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Equivalences of Dessins D'Enfants

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Abstract

Dessins d'enfants are bipartite graphs with a cyclic ordering given to the set of edges that meet at each vertex. Merling and Perlis presented a method by which to construct pairs of dessins d'enfants using the permutations induced by the action of a finite group on the cosets of two locally conjugate subgroups of that group. They called these pairs of dessins Gassmann equivalent and investigated some of their properties. First, we discuss several properties of pairs of dessins that imply Gassmann equivalence. Then, using elementwise conjugate subgroups, we introduce and investigate a weaker type of equivalence of dessins, which we refer to as Kronecker equivalence.

0.1 Introduction

The mathematical objects called dessins d'enfants, or dessins, are a relatively new concept related to several areas of mathematics, including algebra, graph theory, topology, and combinatorics. A dessin is a vertex-edge graph such that the edges of any given vertex have a specific order in their arrangement around the vertex, called a cyclic ordering. In addition, every vertex of the graph is given one of two colors (white or black), and any vertex sharing an edge with it must have the other color [1].

Alexandre Grothendieck gave the objects their name, which means "child's drawing," because of their visual simplicity in that a child's drawing could be taken as a specific example of a dessin. His work with them, beginning around 1984, involved complex geometry [1]. Another way to state the definition of dessin, as it relates to Grothendieck's work, is that it is "a bipartite graph that is embedded in a compact, oriented Riemann surface" [2] (pg 1).

Especially because of their connections with various fields of study in mathematics, it is useful to explore and expand the knowledge on the properties of dessins. Some of the research that has been conducted in the last decade concerning dessins has involved studying pairs of dessins that are similar from a certain point of view.

The strongest type of similarity of two dessins is called isomorphism, which means that the two dessins are essentially the same. An important question regarding isomorphic dessins is to find a set of invariants (for instance, number of white vertices, number of black vertices, number of edges, number of connected components) such that if two dessins have the same set of invariants, they are necessarily isomorphic. It is not clear if such a list of invariants exists, but it would be a huge accomplishment to identify one.

Another type of similarity of dessins, called *Gassmann equivalence*, was introduced by Merling and Perlis [3]. Weaker than isomorphim, Gassmann equivalence of dessins is still strong enough to warrant a thorough investigation. Merling and Perlis associated to any finite group G, subgroup H in G, and elements g_0 and g_1 in G, a dessin, by letting g_0 and g_1 act on the left cosets of H in G. These actions induce permutations on the cosets of H in G that are used to construct the dessin [3]. The idea used to define this type of similarity of dessins is not new in mathematics. Number theorists and topologists have known for nearly half a century that making certain constructions based on a Gassmann triple (G, H, H'), where H and H' are locally conjugate subgroups in G, produces two objects (e.g. number fields, compact manifolds) that are similar from a certain point of view (are alike) but are not necessarily isomorphic. In this way, numbers theorists were able to construct pairs of number fields that have similar arithmetical properties but are not conjugate [2]. It was therefore natural to consider and investigate pairs of Gassmann equivalent dessins, that is, dessins constructed from a Gassmann triple using two fixed elements $g_0, g_1 \in G$, and to expect that they are similar. The first part of my research was aimed at expanding the knowledge of Gassmann equivalent dessins, especially by examining the converse statements of known theorems regarding this kind of equivalence in dessins.

Gassmann equivalence is not the only type of equivalence that is of interest in number theory or topology. Triples (G, H, H') that satisfy the weaker condition that H and H' are elementwise conjugate have been studied by number theorists in the last 50 years. Number theorists call the type of similarity based on such a triple Kronecker equivalence [4]. It appears reasonable to consider pairs of dessins constructed using the Merling and Perlis method but based on triples (G, H, H') where H and H' are elementwise conjugate. There is no reference in the literature to such a type of similarity of dessins, which we will call *Kronecker equivalence.* In addition to studying Gassmann equivalence, the second goal of my research was to introduce and investigate Kronecker equivalence in the context of dessins and compare it to Gassmann equivalence in terms of the implications of invariants. Our investigation of the properties of Kronecker equivalent dessins leads to a number of results that mirror to some extent certain results from number theory (many of which are presented in [4]).

At this point, it is helpful to review the concept of *dessin*, to understand how one is drawn.

Definition 1. *A dessin is a triple of permutations, (τ_0, τ_1, τ_2), of a finite set $S = \{1, 2, ..., 2n\}$, such that:*
i) $\tau_1(\tau_1(s)) = s$ and $\tau_1(s) \neq s$ for any $s \in S$;
ii) $\tau_0(\tau_1(\tau_2(s))) = s$, for any $s \in S$.

The elements of S are called *half-edges.* The orbits of τ_0 form the vertex set, the orbits of τ_1 the edge set, and the orbits of τ_2 the face set of the dessin. [5]

Given a finite group G, a subgroup H in G, and elements $g_0, g_1 \in G$, one can construct a dessin as follows ([6]): let $H, a_1H, ..., a_rH$ be the left cosets of H in G. The element g_0 acts on these cosets and the action induces a permutation σ_0, where $\sigma_0(i) = j$ if $g_0a_iH = a_jH$. Similarly, the action of g_1 on the same cosets induces a permuation σ_1, where $\sigma_1(i) = j$ if $g_1a_iH = a_jH$. The resulting dessin is denoted as $D(G/H, g_0, g_1)$.

The graph associated to this dessin is obtained as follows: draw a white vertex for every cycle in σ_0 and a black vertex for every cycle in σ_1, along with half-edges around each vertex, one for each number in the cycle, labeled counterclockwise with those numbers. These half-edges are then joined according to their labels to form a bipartite graph [3].

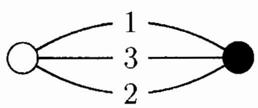
Example 1. *Consider the alternating group A_4 and one of its subgroups*

$$H = \{e, (12)(34), (13)(24), (14)(23)\}.$$

The left cosets of H in A_4 are

1. $\{e, (12)(34), (13)(24), (14)(23)\}$
2. $\{(123), (134), (243), (142)\}$
3. $\{(124), (143), (132), (234)\}$

Let $g_0 = (123)$ and $g_1 = (124)$. The left action of these elements of A_4 on A_4/H produces permutations $\sigma_0 = (123)$ and $\sigma_1 = (132)$, respectively. The dessin then is drawn as follows, with one white vertex for the one 3-cycle of σ_0 and one black vertex for the one 3-cycle of σ_1:



0.2 Gassmann Equivalence

Gassmann equivalence of dessins is a concept introduced by Merling and Perlis in [3].

