, y and z introduced at the beginning of this section.

Lemma 4.14. *The elements xy and z are parabolic quasi-Coxeter elements in W.*

Proof. By Proposition 4.11, we have $\ell_T(z)+\ell_T(w_0z^{-1}) = \ell_T(w_0)$ which yields $z, xy \preceq w_0 \preceq w$. Therefore z as well as xy are parabolic quasi-Coxeter elements in W, see [3, Corollary 6.11]. □

Proposition 4.15. *Let w_0 be a divisor of w of type II or III. Let $x = (x_1, x_2, \ldots, x_p)$, $y = (y_1, y_2, \ldots, y_q)$ and $z = (z_1, z_2, \ldots, z_r)$ be as introduced before. Consider the reduced decomposition of w_0 as described in Procedure* 4.10 *and Proposition* 4.11*.*

- *If p and q are equal to* 1*, then the diagram of the reduced decomposition is the disjoint union of a diagram of type A_{r-1} and two nodes.*
- *If $p = 1$ and $q = 2$ (or $q = 1$ and $p = 2$), then the diagram of the reduced decomposition is a disjoint union of a Coxeter diagram of type A_3 and a Coxeter diagram of type A_{r-1}.*
- *If $p = 1$ and $q > 2$ (or $q = 1$ and $p > 2$), then the diagram of the reduced decomposition is a disjoint union of a Coxeter diagram of type D_{q+1} and a Coxeter diagram of type A_{r-1} (or a disjoint union of diagrams of types D_{p+1} and A_{r-1}, respectively).*

In all these cases, the element w_0 is a Coxeter element in P_{w_0} and therefore a parabolic Coxeter element in W.

Proof. By Lemma 4.14, xy as well as z are parabolic quasi-Coxeter elements in W. Here we prove that xy and z are Coxeter elements in P_{xy} and P_z, respectively.

By Proposition 4.11 and Lemma 3.10 the cycle z is a Coxeter element of type A_{r-1} in P_z. As xy and z are disjoint cycles, the two elements commute.

If $p = q = 1$, then w_0 equals $(u,v)(\overline{u},\overline{v})z$. As (u,v) and $(\overline{u},\overline{v})$ commute, the first bullet of the proposition follows.

If $p = 1$ and $q \geqslant 2$, then it is straightforward to check that the decomposition diagram of the reduced decomposition of xy given in Proposition 4.11 is a Coxeter diagram of type D_{q+1}. Thus, by Lemma 3.10 xy is a Coxeter element of type D_{q+1} in P_{xy}. This yields the other two bullets, as $A_3 = D_3$. □

Proposition 4.16. *Suppose $p, q \geqslant 2$. Then the decomposition diagram of w_0 is a disjoint union of a proper Carter diagram of type D_{p+q} and a Coxeter diagram of type A_{r-1}. Further*

- *the element $z' \coloneqq xy$ is a proper quasi-Coxeter element of type D_{p+q} in $P_{z'}$,*
- *the decomposition of z' is related to its Carter diagram as described in Proposition* 3.8*,*
- *the element z is a Coxeter element of type A_{r-1} in P_z .*

In particular, w_0 is a proper parabolic quasi-Coxeter element in W.

Proof. Since xy and z commute, we can apply Proposition 4.5 to z, and obtain the assertion for z, as well as Procedure 4.10 along with Proposition 4.11 to xy. The latter

yields the decomposition

$$xy = (u, v)(\overline{u}, \overline{v})(v, y_{q-1})(y_{q-1}, y_{q-2}) \dots (y_2, y_1)(v, x_{p-1})(x_{p-1}, x_{p-2}) \dots (x_2, x_1),$$

whose decomposition diagram is the Carter diagram $\Delta_{q-1,n}$ with $p + q$ vertices. In particular the decomposition diagram is connected, which yields that P_{xy} is either of type A_{p+q} or of type D_{p+q}. By [13, Theorem A], $\Delta_{q-1,n}$ is not a Carter diagram in a group of type A_{p+q} (as in the latter type Carter diagrams contain no cycles). Therefore P_{xy} is of type D_{p+q}. All the parabolic subgroups of type D_m, $m \geqslant 4$, are conjugate in W (every parabolic subgroup is conjugate to a standard parabolic subgroup, and there is just one standard parabolic subgroup, which is of type D_4), and all the Coxeter elements in a finite Coxeter group are conjugate. As xy has a different cycle-type than the elements appearing in Proposition 4.15, it is not a Coxeter element in P_{xy}. Thus, in this case xy is a proper quasi-Coxeter element in P_{xy}. Therefore, w_0 is a proper parabolic quasi-Coxeter element in W. □

Example 4.17. Consider Equation (11) for $m + 1 \leqslant i < j \leqslant n$ and n large enough. We have that

$$w(\overline{i}, \overline{n}) = (m, m-1, \dots, \overline{1})(i, i-1, \dots, m+1)(n, n-1, \dots, \overline{i+1}).$$

By Proposition 4.10, a reduced decomposition of $w(\overline{i}, \overline{n})$ is

$$w(\overline{i}, \overline{n}) = (1, i+1)(\overline{1}, \overline{i+1}) s_2 s_3 \dots s_m s_{i+2} \dots s_{n-1} s_n s_{m+2} \dots s_{i-1} s_i.$$

Its decomposition diagram is described in Figure 2.

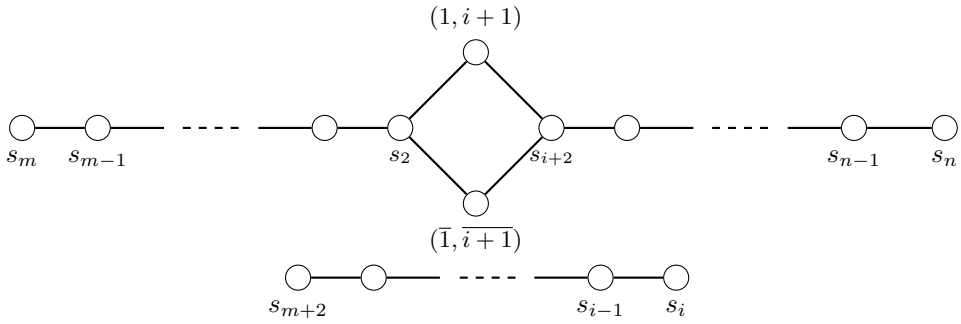


Figure 2. Decomposition diagram in the situation of Equation (11).

5. Decomposition of the reflections and their lifts

5.1. Interval groups and the claimed presentation. Let w be a quasi-Coxeter element in W of type D_n. Consider the interval $[1, w]$ of divisors of w for the absolute order $\preceq$ given in Definition 2.7 and the interval group $G([1, w])$ with its presentation given in Definition 2.9.

We denote by bold symbols the elements in $G([1, w])$. The copy $\boldsymbol{[1, w]}$ of the interval $[1, w]$ contains copies of the reflections (i, j) and $(\overline{i}, \overline{j})$ for $1 \leqslant i < j \leqslant n$, which we denote by $\boldsymbol{(i, j)}$ and $\boldsymbol{(\overline{i}, \overline{j})}$.

By Proposition 2.10, the group $G([1, w])$ is described by a presentation on the set

$$\boldsymbol{T} = \{\boldsymbol{(i, j)}, \boldsymbol{(\overline{i}, \overline{j})} : 1 \leqslant i \neq j \leqslant n\},$$

with relations the dual braid relations. These relations are described as $\boldsymbol{uv} = \boldsymbol{vu}$ if $uv \preceq w$ and $uv = vu$, and as $\boldsymbol{uv} = \boldsymbol{vt} = \boldsymbol{tu}$ ($\boldsymbol{t} \in \boldsymbol{T}$) if $uv = vt \preceq w$ for $u, v \in T$ and $u \neq v$.

It is convenient to reformulate our main result, Theorem A, as the following Theorem, which we shall prove in Section 6. Again we choose $1 \leqslant m \leqslant \lfloor \frac{n}{2} \rfloor$, and consider the quasi-Coxeter element $w = (m, m-1, \dots, \overline{1})(n, n-1, \dots, \overline{m+1})$. Let $\mathcal{S} = \{s_1, \dots, s_n\}$ be the Carter generating set, where $s_{i+1} = (i, i+1)$ for $1 \leqslant i \leqslant n-1$ and $s_1 = (\overline{m}, \overline{m+1})$. Let $\boldsymbol{\mathcal{S}} = \{\boldsymbol{s}_1, \dots, \boldsymbol{s}_n\} \subset \boldsymbol{T}$ be the set in $G([1, w])$ corresponding to $\mathcal{S}$.

Theorem 5.1. *The interval group $G([1, w])$ is isomorphic to the group $\boldsymbol{G}$ defined by the presentation with generating set $\boldsymbol{\mathcal{S}}$ and relations described by the diagram Δ in Figure 3 together with the twisted cycle commutator relator*

$$\mathrm{TC}(\boldsymbol{s}_1, \boldsymbol{s}_m, \boldsymbol{s}_{m+1}, \boldsymbol{s}_{m+2}) = [\boldsymbol{s}_1, \boldsymbol{s}_m^{-1}\boldsymbol{s}_{m+1}\boldsymbol{s}_{m+2}\boldsymbol{s}_{m+1}^{-1}\boldsymbol{s}_m] = [\boldsymbol{s}_1, \boldsymbol{s}_{m+2}^{\boldsymbol{s}_{m+1}^{-1}\boldsymbol{s}_m}],$$

associated with the cycle $(\boldsymbol{s}_1, \boldsymbol{s}_m, \boldsymbol{s}_{m+1}, \boldsymbol{s}_{m+2})$, that is,

$$\boldsymbol{G} = A(\Delta)/\langle\langle \mathrm{TC}(\boldsymbol{s}_1, \boldsymbol{s}_m, \boldsymbol{s}_{m+1}, \boldsymbol{s}_{m+2}) = [\boldsymbol{s}_1, \boldsymbol{s}_m^{-1}\boldsymbol{s}_{m+1}\boldsymbol{s}_{m+2}\boldsymbol{s}_{m+1}^{-1}\boldsymbol{s}_m] = [\boldsymbol{s}_1, \boldsymbol{s}_{m+2}^{\boldsymbol{s}_{m+1}^{-1}\boldsymbol{s}_m}] \rangle\rangle.$$

Note that we will always describe the TC relator by a curved arrow inside the corresponding cycle; see Figure 3.

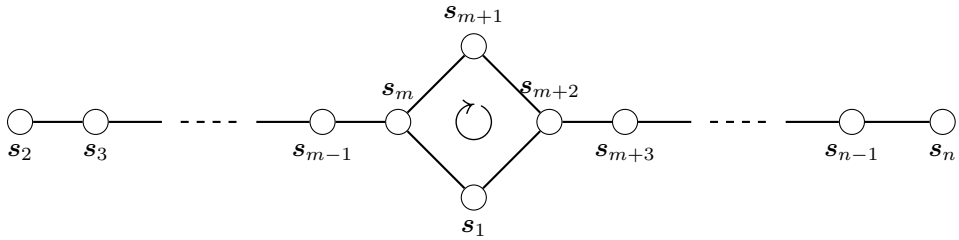


Figure 3. The diagram presentation for the claimed presentation.

Proposition 5.2. *Adding the quadratic relations to the presentation of $\boldsymbol{G}$, we obtain a group that is isomorphic to the Coxeter group W of type D_n.*

Proof. In fact, the twisted cycle commutator relator becomes the cycle commutator relator $[\boldsymbol{s}_1, \boldsymbol{s}_{m+2}^{\boldsymbol{s}_{m+1}\boldsymbol{s}_m}] = (\boldsymbol{s}_1\boldsymbol{s}_m\boldsymbol{s}_{m+1}\boldsymbol{s}_{m+2}\boldsymbol{s}_{m+1}\boldsymbol{s}_m)^2$ introduced in Proposition 3.11. The result follows immediately from the same proposition. □

The proof of the next lemma is easy and left as an exercise.

Lemma 5.3. *The twisted cycle commutator relation $[\boldsymbol{s}_1, \boldsymbol{s}_{m+2}^{\boldsymbol{s}_{m+1}^{-1}\boldsymbol{s}_m}] = 1$ can be written as a relation between positive words as follows:*

$$\boldsymbol{s}_m\boldsymbol{s}_1\boldsymbol{s}_{m+1}\boldsymbol{s}_{m+2}\boldsymbol{s}_m\boldsymbol{s}_{m+1}\boldsymbol{s}_1 = \boldsymbol{s}_1\boldsymbol{s}_{m+1}\boldsymbol{s}_{m+2}\boldsymbol{s}_m\boldsymbol{s}_{m+1}\boldsymbol{s}_1\boldsymbol{s}_m,$$

meaning that $\boldsymbol{s}_m$ commutes with $\boldsymbol{s}_1\boldsymbol{s}_{m+1}\boldsymbol{s}_{m+2}\boldsymbol{s}_m\boldsymbol{s}_{m+1}\boldsymbol{s}_1$.

