General Fuzzy Graphs

Nishad T M[*]
Talal Ali Al-Hawary[†]
B. Mohamed Harif[‡]

Abstract
Fuzzy graphs have many applications in database theory, neural networks, and decision-making problems. Literature survey shows that in the existing definition of fuzzy graph the membership value on links is always less than or equal to the minimum of membership values on the corresponding nodes. This restriction brings a difficulty to address some practical problems where the membership value on links does not depend on that of corresponding nodes. So there exists a gap in the literature on fuzzy graphs. In this article we have developed general fuzzy graph and general weak fuzzy graphs and have proved some properties of general weak fuzzy graphs.
Keywords: Fuzzy graphs, Strong fuzzy graphs, Weak fuzzy graphs, General fuzzy graphs, General weak fuzzy graphs.
2010 AMS subject classification: 03E72,05C72

---

[*]Research Scholar, Department of Mathematics, Rajah Serfoji Government College (Autonomous), (Affiliated to Bharathidasan University), Thanjavur, Tamilnadu, India. nishadtmphd@gmail.com
[†] Professor, Yarmouk University, Jordan. talalhawary@yahoo.com.
[‡]Assistant Professor, Department of Mathematics, Rajah Serfoji Government College, (Autonomous), (Affiliated to Bharathidasan University), Thanjavur-613005, Tamilnadu, India. bmharif@rsgc.ac.in.
[‡]Received on September 15, 2022. Accepted on March 15, 2023. Published on June 30, 2023. DOI:10.23755/rm.v39i0.853. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

The phenomena of uncertainty in real-life situations are described by a mathematical framework by Zadeh in 1965. Azriel Rosenfeld introduced fuzzy graphs in 1975. M S Sunitha and Vijayakumar A defined complement of fuzzy graphs in 2002. A Nagoor Gani and Chandrasekaran introduced $\boldsymbol{\mu}$ complement of a fuzzy graph in 2006. In 2017, T. Al-Hawary discussed certain classes of fuzzy graphs. The existing definition of fuzzy graph in the literature shows that the membership value on links is always less than or equal to the minimum of membership values on the corresponding nodes. This restriction brings difficulty to address some practical problems where the membership value on links does not depend on that of corresponding nodes. To fill this literature gap, in this article, we have developed general fuzzy graphs.

In this article, the second section discusses some existing definitions. In the third section, we introduce the new definitions-general fuzzy graph and general weak fuzzy graph. In the fourth section, the operations-union, c-complement and $|\mu|$complement on general fuzzy graphs are defined. In the fifth section, some properties of general weak fuzzy graphs are proved.

2. Fuzzy Graphs

In this section, we recall some existing definitions of fuzzy graphs, strong fuzzy graphs, weak fuzzy graphs, and complete fuzzy graphs.

Definition 2.1: A fuzzy graph with V as the underlying set is a pair of functions G:(σ, μ) where σ:$V \to [0,1]$ is a fuzzy subset, μ:$V \times V \to [0,1]$ is a fuzzy relation on the fuzzy subset σ such that for all $u, v \in V$, we have $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$ where $\wedge$ stands minimum.

The underlying crisp graph of a fuzzy graph G:(σ, μ) is denoted by G*:(σ^*, μ^*), where $\sigma^* = \{u \in V/\sigma(u) > 0\}$ and $\mu^* = \{(u,v) \in V \times V/\mu(u,v) > 0\}$ .Two nodes u and v are said to be neighbours if $\mu(u,v) > 0$.

Definition 2.2: A fuzzy graph G:(σ, μ) is a strong fuzzy graph if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$for all$(u,v) \in \mu^*$.

Definition 2.3: A fuzzy graph G:(σ, μ) is a weak fuzzy graph if $\mu(u,v) < \sigma(u) \wedge \sigma(v)$for all$(u,v) \in \mu^*$.

Definition 2.4: A fuzzy graph G:(σ, μ) is a complete fuzzy graph if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$for all $(u, v) \in \sigma *$

Definition 2.5: The complement of a fuzzy graph G:(σ, μ) is a fuzzy graph $\bar{G}: (\bar{\sigma}, \bar{\mu})$ where $\bar{\sigma} = \sigma$ and $\bar{\mu}(u, v) = \sigma(u) \wedge \sigma(v)$-$\mu(u, v)$ for all $u, v \in V$.