Definition 2. *Two dessins $D(G/H, g_0, g_1)$ and $D(G/H', g_0, g_1)$ are said to be Gassmann equivalent if (G, H, H') is a Gassmann triple.*

Recall that in a Gassmann triple, G is a group and H and H' are subgroups of G that are locally conjugate in G. This means that there is a bijection $\psi : H \to H'$ such that h and $\psi(h)$ are conjugate in G, for any $h \in H$ [6]. This condition guarantees H and H' have the same index (number of left cosets) in G, as well as the same number of elements in each coset.

In their paper, Merling and Perlis explore various invariants of Gassmann equivalent dessins that are the same. The first such set of invariants is what is known as the *branching data* of the corresponding dessins, $D(G/H, g_0, g_1)$ and $D(G/H', g_0, g_1)$ (see definition 5). Branching data relates to the number of cycles of each length in a given dessin's associated permutations. Other equal invariants of Gassmann equivalent dessins identified by Merling and Perlis are the number of components, or connected parts, and the sum of the genera of the components of each dessin. In essence, the *genus* of a dessin (plural genera) is the maximum number of times any one edge must cross over other edges in the graph to preserve the cycles. In addition, Merling and Parlis showed that the dessins' monodromy groups are isomorphic. A dessin's monodromy group is the subgroup of S_r generated by the permuations σ_0 and σ_1 (which consists of all combinations of powers of σ_0 and σ_1), where r denotes the number of cosets of H and H' in G.

The following is an example of a monodromy group.

Example 2. *Consider the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and the subgroup of G, $H = \langle(1,0,0)\rangle = \{(0,0,0), (1,0,0)\}$. The left cosets are*

1. $H_1 = H = \{(0,0,0), (1,0,0)\}$
2. $H_2 = H + (0,0,1) = \{(0,0,1), (1,0,1)\}$
3. $H_3 = H + (0,1,0) = \{(0,1,0), (1,1,0)\}$
4. $H_4 = H + (0,1,1) = \{(0,1,1), (1,1,1)\}$

The action of element $g_0 = (0,0,1) \in G$ permutes H_1 to $H_1 + (0,0,1) = H_2$, H_2 to $H_2 + (0,0,1) = H + (0,0,0) = H_1$, H_3 to H_4, and H_4 to H_3. This is expressed as permutation $\sigma_0 = (12)(34)$. The action of $g_1 = (0,1,0) \in G$ in a similar manner produces the permutation $\sigma_1 = (13)(24)$. In S_4, the monodromy group is $M = \langle\sigma_0, \sigma_1\rangle = \langle(12)(34), (13)(24)\rangle$, the subgroup generated by the two permutations. Since

$$(12)(34)(13)(24) = (14)(32) = \sigma_0\sigma_1 \text{ and } (13)(24)(12)(34) = (14)(32) = \sigma_1\sigma_0$$

we have that $\sigma_0\sigma_1 = \sigma_1\sigma_0$. Also $\sigma_0^2 = e$ and $\sigma_1^2 = e$. Thus, $M = \{e, \sigma_0, \sigma_1, \sigma_0\sigma_1\}$ since any other combinations of the two permutations could be simplified to these four permutations.

This example raises a point about both Gassmann and Kronecker equivalence: the only interesting (non-trivial) cases of equivalence occur when the group used is not abelian. This is because any two elements of an abelian group commute. If an element h in subgroup H has a conjugate h' in subgroup H', we have $gh'g^{-1} = h$ for some $g \in G$. By commutativity, $gh'g^{-1} = g^{-1}gh' = h'$, so $h' = h$. This means H and H' have the same elements, and the dessins produced by the action on their cosets are trivially isomorphic.

It is helpful to recall the definition of a permutation character:

Definition 3. *Given a group G and finite set S, the permutation character of an action of G on S is $\chi_S : G \to \{0, 1, 2, \dots\}$ such that $\chi_S(g) = |\{x \in S : gx = x\}|$.*

Another definition worth noting is the alternate definition of Gassmann triples given by Merling and Perlis [3].

Definition 4. *(G, H, H'), where H and H' are subgroups of a finite group G, is a Gassmann triple if $\chi_{G/H}(g) = \chi_{G/H'}(g)$ for all $g \in G$.*

In Definition 4, $\chi_{G/H}(g) = |\{aH : gaH = aH\}|$.

We will focus first on the branching data of a dessin, which is defined as follows:

Definition 5. *Let $\mathbf{D} = (G/H, g_0, g_1)$ be a dessin with a pair of permutations, (σ_0, σ_1) from the action of $g_0, g_1 \in G$. Let σ_∞ be defined by the relation $\sigma_0\sigma_1\sigma_\infty = 1$. The branching data of $\mathbf{D}$ are the tuples $(a_1, \dots, a_s)$, $(b_1, \dots, b_t)$, and $(c_1, \dots, c_r)$, where a_i is the number of i-cycles in σ_0, b_i is the number of i-cycles in σ_1, and c_i is the number of i-cycles in σ_∞.*

Merling and Perlis proved the following theorem:

Theorem 1. *If (G, H, H') is a Gassmann triple, then the branching data for $D(G/H, g_0, g_1)$ and $D(G/H', g_0, g_1)$ coincide.*

We begin our investigation of Gassmann equivalent dessins by proving a converse of their result:

Theorem 2. *If the branching data for $D(G/H, g_0, g_1)$ and $D(G/H', g_0, g_1)$ coincide for all $g_0, g_1 \in G$, then (G, H, H') is a Gassmann triple.*

Proof. Suppose all pairs of dessins $\mathbf{D}$ and $\mathbf{D}'$, constructed from the permutations (σ_0, σ_1) and (σ_0', σ_1') aquired from the action a group G and on the left cosets of subgroups H and H' of G, respectively, have the same branching data $(a_1, \dots, a_s)$ and $(b_1, \dots, b_t)$, where a_i is the number of i-cycles in σ_0 (and in σ_0') and b_i is the number of i-cycles in σ_1 (and in σ_1') . a_1 is then the number of 1-cycles in σ_0 and in σ_0'. Thus, the permutation character for all $g \in G$ is the same. By Definition 4, (G, H, H') form a Gassmann triple.

□

Note that the sum $a_1 + \dots + a_s$ is the total number of cycles in σ_0, which is the same as the number of white vertices in $\mathbf{D}$, and $b_1 + \dots + b_t$ is the total number of cycles in σ_1, which translates to the number of black vertices in $\mathbf{D}$, so we could just as well have said that if two dessins $D(G/H, g_0, g_1)$ and $D(G/H', g_0, g_1)$ have the same number of white and black vertices of each degree, (G, H, H') forms a Gassmann triple. We go on to prove the converse of a weaker version of Theorem 1: that only the number of white (or black) vertices of two dessins for all $g_0, g_1 \in G$ must be the same in order for those dessins to be Gassmann equivalent. We begin with a helpful lemma.