Remark 5.4. Consider the cycle $(\boldsymbol{s}_1, \boldsymbol{s}_m, \boldsymbol{s}_{m+1}, \boldsymbol{s}_{m+2})$ of the presentation of $\boldsymbol{G}$. Consider the twisted cycle commutator relators $[\boldsymbol{s}_1, \boldsymbol{s}_{m+2}^{\boldsymbol{s}_{m+1}^{-1}\boldsymbol{s}_m}]$, $[\boldsymbol{s}_m, \boldsymbol{s}_1^{\boldsymbol{s}_{m+2}^{-1}\boldsymbol{s}_{m+1}}]$, $[\boldsymbol{s}_{m+1}, \boldsymbol{s}_{m+2}^{\boldsymbol{s}_1^{-1}}\boldsymbol{s}_m]$, and $[\boldsymbol{s}_{m+2}, \boldsymbol{s}_1^{\boldsymbol{s}_m^{-1}}\boldsymbol{s}_{m+1}]$. It is an easy exercise to check that if one of the twisted cycle commutator relators holds, then the three other relators also hold.

Remark 5.5. Suppose that $m = 1$, i.e. w is a Coxeter element. In this case, the group $\boldsymbol{G}$ is the Artin group of type D_n. Our proof of Theorem 5.1 establishes a new proof of a result of Bessis showing that the interval group related to a Coxeter element in type D_n is isomorphic to the related Artin group (see [7]).

We end the section by showing that the poset $([1,w],\preceq)$ of a proper quasi-Coxeter element w in type D_n is not a lattice. Hence the monoid defined by the same presentation as $G([1,w])$ viewed as a monoid presentation fails to be a Garside monoid. Note that this fact does not mean that the group $G([1,w])$ cannot admit Garside structures.

PROPOSITION 5.6. *Let w be a proper quasi-Coxeter element in type D_n for $n \geqslant 4$. The poset $([1,w],\preceq)$ is not a lattice.*

Proof. We check using `GAP` that for w a proper quasi-Coxeter element in type D_4, there exists a bowtie in $([1,w],\preceq)$, hence it is not a lattice, see [25, Proposition 1.10]. (The bowtie consists of two reflections t_1, t_2 that commute in W such that $t_1 t_2 \notin [1,w]$.)

Let w be a proper quasi-Coxeter element in type D_n for $n > 4$. Then it contains a subword w' of w whose Carter diagram is a 4-cycle. According to [18, Theorem 2.1], all elements below w' in $([1,w],\preceq)$ are in $([1,w'],\preceq)$ read as a poset in the rank 4 parabolic subgroup $P_{w'}$. Therefore there is still the bowtie coming from the case $n = 4$ inside $([1,w],\preceq)$ for $n > 4$. □

5.2. DECOMPOSITION OF THE REFLECTIONS ON CARTER GENERATORS. The purpose of this section is to find decompositions of the elements of T in terms of Carter generators. This corresponds to Step 3 in the strategy of our proof as explained in Section 2.5.

We recall that we fix an integer m with $1 \leqslant m \leqslant \lfloor n/2 \rfloor$. We recall from Proposition 3.11 that a presentation of the Coxeter group W is defined on Carter generators $s_1, s_2, \ldots, s_n$ together with the relations described in the diagram presentation illustrated in Figure 1 together with the quadratic relations.

Recall that the reflections s_2, s_3, …, s_n are the transpositions $(1,2)$, $(2,3)$, …, $(n-1,n)$, respectively, while the reflection s_1 is the marked permutation $(\overline{m}, \overline{m+1})$. In the next proposition, we decompose each reflection (i,j) and $(\overline{i},\overline{j})$ over the Carter generators s_1, s_2, …, s_n.

PROPOSITION 5.7.

(1) *Let $t = (i,j)$ with $1 \leqslant i < j \leqslant n$. We have*

$$t = s_j^{s_{j-1}s_{j-2}\cdots s_{i+1}}. \tag{12}$$

(2) *Let $t = (\overline{i},\overline{j})$ with $1 \leqslant i \leqslant m$ and $m+1 \leqslant j \leqslant n$. We have*

$$(\overline{i},\overline{j}) = s_1^{s_{m+2}s_{m+3}\cdots s_j s_m s_{m-1}\cdots s_{i+1}}. \tag{13}$$

(3) *Let $t = (\overline{i},\overline{j})$ with $1 \leqslant i < j \leqslant m$. We have*

$$(\overline{i},\overline{j}) = s_1^{s_m s_{m-1}\cdots s_{i+1} s_{m+1} s_m \cdots s_{j+1}}. \tag{14}$$

(4) *Let $t = (\overline{i},\overline{j})$ with $m+1 \leqslant i < j \leqslant n$. We have*

$$(\overline{i},\overline{j}) = s_1^{s_{m+2}s_{m+3}\cdots s_j s_{m+1} s_{m+2}\cdots s_i}. \tag{15}$$

Proof. The equations are easily obtained by direct calculation using the definition of the reflections as marked permutations. □

REMARK 5.8.

(1) In Equation (12), if $j = i+1$, we get $t = (i,i+1) = s_{i+1}$ $(1 \leqslant i \leqslant n-1)$ visible among Carter generators.
(2) In Equation (13), if $i = m$ and $j = m+1$, we have $t = s_1$ visible among Carter generators.

(3) The decompositions of (i,j) and $(\bar{i},\bar{j})$ obtained in Equations (12)–(15) are reduced decompositions over Carter generators. This result is not straightforward, but can be established using techniques from [26]. Since this fact is not used in the proof of our main result, we will not include its proof here.

5.3. Lifting the reflections. The purpose of this section is to write each element $\boldsymbol{(i,j)}$ and $\boldsymbol{(\bar{i},\bar{j})}$ $(1 \leqslant i < j \leqslant n)$ of $G([1,w])$ in terms of the generators $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, …, $\boldsymbol{s}_n$ that appear in the presentation of Theorem 5.1. This corresponds to Step 3 in the strategy of our proof as explained in Section 2.5. Recall that $\boldsymbol{(i,j)}$ and $\boldsymbol{(\bar{i},\bar{j})}$ are the copies of the reflections (i,j) and $(\bar{i},\bar{j})$ within $\boldsymbol{T}$, and $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, …, $\boldsymbol{s}_n$ are the copies of the reflections $s_1 = (\overline{m},\overline{m+1})$, s_2, s_3, …, s_n. We employ the decompositions of the reflections in terms of $s_1, s_2, \ldots, s_n$ that we described in Equations (12) to (15) of Section 5.2. These decompositions serve as a guide. In fact, we walk through the reflections in the exponent expressions of these equations and derive our result. The explicitness of these equations enables us to describe in a simple way the main result of this section, Proposition 5.10.

As a preamble, let us illustrate our ideas in the following example.

Example 5.9. Let W be the Coxeter group of type D_4. Let $m = 2$, and let $w = (2,\overline{1})(4,\overline{3})$ be a proper quasi-Coxeter element of length 4 by Proposition 3.6. Consider the reflection $(1,4) \in T$. It is equal to $s_4^{s_3 s_2}$ by Equation (12). We decompose the copy $\boldsymbol{(1,4)} \in \boldsymbol{T}$ in terms of the generators $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, $\boldsymbol{s}_3$, …, $\boldsymbol{s}_n$.

- Since $w' = s_2(1,4)w = (1,\overline{3},\overline{4})(2)$ is of length 2 by Proposition 3.6, then we have $(1,4)s_2 \preceq w$. So we get $(1,4)s_2 = s_2(1,4)^{s_2} = s_2 s_4^{s_3} = s_2(2,4) \preceq w$. Hence this gives $\boldsymbol{(1,4) = s_2(2,4)s_2^{-1}}$.
- Similarly, for $s_4^{s_3} = (2,4)$, we have that $(2,4)s_3 = s_3(3,4) \preceq w$ also by a direct application of Proposition 3.6. Hence we get $\boldsymbol{(2,4) = s_3(3,4)s_3^{-1}} = \boldsymbol{s_3 s_4 s_3^{-1}}$.

It follows that $\boldsymbol{(1,4) = s_2(2,4)s_2^{-1} = s_2 s_3 s_4 s_3^{-1} s_2^{-1} = s_4^{s_3^{-1} s_2^{-1}}}$.

Proposition 5.10. *The copies of the reflections to the interval group $G([1,w])$ decompose on the generators $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, …, $\boldsymbol{s}_n$ as follows.*

$$\boldsymbol{(i,j) = s_j^{s_{j-1}^{-1} s_{j-2}^{-1} \cdots s_{i+1}^{-1}}}, \text{ for } 1 \leqslant i < j \leqslant n, \tag{16}$$

$$\boldsymbol{(\bar{i},\bar{j}) = s_1^{s_{m+2} s_{m+3} \cdots s_j s_m^{-1} s_{m-1}^{-1} \cdots s_{i+1}^{-1}}}, \text{ for } 1 \leqslant i \leqslant m, m+1 \leqslant j \leqslant n. \tag{17}$$

$$\boldsymbol{(\bar{i},\bar{j}) = s_1^{s_m^{-1} s_{m-1}^{-1} \cdots s_{i+1}^{-1} s_{m+1} s_m^{-1} s_{m-1}^{-1} \cdots s_{j+1}^{-1}}}, \text{ for } 1 \leqslant i < j \leqslant m, \tag{18}$$

$$\boldsymbol{(\bar{i},\bar{j}) = s_1^{s_{m+2} s_{m+3} \cdots s_j s_{m+1}^{-1} s_{m+2} \cdots s_i}}, \text{ for } m+1 \leqslant i < j \leqslant n, \tag{19}$$

Proof. We use the decompositions of the reflections (i,j) and $(\bar{i},\bar{j})$ for $1 \leqslant i < j \leqslant n$ in term of Carter generators $s_1, s_2, \ldots, s_n$ that we described in Equations (12) to (15) in Section 5.2. Each of these equations is of the form

$$t = y^{x_1 x_2 \cdots x_p},$$

for $p \geqslant 1$. For k from p down to 1, we proceed as follows. Let $t_k = y^{x_1 x_2 \cdots x_k}$. We have that $t_p = t$.

- If $t_k x_k \preceq w$, then we have $t_k x_k = x_k t_k^{x_k} = x_k t_{k-1} \preceq w$. It follows that $\boldsymbol{t_k x_k = x_k t_{k-1}}$, which implies that $\boldsymbol{t_k = x_k t_{k-1} x_k^{-1}}$.
- If $x_k t_k \preceq w$, then we have $x_k t_k = t_k^{x_k} x_k = t_{k-1} x_k \preceq w$. It follows that $\boldsymbol{x_k t_k = t_{k-1} x_k}$, which gives $\boldsymbol{t_k = x_k^{-1} t_{k-1} x_k}$.

It turns out that for all k from p down to 1, we are in one of the previous two situations in all Equations (12) to (15). It follows that $t = \boldsymbol{y^{x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_p^{\epsilon_p}}}$, where $\boldsymbol{\epsilon_k} = -1$ or 1, for $1 \leqslant k \leqslant p$, depending on whether we apply the first or the second situation, respectively.

