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Blow-up time of solutions to a class of pseudo-parabolic equations

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Abstract. In this paper, we study the Dirichlet problem for a semilinear pseudo-parabolic equation. By using the energy estimates and ordinary differential inequalities, we studied the upper and lower bounds of blow-up time of the solutions. The results of this paper extend and complete the results on this model.

Keywords. Pseudo-parabolic equation, Blow-up, Blow-up time, Potential well method, Global existence.

Mathematical subject classification (2010). 35K70, 35A01.

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1. Introduction

The pseudo-parabolic equation

$$u_t - a\Delta u_t - \Delta u = f(u) \tag{1}$$

with $a > 0$ can be used to describe many interesting physical and biological phenomena [1–6], for example, the non-stationary process in semiconductors in the presence of sources, where $a\Delta u_t - u_t$ stands for the free electron density rate, Δu stands for the linear dissipation of free charge current and the nonlinear term stands for the source of free electron current (see [3, 5]). Showalter and Ting [7] investigated the initial boundary value problem of (1) with $f(u) = 0$, and the global existence, uniqueness and regularity of solutions were studied. When $f(u)$ is a polynomial, i.e., $f(u) = u^{p-1}$ or $f(u) = |u|^{p-2}u$, the initial and boundary value problem of (1) was studied in [8–13], and the existence, asymptotic behavior of the global solutions and global nonexistence of solutions were studied. When $f(u)$ is a logarithmic function, i.e., $f(u) = u\ln|u|$, the initial boundary value problem of (1) was studied in [14], where global existence, infinite-time blow-up of solutions, and behavior of vacuum isolation of solutions were studied. When $f(u)$ is a nonlocal function, i.e., $f(u) = |u|^{p-2}u - \frac{1}{|\Omega|}\int_\Omega |u|^{p-2}u\,dx$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain, problem (1) with homogeneous Neumann boundary and initial value was studied [15],

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and the existence, uniqueness and asymptotic behavior of the global solution and the blow-up phenomena of solution were studied. For the study of global existence and finite time blow-up of solutions for pseudo-parabolic equation with potential terms, we refer the readers to [16, 17].

In this paper, we consider the following initial boundary-value problem (IBVP) for a class of pseudo-parabolic equation proposed by Zhu el al. in [18]:

$$\begin{cases} u_t - \Delta u_t - \Delta u + u = |u|^{p-2}u, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{2}$$

where $p \in (2, 2^*)$ and either $\Omega = \mathbb{R}^N$ $(N \geq 3)$ or $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $u_0 \in H_0^1(\Omega)$. Here

$$2^* = \frac{2N}{N-2}.$$

When $\Omega = \mathbb{R}^N$, the boundary condition that $u(x,t) = 0$ for $x \in \partial\Omega$ and $t > 0$ is ineffective.

Problem (2) was studied by Zhu el al. in [18] via the potential well method (see, for example, [19–21]), where the existence, uniqueness, polynomial decay of the global solutions, and blow-up of solutions were studied. Zhou [22] improve the results of [18] by showing the global solutions decay exponentially. *The main purpose of this paper is to complete the blow-up result given in [18] by estimating the upper and lower bounds of the blow-up time.* To introduce the previous results and give the main results of the present paper, let's firstly recall some notations used in [18]. Throughout this paper, we use $\|\cdot\|_p$ as the usual L^p-norm, $(\cdot,\cdot)$ and $\|\cdot\|$ as the inner product and the associated norm on $H_0^1(\Omega)$ respectively, that is

$$(u, v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\, dx \tag{3}$$

and

$$\|u\| = \sqrt{(u,u)}. \tag{4}$$

By [18, Theorem 1.3], when $p \in (2, 2^*)$ and $u_0 \in H_0^1(\Omega)$, IVBP (2) admits a (weak) solution

$$u \in C^1\left([0, T_{\max}), H_0^1(\Omega)\right), \tag{5}$$

which satisfies $u(0) = u_0 \in H_0^1(\Omega)$ and

$$\left(u'(t), v\right) + \left(u(t), v\right) = \int_\Omega |u(t)|^{p-2} u(t) v\, dx,\ \ \forall\ v \in H_0^1(\Omega) \tag{6}$$

where $T_{\max}$ is the maximum existence time.