Lemma 1. *The number of disjoint cycles in σ^s is $\sum_{m>0} gcd(m,s)a_m$, where σ is some permutation and a_m is the number of disjoint m-cycles in σ.*

Proof. Let $(x_0, x_1, x_2, \ldots, x_{m-1})$ be an m-cycle of σ. When raised to the s power, we have

$$(x_0, x_1, x_2, \ldots, x_{m-1})^s = (x_0, x_s, x_{2s}, \ldots, x_{ks} \ldots).$$

Because it is a cycle, $x_{ks} = x_{ks \ (mod\ m)}$. Let ks be the first multiple of s to be 0 $mod\ m$, that is, $ks \equiv 0(mod\ m)$ such that $k \geq 1$ is minimal. Then $x_{ks} = x_0$ completes the cycle and we have, in part, $(x_0, x_s, x_{2s}, \ldots, x_{(k-1)s})$, a k-length cycle. Thus, we have m/k total k-length cycles from the original cycle:

$$(x_0, x_s, \ldots, x_{(k-1)s})(x_1, x_{s+1}, \ldots, x_{(k-l)s+1}) \ldots (x_{m/k-1}, x_{s+m/k-1}, \ldots, x_{(k-1)s+m/k-1}).$$

Claim: $k = \frac{m}{gcd(m,s)}$. We must show

$$\frac{m}{gcd(m,s)} s \equiv 0(mod\ m)$$

and that $\frac{m}{gcd(m,s)}$ is the least such integer. We have

$$\frac{m}{gcd(m,s)} s = lcm(m,s)$$

and $m | lcm(m,s)$, so

$$\frac{m}{gcd(m.s)} s \equiv 0\ (mod\ m).$$

Suppose $1 \leq t < \frac{m}{gcd(m,s)}$ and $m|ts$. Since $m|ts$ and $s|ts$,

$$lcm(m,s) = \frac{ms}{gcd(m,s)} | ts \text{ and thus } \frac{m}{gcd(m,s)} | t,$$

a contradiction. Thus, the number of disjoint cycles made by raising an m-cycle to the s power is

$$\frac{m}{k} = \frac{m}{\frac{m}{gcd(m,s)}} = gcd(m,s).$$

If there are a_m m-cycles, they become $gcd(m,s)a_m$ disjoint cycles. The total number of cycles of σ, then, is

$$\sum_{m>0} gcd(m,s)a_m.$$

$\square$

This lemma is useful in proving our next theorem.

Theorem 3. *Let H and H' be subgroups in G. If $D(G/H, g_0, g_1)$ and $D(G/H', g_0, g_1)$ have the same number of white (or black) vertices for all $g_0, g_1 \in G$, then (G, H, H') is a Gassmann triple.*

Proof. Since the number of white vertices correspond with the number of disjoint cycles of σ_1 and σ_1', the permutations induced by the action of G on the cosets G/H and G/H', respectively, let

$$a = \sum_{m>0} a_m$$

where a_m denotes the number of m-length disjoint cycles of σ_1, and similarly define

$$a' = \sum_{m>0} a'_m.$$

Note that G is a finite group, so these summations are finite. Fix g_0. $D(G/H, g_0, g_1)$ and $D(G/H', g_0, g_1)$ have the same number of white vertices for all $s \in \mathbb{Z}$. Equivalently, σ_1^s has the same number of disjoint cycles as $(\sigma_1')^s$ for all s. Thus from the above lemma, we have

$$\sum_{m>0} gcd(m,\ s)a_m = \sum_{m>0} gcd(m,\ s)a'_m \text{ for all } s.$$

From this, we have a system of linear equations:

$$\sum_{m>0} gcd(m,\ 1)(a_m - a'_m) = 0$$
$$\sum_{m>0} gcd(m,\ 1)(a_m - a'_m) = 0$$
$$\vdots$$
$$\sum_{m>0} gcd(m,\ 1)(a_m - a'_m) = 0$$

Written in matrix form, where each $x_m = a_m - a'_m$ and n is an integer such that $a_k = a'_k = 0$ for all $k > n$, $k \in \mathbb{Z}$, we have an $n \times n$ gcd matrix, A, multiplied by vector $\vec{x} = \langle x_1,\ x_2, \ldots, x_n\rangle$, $A\vec{x} = \vec{0}$. From [7],

$$det(A) = \phi(1) \cdot \phi(2) \ldots \phi(n),$$

where ϕ is Euler's phi (totient) function. Thus, $det(A) \neq 0$, so there is only the trivial solution, meaning $a_m = a'_m$ for all m. Then $a_1 = a'_1$ for all g_0, so $\chi_{G/H}(g) = \chi_{G/H'}(g)$ for all $g \in G$, and (G, H, H') is a Gassmann triple. □

Another result from Merling and Perlis was that Gassmann equivalence implies that two dessins will have the same number of components. Here we prove the converse.

Theorem 4. *Let $\mathbf{D} = D(G/H, g_0, g_1)$ and $\mathbf{D'} = D(G/H', g_0, g_1)$ be dessins such that $\mathbf{D}$ and $\mathbf{D'}$ have the same number of components for all $g_0, g_1 \in G$. Then (G, H, H') is a Gassmann triple.*

Proof. Let $g_0 \in G$ be arbitrary and let $g_1 = e$. Then every coset is fixed by the action of g_1, so all of the black vertices of $\mathbf{D}$ and $\mathbf{D}'$ are danglers. Thus for both dessins, the number of components is entirely determined by the action of g_0, namely, the number of components of each is equal to the number of white vertices it has. $\mathbf{D}$ and $\mathbf{D}'$ then have the same number of white vertices for all $g_0 \in G$. Therefore, (G, H, H') is a Gassmann triple. Similarly, (G, H, H') can be proven to be a Gassmann triple given the dessins have the same number of black vertices. □

We conclude this section with the following example of Gassmann equivalent dessins which illustrates some of their properties:

Let $G = GL_3(\mathbf{F_2})$, the set of all 3×3 invertible matrices over $\mathbf{F_2}$. G is a group of order 168. The following are subgroups of G:

$$H \text{ has the form } \left[\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right] \text{ and } H' \text{ has the form } \left[\begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}\right]$$

where $*$ indicates any choice of 0 or 1, so that all matrices in H have the same first column and those in H' have the same first row. Note that H and H' both consist of 24 matrices. Let $g \in G$. Then gH consists of 24 matrices where the first column is the same as the first column of g. Since $[G : H] = [G : H'] = 168/24 = 7$, H and H' have 7 left cosets in G. All elements in a coset gH have the same first column, so to keep track of them, we denote them as g_1H, g_2H, g_3H, g_4H, g_5H, g_6H, g_7H, where the first column of matrix g_j is j in binary. The first column of g_3 is, for instance, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, while $g_4H = H$. When an abritrary element $g \in G$ acts on the above cosets, it permutes them: $(gg_1)H$, $(gg_2)H$, $(gg_3)H$, $(gg_4)H$, $(gg_5)H$, $(gg_6)H$, $(gg_7)H$. When a fixed g permutes the left cosets of H in G, the first column of gg_j only depends on the first column of g_j. For example, if $g = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then

$$gg_3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & * & * \\ 1 & * & * \\ 1 & * & * \end{pmatrix} = \begin{pmatrix} 1 & * & * \\ 1 & * & * \\ 1 & * & * \end{pmatrix}$$

so g maps g_3H to g_7H.

Let $x = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Now we find the permutations induced by the action of these elements on the left cosets of H in G.

$$xg_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 1 & * & * \end{pmatrix} = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

so $xg_1H = g_4H$. Continuing in this manner, we find the permutation σ_0 induced by the action of x to be $(1436)(2)(57)$ and the permutation induced by the action of y, $\sigma = (132)(4)(576)$ Next, we would like to find the permutations arising from the action of x and y on G/H'. Note that the inverse of any matrix in H is a matrix in H (as H is a subgroup) so the transpose of any such inverse is in H'. Define a mapping $\phi : G \to G$, $\phi(g) = (g^{-1})^t$.

Claim: ϕ is an isomorphism.

Proof. One-to-one: Suppose for some $g_1, g_2 \in G$, $\phi(g_1) = \phi(g_2)$ Then by definition of ϕ we have $(g_1^{-1})^t = (g_2^{-1})^t$. Thus by taking the transpose of both sides, $g_1^{-1} = g_2^{-1}$. Since g is invertible for all $g \in G$, we can take the inverse of both sides to arrive at $g_1 = g_2$.

Onto: Let $g \in G$ and let h be such that $(h^{-1})^t = g$. Then $((h^{-1})^t)^t = g^t$, and so $(h^{-1})^{-1} = (g^t)^{-1}$. Since g is invertible, g^t is invertible, and $(g^t)^{-1} \in G$. Thus, $h \in G$.

Homorphism: Let $g_1, g_2 \in G$. $\phi(g_1g_2) = ((g_1g_2)^{-1})^t = (g_2^{-1}g_1^{-1})^t = (g_1^{-1})^t(g_2^{-1})^t = \phi(g_1)\phi(g_2)$.

Therefore, ϕ is indeed an isomorphism. □

A second claim: $\phi(H) = H'$.

Proof. $\phi(H) \subseteq H'$: Let $h \in \phi(H)$. Then $h = (k^{-1})^t$ for some $k \in H$. Then k is an invertible matrix of the form $\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$ and

$$k^{-1} = \frac{1}{det(k)} \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

as $det(k) = 1$ since k is invertible. $(k^{-1})^t$ has the form $\begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$. Thus, $(k^{-1})^t = h \in H'$.

$H' \subseteq \phi(H)$: Let $h \in H'$. Then h has the form $\begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$. Let $h = (k^{-1})^t$ for some matrix k. Taking the transpose of both sides, we have $k^{-1} = h^t$, which has the form $\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$.

Since h is invertible, so is h^t. Taking the inverse of both sides gives us

$$k = (h^t)^{-1} = \frac{1}{det(h^t)} \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Then k has the form $\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$ since h^t is invertible, causing $det(h^t) = 1$. Thus $k \in H$ and $h = (k^{-1})^t \in \phi(H)$. $\square$

With these claims proven, we have $\phi(g_j H) = [(g_j H)^{-1}]^t = (H' g_j^{-1})^t = (g_j^{-1})^t H'$, so ϕ maps the seven cosets of H in G to the seven cosets

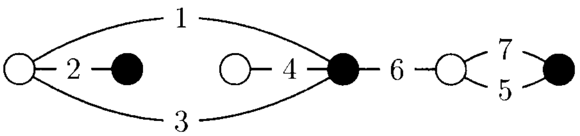
$$(g_1^{-1})^t H', g_1^{-1})^t H', \ldots, g_1^{-1})^t H'$$

of H' in G.

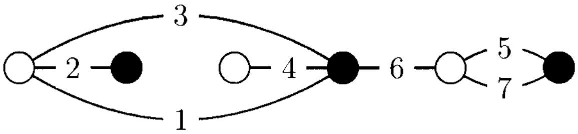
Therefore, we can use ϕ to check how an arbitrary $g \in G$ permutes the cosets of H'.

$$\phi((g^{-1})^t g_j H) = (H^{-1})^t (g_j^{-1})^t g = g(g_j^{-1})^t H'$$

Hence the action of g on the cosets of H' in G is given by the action of $(g^{-1})^t$ on the cosets of H in G. Thus we can use $(x^{-1})^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $(y^{-1})^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ on G/H to find the permutations, σ_0' and σ_1', induced by x and y on G/H', respectively. We find that $\sigma_0' = (1634)(2)(57)$ and $\sigma_1' = (123)(4)(567)$. Now, we use the permutations to draw the corresponding dessins. Note that the only difference between these two dessins is the labeling on the edges.



$\mathbf{D} = D(G/H, x, y)$



$\mathbf{D}' = D(G/H', x, y)$

0.3 Kronecker Equivalence

As stated above, Kronecker equivalence is a type of similarity of dessins that has not yet been studied. Triples (G, H, H') that are Kronecker equivalent have the property that subgroups H and H' of G are elementwise conjugate, that is, every element in H is conjugate to some element in H' and vice versa. The conjugate of the element h of H with respect to some element g in G is $g^{-1}hg$ [4]. Klingen [4] provides a survey of properties of Kronecker equivalent number fields. Some of these properties can be formulated in terms of the action of a certain group G on the (left) cosets of two fixed subgroups H and H' that are elementwise conjugate. This action induces pairs of permutations. As shown in Klingen's book, Kronecker equivalence of number fields forces these permutations to have similar (but not necessarily identical) cycle decomposition types [4]. In particular, it is known that these permutations can be decomposed into disjoint cycles in such a way that the cycle lengths that are minimal with respect to divisibility are the same. It is this analogy between Kronecker equivalence of number fields and of dessins that we explore in this paper.