Let us explain this for the copy $\boldsymbol{(\bar{i}, \bar{j})}$ of $(\bar{i}, \bar{j})$ for $1 \leqslant i < j \leqslant m$. The same argument applies for the other equations. For $1 \leqslant i < j \leqslant m$, by Equation (14), we have that

$$(\bar{i}, \bar{j}) = s_1^{s_m s_{m-1} \cdots s_{i+1} s_{m+1} s_m \cdots s_{j+1}}.$$

Applying Proposition 3.6, we have that $(\bar{i}, \bar{j}) s_{j+1} \preceq w$, so that $(\bar{i}, \bar{j}) s_{j+1} = s_{j+1}(\bar{i}, \bar{j})^{s_{j+1}} = s_{j+1}(\bar{i}, \overline{j+1}) \preceq w$. Then we get

$$\boldsymbol{(\bar{i}, \bar{j}) = s_{j+1}(\bar{i}, \overline{j+1}) s_{j+1}^{-1}}.$$

Next, applying Proposition 3.6, we have $(\bar{i}, \overline{j+1}) s_{j+2} \preceq w$, meaning that $(\bar{i}, \overline{j+1}) s_{j+2} = s_{j+2}(\bar{i}, \overline{j+1})^{s_{j+2}} = s_{j+2}(\bar{i}, j+2) \preceq w$. So, we get

$$\boldsymbol{(\bar{i}, \overline{j+1}) = s_{j+2}(\bar{i}, \overline{j+2}) s_{j+2}^{-1}}.$$

Hence we have

$$\boldsymbol{(\bar{i}, \bar{j}) = (\bar{i}, \overline{j+2})^{s_{j+2}^{-1} s_{j+1}^{-1}}}.$$

And so on, we apply the same computations for s_{j+2}, s_{j+3}, …, s_m appearing in the exponent part of Equation (14), and get

$$\boldsymbol{(\bar{i}, \bar{j}) = (\bar{i}, \overline{m})^{s_m^{-1} \cdots s_{j+2}^{-1} s_{j+1}^{-1}}}.$$

Next, applying Proposition 3.6, we have $s_{m+1}(\bar{i}, \overline{m}) \preceq w$ (and not $(\bar{i}, \overline{m}) s_{m+1} \preceq w$), meaning that $s_{m+1}(\bar{i}, \overline{m}) = (\bar{i}, \overline{m+1}) s_{m+1} \preceq w$. Hence we get

$$\boldsymbol{(\bar{i}, \overline{m}) = s_{m+1}^{-1}(\bar{i}, \overline{m+1}) s_{m+1}}.$$

Then we obtain

$$\boldsymbol{(\bar{i}, \bar{j}) = (\bar{i}, \overline{m+1})^{s_{m+1} s_m^{-1} \cdots s_{j+2}^{-1} s_{j+1}^{-1}}}.$$

Similarly, we have $(\bar{i}, \overline{m+1}) s_{i+1} \preceq w$, so $(\bar{i}, \overline{m+1}) s_{i+1} = s_{i+1}(\overline{i+1}, \overline{m+1}) \preceq w$. Thus, we get

$$\boldsymbol{(\bar{i}, \overline{m+1}) = s_{i+1}(\overline{i+1}, \overline{m+1}) s_{i+1}^{-1}}.$$

And so on, we apply the same calculation for s_{i+1}, s_{i+2}, …, s_m until we obtain the desired equation:

$$\text{For } 1 \leqslant i < j \leqslant m, \boldsymbol{(\bar{i}, \bar{j}) = s_1^{s_m^{-1} s_{m-1}^{-1} \cdots s_{i+1}^{-1} s_{m+1} s_m^{-1} s_{m-1}^{-1} \cdots s_{j+1}^{-1}}}.$$

□

We provide an example where the decomposition that we obtain will appear in the twisted cycle commutator relators in the next section.

Example 5.11. Let W be a Coxeter group of type D_4. Let $m = 2$ and $w = (2, \bar{1})(4, \bar{3})$ be a proper quasi-Coxeter element. Consider the reflection $t = (\bar{1}, \bar{2})$ of type II. By Equation (14), we have that $(\bar{1}, \bar{2}) = s_1^{s_2 s_3}$.

- Since $w' = (\bar{1}, \bar{2}) s_3 w = (2)(4, 3, 1)$ is of length 2 by Proposition 3.6, then we have that $s_3(\bar{1}, \bar{2}) \preceq w$. Note that we are in the situation of the second bullet in the proof of Proposition 5.10. Hence we get $s_3(\bar{1}, \bar{2}) = (\bar{1}, \bar{2})^{s_3} s_3 = s_1^{s_2} s_3 = (\bar{1}, \bar{3}) s_3 \preceq w$. Therefore, we obtain $\boldsymbol{s_3(\bar{1}, \bar{2}) = (\bar{1}, \bar{3}) s_3}$, which gives $\boldsymbol{(\bar{1}, \bar{2}) = s_3^{-1}(\bar{1}, \bar{3}) s_3}$.
- Next, we consider $(\bar{1}, \bar{3}) = s_1^{s_2}$. We have that $w' = s_2(\bar{1}, \bar{3}) w = (1)(4, 3, 2)$ is of length 2. Thus, we have $(\bar{1}, \bar{3}) s_2 \preceq w$, which says that $(\bar{1}, \bar{3}) s_2 = s_2(\bar{1}, \bar{3})^{s_2} = s_2 s_1 \preceq w$. We obtain $\boldsymbol{(\bar{1}, \bar{3}) s_2 = s_2 s_1}$, which gives $\boldsymbol{(\bar{1}, \bar{3}) = s_2 s_1 s_2^{-1}}$.

Therefore, we obtain

$$\boldsymbol{(\overline{1},\overline{2}) = s_3^{-1}(\overline{1},\overline{3})s_3 = s_3^{-1}s_2s_1s_2^{-1}s_3 = s_1^{s_2^{-1}s_3}}.$$

We finish this section by the next lemma which is used to show that f is a homomorphism (see Step 1 of the strategy of the proof in Section 2.5). The proof of the lemma uses Proposition 5.10.

Lemma 5.12. *The braid relators* $\mathrm{B}(\boldsymbol{s}_i, \boldsymbol{s}_j)$ *and the twisted cycle commutator relator* $\mathrm{TC}(\boldsymbol{s}_{m+2}, \boldsymbol{s}_{m+1}, \boldsymbol{s}_m, \boldsymbol{s}_1)$ *specified by the presentation given for* $\boldsymbol{G}$ *hold in* $G([1,w])$.

Proof. Consider case (1) of Proposition 3.12. It implies a commuting braid relation, which lifts to $\boldsymbol{s_is_j = s_js_i}$ for $|i-j|>1$.

Consider case (2) of Proposition 3.12. It implies a dual braid relation $\boldsymbol{s_is_{i+1}} = \boldsymbol{s_{i+1}t}$, for $2 \leqslant i \leqslant n-1$, where $\boldsymbol{t = (i-1, i+1)}$. Applying Equation (16) of Proposition 5.10, we have that $\boldsymbol{t}$ is equal to $\boldsymbol{s_{i+1}^{s_i^{-1}}}$. The dual braid relation becomes $\boldsymbol{s_is_{i+1} = s_{i+1}s_{i+1}^{s_i^{-1}}}$, that is $\boldsymbol{s_is_{i+1}s_i = s_{i+1}s_is_{i+1}}$.

Case (3) of Proposition 3.12 is treated similarly. We will give details on case (4) where the TC relator will appear.

Consider then Case (4) of Proposition 3.12. It implies that $ts_{m+2} \preceq w$ with ts_{m+2} of order 2. So we get a commuting dual braid relation. Now, we have to lift the relation to prove that it lives in $G([1,w])$. Applying Equation (18) of Proposition 5.10 for $i = m-1$ and $j = m$, we have that $\boldsymbol{t}$ is equal to $\boldsymbol{s_1^{s_m^{-1}s_{m+1}}}$. Then the dual braid relation becomes $[\boldsymbol{s_{m+2}}, \boldsymbol{s_1^{s_m^{-1}s_{m+1}}}] = 1$, which is exactly the TC relator (see Remark 5.4). □

6. The proof of the main theorem

6.1. The case $n = 4$. Let W be the Coxeter group of type D_4. By Proposition 3.11, W has a presentation on the four generators $s_2 = (1,2)$, $s_3 = (2,3)$, $s_4 = (3,4)$, and $s_1 = (\overline{2},\overline{3})$. We have $m = 2$ and the corresponding proper quasi-Coxeter element is $w = (2,\overline{1})(4,\overline{3})$.

We prove that the interval group $G([1,w])$ is isomorphic to the group $\boldsymbol{G}$ with four generators $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4$, corresponding to reflections s_1, s_2, s_3, s_4, with relations described by the corresponding Carter diagram, along with the twisted cycle commutator relator

$$\mathrm{TC}(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4) = [\boldsymbol{s}_1, \boldsymbol{s}_4^{\boldsymbol{s}_3^{-1}\boldsymbol{s}_2}].$$

We consider a reflection t in T to be of type I, II, or III according to the type of the maximal divisor $w_0 = wt$ of w, and assign the same types to the elements of $\boldsymbol{T}$. We collect the information we need in Tables 4 and 5.

Table 4 provides decompositions for the elements of $\boldsymbol{T}$ by applying Proposition 5.10.

The second column of Table 5 contains the 12 divisors (the w_0's) of length 3 of the quasi-Coxeter element w of types I, II, and III, where we separate each type by two lines. We follow Section 4.1 in order to produce them. We also follow Sections 4.2 and 4.3 to produce reduced decompositions of these elements and their decomposition diagrams (the Δ_0's). The last column produces a Coxeter-like diagram related to Δ_0 that we call the lift of Δ_0. It encodes the relations between the lift to the interval group of two reflections that appear in the reduced decomposition of each w_0.

Proposition 6.1. *All the relations that describe the type* A_3 *diagrams on the last column of Table* 5 *are consequences of the relations described by the diagram presentation over* $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4$ *illustrated in Figure* 3.

Number	Decomposition of $\boldsymbol{t}$ (type I)
1.	$\boldsymbol{(1,3)} = \boldsymbol{s_3^{s_2^{-1}}}$
2.	$\boldsymbol{(1,4)} = \boldsymbol{s_4^{s_3^{-1}s_2^{-1}}}$
3.	$\boldsymbol{(2,3)} = \boldsymbol{s_3}$
4.	$\boldsymbol{(2,4)} = \boldsymbol{s_4^{s_3^{-1}}}$
5.	$\boldsymbol{(\overline{1},\overline{3})} = \boldsymbol{s_1^{s_2^{-1}}}$
6.	$\boldsymbol{(\overline{1},\overline{4})} = \boldsymbol{s_1^{s_4 s_2^{-1}}}$
7.	$\boldsymbol{(\overline{2},\overline{3})} = \boldsymbol{s_1}$
8.	$\boldsymbol{(\overline{2},\overline{4})} = \boldsymbol{s_3^{s_4}}$
	Decomposition of $\boldsymbol{t}$ (type II)
9.	$\boldsymbol{(1,2)} = \boldsymbol{s_2}$
10.	$\boldsymbol{(\overline{1},\overline{2})} = \boldsymbol{s_1^{s_2^{-1}s_3}}$
	Decomposition of $\boldsymbol{t}$ (type III)
11.	$\boldsymbol{(3,4)} = \boldsymbol{s_4}$
12.	$\boldsymbol{(\overline{3},\overline{4})} = \boldsymbol{s_1^{s_4 s_3^{-1}}}$

TABLE 4. Decompositions of $\boldsymbol{t}$ in the case $n = 4$.

We prove the proposition by showing using `kbmag` [24] within `GAP` [22] that the relations appearing in the last column of Table 5 are consequences of the relations between $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4$. For example, let us consider a diagram where some twisted cycle commutator relators appear. Consider the element Number 6 of the table. We have to show that $\boldsymbol{(\overline{1},\overline{2})}$ commutes with $\boldsymbol{s}_4$, where $\boldsymbol{(\overline{1},\overline{2})} = \boldsymbol{s_1^{s_2^{-1}s_3}}$. The commuting relation between $\boldsymbol{(\overline{1},\overline{2})}$ and $\boldsymbol{s}_4$ is precisely the twisted cycle commutator relator $[\boldsymbol{s}_4, \boldsymbol{s_1^{s_2^{-1}s_3}}]$ that is a consequence of the relations of the claimed presentation.

Now we can show that $G([1,w])$ is isomorphic to $\boldsymbol{G}$, that is Theorem 5.1 in the case where $n = 4$.

PROPOSITION 6.2. *In the case $n = 4$, the groups $\boldsymbol{G}$ and $G([1,w])$ are isomorphic.*

Proof. By transitivity of the Hurwitz action on the reduced decompositions over $\boldsymbol{T}$ of w, the group $G([1,w])$ is generated by a copy

$$\boldsymbol{T} = \{\boldsymbol{(i,j)}, \boldsymbol{(\overline{i},\overline{j})} : 1 \leqslant i < j \leqslant 4\}$$

of the set of reflections in T, and subject to the dual braid relations $\boldsymbol{tt' = t't''}$ that correspond to relations $tt' = t't''$ in W where $tt' = t't'' \preceq w$.

Consider the map $f : \boldsymbol{G} \longrightarrow G([1,w]) : \boldsymbol{s}_i \longmapsto \boldsymbol{s}_i$. By Lemma 5.12, the relations of the presentation of $\boldsymbol{G}$ hold in $G([1,w])$.

Now consider the map $g : G([1,w]) \longrightarrow \boldsymbol{G}$ that maps each generator $\boldsymbol{t}$ of $G([1,w])$ to the expression for it over the generators $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, $\boldsymbol{s}_3$, and $\boldsymbol{s}_4$ that is given by Proposition 5.10. Let $\boldsymbol{tt'}$, $\boldsymbol{t't''}$ be the two sides of a dual braid relation. Then there exists $w_0 \preceq w$ of length 3 such that $tt' = t't'' \preceq w_0$. By [7], we know that the group $G([1,w_0])$ is isomorphic to the group defined by a presentation that we have described by Coxeter diagrams of type A_3 in the last column of Table 5. Hence we obtain that the dual braid relation $\boldsymbol{tt' = t't''}$ is a consequence of the relations of the corresponding diagram in the table.