Definition 2.6: The μ −complement of a fuzzy graph G:(σ, μ) is a fuzzy graph $G^{\mu}: (\sigma, \mu^{\mu})$ where μ^{μ} is defined as $\mu^{\mu}(u, v) = 0$ if $\mu(u, v) = 0$ and $\mu^{\mu}(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v)$ if $\mu(u, v) > 0$ for all $u, v \in \sigma^*$.

3. General Fuzzy Graph

In this section, we introduce the relatively new definitions of general fuzzy graph and general weak fuzzy graph.

Definition 3.1: A general fuzzy graph with V as the underlying set is a pair of functions G:(σ, μ) where σ:$V \rightarrow [0,1]$ is a fuzzy subset of V and $\mu : V \times V \rightarrow [0,1]$, is a fuzzy subset of $V \times V$. To illustrate our definition, we provide the following example:

Example:3.1. The graph G: (σ, μ) is a general fuzzy graph.

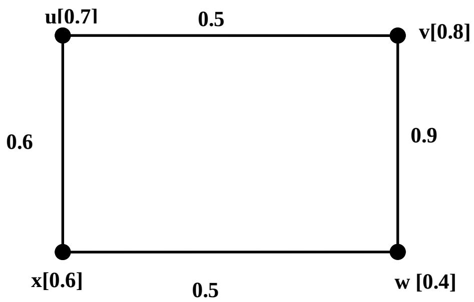

Figure 3.1, General fuzzy graph G: (σ, μ)

Definition 3.2: A general weak fuzzy graph with V as the underlying set is a pair of functions G:(σ, μ) where $\sigma : V \rightarrow [0,1]$ is a fuzzy subset of V and $\mu : V \times V \rightarrow [0,1]$, is a fuzzy subset of $V \times V$ such that for all $(u, v) \in \mu^*$, $\mu(u, v) \neq \sigma(u) \wedge \sigma(v)$.

4. Some Operations on General Fuzzy Graphs

Union: Let $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ be two general fuzzy graphs with the underlying crisp graphs $G_1: (V_1, X_1)$ and $G_2: (V_2, X_2)$ respectively. The union $G: (\sigma_1 \cup \sigma_2,\ \mu_1 \cup \mu_2)$ is defined by:
$(\sigma_1 \cup \sigma_2)(u) = \sigma_1(u)$ if $u \in V_1 - V_2$, $(\sigma_1 \cup \sigma_2)(u) = \sigma_2(u)$ if $u \in V_2 - V_1$, and
$(\sigma_1 \cup \sigma_2)(u) = \max\{\sigma_1(u), \sigma_2(u)\}$ if $u \in V_1 \cap V_2$.
$(\mu_1 \cup \mu_2)(u, v) = \mu_1(u, v)$ if $(u, v) \in X_1 - X_2$, $(\mu_1 \cup \mu_2)(u, v) = \mu_2(u, v)$ if $(u, v) \in X_2 - X_1$ and $(\mu_1 \cup \mu_2)(u, v) = \max\{\mu_1(u, v), \mu_2(u, v)\}$ if $(u, v) \in X_1 \cap X_2$.

c-complement: The c-complement of a general fuzzy graph $G:(\sigma, \mu)$ is a fuzzy graph $G^c: (\sigma^c, \mu^c)$ where $\sigma^c = \sigma$, $\mu^c(u, v) = 0$ if $\mu(u, v) = 0$ and $\mu^c(u, v) = 1 - \mu(u, v)$, for all $u, v \in \sigma^*$.

Example:4.1. The c-complement of the general fuzzy graph $G:(\sigma, \mu)$ in example 3.1, $G^c: (\sigma^c, \mu^c)$ is represented in the following figure.

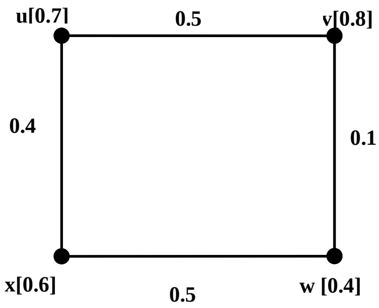

Figure 4.1, c-complement $G^c: (\sigma^c, \mu^c)$

$|\boldsymbol{\mu}|$ **complement:** The $|\mu|$ complement of a general fuzzy graph $G:(\sigma, \mu)$ is a fuzzy graph $G^{|\mu|}: (\sigma, \mu^{|\mu|})$ where $\mu^{|\mu|}$ is defined as $\mu^{|\mu|}(u, v) = 0$ if $\mu(u, v) = 0$ and $\mu^{|\mu|}(u, v) = |\sigma(u) \wedge \sigma(v) - \mu(u, v)|$ if $\mu(u, v) > 0$ for all $u, v \in \sigma^*$.