We define Kronecker equivalence between two dessins, $\mathbf{D}$ and $\mathbf{D}'$, as follows. Note that we refer to a leaf, that is, a vertex with exactly one edge coming from it, of a vertex-edge graph of a dessin as a dangler.

Definition 6. $\mathbf{D}$ *and* $\mathbf{D}'$ *are said to be* Kronecker equivalent *if* $\mathbf{D}$ *has danglers of a certain color if and only if* $\mathbf{D}'$ *has danglers of the same color.*

Kronecker equivalence of dessins is related to elementwise conjugate subgroups in this way:

Theorem 5. *H and H' are elementwise conjugate in G if and only if $D(G/H, g_0, g_1)$ and $D(G/H', g_0, g_1)$ are Kronecker equivalent, for all choices of $g_0, g_1 \in G$.*

Proof. Let $\mathbf{D}$ and $\mathbf{D}'$ be dessins defined by the action of elements of group G on the left cosets of subgroups H and H' in G, respectively. Note that D has a white dangler $\Leftrightarrow$ the action of g_0 on the cosets of H fixes some coset $Hg_i \Leftrightarrow g_iH = g_0g_iH \Leftrightarrow g_ig_0g_i^{-1} = h$ for some $h \in H$.

Elementwise conjugacy $\Rightarrow$ Kronecker equivalence.

Suppose H and H' are elementwise conjugate. Then for all $h \in H$, $h = g^{-1}h'g$ for some $h' \in H'$ and $g \in G$. Suppose $\mathbf{D}$ has a white dangler. Then there is $h \in H$ and $g_i \in G$ such that $g_ig_0g_i^{-1} = h = g^{-1}h'g$, so by letting g act on the left and g^{-1} act on the right, we find $(gg_i)g_0(gg_i)^{-1} = h'$ in H'. Thus, g_0 fixes the coset $(gg_i)H'$. Then $\mathbf{D}'$ has a white dangler. If we start with $\mathbf{D}'$ having a white dangler, we can show in a similar manner that this implies D has a white dangler, since for all $h' \in H$, $h' = g^{-1}hg$ for some $h \in H$. So either both $\mathbf{D}$ and $\mathbf{D}'$ have white danglers, or neither do. The same can be shown for black danglers, working instead with g_1.

Kronecker equivalence $\Rightarrow$ elementwise conjugacy.

Suppose $\mathbf{D}$ and $\mathbf{D}'$ are Kronecker equivalent. Let $h \in H$. Pick $g_0 = h$. Then $\mathbf{D}$ has a white dangler, since g_0 then fixes the coset H. Thus, $\mathbf{D}'$ must also have a white dangler, so there is an element $g_i \in G$ such that $g_ig_0g_i^{-1} \in H'$. Then by substitution, the conjugate $g_ihg_i^{-1}$ of h is an element of H'.

□

Here, we note various necessary and sufficient conditions for Kronecker equivalence.

Theorem 6. *Let H and H' be subgroups of G. For $g_0, g_1 \in G$, consider the dessins $\mathbf{D} = D(G/H,\ g_0,\ g_1)$ and $\mathbf{D}' = D(G/H',\ g_0,\ g_1)$. Let*

$$d_1 \leq d_2 \leq \cdots \leq d_m \text{ and } \delta_1 \leq \delta_2 \leq \cdots \leq \delta_n$$

be the degrees of the white and black vertices, respectively, of $\mathbf{D}$ and

$$d'_1 \leq d'_2 \leq \cdots \leq d'_{m'} \text{ and } \delta'_1 \leq \delta'_2 \leq \cdots \leq \delta'_{n'}$$

be the degrees of the white and black vertices, respectively, of $\mathbf{D}'$. Then the following are equivalent conditions:

1. $\mathbf{D}$ *and* $\mathbf{D}'$ *are Kronecker equivalent for any* $g_0, g_1 \in G$.

2. $d_1 = d'_1$ *and* $\delta_1 = \delta'_1$*, for any* $g_0, g_1 \in G$.

3. $\mathbb{N}d_1 + \mathbb{N}d_2 + \cdots + \mathbb{N}d_m = \mathbb{N}d'_1 + \mathbb{N}d'_2 + \cdots + \mathbb{N}d'_{m'}$ *and* $\mathbb{N}\delta_1 + \mathbb{N}\delta_2 + \cdots + \mathbb{N}\delta_n = \mathbb{N}\delta'_1 + \mathbb{N}\delta'_2 + \cdots + \mathbb{N}\delta'_{n'}$*, for any* $g_0, g_1 \in G$.

4. *The degrees of the white vertices of* $\mathbf{D}$ *that are minimal with respect to divisibility are the same as the degrees of the white vertices of* $\mathbf{D}'$ *that are minimal with respect to divisibility, for any* $g_0, g_1 \in G$*. The same applies to the black vertices.*

5. *Every white vertex degree* d_i *in* $\mathbf{D}$ *is divisible by a white vertex degree* d'_j *in* $\mathbf{D}'$ *and vice-versa, for any* $g_0, g_1 \in G$.

6. *For all* $s \geq 1$, $g_0, g_1 \in G$, $\sum_{d_i|s} d_i > 0 \Longleftrightarrow \sum_{d'_i|s} d'_i > 0$ *and* $\sum_{\delta_i|s} \delta_i > 0 \Longleftrightarrow \sum_{\delta'_i|s} \delta'_i > 0$.

Proof. We will show these conditions to be equivalent by proving $6 \Rightarrow 5 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 6$.

$6 \Rightarrow 5$. Let d_j be an arbitrary white vertex degree in $\mathbf{D}$. Then $d_j \geq 1$ and

$$\sum_{d_i|d_j} d_i \geq d_j > 0.$$

Thus, $\sum_{d'_i|d'_j} d'_i > 0$. So there exists at least one vertex degree, d'_k, in $\mathbf{D}'$ such that $d'_k|d_j$. Similarly, for any $d'_m \in \mathbf{D}$, there is $d_n \in \mathbf{D}$ such that $d_n|d'_m$. The same can be said concerning the black vertices.