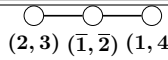
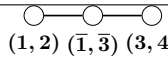
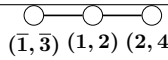
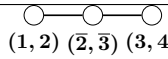
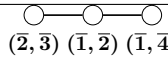
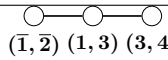
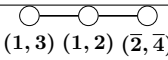
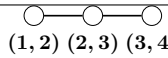
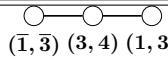
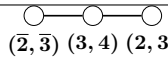
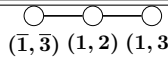
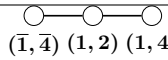
Number	Maximal divisor w_0	Reduced Decomposition	Lift of Δ_0
1.	$w(1,3) = (4,\overline{1},2,\overline{3})$	$(2,3)(\overline{1},\overline{2})(1,4)$	
2.	$w(1,4) = (4,\overline{3},\overline{1},2)$	$(1,2)(\overline{1},\overline{3})(3,4)$	
3.	$w(2,3) = (4,2,\overline{1},\overline{3})$	$(\overline{1},\overline{3})(1,2)(2,4)$	
4.	$w(2,4) = (4,\overline{3},2,\overline{1})$	$(1,2)(\overline{2},\overline{3})(3,4)$	
5.	$w(\overline{1},\overline{3}) = (\overline{4},\overline{1},\overline{2},\overline{3})$	$(\overline{2},\overline{3})(\overline{1},\overline{2})(\overline{1},\overline{4})$	
6.	$w(\overline{1},\overline{4}) = (4,3,\overline{1},\overline{2})$	$(\overline{1},\overline{2})(1,3)(3,4)$	
7.	$w(\overline{2},\overline{3}) = (\overline{4},2,1,\overline{3})$	$(1,3)(1,2)(\overline{2},\overline{4})$	
8.	$w(\overline{2},\overline{4}) = (4,3,2,1)$	$(1,2)(2,3)(3,4)$	
9.	$w(1,2) = (\overline{1})(2)(4,\overline{3})$	$(1,3)(\overline{1},\overline{3})(3,4)$	
10.	$w(\overline{1},\overline{2}) = (1)(\overline{2})(4,\overline{3})$	$(2,3)(\overline{2},\overline{3})(3,4)$	
11.	$w(3,4) = (2,\overline{1})(\overline{3})(4)$	$(1,3)(\overline{1},\overline{3})(1,2)$	
12.	$w(\overline{3},\overline{4}) = (2,\overline{1})(3)(\overline{4})$	$(1,4)(\overline{1},\overline{4})(1,2)$	

Table 5. Reduced decompositions and diagram lifts in the case $n = 4$.

In addition, we have already shown in Proposition 6.1 that the relations of these diagrams are consequences of the relations we have associated with the diagram Δ. Hence the map g is a homomorphism.

Clearly, the composition $f \circ g$ is equal to $\mathrm{id}_{G([1,w])}$ and $g \circ f$ equal to $\mathrm{id}_{\boldsymbol{G}}$. Therefore, the groups $\boldsymbol{G}$ and $G([1,w])$ are isomorphic. □

6.2. The case $n = 5$. Let W be the Coxeter group of type D_5. It has a presentation on the five generators $s_2 = (1,2)$, $s_3 = (2,3)$, $s_4 = (3,4)$, $s_5 = (4,5)$, and $s_1 = (\overline{2},\overline{3})$ (see Proposition 3.11). Here m is equal to 2 and the corresponding proper quasi-Coxeter element is $w = (2,\overline{1})(5,4,\overline{3})$.

We prove that the interval group $G([1,w])$ is isomorphic to the group $\boldsymbol{G}$ with five generators $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4$, and $\boldsymbol{s}_5$ corresponding to s_1, s_2, s_3, s_4, s_5, with relations described by the diagram of Figure 3, where the curved arrow describes the twisted cycle commutator relator: $\mathrm{TC}(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4) = [\boldsymbol{s}_1, \boldsymbol{s}_4^{\boldsymbol{s}_3^{-1}\boldsymbol{s}_2}]$.

Similarly to Table 4 in the case $n = 4$, we provide in Table 6 decompositions of $\boldsymbol{t}$ in the case $n = 5$ by applying Proposition 5.10 and we divide them according to the three types I, II, III of elements in $\boldsymbol{T}$.

Table 7 contains for the case $n = 5$ the information that is displayed in Table 5 for the case $n = 4$. We have 20 divisors of w of length 4 (the w_0's) obtained by multiplying w from the right by (i,j) and $(\overline{i},\overline{j})$ for $1 \leqslant i < j \leqslant 5$. These divisors belong to types I, II, and III. We separate each type by 2 lines in the table. The third column produces the reduced decomposition from Sections 4.2 and 4.3. The last column describes the

Number	Decomposition of $\boldsymbol{t}$ (type I)
1.	$\boldsymbol{(1,3) = s_3^{s_2^{-1}}}$
2.	$\boldsymbol{(1,4) = s_4^{s_3^{-1}s_2^{-1}}}$
3.	$\boldsymbol{(1,5) = s_5^{s_4^{-1}s_3^{-1}s_2^{-1}}}$
4.	$\boldsymbol{(2,3) = s_3}$
5.	$\boldsymbol{(2,4) = s_4^{s_3^{-1}}}$
6.	$\boldsymbol{(2,5) = s_5^{s_4^{-1}s_3^{-1}}}$
7.	$\boldsymbol{(\bar{1},\bar{3}) = s_1^{s_2^{-1}}}$
8.	$\boldsymbol{(\bar{1},\bar{4}) = s_1^{s_4s_2^{-1}}}$
9.	$\boldsymbol{(\bar{1},\bar{5}) = s_1^{s_4s_5s_2^{-1}}}$
10.	$\boldsymbol{(\bar{2},\bar{3}) = s_1}$
11.	$\boldsymbol{(\bar{2},\bar{4}) = s_1^{s_4}}$
12.	$\boldsymbol{(\bar{2},\bar{5}) = s_1^{s_4s_5}}$
	Decomposition of $\boldsymbol{t}$ (type II)
13.	$\boldsymbol{(1,2) = s_2}$
14.	$\boldsymbol{(\bar{1},\bar{2}) = s_1^{s_2^{-1}s_3}}$
	Decomposition of $\boldsymbol{t}$ (type III)
15.	$\boldsymbol{(3,4) = s_4}$
16.	$\boldsymbol{(3,5) = s_5^{s_4^{-1}}}$
17.	$\boldsymbol{(4,5) = s_5}$
18.	$\boldsymbol{(\bar{3},\bar{4}) = s_1^{s_4s_3^{-1}}}$
19.	$\boldsymbol{(\bar{3},\bar{5}) = s_1^{s_4s_5s_3^{-1}}}$
20.	$\boldsymbol{(\bar{4},\bar{5}) = s_1^{s_4s_5s_3^{-1}s_4}}$

TABLE 6. Decompositions of $\boldsymbol{t}$ in the case $n = 5$.

lift of the diagram Δ_0 that encodes the relations between the lift to the interval group of two reflections that appear in the reduced decomposition of each w_0.

We showed, using `kbmag` within `GAP` that all the relations described in the diagrams of the last column are consequences of the relations of the claimed presentation (see Theorem 5.1). The only cases that correspond to proper quasi-Coxeter elements are numbers 15, 17, and 19 of Table 7. Let w_0 be one of these elements. We know from Proposition 6.2 that $G([1, w_0])$ is isomorphic to the group defined by a presentation associated to the square diagram with the twisted cycle commutator relator. We conclude with the statement of the result for $n = 5$, whose proof we omit since it is similar to the proof of Proposition 6.2.

PROPOSITION 6.3. *In the case $n = 5$, the groups $\boldsymbol{G}$ and $G([1, w])$ are isomorphic.*

6.3. LIFTING THE REDUCED DECOMPOSITIONS. This section establishes Step 4 in our strategy that we have described in Section 2.5.

Let w be the quasi-Coxeter element $(m, m-1, \ldots, 2, \bar{1})(n, n-1, \ldots, \overline{m+1})$ in type D_n. We define g to be the map from $G([1, w])$ to $\boldsymbol{G}$ that sends t_i to its decomposition over the generating set $\boldsymbol{\mathcal{S}}$ of $\boldsymbol{G}$ given by Proposition 5.10. In this section, we prove the following.

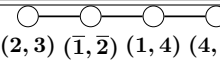
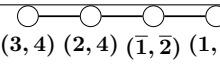
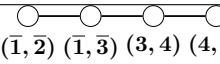
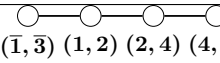
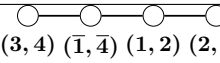
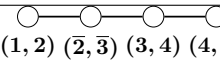
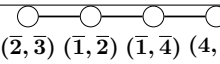
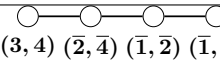
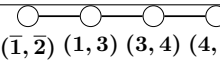
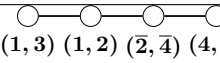
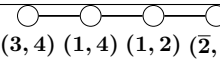
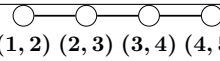
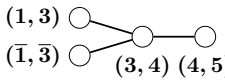
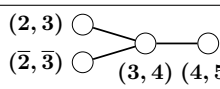
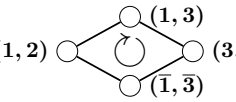
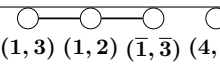
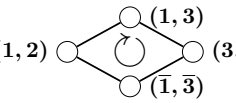
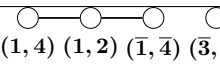
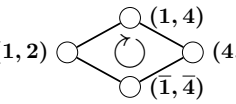
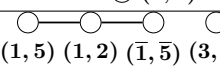
	Maximal divisor w_0	Reduced decomposition	Lift of Δ_0
1.	$w(1,3) = (5,4,\overline{1},2,\overline{3})$	$(2,3)(\overline{1},\overline{2})(1,4)(4,5)$	
2.	$w(1,4) = (5,\overline{1},2,4,\overline{3})$	$(3,4)(2,4)(\overline{1},\overline{2})(1,5)$	
3.	$w(1,5) = (5,4,\overline{3},\overline{1},2)$	$(\overline{1},\overline{2})(\overline{1},\overline{3})(3,4)(4,5)$	
4.	$w(2,3) = (5,4,2,\overline{1},\overline{3})$	$(\overline{1},\overline{3})(1,2)(2,4)(4,5)$	
5.	$w(2,4) = (5,2,\overline{1},4,\overline{3})$	$(3,4)(\overline{1},\overline{4})(1,2)(2,5)$	
6.	$w(2,5) = (5,4,\overline{3},2,\overline{1})$	$(1,2)(\overline{2},\overline{3})(3,4)(4,5)$	
7.	$w(\overline{1},\overline{3}) = (5,\overline{4},\overline{1},\overline{2},\overline{3})$	$(\overline{2},\overline{3})(\overline{1},\overline{2})(\overline{1},\overline{4})(4,5)$	
8.	$w(\overline{1},\overline{4}) = (\overline{5},\overline{1},\overline{2},4,\overline{3})$	$(3,4)(\overline{2},\overline{4})(\overline{1},\overline{2})(\overline{1},\overline{5})$	
9.	$w(\overline{1},\overline{5}) = (5,4,3,\overline{1},\overline{2})$	$(\overline{1},\overline{2})(1,3)(3,4)(4,5)$	
10.	$w(\overline{2},\overline{3}) = (5,\overline{4},2,1,\overline{3})$	$(1,3)(1,2)(\overline{2},\overline{4})(4,5)$	
11.	$w(\overline{2},\overline{4}) = (\overline{5},2,1,4,\overline{3})$	$(3,4)(1,4)(1,2)(\overline{2},\overline{5})$	
12.	$w(\overline{2},\overline{5}) = (5,4,3,2,1)$	$(1,2)(2,3)(3,4)(4,5)$	
13.	$w(1,2) = (\overline{1})(2)(5,4,\overline{3})$	$(1,3)(\overline{1},\overline{3})(3,4)(4,5)$	
14.	$w(\overline{1},\overline{2}) = (1)(\overline{1})(5,4,\overline{3})$	$(2,3)(\overline{2},\overline{3})(3,4)(4,5)$	
15.	$w(3,4) = (2,\overline{1})(5,\overline{3})(4)$	$(1,3)(\overline{1},\overline{3})(1,2)(3,5)$	
16.	$w(3,5) = (2,\overline{1})(\overline{3})(5,4)$	$(1,3)(\overline{1},\overline{3})(1,2)(4,5)$	
17.	$w(4,5) = (2,\overline{1})(4,\overline{3})(5)$	$(1,3)(\overline{1},\overline{3})(1,2)(3,4)$	
18.	$w(\overline{3},\overline{4}) = (2,\overline{1})(\overline{5},\overline{3})(\overline{4})$	$(1,4)(\overline{1},\overline{4})(1,2)(\overline{3},\overline{5})$	
19.	$w(\overline{3},\overline{5}) = (2,\overline{1})(3)(5,\overline{3})$	$(1,4)(\overline{1},\overline{4})(1,2)(4,5)$	
20.	$w(\overline{4},\overline{5}) = (2,\overline{1})(4,3)(\overline{5})$	$(1,5)(\overline{1},\overline{5})(1,2)(3,4)$	

TABLE 7. Reduced decompositions and diagram lifts in the case $n = 5$.