Example:4.2. The $|\mu|$ complement of the general fuzzy graph $G:(\sigma, \mu)$ in example 3.1, $G^{|\mu|}: (\sigma, \mu^{|\mu|})$ is represented in the following figure.

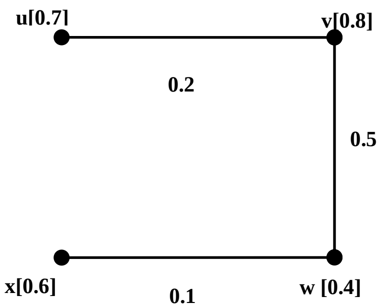

Figure 4.2, $|\mu|$complement $G^{|\mu|}: (\sigma, \mu^{|\mu|})$

5. Some Properties of General Weak Fuzzy Graphs

In this section, we discuss some properties of general fuzzy weak graphs.

Theorem 5.1. General fuzzy graphs are closed under c-complementation and $|\mu|$ complementation.
The proof of this theorem directly follows from the definitions.

Theorem 5.2. The union of two general weak fuzzy graphs need not be a general weak fuzzy graph.
Proof. Let $F_1: (m_1, r_1)$ and $F_2: (m_2, r_2)$ be two general weak fuzzy graphs with the underlying crisp graphs $F_1^*: (N_1, X_1)$ and $F_2^*: (N_2, X_2)$ respectively where $N_1=m_1^*$, $N_2=m_2^*$,$X_1=r_1^*$ and $X_2=r_2^*$. The union $F: (m_1 \cup m_2, r_1 \cup r_2)$ is defined by
$(m_1 \cup m_2)(a) = m_1(a)$ if $a \in N_1 - N_2$.
$(m_1 \cup m_2)(a) = m_2(a)$ if $a \in N_2 - N_1$, and
$(m_1 \cup m_2)(a) = \max\{ m_1(a), m_2(a)\}$ if $a \in N_1 \cap N_2$.
$(r_1 \cup r_2)(a, b) = r_1(a, b)$ if $(a, b) \in X_1 - X_2$.
$(r_1 \cup r_2)(a, b) = r_2(a, b)$ if $(a, b) \in X_2 - X_1$, and
$(r_1 \cup r_2)(a, b) = \max\{r_1(a, b), r_2(a, b)\}$ if $(a, b) \in X_1 \cap X_2$.
Case 1: When $(a, b) \in X_1 - X_2$
Since $F_1: (m_1, r_1)$ is a general weak fuzzy graph$(r_1 \cup r_2)(a, b) = r_1(a, b) \neq m_1(a) \wedge m_1(b)$. Since $m_2(a) = 0$ for all $a \in N_1 - N_2$, $(r_1 \cup r_2)(a, b) = r_1(a, b) \neq ((m_1 \cup m_2)(a) \wedge (m_1 \cup m_2)(b))$.
Case 2: When $(a, b) \in X_2 - X_1$.
Since $F_2: (m_2, r_2)$ is a general weak fuzzy graph, $(r_1 \cup r_2)(a, b) = r_2(a, b) \neq m_2(a) \wedge m_2(b)$. Since $m_1(a) = 0$ for all $a \in N_2 - N_1$,$(r_1 \cup r_2)(a, b) = r_2(a, b) \neq ((m_1 \cup m_2)(a) \wedge (m_1 \cup m_2)(b))$.
Case 3: When $(a, b) \in X_1 \cap X_2$
Suppose $m_2(a) > m_2(b)$ and $r_1(a, b) = m_1(a) \wedge m_1(b) = m_2(a) = r_2(a, b)$, then $(r_1 \cup r_2)(a, b) = r_1(a, b) = r_2(a, b)$. In this case $F: (m_1 \cup m_2, r_1 \cup r_2)$ is not a general weak fuzzy graph