$5 \Rightarrow 4$. Let d_i be the degree of a vertex in $\mathbf{D}$ that is minimal with respect to divisibility. Then $d'_j|d_i$ for some $d'_j \in \mathbf{D}$. Assume d'_j is not minimal with respect to divisiblility. Then there is $d'_k \in \mathbf{D}$ such that $d'_k|d'_j$ and $d'_k < d'_j$. By the assumption of (5), there exists $d_m \in \mathbf{D}$ such that $d_m|d'_k$. By transitivity, $d_m|d_i$. Since $d_m \leq d'_k < d'_j \leq d_i$, we have that $d_m < d_i$, but d_i is minimal. Thus d'_j must be minimal with respect to divisibility in $\mathbf{D}$. Similarly, the degrees of black vertices have the same property.

$4 \Rightarrow 3$. Since any linear combination of degrees that are not minimal with respect to divisibility are linear combinations of degrees that are minimal with respect to divisibility, and since $\mathbf{D}$ and $\mathbf{D}'$ have the same minimal degrees, (3) holds.

$3 \Rightarrow 2$. Choose the linear combination

$$1d_1 + 0d_2 + \cdots + 0d_m.$$

Then some combination

$$\mathbb{N}d'_1 + \mathbb{N}d'_2 + \cdots + \mathbb{N}d'_{m'} = d_1,$$

and similarly, some combination

$$\mathbb{N}d_1 + \mathbb{N}d_2 + \cdots + \mathbb{N}d_m = d'_1.$$

Assume $d_1 \neq d'_1$. Without loss of generality, asume $d_1 < d'_1$. Then $d_1 < d'_i$ for all $1 \leq i \leq m'$, so any linear combination

$$\mathbb{N}d'_1 + \mathbb{N}d'_2 + \cdots + \mathbb{N}d'_{m'} > d_1.$$

Thus $d_1 = d'_1$, and similarly $\delta_1 = \delta'_1$.

2 $\Rightarrow$ 1. From (2), we have $d_1 = 1$ iff $d'_1 = 1$, and $\delta_1 = 1$ iff $\delta'_1 = 1$, so $\mathbf{D}$ and $\mathbf{D}'$ are Kronecker equivalent.

1 $\Rightarrow$ 6. Let $s \in \mathbb{N}$ be arbitrarily fixed. Suppose $\sum_{d_i|s} d_i > 0$. Consider the dessins $\mathbf{D}_1 = D(G/H, g_0^s, g_1)$ and $\mathbf{D}'_1 = D(G/H', g_0^s, g_1)$ with white vertex degrees

$$b_1 \leq b_2 \leq \cdots \leq b_p \text{ and } b'_1 \leq b'_2 \leq \cdots \leq b'_{p'},$$

respectively. Then $\sum_{b_j|s} b_j > 0$ and $b_1 = 1$ since the cycles whose lengths are factors of s in $\mathbf{D}$ are reduced to 1-cycles in $\mathbf{D}_1$. Thus $b'_1 = 1$. For $\mathbf{D}'_1$ to have a one-cycle, $\mathbf{D}'$ must have a vertex degree d'_k such that $d'_k|s$. Similarly,

$$\sum_{d'_i|s} d'_i > 0 \Longrightarrow \sum_{d_i|s} d_i > 0$$

and

$$\sum_{\delta_i|s} \delta_i > 0 \Longleftrightarrow \sum_{\delta'_i|s} \delta'_i > 0.$$

□

Unlike the case with Gassmann equivalence, Kronecker equivalence does not guarantee two dessins have the same number of components or even that two dessins will be both connected or both not connected. Consider the dessins $D(G/H, g_0, e)$ and $D(G/H', g_0, e)$ where $G = A_5$,

$$H = \{e, (12)(34), (125), (152), (15)(34), (25)(34)\}, \text{ and}$$

$$H' = \{e, (23)(45), (24)(35), (25)(34), (234), (235), (243), (245), (253), (254), (345), (354)\}$$

with $g_0 = (12345)$. As can be verified by computation, H and H' are elementwise conjugate, so the dessins are Kronecker equivalent. The left cosets G/H are

1. $\{e, (12)(34), (125), (152), (15)(34), (25)(34)\}$
2. $\{(12)(35), (345), (235), (135), (13452), (12345)\}$
3. $\{(12)(45), (354), (245), (145), (14352), (12435)\}$
4. $\{(13)(24), (14)(23), (14253), (15432), (15324), (13254)\}$
5. $\{(13)(25), (15234), (153), (123), (12534), (134)\}$
6. $\{(13)(45), (12354), (12453), (14523), (14)(35), (13524)\}$

7. $\{(14)(25), (15243), (154), (124), (12543), (143)\}$

8. $\{(15)(23), (13425), (132), (253), (234), (15342)\}$

9. $\{(15)(24), (14325), (142), (254), (243), (15432)\}$

10. $\{(23)(45), (13542), (13245), (14532), (14235), (24)(35)\}$

The left cosets G/H' are

1. $\{e, (23)(45), (24)(35), (25)(34), (234), (235), (243), (245), (253), (254), (345), (354)\}$

2. $\{(12)(34), (12453), (12354), (125), (124), (12435), (123), (12345), (12543), (12534), (12)(45), (12)(35)\}$

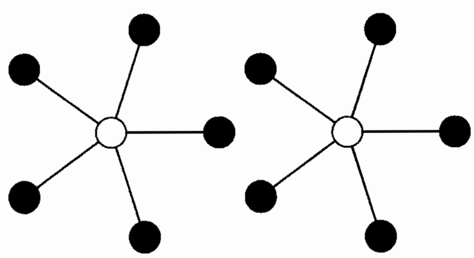
3. $\{(13)(24), (13452), (135), (13254), (132), (13542), (134), (13)(45), (13425), (13)(25), (13245), (13524)\}$

4. $\{(14)(23), (145), (14352), (14253), (143), (14)(35), (142), (14532), (14)(25), (14325), (14523), (14235)\}$

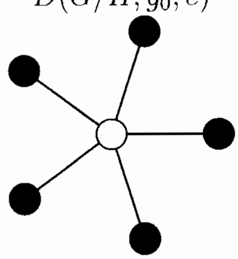
5. $\{(15)(23), (154), (15243), (15342), (15)(34), (153), (15)(24), (15324), (152), (15432), (15234), (15423)\}$

The action of g_0 on G/H produces a permutation of exactly two 5-cycles, $(1\,2\,6\,4\,9)(3\,5\,10\,7\,8)$ while its action on G/H' produces a permutation of exactly one 5-cycle, $(1\ 2\ 3\ 4\ 5)$. Note that e fixes all the cosets of H and H' in G, so all black vertices are danglers. Then $D(G/H, g_0, e)$ has two white vertices, and so two components, while $D(G/H', g_0, e)$ has one white vertex, and so one component.