PROPOSITION 6.4. *Let w_0 be a divisor of length $n-1$ of the quasi-Coxeter element w, let $t_1t_2\dots t_{n-1}$ be the reduced decomposition of w_0 obtained using the results of Sections* 4.2, 4.3*, and let Δ_0 be the associated decomposition diagram (described in Propositions* 4.5, 4.15 *and* 4.16*). Then for each of the relators* $\mathrm{B}(t_i,t_j)$ *and* $\mathrm{TC}(t_i,t_j,t_k,t_l)$ *between the reflections t_i that is implied by the diagram Δ_0 the corresponding relators* $\mathrm{B}(g(\boldsymbol{t}_i),g(\boldsymbol{t}_j))$ *or* $\mathrm{TC}(g(\boldsymbol{t}_i),g(\boldsymbol{t}_j),g(\boldsymbol{t}_k),g(\boldsymbol{t}_l))$ *can be derived from the relations of the presentation of $\boldsymbol{G}$ given in Theorem* 5.1.

Proof. The proof is by induction on n. The proposition is proved for $n=4$ and $n=5$ in Sections 6.1 and 6.2 within the proofs of Propositions 6.2 and 6.3.

Now let $n\geqslant 6$. Set $w' \coloneqq (m,m-1,\dots,2,\overline{1})(n-1,n-2,\dots,\overline{m+1})$. Then

$$w' = s_2s_3\dots s_ms_1s_{m+1}s_{m+2}s_{m+3}s_{m+4}\dots s_{n-1}$$

has the diagram $\Delta_{m,n-1}$ by Proposition 3.8, and $P \coloneqq P_{w'} = \langle s_1,\dots,s_{n-1}\rangle$ is a parabolic subgroup of W, which is of type D_{n-1}. Moreover, the braid relators in the generators $g(\boldsymbol{s}_i)$ and $g(\boldsymbol{s}_j)$ as well as the twisted cycle relator for w', which we will call the w'-relators, are a subset of the $\boldsymbol{G}$-relators.

There are 11 different possibilities for w_0 that are described in Section 4.1 by Equations (1)–(11). For the 11-th equation we need to deal separately with the cases $n>i+1$ and $n=i+1$.

Suppose first that w_0 is either as in one of Equations $1-10$ or as in Equation 11 with $n>i+1$. In any of these cases, in the cycle decomposition of w_0 the number n is only overlined in a cycle that has an even number of overlined entries. This implies that at most one of the reflections $t_1,\dots,t_{n-1}$ is not contained in P and that this reflection corresponds to an end node of Δ_0 and is without loss of generality t_{n-1}. Thus we have $t_1,\dots,t_{n-2}\in P$. Set $w_1 \coloneqq w_0t_{n-1} = t_1\cdots t_{n-2}\in P$. Then we get by [10, Lemma 5.3] that w_1 is a divisor of length $\ell_T(w_1)=n-2$ of w' in P. Further w' is of length $n-1$. By induction on (P,w'), the relators $\mathrm{B}(g(\boldsymbol{t}_i),g(\boldsymbol{t}_j))$ and $\mathrm{TC}(g(\boldsymbol{t}_i),g(\boldsymbol{t}_j),g(\boldsymbol{t}_k),g(\boldsymbol{t}_l))$ are a consequence of the w'-relators, which are $\boldsymbol{G}$-relators, for $1\leqslant i,j,k,l\leqslant n-2$.

Hence it only remains to show that $\mathrm{B}(g(\boldsymbol{t}_i),g(\boldsymbol{t}_j))$ and $\mathrm{TC}(g(\boldsymbol{t}_i),g(\boldsymbol{t}_j),g(\boldsymbol{t}_k),g(\boldsymbol{t}_l))$ are consequences of the $\boldsymbol{G}$-relators under the assumption that $i=n-1$. This is done in the appendix in Lemmas A.1, …, A.6. Thereby notice, as t_{n-1} corresponds to an end node of Δ_0, it is not contained in a cycle of Δ_0 and the relator $\mathrm{TC}(t_{n-1},t_j,t_k,t_l)$ does not appear.

Now suppose that w_0 is as in Equation (11) and that $n=i+1$. Then Δ_0 is the union of three strings, one of length 1. The reflections t_i are in P beside that one corresponding to the single vertex and one of the other four end nodes of the strings of Δ_0. By induction it remains to derive the braid relators for the two reflections just mentioned from the $\boldsymbol{G}$-relators, which is treated in Lemma A.6. □

In Appendix A we establish the proof of the lemmas we refer to in the proof of Proposition 6.4.

6.4. THE PROOF FOR $n>5$. We are in position to prove Theorem 5.1. The details of the proof are discussed and commented in our strategy developed in Section 2.5.

Consider the map $f : \boldsymbol{G} \longrightarrow G([1,w]) : \boldsymbol{s}_i \longmapsto \boldsymbol{s}_i$. By Proposition 3.12 and Lemma 5.12, the relations of the presentation of $\boldsymbol{G}$ hold in $G([1,w])$. This is Step 1 in the strategy of the proof.

Consider the map $g : G([1,w]) \longrightarrow \boldsymbol{G}$ that maps each generator $\boldsymbol{t}$ of the generating set $\boldsymbol{T}$ of $G([1,w])$ to its decomposition on the generators $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, …, $\boldsymbol{s}_n$ that we

described in Equations (16) to (19) within Proposition 5.10. This is Step 3 in the strategy of the proof that was established within Section 5.

We want to show that g is a homomorphism. Consider a dual braid relation of $G([1,w])$, meaning a relation of the form $\boldsymbol{t}\boldsymbol{t}' = \boldsymbol{t}'\boldsymbol{t}''$ for $\boldsymbol{t},\boldsymbol{t}',\boldsymbol{t}'' \in \boldsymbol{T}$. It corresponds to the fact that $tt' = t't'' \preceq w$. Then we have to prove that the relation $g(\boldsymbol{t})g(\boldsymbol{t}') = g(\boldsymbol{t}')g(\boldsymbol{t}'')$ is a consequence of the relations of the presentation of $\boldsymbol{G}$. As $([1,w],\preceq)$ is a graded poset in which all the maximal flags have the same length, there are divisors w_0 of length $n-1$ of w such that $tt' \preceq w_0$. By Proposition 6.4, the braid relations and the twisted cycle commutator relator that correspond to the reduced decomposition and the diagram Δ_0 for w_0 (produced in Sections 4.2 and 4.3) are a consequence of the $\boldsymbol{G}$-relations (see Step 4 of our strategy). By induction, the relation $g(\boldsymbol{t})g(\boldsymbol{t}') = g(\boldsymbol{t}')g(\boldsymbol{t}'')$ is a consequence of the braid relations and the twisted cycle commutator relator related to the reduced decomposition in the g-image of the lift of P_{w_0} to $\boldsymbol{G}$. Therefore, g is a homomorphism.

Clearly, the composition $f \circ g$ is equal to $\mathrm{id}_{G([1,w])}$ and $g \circ f$ is equal to $\mathrm{id}_{\boldsymbol{G}}$. Therefore, the groups $\boldsymbol{G}$ and $G([1,w])$ are isomorphic.

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Appendix A. Proof of the lemmas

In the calculations within the proofs of the next lemmas, we underline an expression being manipulated for emphasis.

Lemma A.1. *Suppose that $w_0 = w(i,j)$, with $1 \leqslant i \leqslant m$, $m+1 \leqslant j \leqslant n$, so that we are in the situation of Equation* (1)*. Let $t_1 t_2 \dots t_{n-1}$ be the reduced decomposition of w_0 described in Lemma* 4.6*. Then we can deduce from the relations of the presentation of $\boldsymbol{G}$ that $g(\boldsymbol{t}_{n-1})g(\boldsymbol{t}_{n-2})g(\boldsymbol{t}_{n-1}) = g(\boldsymbol{t}_{n-2})g(\boldsymbol{t}_{n-1})g(\boldsymbol{t}_{n-2})$ and $g(\boldsymbol{t}_{n-1})$ commutes with each of the elements $g(\boldsymbol{t}_k)$ with $k < n-2$.*

Proof. We consider the seven different possible decompositions of w_0 that are described in Lemma 4.6. In each case we do not need to consider the relations between $g(\boldsymbol{t}_n)$ and $g(\boldsymbol{t}_k)$ if both elements are within the set $\boldsymbol{\mathcal{S}}$.

(1) Where $i \neq m$ and $j \neq n-2, n-1, n$, we need to check that the element $g(\boldsymbol{s}_n) = g(\boldsymbol{(n-1,n)})$ commutes with each of $g(\boldsymbol{(i+1,j)})$, $g(\boldsymbol{(i,j+1)})$, and $g\left(\boldsymbol{(\overline{1},\overline{m})}\right)$.

By Proposition 5.10, we have that

$$g(\boldsymbol{(i+1,j)}) = \boldsymbol{s}_j^{\boldsymbol{s}_{j-1}^{-1}\boldsymbol{s}_{j-2}^{-1}\cdots\boldsymbol{s}_{i+2}^{-1}} \text{ and } g(\boldsymbol{(i,j+1)}) = \boldsymbol{s}_{j+1}^{\boldsymbol{s}_j^{-1}\boldsymbol{s}_{j-1}^{-1}\cdots\boldsymbol{s}_{i+1}^{-1}}.$$

It follows from the relations of the presentation of $\boldsymbol{G}$ that $g(\boldsymbol{s}_n)$ commutes with both $g(\boldsymbol{(i+1,j)})$ and $g(\boldsymbol{(i,j+1)})$.

We also have that $g\left(\boldsymbol{(\overline{1},\overline{m})}\right) = s_1^{\boldsymbol{s}_m^{-1}\boldsymbol{s}_{m-1}^{-1}\cdots\boldsymbol{s}_2^{-1}\boldsymbol{s}_{m+1}}$. Then it also follows from the relations of the presentation of $\boldsymbol{G}$ that $g(\boldsymbol{s}_n)$ commutes with $g\left(\boldsymbol{(\overline{1},\overline{m})}\right)$.

(2) Where $i \neq m$ and $j = n-2$, we need to check that the element $g(\boldsymbol{s}_n) = g(\boldsymbol{(n-1,n)})$ commutes with each of $g(\boldsymbol{(i+1,n-2)})$, $g\left(\boldsymbol{(\overline{1},\overline{m})}\right)$, and that the relation $g(\boldsymbol{s}_n)\,g(\boldsymbol{(i,n-1)})\,g(\boldsymbol{s}_n) = g(\boldsymbol{(i,n-1)})\,g(\boldsymbol{s}_n)\,g(\boldsymbol{(i,n-1)})$ holds.

We have already shown in item (1) of this proof that $g(\boldsymbol{s}_n)$ commutes with $g\left(\boldsymbol{(\overline{1},\overline{m})}\right)$.

We have that $g\left((\boldsymbol{i+1, n-2})\right) = \boldsymbol{s}_{n-2}^{\boldsymbol{s}_{n-3}^{-1}\boldsymbol{s}_{n-4}^{-1}\cdots\boldsymbol{s}_{i+2}^{-1}}$. It follows directly from the relations of $\boldsymbol{G}$ that $g\left(\boldsymbol{s}_n\right)$ commutes with $g\left((\boldsymbol{i+1, n-2})\right)$.