As in [18], the energy functional $J(u)$ can be defined through

$$J(u) := \frac{p-2}{2p}\|u\|^2 + \frac{1}{p} I(u),\ \ \forall\ u \in H_0^1(\Omega), \tag{7}$$

where

$$I(u) := \|u\|^2 - \|u\|_p^p,\ \ \forall\ u \in H_0^1(\Omega). \tag{8}$$

By (5), $J(u(t))$ and $I(u(t))$ are well-defined and are continuous differentiable with respect to t.

Let

$$\mathcal{N} := \left\{u \in H_0^1(\Omega) \setminus \{0\} : I(u) = 0\right\} \tag{9}$$

denote the Nehari manifold. We also define two sets $\mathcal{N}_\pm$ related to $\mathcal{N}$ as follows:

$$\begin{aligned} \mathcal{N}_+ &:= \left\{u \in H_0^1(\Omega) : I(u) > 0\right\}, \\ \mathcal{N}_- &:= \left\{u \in H_0^1(\Omega) : I(u) < 0\right\}. \end{aligned} \tag{10}$$

The main result on global existence and blow-up of [18] is the following theorem (see [18, Theorem 1.4]):

Theorem 1. *Let $u = u(t)$ be a solution of IBVP (2), whose maximal existence time is $T_{\max}$. Then $T_{\max} < \infty$ if and only if there exists a $t_0 \in [0, T_{\max})$ such that $u(t_0) \in \mathcal{N}_-$. Moreover, if $T_{\max} < \infty$, then $\lim_{t\uparrow T_{\max}} \|u(t)\| = \infty$.*

When blow-up occurs, the blow-up time $T_{\max}$ cannot usually be computed exactly. It is therefore of great importance in practice to determine the upper and lower bounds for $T_{\max}$. *So we complete Theorem 1 by showing the upper and lower bounds of $T_{\max}$:*

Theorem 2. *Let $u = u(t)$ be a solution of IBVP (2) with initial value $u_0 \in \mathcal{N}_-$, i.e., $I(u_0) = \|u_0\|^2 - \|u_0\|_p^p < 0$. Then*

$$\frac{1}{2}\int_{\|u_0\|^2}^{\infty} \frac{d\xi}{\kappa^p \xi^{\frac{p}{2}} - \xi} \le T_{\max} \le \frac{4(p-1)\|u_0\|^2}{(p-2)^2\left(\|u_0\|_p^p - \|u_0\|^2\right)}, \tag{11}$$

where κ is the optimal constant of the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, i.e.,

$$\kappa = \sup_{\substack{u\in H_0^1(\Omega)\\ \|u\|=1}} \|u\|_p. \tag{12}$$

Remark 3.

(1) Since $p \in (2, 2^*)$, it follows $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$;

(2) We show that

$$\frac{1}{2}\int_{\|u_0\|^2}^{\infty} \frac{d\xi}{\kappa^p \xi^{\frac{p}{2}} - \xi} < \frac{4(p-1)\|u_0\|^2}{(p-2)^2\left(\|u_0\|_p^p - \|u_0\|^2\right)}. \tag{13}$$

So, (11) makes sense. By using $I(u_0) < 0$, it follows from (8) and (12) that

$$\|u_0\|^2 < \|u_0\|_p^p \le \kappa^p \|u_0\|^p,$$

which implies

$$\|u_0\| > \kappa^{\frac{-p}{p-2}}. \tag{14}$$

Let $f(\xi) = \sigma\kappa^p \xi^{\frac{p}{2}} - \xi$ for $\xi \ge 0$, where

$$\sigma = \kappa^{-p}\|u_0\|^{2-p}.$$

Since $p > 2$, by (14),

$$\sigma < \kappa^{-p}\left(\kappa^{\frac{-p}{p-2}}\right)^{2-p} = 1.$$

Then, for $\xi \in [\|u_0\|^2, \infty)$,

$$\begin{aligned} f'(\xi) &= \frac{p}{2}\sigma\kappa^p \xi^{\frac{p}{2}-1} - 1 = \frac{p}{2}\|u_0\|^{2-p}\xi^{\frac{p}{2}-1} - 1 \\ &\ge \frac{p}{2}\|u_0\|^{2-p}\left(\|u_0\|^2\right)^{\frac{p}{2}-1} - 1 \\ &= \frac{p}{2} - 1 > 0. \end{aligned} \tag{15}$$