Theorem 5.3. The $|\mu|$ complement of a general weak fuzzy graph is a general weak fuzzy graph.
Proof. Let $F^{|\boldsymbol{\mu}|}: (m, r^{|\boldsymbol{\mu}|})$ be the $|\mu|$ complement of a general weak fuzzy graph F: (m, r),where $r^{|\boldsymbol{\mu}|}$ is defined as
$r^{|\boldsymbol{\mu}|}(a, b) = |m(a) \wedge m(b) - r(a, b)|$ if $r(a, b) > 0$ for all $a, b \in m^* \rightarrow (1)$
and $r^{|\boldsymbol{\mu}|}(a, b) = 0$ if $r(a, b) = 0 \rightarrow (2)$

Since F:(m, r) is a general weak fuzzy graph for all $(a, b) \in r^*$, $r(a, b) \neq m(a) \wedge m(b) \rightarrow (3)$

To prove that $F^{|\mu|}: (m, r^{|\mu|})$ is a general weak fuzzy graph, it is enough to prove that for all $(a, b) \in r^{|\mu|^*}$,$r^{|\mu|}(a, b) \neq m(a) \wedge m(b)$

Case 1. Let$(a, b) \in r^{|\mu|^*}$ such that $r(a, b) > 0$.Then by the equations(1) and(3)

$r^{|\mu|}(a, b) = |m(a) \wedge m(b) - r(a, b)| \neq m(a) \wedge m(b)$

Case 2. Let $(a, b) \in r^{|\mu|^*}$ such that $r(a, b) = 0$.Then by the equation (2),$r^{|\mu|}(a, b) = 0$

Therefore, in both cases $r^{|\mu|}(a, b) \neq m(a) \wedge m(b)$for all $(a, b) \in r^{|\mu|^*}$.

Remark 5.4. If G is a general weak fuzzy graph then $(G^{|\mu|})^{|\mu|}$ need not be isomorphic to G.

Proof. A particular example is enough to prove this.

Consider G: (σ, μ)

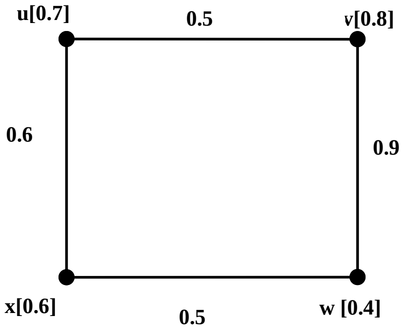

Figure 5.1, G :(σ, μ)

$G^{|\mu|}: (\sigma, \mu^{|\mu|})$ is as follows

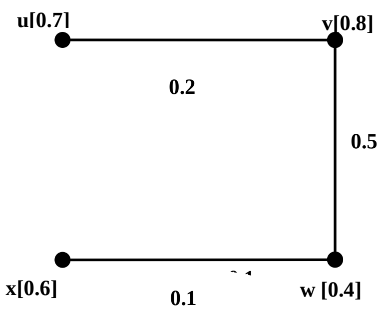

Figure 5.2, $G^{|\mu|}: (\sigma, \mu^{|\mu|})$

$(G^{|\mu|})^{|\mu|}$ is

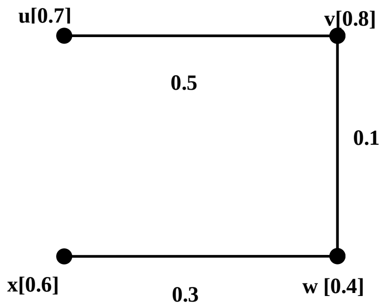

Figure 5.3, $(G^{|\mu|})^{|\mu|}$

This example shows that$(G^{|\mu|})^{|\mu|}$ need not be isomorphic to G.

Theorem 5.4. If *F:* (m, r) is a general weak fuzzy graph such that $m(a) \wedge m(b) > r(a, b)$ for all $(a, b) \in r^*$, then $(F^{|\mu|})^{|\mu|} = F$.

Proof. Let $F^{|\mu|}: (m, r^{|\mu|})$ be the $|\mu|$complement of *F:* (m, r). Since *F:* (m, r) is a general weak fuzzy graph,for all $(a, b) \in r^*, r(a, b) \neq m(a) \wedge m(b)$

We shall consider two cases.