Now we prove that $g\left(\boldsymbol{s}_n\right) g\left((\boldsymbol{i, n-1})\right) g\left(\boldsymbol{s}_n\right) = g\left((\boldsymbol{i, n-1})\right) g\left(\boldsymbol{s}_n\right) g\left((\boldsymbol{i, n-1})\right)$:

$$\begin{aligned}
g\left(\boldsymbol{s}_n\right) g\left((\boldsymbol{i, n-1})\right) g\left(\boldsymbol{s}_n\right) =& \boldsymbol{s}_{i+1}\ldots\boldsymbol{s}_{n-3}\boldsymbol{s}_{n-2}\underline{\boldsymbol{s}_n\boldsymbol{s}_{n-1}\boldsymbol{s}_n}\boldsymbol{s}_{n-2}^{-1}\boldsymbol{s}_{n-3}^{-1}\ldots\boldsymbol{s}_{i+1}^{-1}\\
=& \boldsymbol{s}_{i+1}\ldots\boldsymbol{s}_{n-3}\boldsymbol{s}_{n-2}\boldsymbol{s}_{n-1}\boldsymbol{s}_n\boldsymbol{s}_{n-1}\boldsymbol{s}_{n-2}^{-1}\boldsymbol{s}_{n-3}^{-1}\ldots\boldsymbol{s}_{i+1}^{-1}\\
=& (\boldsymbol{s}_{i+1}\ldots\boldsymbol{s}_{n-3}\boldsymbol{s}_{n-2}\boldsymbol{s}_{n-1}\boldsymbol{s}_{n-2}^{-1}\boldsymbol{s}_{n-3}^{-1}\ldots\boldsymbol{s}_{i+1}^{-1})\\
& (\boldsymbol{s}_{i+1}\ldots\boldsymbol{s}_{n-3}\boldsymbol{s}_{n-2}\underline{\boldsymbol{s}_n}\boldsymbol{s}_{n-1}\boldsymbol{s}_{n-2}^{-1}\boldsymbol{s}_{n-3}^{-1}\ldots\boldsymbol{s}_{i+1}^{-1})\\
=& (\boldsymbol{s}_{i+1}\ldots\boldsymbol{s}_{n-3}\boldsymbol{s}_{n-2}\boldsymbol{s}_{n-1}\boldsymbol{s}_{n-2}^{-1}\boldsymbol{s}_{n-3}^{-1}\ldots\boldsymbol{s}_{i+1}^{-1})\\
& \boldsymbol{s}_n(\boldsymbol{s}_{i+1}\ldots\boldsymbol{s}_{n-3}\boldsymbol{s}_{n-2}\boldsymbol{s}_{n-1}\boldsymbol{s}_{n-2}^{-1}\boldsymbol{s}_{n-3}^{-1}\ldots\boldsymbol{s}_{i+1}^{-1})\\
=& g\left((\boldsymbol{i, n-1})\right) g\left(\boldsymbol{s}_n\right) g\left((\boldsymbol{i, n-1})\right).
\end{aligned}$$

(3) Where $i = m$ and $j = n-2$, we need to check that $g\left(\boldsymbol{s}_n\right)$ commutes with $g\left(\overline{(\boldsymbol{1}, \boldsymbol{n-2})}\right)$ and the relation

$$g\left(\boldsymbol{s}_n\right) g\left((\boldsymbol{m, n-1})\right) g\left(\boldsymbol{s}_n\right) = g\left((\boldsymbol{m, n-1})\right) g\left(\boldsymbol{s}_n\right) g\left((\boldsymbol{m, n-1})\right).$$

We already proved this last relation in item (2). For the commuting relation, we have that $g\left(\overline{(\boldsymbol{1}, \boldsymbol{n-2})}\right) = \boldsymbol{s}_1^{\boldsymbol{s}_{m+2}\boldsymbol{s}_{m+3}\cdots\boldsymbol{s}_{n-2}\boldsymbol{s}_m^{-1}\boldsymbol{s}_{m-2}^{-1}\cdots\boldsymbol{s}_2^{-1}}$ for $n > 5$. It follows that $g\left(\boldsymbol{s}_n\right)$ commutes with $g\left(\overline{(\boldsymbol{1}, \boldsymbol{n-2})}\right)$.

(4) Where $i \neq m$ and $j = n-1$, we need to prove that $g\left(\boldsymbol{s}_i\right) g\left((\boldsymbol{i, n})\right) g\left(\boldsymbol{s}_i\right) = g\left((\boldsymbol{i, n})\right) g\left(\boldsymbol{s}_i\right) g\left((\boldsymbol{i, n})\right)$ and $g\left((\boldsymbol{i, n})\right)$ commutes with all the images by g of elements of $\boldsymbol{T}$ that correspond to reflections in the string of item (4) in Lemma 4.6:

$$s_{m+2}\ldots s_{n-2}s_{n-1}(i+1, n-1)s_{i+2}\ldots s_{m-1}s_m(\overline{1}, \overline{m})s_2\ldots s_{i-1}.$$

We have that $g\left((\boldsymbol{i, n})\right) = \boldsymbol{s}_n^{\boldsymbol{s}_{n-1}^{-1}\boldsymbol{s}_{n-2}^{-1}\cdots\boldsymbol{s}_{i+1}^{-1}}$. Then, we get

$$\begin{aligned}
g\left(\boldsymbol{s}_i\right) g\left((\boldsymbol{i, n})\right) g\left(\boldsymbol{s}_i\right) =& \boldsymbol{s}_i(\boldsymbol{s}_{i+1}\ldots\boldsymbol{s}_{n-2}\underline{\boldsymbol{s}_{n-1}\boldsymbol{s}_n\boldsymbol{s}_{n-1}^{-1}}\boldsymbol{s}_{n-2}^{-1}\ldots\boldsymbol{s}_{i+1}^{-1})\boldsymbol{s}_i\\
=& \boldsymbol{s}_i(\boldsymbol{s}_{i+1}\ldots\boldsymbol{s}_{n-2}\underline{\boldsymbol{s}_n^{-1}}\boldsymbol{s}_{n-1}\underline{\boldsymbol{s}_n}\boldsymbol{s}_{n-2}^{-1}\ldots\boldsymbol{s}_{i+1}^{-1})\boldsymbol{s}_i\\
=& \boldsymbol{s}_n^{-1}\boldsymbol{s}_i(i, n-1)\boldsymbol{s}_i\boldsymbol{s}_n\\
=& \boldsymbol{s}_n^{-1}(\boldsymbol{i, n-1})\boldsymbol{s}_i(\boldsymbol{i, n-1})\boldsymbol{s}_n \text{ (by induction hypothesis)}\\
=& \boldsymbol{s}_n^{-1}(\boldsymbol{i, n-1})\boldsymbol{s}_n\boldsymbol{s}_i\boldsymbol{s}_n^{-1}(\boldsymbol{i, n-1})\boldsymbol{s}_n,
\end{aligned}$$

with $\boldsymbol{s}_n^{-1}g\left((\boldsymbol{i, n-1})\right)\boldsymbol{s}_n = g\left((\boldsymbol{i, n})\right)$. Hence we get

$$g\left(\boldsymbol{s}_i\right) g\left((\boldsymbol{i, n})\right) g\left(\boldsymbol{s}_i\right) = g\left((\boldsymbol{i, n})\right) g\left(\boldsymbol{s}_i\right) g\left((\boldsymbol{i, n})\right).$$

It is clear that $g\left((\boldsymbol{i, n})\right)$ commutes with $\boldsymbol{s}_2$, $\boldsymbol{s}_3$, …, $\boldsymbol{s}_{i-1}$.

Let us show that $g\left((\boldsymbol{i, n})\right)$ commutes with $\boldsymbol{s}_m$. We have seen that $g\left((\boldsymbol{i, n})\right) = \boldsymbol{s_n^{-1}}g\left((\boldsymbol{i, n-1})\right)\boldsymbol{s_n}$. Then we get

$$\begin{aligned}
g\left((\boldsymbol{i, n})\right)\boldsymbol{s}_m =& \boldsymbol{s}_n^{-1}g\left((\boldsymbol{i, n-1})\right)\underline{\boldsymbol{s}_n\boldsymbol{s}_m}\\
=& \boldsymbol{s}_n^{-1}g\left((\boldsymbol{i, n-1})\right)\boldsymbol{s}_m\boldsymbol{s}_n\\
=& \underline{\boldsymbol{s}_n^{-1}\boldsymbol{s}_m}g\left((\boldsymbol{i, n-1})\right)\boldsymbol{s}_n \quad \text{(by induction hypothesis)}\\
=& \boldsymbol{s}_m\boldsymbol{s}_n^{-1}g\left((\boldsymbol{i, n-1})\right)\boldsymbol{s}_n\\
=& \boldsymbol{s}_m g\left((\boldsymbol{i, n})\right).
\end{aligned}$$

Similarly $g\left((\boldsymbol{i, n})\right)$ commutes with $\boldsymbol{s}_{m-1}$, $\boldsymbol{s}_{m-2}$, …, $\boldsymbol{s}_{i+2}$, $\boldsymbol{s}_{n-2}$, $\boldsymbol{s_{n-3}}$, …, $\boldsymbol{s}_{m+2}$, and $g\left(\overline{(\boldsymbol{1}, \boldsymbol{m})}\right)$. It is done by just replacing $\boldsymbol{s}_m$ by each of the previous elements.

Let us now show that $g\left((\boldsymbol{i},\boldsymbol{n})\right)$ commutes with $\boldsymbol{s}_{n-1}$. We have that

$$\begin{aligned}
g\left((\boldsymbol{i},\boldsymbol{n})\right)\boldsymbol{s}_{n-1} =&(\boldsymbol{s}_{i+1}\dots\boldsymbol{s}_{n-2}\boldsymbol{s}_{n-1})\boldsymbol{s}_n(\boldsymbol{s}_{n-1}^{-1}\boldsymbol{s}_{n-2}^{-1}\dots\boldsymbol{s}_{i+1}^{-1})\underline{\boldsymbol{s}_{n-1}}\\
=&(\boldsymbol{s}_{i+1}\dots\boldsymbol{s}_{n-2}\boldsymbol{s}_{n-1})\boldsymbol{s}_n(\underline{\boldsymbol{s}_{n-1}^{-1}\boldsymbol{s}_{n-2}^{-1}\boldsymbol{s}_{n-1}}\boldsymbol{s}_{n-3}^{-1}\dots\boldsymbol{s}_{i+1}^{-1})\\
=&(\boldsymbol{s}_{i+1}\dots\boldsymbol{s}_{n-2}\boldsymbol{s}_{n-1})\boldsymbol{s}_n(\underline{\boldsymbol{s}_{n-2}}\boldsymbol{s}_{n-1}^{-1}\boldsymbol{s}_{n-2}^{-1}\dots\boldsymbol{s}_{i+1}^{-1})\\
=&(\boldsymbol{s}_{i+1}\dots\underline{\boldsymbol{s}_{n-2}\boldsymbol{s}_{n-1}\boldsymbol{s}_{n-2}})\boldsymbol{s}_n(\boldsymbol{s}_{n-1}^{-1}\boldsymbol{s}_{n-2}^{-1}\dots\boldsymbol{s}_{i+1}^{-1})\\
=&(\boldsymbol{s}_{i+1}\dots\boldsymbol{s}_{n-3}\underline{\boldsymbol{s}_{n-1}}\boldsymbol{s}_{n-2}\boldsymbol{s}_{n-1})\boldsymbol{s}_n(\boldsymbol{s}_{n-1}^{-1}\boldsymbol{s}_{n-2}^{-1}\dots\boldsymbol{s}_{i+1}^{-1})\\
=&\boldsymbol{s}_{n-1}g\left((\boldsymbol{i},\boldsymbol{n})\right).
\end{aligned}$$

Finally, we show that $g\left((\boldsymbol{i},\boldsymbol{n})\right)$ commutes with $g\left((\boldsymbol{i+1},\boldsymbol{n-1})\right)$. Since

$$g\left((\boldsymbol{i+1},\boldsymbol{n-1})\right)=\boldsymbol{s}_{n-1}^{\boldsymbol{s}_{n-2}^{-1}\boldsymbol{s}_{n-3}^{-1}\dots\boldsymbol{s}_{i+2}^{-1}},$$

we get $g\left((\boldsymbol{i+1},\boldsymbol{n-1})\right)=\boldsymbol{s}_{n-1}^{-1}g\left((\boldsymbol{i+1},\boldsymbol{n-2})\right)\boldsymbol{s}_{n-1}$. We obtain

$$\begin{aligned}
g\left((\boldsymbol{i+1},\boldsymbol{n-1})\right)g\left((\boldsymbol{i},\boldsymbol{n})\right) =&\boldsymbol{s}_{n-1}^{-1}g\left((\boldsymbol{i+1},\boldsymbol{n-2})\right)\underline{\boldsymbol{s}_{n-1}g\left((\boldsymbol{i},\boldsymbol{n})\right)}\\
=&\boldsymbol{s}_{n-1}^{-1}g\left((\boldsymbol{i+1},\boldsymbol{n-2})\right)\underline{g\left((\boldsymbol{i},\boldsymbol{n})\right)}\boldsymbol{s}_{n-1}\\
&\text{(by the previous case)}\\
=&\boldsymbol{s}_{n-1}^{-1}\underline{g\left((\boldsymbol{i+1},\boldsymbol{n-2})\right)\boldsymbol{s}_n^{-1}}g\left((\boldsymbol{i},\boldsymbol{n-1})\right)\boldsymbol{s}_n\boldsymbol{s}_{n-1}\\
=&\boldsymbol{s}_{n-1}^{-1}\boldsymbol{s}_n^{-1}\underline{g\left((\boldsymbol{i+1},\boldsymbol{n-2})\right)g\left((\boldsymbol{i},\boldsymbol{n-1})\right)}\boldsymbol{s}_n\boldsymbol{s}_{n-1}\\
=&\boldsymbol{s}_{\boldsymbol{n-1}}^{\boldsymbol{-1}}\boldsymbol{s}_{\boldsymbol{n}}^{\boldsymbol{-1}}g\left((\boldsymbol{i},\boldsymbol{n-1})\right)\underline{g\left((\boldsymbol{i+1},\boldsymbol{n-2})\right)\boldsymbol{s}_n}\boldsymbol{s}_{n-1}\\
&\text{(by induction hypothesis)}\\
=&\boldsymbol{s}_{n-1}^{-1}\underline{\boldsymbol{s}_n^{-1}g\left((\boldsymbol{i},\boldsymbol{n-1})\right)\boldsymbol{s}_n}g\left((\boldsymbol{i+1},\boldsymbol{n-2})\right)\boldsymbol{s}_{n-1}\\
=&\underline{\boldsymbol{s}_{n-1}^{-1}g\left((\boldsymbol{i},\boldsymbol{n})\right)}g\left((\boldsymbol{i+1},\boldsymbol{n-2})\right)\boldsymbol{s}_{n-1}\\
=&g\left((\boldsymbol{i},\boldsymbol{n})\right)\underline{\boldsymbol{s}_{n-1}^{-1}g\left((\boldsymbol{i+1},\boldsymbol{n-2})\right)\boldsymbol{s}_{n-1}}\\
=&g\left((\boldsymbol{i},\boldsymbol{n})\right)g\left((\boldsymbol{i+1},\boldsymbol{n-1})\right).
\end{aligned}$$