So,

$$f(\xi) \ge f\left(\|u_0\|^2\right) = 0, \; \xi \in [\|u_0\|^2, \infty).$$

Then, note $\sigma \in (0,1)$, we can estimate the integral to the left of (13) as

$$\begin{aligned}\frac{1}{2}\int_{\|u_0\|^2}^{\infty}\frac{d\xi}{\kappa^p\xi^{\frac{p}{2}}-\xi} &= \frac{1}{2}\int_{\|u_0\|^2}^{\infty}\frac{d\xi}{(1-\sigma)\kappa^p\xi^{\frac{p}{2}}+f(\xi)}\\ &\le \frac{1}{2(1-\sigma)\kappa^p}\int_{\|u_0\|^2}^{\infty}\frac{d\xi}{\xi^{\frac{p}{2}}} = \frac{1}{(1-\sigma)\kappa^p(p-2)}\|u_0\|^{2-p}\\ &= \frac{1}{\left(1-\kappa^{-p}\|u_0\|^{2-p}\right)\kappa^p(p-2)}\|u_0\|^{2-p}\\ &= \frac{1}{(p-2)\left(\kappa^p\|u_0\|^{p-2}-1\right)}.\end{aligned} \tag{16}$$

By (12), we can estimate the right of (13) as

$$\begin{aligned}\frac{4(p-1)\|u_0\|^2}{(p-2)^2\left(\|u_0\|_p^p-\|u_0\|^2\right)} &\ge \frac{4(p-1)\|u_0\|^2}{(p-2)^2\left(\kappa^p\|u_0\|^p-\|u_0\|^2\right)}\\ &= \frac{4(p-1)}{(p-2)^2\left(\kappa^p\|u_0\|^{p-2}-1\right)}.\end{aligned} \tag{17}$$

Since $4(p-1)/(p-2) > 4$, (13) follows from (16) and (17).

(3) It obvious that

$$\int_{\|u_0\|^2}^{\infty}\frac{d\xi}{\kappa^p\xi^{\frac{p}{2}}-\xi} \ge \int_{\|u_0\|^2}^{\infty}\frac{d\xi}{\kappa^p\xi^{\frac{p}{2}}} = \frac{2}{\kappa^p(p-2)}\|u_0\|^{2-p}.$$

Then it follows from (11) that

$$\frac{1}{\kappa^p(p-2)}\|u_0\|^{2-p} \le T_{\max} \le \frac{4(p-1)\|u_0\|^2}{(p-2)^2\left(\|u_0\|_p^p-\|u_0\|^2\right)}.$$

2. Proof of Theorem 2

In this section, we give the proof of Theorem 2 by using the following lemma:

Lemma 4 ([23]). *Suppose that* $0 < T \le +\infty$ *and suppose a nonnegative function* $F(t) \in C^2[0,T)$ *satisfies*

$$F''(t)F(t) - (1+\gamma)\left(F'(t)\right)^2 \ge 0$$

for some constant $\gamma > 0$*. If* $F(0) > 0$, $F'(0) > 0$*, then*

$$T \le \frac{F(0)}{\gamma F'(0)} < \infty$$

and $F(t) \to +\infty$ *as* $t \uparrow T$.

Proof of Theorem 2. Let $u = u(t)$ be a solution of problem (2) with initial value $u_0 \in \mathcal{N}_-$. By Theorem 1, the maximal existence $T_{\max} < \infty$. So we only need to prove the inequality (11).