This example is depicted below:

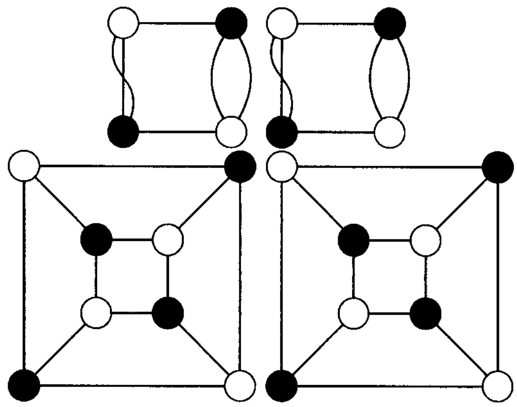


$D(G/H, g_0, e)$



$D(G/H', g_0, e)$

Another interesting example is with group $A_4 \times A_4$. Let $H = \{e, (12)(34)\}$ and $H' = \{e, (12)(34), (13)(24), (14)(23)\}$, both subgroups of A_4. Then there are 36 left cosets of $H \times H$ in $A_4 \times A_4$ with elements of the form (h_1, h_2), where h_1 is an element of some particular coset of H in A_4 and h_2 is an element of another (not necessarily the same) coset of H in A_4. In a similar fashion, there are 9 left cosets of $H' \times H'$ in $A_4 \times A_4$. To find all of these cosets, find all ordered choices of two of the left cosets of H (H') in A_4 ($H_1 \times H_1, H_1 \times H_2, \ldots, H_6 \times H_6$). Let $g_0 = (123)$ and $g_1 = (124)$. Then, consider the action of (g_0, g_0) and (g_1, g_1), elements of $A_4 \times A_4$, on $A_4 \times A_4/H \times H$ and $G/H \times H$. The resulting permutations are made up entirely of 3-cycles. They are drawn as follows:



$\mathbf{D} = D(A_4 \times A_4/H \times H, (g_0, g_0), (g_1, g_1))$

$\mathbf{D} = D(A_4 \times A_4/H' \times H', (g_0, g_0), (g_1, g_1))$

What is interesting to note is that $\mathbf{D}$ has four components while $\mathbf{D}'$ has three, and $\mathbf{D}$ has two crossings of edges while $\mathbf{D}'$ does not have any ($\mathbf{D}$ has a genus of 2, and $\mathbf{D}'$ has a genus of 0). These examples leave properties of the components of dessins and their genera up to further exploration, in addition to the topic of monodromy groups.

In researching Gassmann and Kronecker equivalent dessins, I was able to contribute knowledge in regards to their properties. Namely, my work built on previous research on invariants (components, number of vertices of a certain color, and branching data) that imply or are implied by these types of equivalences. What I found was that the converses of theorems proven by Merling and Perlis about Gassmann triples often held, and surprisingly that the weaker condition of always having the same number of vertices of one color led to Gassmann equivalence.

Regarding Kronecker equivalence of dessins, I was able to provide several different characterizations of Kronecker equivalence. Since any Gassmann triple involves elementwise conjugate subgroups H and H' in G, it is clear that any Gassmann equivalent dessins are Kronecker equivalent (the converse does not hold). Therefore, all properties of Gassmann equivalent dessins apply to Kronecker equivalent dessins. I showed with an example, however, that the properties of connectedness and number of components, as well as branching data, do not hold the same implications for Kronecker equivalence, as it is a weaker condition than Gassmann equivalence.

Future research could be conducted concerning how the properties Merling and Perlis studied in Gassmann triples apply, in modified form, to Kronecker equivalent dessins. The implications of the additional properties of genera and monodromy groups could be studied for both Gassmann and Kronecker equivalent dessins. My research has further opened the door to a vast, new area of mathematics waiting to be explored.

0.4 Appendix

The following is code I wrote in Java to simplify the task of finding permutations. There are four classes: Cycle, Permutation, PermutationApp (contains the main method), and PermutationFrame (for the user interface).

0.4.1 Cycle Class

```
// File:   Cycle.java
// Author: Rachel Volkert
// Date:   2/4/2013
// Modified: new



import java.util.Vector;

public class Cycle
{
  private Vector<Integer> data;
  private int index;

  public Cycle()
  {
    data = new Vector<Integer>();
    index = 0;
  }

  public Cycle(String numbers)
  {
    // numbers is expected to be a string of the elements
    //separated by non-alphanumeric characters (e.g. spaces, commas)
    data = new Vector<Integer>();
    index = 0;
    String[] list = numbers.split("\\W+");
    for (int i = 0; i < list.length; i++)
    {
      data.add(Integer.parseInt(list[i]));
      index += 1;
    }
  }

  public String display()
```

```
{
  String string = "(";
  for (int i = 0; i < index; i++)
  {
    string += ("" + data.elementAt(i));
  }
  string += ")";
  return string;
}

public void append(int i)
{
  // adds element i to the end of the cycle
  data.add(i);
  index += 1;
}

public int elementAt(int i)
{
  // returns the i-th element, or 0 if i is out of range
  if (i >= index)
    return 0;
  return data.elementAt(i);
}

public int index()
{
  return index;
}

public int nextAfter(int i)
{
  // returns the next integer in the cycle after i,
  // or i if it is not in the cycle (1-cycle)
  if (! this.contains(i))
    return i;
  return data.elementAt((data.indexOf(i) + 1) % index);
}

public Boolean contains(int i)
{
  // returns whether i is in the cycle
  return data.contains(i);
}
```

```
}
```