(5) Where $i=m$ and $j=n-1$, we need to check that $g\left(\boldsymbol{s}_m\right)g\left((\boldsymbol{m},\boldsymbol{n})\right)g\left(\boldsymbol{s}_m\right)=g\left((\boldsymbol{m},\boldsymbol{n})\right)g\left(\boldsymbol{s}_m\right)g\left((\boldsymbol{m},\boldsymbol{n})\right)$ and that $g\left((\boldsymbol{m},\boldsymbol{n})\right)$ commutes with all the images by g of the elements of $\boldsymbol{T}$ that correspond to the reflections in the string from item (5) in Lemma 4.6:

$$s_{m+2}\dots s_{n-2}s_{n-1}(\overline{1},\overline{n-1})s_2\dots s_{m-1}.$$

This is shown by following the same arguments as in (4).

(6) Where $i\neq m$ and $j=n$, we need to check that $\boldsymbol{s}_n$ commutes with $g\left((\overline{\boldsymbol{1}},\overline{\boldsymbol{m}})\right)$ and $g\left((\overline{\boldsymbol{i}},\overline{\boldsymbol{m+1}})\right)$ This is straightforward to show.

(7) Where $i=m$ and $j=n$, the image by g of the copies of the reflections in the decomposition in item (7) of Lemma 4.6 are only elements of $\boldsymbol{\mathcal{S}}$, so there is nothing to check. □

Lemma A.2. *Suppose that $w_0=w(\overline{i},\overline{j})$, with $1\leqslant i<m$, $m+1\leqslant j<n$, so that we are in the situation of Equation* (2)*. Let $t_1t_2\dots t_{n-1}$ be the reduced decomposition of w_0 described in Lemma* 4.7*. Then we can deduce from the relations of the presentation of $\boldsymbol{G}$ given in Theorem* 5.1 *that $g\left(\boldsymbol{t}_{n-1}\right)g\left(\boldsymbol{t}_{n-2}\right)g\left(\boldsymbol{t}_{n-1}\right)=g\left(\boldsymbol{t}_{n-2}\right)g\left(\boldsymbol{t}_{n-1}\right)g\left(\boldsymbol{t}_{n-2}\right)$, and that $g\left(\boldsymbol{t}_{n-1}\right)$ commutes with each of the elements $g\left(\boldsymbol{t}_k\right)$ with $k<n-2$.*

Proof. We consider the three different possible decompositions of w_0 that are described in Lemma 4.7. In each case we do not need to consider the relations between $g(\boldsymbol{t}_n)$ and $g(\boldsymbol{t}_k)$ if both elements are within the set $\boldsymbol{\mathcal{S}}$.

(1) Where $i \neq m$ and $j \neq n-2, n-1, n$, we need to check that $g(\boldsymbol{s}_n)$ commutes with $g\left(\overline{(\boldsymbol{1}, \boldsymbol{m})}\right)$, $g\left(\overline{(\boldsymbol{i}, \boldsymbol{j+1})}\right)$, and $g\left(\overline{(\boldsymbol{i+1}, \boldsymbol{j})}\right)$.

The fact that $g(\boldsymbol{s}_n) = \boldsymbol{s}_n$ commutes with $g\left(\overline{(\boldsymbol{1}, \boldsymbol{m})}\right)$ is done in Lemma A.1(1). We have that $g\left(\overline{(\boldsymbol{i}, \boldsymbol{j+1})}\right) = \boldsymbol{s}_1^{\boldsymbol{s}_{m+2}\boldsymbol{s}_{m+3}\cdots\boldsymbol{s}_{j+1}\boldsymbol{s}_m^{-1}\boldsymbol{s}_{m-1}^{-1}\cdots\boldsymbol{s}_{i+1}^{-1}}$, and $g\left(\overline{(\boldsymbol{i+1}, \boldsymbol{j})}\right) = \boldsymbol{s}_1^{\boldsymbol{s}_{m+2}\boldsymbol{s}_{m+3}\cdots\boldsymbol{s}_j\boldsymbol{s}_m^{-1}\boldsymbol{s}_{m-1}^{-1}\cdots\boldsymbol{s}_{i+2}^{-1}}$ that both obviously commute with $\boldsymbol{s}_n$.

(2) Where $i \neq m$ and $j = n-2$, we need to check that $g(\boldsymbol{s}_n) = \boldsymbol{s}_n$ commutes with $g\left(\overline{(\boldsymbol{1}, \boldsymbol{m})}\right)$, $g\left(\overline{(\boldsymbol{i+1}, \boldsymbol{n-2})}\right)$, and check that $g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-1})}\right) g(\boldsymbol{s}_n)\, g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-1})}\right) = g(\boldsymbol{s}_n)\, g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-1})}\right) g(\boldsymbol{s}_n)$.

The fact that $\boldsymbol{s}_n$ commutes with $g\left(\overline{(\boldsymbol{1}, \boldsymbol{m})}\right)$ and $g\left(\overline{(\boldsymbol{i+1}, \boldsymbol{n-2})}\right)$ is done in Lemma A.2(1). For the last relation, we have that

$$\begin{aligned}
g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-1})}\right) \boldsymbol{s}_n g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-1})}\right) &= \boldsymbol{s}_{n-1}^{-1} g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-2})}\right) \underline{\boldsymbol{s}_{n-1}\boldsymbol{s}_n\boldsymbol{s}_{n-1}^{-1}} g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-2})}\right) \boldsymbol{s}_{n-1} \\
&= \boldsymbol{s}_{n-1}^{-1} \underline{g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-2})}\right) \boldsymbol{s}_n^{-1}} \boldsymbol{s}_{n-1} \underline{\boldsymbol{s}_n g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-2})}\right)} \boldsymbol{s}_{n-1} \\
&= \boldsymbol{s}_{n-1}^{-1}\boldsymbol{s}_n^{-1} g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-2})}\right) \boldsymbol{s}_{n-1} \underline{g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-2})}\right) \boldsymbol{s}_n} \boldsymbol{s}_{n-1} \\
&= \underline{\boldsymbol{s}_{n-1}^{-1}\boldsymbol{s}_n^{-1}\boldsymbol{s}_{n-1}} g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-2})}\right) \underline{\boldsymbol{s}_{n-1}\boldsymbol{s}_n\boldsymbol{s}_{n-1}} \\
&\quad \text{(by induction hypothesis)} \\
&= \boldsymbol{s}_n\boldsymbol{s}_{n-1}^{-1} \underline{\boldsymbol{s}_n^{-1} g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-2})}\right)} \boldsymbol{s}_n\boldsymbol{s}_{n-1}\boldsymbol{s}_n \\
&= \boldsymbol{s}_n\boldsymbol{s}_{n-1}^{-1} g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-2})}\right) \boldsymbol{s}_n^{-1} \boldsymbol{s}_n\boldsymbol{s}_{n-1}\boldsymbol{s}_n \\
&= \boldsymbol{s}_n g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-1})}\right) \boldsymbol{s}_n.
\end{aligned}$$

(3) Where $i \neq m$ and $j = n-1$, we need to check that $\boldsymbol{s}_i g\left(\overline{(\boldsymbol{i}, \boldsymbol{n})}\right) \boldsymbol{s}_i = g\left(\overline{(\boldsymbol{i}, \boldsymbol{n})}\right) \boldsymbol{s}_i g\left(\overline{(\boldsymbol{i}, \boldsymbol{n})}\right)$ and that $g\left(\overline{(\boldsymbol{i}, \boldsymbol{n})}\right)$ commutes with the images by g of elements that correspond to the reflections in the string from item (3) in Lemma 4.7:

$$s_{m+2} \dots s_{n-2} s_{n-1} (\overline{i+1}, \overline{n-1}) s_{i+2} \dots s_{m-1} s_m (\overline{1}, \overline{m}) s_2 \dots s_{i-1}.$$

First, we have

$$\begin{aligned}
\boldsymbol{s}_i g\left(\overline{(\boldsymbol{i}, \boldsymbol{n})}\right) \boldsymbol{s}_i &= \underline{\boldsymbol{s}_i\boldsymbol{s}_n^{-1}} g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-1})}\right) \underline{\boldsymbol{s}_n\boldsymbol{s}_i} \\
&= \boldsymbol{s}_n^{-1} \underline{\boldsymbol{s}_i g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-1})}\right) \boldsymbol{s}_i} \boldsymbol{s}_n \\
&= \boldsymbol{s}_n^{-1} g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-1})}\right) \boldsymbol{s}_i g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-1})}\right) \boldsymbol{s}_n \ \text{(by induction hypothesis)} \\
&= \underline{\boldsymbol{s}_n^{-1} g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-1})}\right) \boldsymbol{s}_n} \boldsymbol{s}_i \underline{\boldsymbol{s}_n^{-1} g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-1})}\right) \boldsymbol{s}_n} \\
&= g\left(\overline{(\boldsymbol{i}, \boldsymbol{n})}\right) \boldsymbol{s}_i g\left(\overline{(\boldsymbol{i}, \boldsymbol{n})}\right).
\end{aligned}$$

Since $\boldsymbol{s}_2, \dots, \boldsymbol{s}_{i-2}, \boldsymbol{s}_{i-1}, \boldsymbol{s}_{i+2}, \dots, \boldsymbol{s}_{m-1}, \boldsymbol{s}_m, \boldsymbol{s}_{m+2}, \dots, \boldsymbol{s}_{n-3}, \boldsymbol{s}_{n-2}$ commute with $\boldsymbol{s}_n$, then they commute with $g\left(\overline{(\boldsymbol{i}, \boldsymbol{n})}\right) = \boldsymbol{s}_n^{-1} g\left(\overline{(\boldsymbol{i}, \boldsymbol{n-1})}\right) \boldsymbol{s}_n$ by applying the induction hypothesis. Because $g\left(\overline{(\boldsymbol{1}, \boldsymbol{m})}\right)$ commutes with $\boldsymbol{s}_n$, we also get that $g\left(\overline{(\boldsymbol{i}, \boldsymbol{n})}\right)$ commutes with $g\left(\overline{(\boldsymbol{1}, \boldsymbol{m})}\right)$ by applying the same argument. Now, we prove that $g\left(\overline{(\boldsymbol{i}, \boldsymbol{n})}\right)$

commutes with $\boldsymbol{s}_{n-1}$. Actually, we have

$$\begin{aligned}
g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n}}\right)\right)\boldsymbol{s}_{n-1} =& \boldsymbol{s}_n^{-1}\boldsymbol{s}_{n-1}^{-1} g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-2}}\right)\right)\underline{\boldsymbol{s}_{n-1}\boldsymbol{s}_n\boldsymbol{s}_{n-1}}\\
=& \boldsymbol{s}_n^{-1}\boldsymbol{s}_{n-1}^{-1}\underline{g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-2}}\right)\right)\boldsymbol{s}_n}\boldsymbol{s}_{n-1}\boldsymbol{s}_n\\
=& \underline{\boldsymbol{s}_n^{-1}\boldsymbol{s}_{n-1}^{-1}\boldsymbol{s}_n} g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-2}}\right)\right)\boldsymbol{s}_{n-1}\boldsymbol{s}_n\\
=& \boldsymbol{s}_{n-1}\boldsymbol{s}_n^{-1}\underline{\boldsymbol{s}_{n-1}^{-1} g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-2}}\right)\right)\boldsymbol{s}_{n-1}}\boldsymbol{s}_n\\
=& \boldsymbol{s}_{n-1}\underline{\boldsymbol{s}_n^{-1} g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-1}}\right)\right)\boldsymbol{s}_n}\\
=& \boldsymbol{s}_{n-1} g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n}}\right)\right).
\end{aligned}$$