(1) **Upper bound estimate**. In the part, we will give the upper bound estimate, i.e., show that

$$T_{\max} \le \frac{4(p-1)\|u_0\|^2}{(p-2)^2\left(\|u_0\|_p^p-\|u_0\|^2\right)}, \tag{18}$$

For $t \in [0, T_{\max})$, let

$$\xi(t) := \left(\int_0^t \|u(s)\|^2 ds \right)^{\frac{1}{2}},$$

$$\eta(t) := \left(\int_0^t \|u'(s)\|^2 ds \right)^{\frac{1}{2}}.$$

The following facts can be found in [18, Lemma 3.2 and the proof of Theorem 5.1]:

(R_1) $J(u(t)) + \eta^2(t) = J(u_0),\ t \in [0, T_{\max})$;
(R_2) $\frac{d}{dt}\|u(t)\|^2 = -2I(u(t)),\ t \in [0, T_{\max})$;
(R_3) $u(t) \in \mathcal{N}_-$, i.e., $I(u(t)) < 0,\ t \in [0, T_{\max})$.

By (R_1)-(R_3), it is easy to see

(R_4) $\|u(t)\|^2$ is strictly increasing with respect to t;
(R_5) the function $\phi(t) := (p-2)\|u(t)\|^2 - 2p(J(u(t)))$ satisfies

$$\begin{aligned} \phi(t) &\ge (p-2)\|u_0\|^2 - 2p\left(J(u_0) - \eta^2(t)\right) \\ &= \underbrace{2\left(\|u_0\|_p^p - \|u_0\|^2\right) + 2p\eta^2(t),}_{\text{by (7) and (8)}} \quad t \in [0, T_{\max}). \end{aligned}$$

Consider the following functional

$$F(t) := \xi^2(t) + (T_{\max} - t)\|u_0\|^2 + \beta(t+\alpha)^2,\ t \in [0, T_{\max}), \tag{19}$$

where α and β are two positive constants to be determined later. Then by (R_4), we have

$$\begin{aligned} F'(t) &= \|u(t)\|^2 - \|u_0\|^2 + 2\beta(t+\alpha) \\ &\ge 2\beta(t+\alpha) > 0,\ t \in [0, T_{\max}), \end{aligned} \tag{20}$$

which implies

$$F(t) \ge F(0) = T_{\max}\|u_0\|^2 + \beta\alpha^2 > 0,\ t \in [0, T_{\max}) \tag{21}$$

and (by (R_2), (7) and (R_5))

$$\begin{aligned} F''(t) &= -2I((t)) + 2\beta \\ &= \phi(t) + 2\beta \\ &\ge 2\left(\|u_0\|_p^p - \|u_0\|^2\right) + 2p\eta^2(t) + 2\beta,\ t \in [0, T_{\max}). \end{aligned} \tag{22}$$

By (3), the Cauchy–Schwarz inequality and Hölder's inequality, we have

$$\begin{aligned} \frac{1}{2}\int_0^t \frac{d}{ds}\|u(s)\|^2 ds &= \int_0^t (u(s), u'(s)) ds \\ &\le \int_0^t \|u(s)\|\|u'(s)\| ds \\ &\le \xi(t)\eta(t),\ t \in [0, T_{\max}), \end{aligned}$$

which, together with (19), implies

$$\begin{aligned}\left(F(t)-(T_{\max}-t)\|u_0\|_2^2\right)\left(\eta^2(t)+\beta\right) &= \left(\xi^2(t)+\beta(t+\alpha)^2\right)\left(\eta^2(t)+\beta\right)\\ &= \xi^2(t)\eta^2(t)+\beta\xi^2(t)+\beta(t+\alpha)^2\eta^2(t)+\beta^2(t+\alpha)^2\\ &\geq \xi^2(t)\eta^2(t)+2\xi(t)\eta(t)\beta(t+\alpha)+\beta^2(t+\alpha)^2\\ &\geq \left(\xi(t)\eta(t)+\beta(t+\alpha)\right)^2\\ &\geq \left(\frac{1}{2}\int_0^t \frac{d}{ds}\|u(s)\|^2 ds+\beta(t+\alpha)\right)^2,\ t\in[0,T_{\max}).\end{aligned}$$

Then it follows from (20) and the above inequality that

$$\begin{aligned}\left(F'(t)\right)^2 &= 4\left(\frac{1}{2}\int_0^t \frac{d}{ds}\|u(s)\|^2 ds+\beta(t+\alpha)\right)^2\\ &\leq 4F(t)\left(\eta^2(t)+\beta\right),\ t\in[0,T_{\max}).\end{aligned} \tag{23}$$