0.4.2 Permutation Class

```
// File:   Permutation.java
// Author: Rachel Volkert
// Date:   2/4/2013
// Modified: new

import java.util.Vector;

public class Permutation
{
  // stores the cycles in the permutation in order
  private Vector<Cycle> cycles;
  private int numOfCycles; // this is the number of cycles besides 1-cycles

  public Permutation()
  {
    // creates a new Permutation
    cycles = new Vector<Cycle>();
    numOfCycles = 0;
  }

  public Permutation(Vector<Cycle> cycleVector)
  {
    // creates a new Permutation with an existing vector of cycles
    cycles = cycleVector;
    numOfCycles = cycles.size();
  }

  public void append(Cycle newCycle)
  {
    // adds the new cycle to the end of the permutation
    cycles.add(newCycle);
    numOfCycles += 1;
  }

  public String display()
  {
    String string = "";
    for (int i = 0; i < numOfCycles; i++)
    {
```

```
      string += cycles.elementAt(i).display();
    }
    if (string.equals(""))
      return "e";
    return string;
  }

  public void simplify()
  {
    // composes all the cycles together in order and takes out any 1-cycles

    Vector<Integer> usedNumbers = new Vector<Integer>();
    Boolean completeCycle;
    int startOfCycle;
    int element;
    Cycle temp;
    Vector<Cycle> newCycles = new Vector<Cycle>();

    for (int k = numOfCycles - 1; k >= 0; k--)
    {
      for (int i = 0; i < cycles.elementAt(k).index(); i++)
      {
        element = cycles.elementAt(k).elementAt(i);
        if (! usedNumbers.contains(element))
        {
          completeCycle = false;
          temp = new Cycle();
          startOfCycle = element;

          while (! completeCycle)
          {
            temp.append(element);
            usedNumbers.add(element);
            for (int j = k; j >= 0; j--)
            {
              element = cycles.elementAt(j).nextAfter(element);
            }
            if (element == startOfCycle)
            {
              if (temp.index() > 1)
              {
                newCycles.add(temp);
              }
              completeCycle = true;
```

```
            }
          }
        }
      }
    }
    cycles = newCycles;
    numOfCycles = cycles.size();
  }

  public void refill(Vector<Cycle> newCycles)
  {
    cycles = newCycles;
    numOfCycles = newCycles.size();

  }
}
```

0.4.3 PermutationApp Class

```
// File:   PermuationApp.java
// Author: Rachel Volkert
// Date:   2/5/2013
// Modified: new

public class PermutationApp
{
  public static void main(String[] args)
  {
    PermutationFrame p = new PermutationFrame();
    p.show();
  }
}
```

0.4.4 PermutationFrame Class

```
// File:   PermutationFrame.java
// Author: Rachel Volkert
// Date:   2/5/2013
// Modified: 2/12/2013
//
// Change: added coset feature: input for subgroup
```

```
import java.awt.*;
import java.awt.event.*;
import java.util.Vector;

public class PermutationFrame extends Frame
{

  private Label inputCycleLabel;
  private Label subgroupLabel;
  private TextArea answerArea;
  private TextField inputField;
  private TextField subgroupField;
  private Permutation myPermutation;

  public PermutationFrame()
  {
    setSize(500, 400);
    setTitle("Permutation App");

    myPermutation = new Permutation();

    inputCycleLabel = new Label("Enter cycles: ");
    // these are to be cycles of terms separated by non-alphanumeric
    // chars (except ,) with cycles separated by commas
    // for example: (123)(45) could be typed as 1 2 3,4 5 or 1@2@3*,4*5
    subgroupLabel = new Label("Enter a subgroup: ");
    // these are to be permutations in a subgroup made up of
    // cycles (separated by ,s) which are separated by ;s
    // for example: {e, (12)(34), (13)(24), (14)(23)}
    // could be entered as 1; 1 2, 3 4; 1 3, 2 4; 1 4, 2 3

    // if this field is filled, the program will permute (left permutation)
    // each of the elements in the subgroup with the permutation
    // entered in the top field
    // the result of the example entries is:
    // {(123)(45), (3541), (2543), (2154)}

    // else if the bottom field is left blank, the program will find
    // the simplification of the cycles entered above
    // the result of only typing the first entry is: (45)(123)

    inputField = new TextField();
    subgroupField = new TextField();
```

```
    Panel inputPanel = new Panel();
    inputPanel.setLayout(new GridLayout(2, 2));
    inputPanel.add(inputCycleLabel);
    inputPanel.add(inputField);
    inputPanel.add(subgroupLabel);
    inputPanel.add(subgroupField);
    add("North", inputPanel);

    answerArea = new TextArea();
    add("Center", answerArea);

    Button simplifyButton = new Button("Simplify");
    simplifyButton.addActionListener(new SimplifyButtonListener());
    add("South", simplifyButton);

    addWindowListener(
    // taken from example in Intermediate Computing (Dr. Wallingford)
         new WindowAdapter()
         {
            public void windowClosing( WindowEvent e )
            {
               System.exit(0);
            }
         });
  }

  private class SimplifyButtonListener implements ActionListener
   {
      public void actionPerformed( ActionEvent e )
      {
        String input = inputField.getText();
        String subgroupString = subgroupField.getText();

        if (! input.equals(""))
        // if there is a cycle to permute do the following, else do nothing
        {
          String[] inputCycleList = input.split(",");
          if (! subgroupString.equals(""))
          // if there are elements for a subgroup listed below
          {
            answerArea.setText(
                findCoset(subgroupString.split(";"), inputCycleList)  );
          }
```

```
        else // else if there is no subgroup listed
        {
          answerArea.setText(simplify(inputCycleList));
        }
        //inputField.setText("");  // optional clearing of the input field
      }
    }
  }

private String findCoset(String[] subgroupList, String[] inputCycleList)
{
  String outputString = "";

  String[] subgroupCyclesList;
  for (int i = 0; i < subgroupList.length; i++)
  {
    subgroupCyclesList = subgroupList[i].split(",");
    String[] newCycleList = new String[subgroupCyclesList.length +
                                        inputCycleList.length];
    for (int j = 0; j < inputCycleList.length; j++)
    { // load the new list with the generating
      // cycle (the input cycles) on the left
      newCycleList[j] = inputCycleList[j];
    }
    for (int k = inputCycleList.length;
           k < subgroupCyclesList.length + inputCycleList.length; k++)
    {
      newCycleList[k] = subgroupCyclesList[k - inputCycleList.length];
    }
    outputString += simplify(newCycleList) + ", ";
  }
  return "{" + outputString.substring(0, outputString.length() - 2) + "}";
}

private String simplify(String[] inputCycleList)
{
  Vector<Cycle> cycleList = new Vector<Cycle>();
  String elements;

  for (int i = 0; i < inputCycleList.length; i++)
  {
    elements = inputCycleList[i];
    cycleList.add(new Cycle(elements.trim()));
  }
```

```
    myPermutation.refill(cycleList);
    //String before = myPermutation.display(); // (*) option for
    // a before/after equation way of displaying results
    // for example: (123)(45) for the permutation cycles and
    // {e, (12)(34), (13)(24), (14)(23)}
    // for a subgroup as seen above would produce the answer
    // {(123)(45)(1) = (123)(45), (123)(45)(12)(34) = (3541),
    //    (123)(45)(13)(24) = (2543), (123)(45)(14)(23) = (2154)}
    // only entering the top permutation gives: (123)(45) = (45)(123)

    myPermutation.simplify();

    //return before + " = " + myPermutation.display(); // use this with
    // option (*) and comment out return statement below
    return myPermutation.display();
  }
}
```

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