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Complete Bell Polynomials and Recurrence Relations for Arithmetic Functions

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Abstract:

We use the Z-transform to solve a type of recurrence relation satisfied by the number of representations of an integer *n* as the sum of squares and as the sum of triangular numbers, and also by the color partitions of *n*; the corresponding solution is in terms of the complete Bell polynomials.

Keywords: *sum of divisors function, Z-transform, sums of squares, color partitions, recurrence relations, sums of triangular numbers, bell polynomials, sum of inverses of odd divisors of an integer.*

Introduction

We know the following recurrence relations (Andrews, Jha & L´opez-Bonilla, 2023; Pathan et al., 2023):

$$n\, r_k(n) = -2\, k \sum_{j=1}^{n} (-1)^j\, j\, D(j) r_k(n-j),$$

$$n\, t_k(n) = -k\, \sum_{j=1}^{n} j\ T(j) t_k(n-j), \qquad (1)$$

$$n\, p_k(n) = -k \sum_{j=1}^{n} \sigma(j)\, p_k(n-j),$$

where σ is the sum of divisors function (Apostol, 1976; Hardy & Wright, 1979; Sivaramakrishnan, 1989; Sivaraman & López-Bonilla, 2023); $r_k(n)$ and $t_k(n)$ are the number of representations of n as a sum of k squares and k triangular numbers, respectively (Taussky, 1970; Grosswald, 1984; Grosswald, 1985; Ono et al., 1995; Cooper, 2001; Moreno & Wagstaff, 2006), besides, $p_k(n)$ is the number of color partitions of n (Lazarev et al., 2010; López-Bonilla & Yaljá Montiel-Pérez, 2021; López-Bonilla & Morales-García, 2022; López-Bonilla et al. 2021); finally, in (1) we have the expressions:

$D(j) = \sum_{odd\ d|j} \frac{1}{d}$, $T(j) = \sum_{d|j} \frac{1+2\,(-1)^d}{d} = \frac{1}{j} \sum_{d|j} (-1)^d\, d$. (2)

Then it is natural to study recurrence relations with the structure of a Cauchy convolution (Sivaramakrishnan, 1989):

$$n\, f_k(n) = \sum_{j=1}^{n} h(j)\, f_k(n-j), \quad k \geq 1, \quad n \geq 0, \tag{3}$$

verifying the properties $f_k(0) = 1 \quad \forall\, k$ and $h(0) = 0$.

In Sec. 2 we apply the Z- transform to (3) to obtain the following solution (Debnath & Bhatta, 2007; Patra, 2018; López-Bonilla, Vidal-Beltrán & Zúñiga-Segundo, 2018a; Vidal-Beltrán & Zúñiga-Segundo, 2018b; López-Bonilla, López-Vázquez & Vidal-Beltrán, 2018):

$$f_k(n) = \frac{1}{n!} B_n\big(0!\, h(1), 1!\, h(2), 2!\, h(3), \ldots, (n-1)!\, h(n)\big), \tag{4}$$

in terms of the complete Bell polynomials (Kölbig, 1994; Johnson, 2002; Connon, 2010; Wikipedia, N/A).

Solution of (3)

If $F(z)$ and $G(z)$ are the Z-transforms of the sequences $\{ f_k(0), f_k(1), f_k(2), \ldots \}$ and $\{h(0), h(1), \ldots\}$, respectively:

$$F(z) = 1 + \frac{f_k(1)}{z} + \frac{f_k(2)}{z^2} + \cdots \quad , H(z) = \frac{h(1)}{z} + \frac{h(2)}{z^2} + \cdots , \tag{5}$$

then (3) implies the differential equation:

$$- z\, \frac{d}{dz} F = H(z)\, F(z), \tag{6}$$

whose integration gives the solution:

$$Ln\ F = \frac{h(1)}{z} + \frac{h(2)}{2\, z^2} + \frac{h(3)}{3\, z^3} + \cdots , \tag{7}$$

that is:

$$F(z) \coloneqq \sum_{n=0}^{\infty} f_k(n) \frac{1}{z^n} = \exp\Big(\sum_{j=1}^{\infty} \frac{h(j)}{j} \frac{1}{z^j} \Big) . \tag{8}$$

On the other hand, we have the generating function of the complete Bell polynomials (Wikipedia, N/A):

$$\sum_{n=0}^{\infty} \frac{1}{n!} B_n(x_1, x_2, \ldots, x_n) \frac{1}{z^n} = \exp\Big(\sum_{j=1}^{\infty} \frac{x_j}{j!} \frac{1}{z^j} \Big), \tag{9}$$

whose comparison with (8) implies (4).

This completes the proof.

Application of (4) to $r_k(n), t_k(n)$ and $p_k(n)$.

The result (4) gives the solution of recurrence relations with the structure (3), then its application to (1) allows obtain the following interesting explicit expressions:

$$r_k(n) = \frac{1}{n!} B_n\big(2k \cdot 1!\, D(1), -2k \cdot 2!\, D(2), 2k \cdot 3!\, D(3), \ldots, -2k\, (-1)^n \cdot n!\, D(n)\big),$$

$$t_k(n) = \frac{1}{n!} B_n\big(-k \cdot 1!\, T(1), -k \cdot 2!\, T(2), -k \cdot 3!\, T(3), \ldots, -k \cdot n!\, T(n)\big), \tag{10}$$

$$p_k(n) = \frac{1}{n!} B_n\big(-k \cdot 0!\ \sigma(1), -k \cdot 1!\ \sigma(2), -k \cdot 2!\ \sigma(3), \ldots, -k \cdot (n-1)!\ \sigma(n)\big).$$

Besides, we know the recurrence relation (Osler, 2007; Flajolet & Sedgewick, 2008):

$$n\, p(n) = \sum_{j=1}^{n} \sigma(j)\, p(n-j), \qquad (11)$$

for the partition function (Apostol, 1976; Hardy & Wright, 1979; Sivaramakrishnan, 1989), hence from (4):

$$p(n) = \frac{1}{n!} B_n\big(0!\ \sigma(1), 1!\ \sigma(2), 2!\ \sigma(3), \ldots, (n-1)!\ \sigma(n)\big). \qquad (12)$$

As a last application, Robbins deduced the following recurrence (Robbins, 1999; Robbins, 2002):

$$n\, p_D(n) = \sum_{j=1}^{n} \sigma_O(j)\, p_D(n-j), \sigma_O(n) = \sum_{odd\ d|n} d = \sum_{d|n} (-1)^{d-1} \frac{n}{d}, \qquad (13)$$

where $p_D(n)$ is the number of partitions of n using only distinct parts, therefore:

$$p_D(n) = \frac{1}{n!} B_n\big(0!\ \sigma_O(1), 1!\ \sigma_O(2), 2!\ \sigma_O(3), \ldots, (n-1)!\ \sigma_O(n)\big). \qquad (14)$$

Conclusion

In this short paper, we have obtained nice expressions for obtaining number of representations of a positive integer *n* in terms of *k* triangular numbers and *k* square numbers. We have also obtained a nice expression for expressing color partitions of a positive integer *n* in terms of complete Bell polynomials. We have used Cauchy convolution and Z - transform technique in order to obtain such expressions.

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