In view of (21), (22) and (23), we have

$$\begin{aligned}F(t)F''(t)-\frac{p}{2}\left(F'(t)\right)^2 &\geq F(t)\left(2\left(\|u_0\|_p^p-\|u_0\|^2\right)+2p\eta^2(t)+2\beta-2p\left(\eta^2(t)+\beta\right)\right)\\ &= F(t)\left(2\left(\|u_0\|_p^p-\|u_0\|^2\right)-2(p-1)\beta\right),\ t\in[0,T_{\max}),\end{aligned}$$

which is nonnegative if we take β small enough such that

$$0<\beta\leq\frac{\|u_0\|_p^p-\|u_0\|^2}{p-1}. \tag{24}$$

Then it follows from Lemma 4 that

$$\begin{aligned}T_{\max} &\leq \frac{F(0)}{\left(\frac{p}{2}-1\right)F'(0)}\\ &= \frac{1}{p-2}\left(\alpha+\frac{\|u_0\|^2}{\beta\alpha}T_{\max}\right).\end{aligned} \tag{25}$$

By taking α large enough such that

$$\alpha>\frac{\|u_0\|^2}{(p-2)\beta}, \tag{26}$$

we get from (25) that

$$T_{\max}\leq\frac{\beta\alpha^2}{(p-2)\beta\alpha-\|u_0\|^2}. \tag{27}$$

The above analysis shows that ($\rho:=\beta\alpha$)

$$T_{\max}\leq\inf_{(\rho,\alpha)\in\Phi} f\left(\rho,\alpha\right), \tag{28}$$

where

$$\begin{aligned}\Phi &:= \left\{\left(\rho,\alpha\right):\rho>\frac{\|u_0\|^2}{p-2},\ \alpha\geq\frac{(p-1)\rho}{\|u_0\|_p^p-\|u_0\|^2}\right\},\\ f\left(\rho,\alpha\right) &:= \frac{\rho\alpha}{(p-2)\rho-\|u_0\|^2}.\end{aligned}$$

Since $f(\rho,\cdot)$ is increasing, we get

$$\begin{aligned}T_{\max} &\le \inf_{\rho>\frac{\|u_0\|^2}{p-2}} f\left(\rho, \frac{(p-1)\rho}{\|u_0\|_p^p-\|u_0\|^2}\right)\\ &= \inf_{\rho>\frac{\|u_0\|^2}{p-2}} \frac{(p-1)\rho^2}{\left((p-2)\rho-\|u_0\|^2\right)\left(\|u_0\|_p^p-\|u_0\|^2\right)}\\ &= \left.\frac{(p-1)\rho^2}{\left((p-2)\rho-\|u_0\|^2\right)\left(\|u_0\|_p^p-\|u_0\|^2\right)}\right|_{\rho=\frac{2\|u_0\|^2}{p-2}}\\ &= \frac{4(p-1)\|u_0\|^2}{(p-2)^2\left(\|u_0\|_p^p-\|u_0\|^2\right)}.\end{aligned}$$

So, (18) is true.

(2) **Lower bound estimate**. In the part, we will give the low bound estimate, i.e., show that

$$T_{\max} \ge \frac{1}{2}\int_{\|u_0\|^2}^{\infty} \frac{d\xi}{\kappa^p\xi^{\frac{p}{2}2}-\xi}. \tag{29}$$

By (R_2), (8), and (12), we have

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2 &= -2I(u(t))\\ &= -2\|u(t)\|^2+2\|u(t)\|_p^p\\ &\le -2\|u(t)\|^2+2\kappa^p\|u(t)\|^p, \quad t\in[0,T_{\max}).\end{aligned}$$

Since $I(u(t))<0$ (see (R_3)), we get $-2\|u(t)\|^2+2\kappa^p\|u(t)\|^p \ge -2I(u(t))>0$, then the above inequality can be rewritten as

$$\frac{\frac{d}{dt}\|u(t)\|^2}{-2\|u(t)\|^2+2\kappa^p\|u(t)\|^p} \le 1,\ t\in[0,T_{\max}). \tag{30}$$

Since u blow up at $T_{\max}$, by Theorem 1,

$$\lim_{t\uparrow T_{\max}} \|u(t)\|^2=\infty,$$

integrating (30) from 0 to $T_{\max}$, we get (29). □

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