Finally, we show that $g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n}}\right)\right)$ commutes with $g\left(\left(\overline{\boldsymbol{i+1}},\overline{\boldsymbol{n-1}}\right)\right)$. We have that $g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n}}\right)\right) g\left(\left(\overline{\boldsymbol{i+1}},\overline{\boldsymbol{n-1}}\right)\right)$ is equal to

$$\begin{aligned}
&\boldsymbol{s}_n^{-1} g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-1}}\right)\right)\boldsymbol{s}_n\boldsymbol{s}_{n-1}^{-1} g\left(\left(\overline{\boldsymbol{i+1}},\overline{\boldsymbol{n-2}}\right)\right)\boldsymbol{s}_{n-1} =\\
&\boldsymbol{s}_n^{-1}\boldsymbol{s}_{n-1}^{-1} g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-2}}\right)\right)\underline{\boldsymbol{s}_{n-1}\boldsymbol{s}_n\boldsymbol{s}_{n-1}^{-1}} g\left(\left(\overline{\boldsymbol{i+1}},\overline{\boldsymbol{n-2}}\right)\right)\boldsymbol{s}_{n-1} =\\
&\boldsymbol{s}_n^{-1}\boldsymbol{s}_{n-1}^{-1}\underline{g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-2}}\right)\right)\boldsymbol{s}_n^{-1}}\boldsymbol{s}_{n-1}\underline{\boldsymbol{s}_n g\left(\left(\overline{\boldsymbol{i+1}},\overline{\boldsymbol{n-2}}\right)\right)}\boldsymbol{s}_{n-1} =\\
&\underline{\boldsymbol{s}_n^{-1}\boldsymbol{s}_{n-1}^{-1}\boldsymbol{s}_n^{-1}} g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-2}}\right)\right)\boldsymbol{s}_{n-1} g\left(\left(\overline{\boldsymbol{i+1}},\overline{\boldsymbol{n-2}}\right)\right)\boldsymbol{s}_n\boldsymbol{s}_{n-1} =\\
&\boldsymbol{s}_{n-1}^{-1}\boldsymbol{s}_n^{-1}\underline{\boldsymbol{s}_{n-1}^{-1} g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-2}}\right)\right)\boldsymbol{s}_{n-1}} g\left(\left(\overline{\boldsymbol{i+1}},\overline{\boldsymbol{n-2}}\right)\right)\boldsymbol{s}_n\boldsymbol{s}_{n-1} =\\
&\boldsymbol{s}_{n-1}^{-1}\boldsymbol{s}_n^{-1}\underline{g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-1}}\right)\right) g\left(\left(\overline{\boldsymbol{i+1}},\overline{\boldsymbol{n-2}}\right)\right)}\boldsymbol{s}_n\boldsymbol{s}_{n-1} = \quad \text{(by induction hypothesis)}\\
&\boldsymbol{s}_{n-1}^{-1}\underline{\boldsymbol{s}_n^{-1} g\left(\left(\overline{\boldsymbol{i+1}},\overline{\boldsymbol{n-2}}\right)\right) g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-1}}\right)\right)}\boldsymbol{s}_n\boldsymbol{s}_{n-1} =\\
&\boldsymbol{s}_{n-1}^{-1} g\left(\left(\overline{\boldsymbol{i+1}},\overline{\boldsymbol{n-2}}\right)\right)\underline{\boldsymbol{s}_n^{-1} g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-1}}\right)\right)\boldsymbol{s}_n}\boldsymbol{s}_{n-1} =\\
&\boldsymbol{s}_{n-1}^{-1} g\left(\left(\overline{\boldsymbol{i+1}},\overline{\boldsymbol{n-2}}\right)\right)\underline{g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n}}\right)\right)\boldsymbol{s}_{n-1}} = \quad \text{(by the previous case)}\\
&\boldsymbol{s}_{n-1}^{-1} g\left(\left(\overline{\boldsymbol{i+1}},\overline{\boldsymbol{n-2}}\right)\right)\boldsymbol{s}_{n-1} g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n}}\right)\right) =\\
&g\left(\left(\overline{\boldsymbol{i+1}},\overline{\boldsymbol{n-1}}\right)\right) g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n}}\right)\right).
\end{aligned}$$

□

Lemma A.3. *Suppose that $w_0 = w(\overline{m},\bar{j})$, with $m+1 \leqslant j < n$, so that we are in the situation of Equation* (3)*. Let $t_1 t_2 \dots t_{n-1}$ be the reduced decomposition of w_0 described in Lemma* 4.8*. Then we can deduce from the relations of the presentation of $\boldsymbol{G}$ given in Theorem* 5.1 *that $g(\boldsymbol{t}_{n-1})\, g(\boldsymbol{t}_{n-2})\, g(\boldsymbol{t}_{n-1}) = g(\boldsymbol{t}_{n-2})\, g(\boldsymbol{t}_{n-1})\, g(\boldsymbol{t}_{n-2})$, and that $g(\boldsymbol{t}_{n-1})$ commutes with each of the elements $g(\boldsymbol{t}_k)$ with $k < n-2$.*

Proof. We consider the three cases of Lemma 4.8.

(1) Where $j < n-2$, we need to check that $\boldsymbol{s}_n$ commutes with $g\left(\left(\boldsymbol{1},\boldsymbol{j}\right)\right)$ and $g\left(\left(\overline{\boldsymbol{m}},\overline{\boldsymbol{j+1}}\right)\right)$. This is readily checked since

$$g\left(\left(\boldsymbol{1},\boldsymbol{j}\right)\right) = \boldsymbol{s}_j^{\boldsymbol{s}_{j-1}^{-1}\boldsymbol{s}_{j-2}^{-1}\cdots\boldsymbol{s}_2^{-1}} \text{ and } g\left(\left(\overline{\boldsymbol{m}},\overline{\boldsymbol{j+1}}\right)\right) = \boldsymbol{s}_1^{\boldsymbol{s}_{m+2}\boldsymbol{s}_{m+3}\cdots\boldsymbol{s}_{j+1}}.$$

(2) Where $j = n-2$, we need to check that $\boldsymbol{s}_n$ commutes with $g\left(\left(\boldsymbol{1},\boldsymbol{n-2}\right)\right)$ and $g\left(\left(\overline{\boldsymbol{m}},\overline{\boldsymbol{n-1}}\right)\right)$. The first check is straightforward. For the second, we have that $g\left(\left(\overline{\boldsymbol{m}},\overline{\boldsymbol{n-1}}\right)\right) = \boldsymbol{s}_{n-1}^{-1} g\left(\left(\overline{\boldsymbol{m}},\overline{\boldsymbol{n-2}}\right)\right)\boldsymbol{s}_{n-1}$ and one shows that

$$g\left(\left(\overline{\boldsymbol{m}},\overline{\boldsymbol{n-1}}\right)\right)\boldsymbol{s}_n g\left(\left(\overline{\boldsymbol{m}},\overline{\boldsymbol{n-1}}\right)\right) = \boldsymbol{s}_n g\left(\left(\overline{\boldsymbol{m}},\overline{\boldsymbol{n-1}}\right)\right)\boldsymbol{s}_n$$

similarly to the case $g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-1}}\right)\right)\boldsymbol{s}_n g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-1}}\right)\right) = \boldsymbol{s}_n g\left(\left(\bar{\boldsymbol{i}},\overline{\boldsymbol{n-1}}\right)\right)\boldsymbol{s}_n$ in the proof of Lemma A.2(1).

(3) Where $j = n-1$, we have $g\left((\overline{\boldsymbol{m}}, \overline{\boldsymbol{n}})\right) = \boldsymbol{s}_1^{\boldsymbol{s}_{m+2}\boldsymbol{s}_{m+3}\cdots\boldsymbol{s}_n} = \boldsymbol{s}_n^{-1} g\left((\overline{\boldsymbol{m}}, \overline{\boldsymbol{n-1}})\right)\boldsymbol{s}_n$. All the commuting relations between $g\left((\overline{\boldsymbol{m}}, \overline{\boldsymbol{n}})\right)$ and $\boldsymbol{s}_{m-1}, \boldsymbol{s}_{m-2}, \ldots, \boldsymbol{s}_2$ are obvious. We are left to prove that $g\left((\overline{\boldsymbol{m}}, \overline{\boldsymbol{n}})\right)\boldsymbol{s}_m g\left((\overline{\boldsymbol{m}}, \overline{\boldsymbol{n}})\right) = \boldsymbol{s}_m g\left((\overline{\boldsymbol{m}}, \overline{\boldsymbol{n}})\right)\boldsymbol{s}_m$, and $g\left((\overline{\boldsymbol{m}}, \overline{\boldsymbol{n}})\right)$ commutes with $g\left((\boldsymbol{1}, \boldsymbol{n-1})\right)$, $\boldsymbol{s}_{n-1}, \boldsymbol{s}_{n-2}, \ldots, \boldsymbol{s}_{m+2}$. This is done similarly to the proof of Lemma A.2(3). □

The next lemma is readily checked.

LEMMA A.4. *Suppose that $w_0 = w(\overline{i}, \overline{n})$, with $1 \leqslant i \leqslant m$, so that we are in the situation of Equation* (4) *or Equation* (5)*. An identical result to the previous lemmas holds in this situation.*

This finishes the situation where w_0 is of type I. The next two lemmas are for types II and III, respectively.

LEMMA A.5. *Suppose that $w_0 = w(i, j)$ or $w_0 = w(\overline{i}, \overline{j})$ with $1 \leqslant i < j \leqslant m$, so that we are in the situation of one of Equations* (6)-(8)*. Let $t_1 t_2 \ldots t_{n-1}$ be the reduced decomposition of w_0 as described in Section* 4.3*. Then we can deduce from the relations of the presentation of $\boldsymbol{G}$ that $g\left(\boldsymbol{t}_{n-1}\right) g\left(\boldsymbol{t}_{n-2}\right) g\left(\boldsymbol{t}_{n-1}\right) = g\left(\boldsymbol{t}_{n-2}\right) g\left(\boldsymbol{t}_{n-1}\right) g\left(\boldsymbol{t}_{n-2}\right)$, and that $g\left(\boldsymbol{t}_{n-1}\right)$ commutes with each of the elements $g\left(\boldsymbol{t}_k\right)$ with $k < n-2$.*

Proof. In the case of this lemma, we have that $g\left(\boldsymbol{t}_{n-1}\right) = \boldsymbol{s}_n$ and $g\left(\boldsymbol{t}_{n-2}\right) = \boldsymbol{s}_{n-1}$, so the braid relation $\boldsymbol{s}_n\boldsymbol{s}_{n-1}\boldsymbol{s}_n = \boldsymbol{s}_{n-1}\boldsymbol{s}_n\boldsymbol{s}_{n-1}$ is clearly a consequence of the relations of $\boldsymbol{G}$. The commuting relations are also obvious since in the situation of Equations (6)–(8), the indices i, j are such that $1 \leqslant i < j \leqslant m$, so that they are far away from n (we have $i, j < n-2$). Hence $\boldsymbol{s}_n$ obviously commutes with the image by g of the elements $\boldsymbol{t}_i$ corresponding to the reflections in the reduced decomposition of w_0 described in Section 4.3. □

LEMMA A.6. *Suppose that $w_0 = w(i, j)$ or $w_0 = w(\overline{i}, \overline{j})$ with $m+1 \leqslant i < j \leqslant n$, so that we are in the situation of one of Equations* (9)–(11)*. Let $t_1 t_2 \ldots t_{n-1}$ be the reduced decomposition of w_0 as described in Section* 4.3*. Then we can deduce from the relations of the presentation of $\boldsymbol{G}$ that $g\left(\boldsymbol{t}_{n-1}\right) g\left(\boldsymbol{t}_{n-2}\right) g\left(\boldsymbol{t}_{n-1}\right) = g\left(\boldsymbol{t}_{n-2}\right) g\left(\boldsymbol{t}_{n-1}\right) g\left(\boldsymbol{t}_{n-2}\right)$, and that $g\left(\boldsymbol{t}_{n-1}\right)$ commutes with each of the elements $g\left(\boldsymbol{t}_k\right)$ with $k < n-2$, except for Equation* (11) *(with $i = n-1$) where we need to show one additional non-commuting relation.*

Proof. The argument of the proof is identical to the situation of Equations (1)–(5) treated in Lemmas A.1–A.4, which is appropriate to leave as an exercise. □

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