Case 1. For all $(a, b) \in r^*$.As per definition of $r^*, r(a, b) > 0$. In this case $r^{|\mu|}(a, b) = |m(a) \wedge m(b) - r(a, b)|$. Now $m(a) \wedge m(b) > r(a, b)$ for all $(a, b) \in r^*$, implies that $(r^{|\mu|})^{|\mu|}(a, b) = \left|m(a) \wedge m(b) - r^{|\mu|}(a, b)\right| = r(a, b)$

Case 2. For all $(a, b) \notin r^*$. As per definition of $r^*, r(a, b) = 0$.

Here $r^{|\mu|}(a, b) = 0$ and $(r^{|\mu|})^{|\mu|}(a, b) = 0$. Therefore $(F^{|\mu|})^{|\mu|} = F$.

6 Conclusions

In this article we discussed the literature gap in existing research in fuzzy graph theory. To fill that gap, we have introduced general fuzzy graphs. The literature review shows that various authors describe the properties of strong fuzzy graph. But the studies on weak fuzzy graphs have not been given importance. In this article we defined general weak fuzzy graphs and proved some properties. We wish to extend our further studies on operations, algebraic properties and applications of general weak fuzzy graphs, weak fuzzy graphs, interval-valued weak fuzzy graphs, and various types of intuitionistic weak fuzzy graphs.

References

A. Nagoor Gani and Chandrasekaran. V.T, Free nodes and busy nodes of a fuzzy graph. East Asian Math J.22(2006), No.2.pp 163-170.

A. Nagoor Gani. A and Chandrasekaran. V. T, A first look at Fuzzy Graph Theory, Allied publishers Pvt. Ltd (2010).

A Prasanna and T. M. Nishad, Some properties of weak fuzzy graphs, J. Math. Comput.Sci.11(3)(2021),3594-3601.

B. Mohamed Harif and Nishad T M, Interval valued weak fuzzy graphs and Fermat's weak fuzzy Graphs, Advances and Applications in Mathematical Sciences, Volume 22,Issue 2, Dec.2022, 487-501.

G. Nirmala and Vijaya. M, Fuzzy Graphs on Composition, Normal and Tensor Products, IJSR Publications, 2 (2012) 1-7.

J. N Mordeson and Chang-Shyh Peng, Operations on fuzzy graphs, Information Sciences, 79 (1994) 169-170.

J. N Mordeson and Nair P.S. Fuzzy Graphs and Fuzzy Hypergraphs, Physica-Verlag (2000).

K. R. Bhutani and Rosenfeld A, Strong arcs in fuzzy graphs, Information Sciences, 152 (2003) 319-322.

L A Zadeh., Fuzzy Sets, Inform. and Control 8 (1965) 338-353.

M L N Mcallister, Fuzzy Intersection Graphs, Intl. J. Computers in Mathematics with Applications,15 (1988) 871-886.

M S Sunitha and Vijayakumar. A, Complement of fuzzy graphs, Indian J, Pure appl Math, 33(9); 1451-1464 Sept 2002.

Nishad T. M, B. M Harif, Group decision making in conditions of uncertainty using Fermat's weak fuzzy graph and Beal's weak fuzzy graph, Ratio Mathematica, Vol. 43, Dec. 2022.

P. S Nair, Triangle and parallelogram laws on fuzzy graphs, Pattern Recognition letters, 15 (1994) 803-805.

Talal Ali Al-Hawary, Certain Matrices and Energies of Fuzzy Graphs,TWMS J. App.Eng.Math.V.11(3), 2021,1-17.

Talal Ali Al-Hawary, Certain classes of fuzzy graphs, Eur. J. Pure Appl.Math.10(3) (2017), 552-560.

Talal Ali Al-Hawary, Complete fuzzy graphs, Inter.J.Math.Cobin.Vol.4 (2011),26-34.

Talal Ali Al-Hawary, Complete Hamacher fuzzy graphs, Journal of Applied and Informatics. 2022, 40(5-6), pp.1043—1052.

Talal Ali Al-Hawary, Density Results for Perfectly Regular and Perfectly Edge-regular Fuzzy Graphs, J. Disc. Math. Scie. & Cryptography 2(1) (2022), 1-10.

Talal Ali Al-Hawary, On Intuitionistic Product Fuzzy Graphs, Ital. J. Pure. Appl. Math.38(2017),113-126.

Talal Ali Al-Hawary, Strong Modular Product and Complete Fuzzy Graphs, Ital. J. Pure Appl. Math.