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Combinatorial Game and Number Triangle

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Abstract:

By considering a triangular array of numbers for first six rows, we introduce a combinatorial game, whose solution depends on a number triangle. The conclusion brings us with a surprising consequence in deciding the result of the game. This paper analyzes the game and present the solution in detail using the number triangle which resembles the famous Leibniz Triangle of unit fractions.

Keywords: *combinatorial game, generation rule, leibniz harmonic triangle, modified number triangle, general binomial theorem.*

Introduction

Among several combinatorial games that exist in mathematics, this paper discusses one such game in the form of a classic solitaire puzzle. The whole game is played in a triangular array whose first six rows are shown in Figure 1.

Using this triangular array and three red coins, we frame the rules for the game and discuss the possibility of winning the game, which is the focus of this paper. Interestingly the solution depends on a number triangle similar to that of the famous Leibniz Harmonic Triangle.

Describing the Game

With reference to Figure 1, the game begins with three coins at the first two rows such that, the first coin placed at first row and other two coins at second row. These three coins were shown as red color forming the apex (shown in small yellow triangle) of the triangular array of Figure 1.

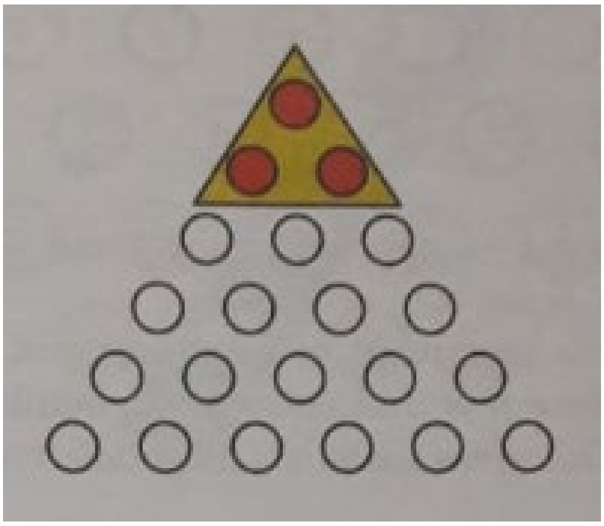

Figure 1. Triangular Array with First Six Rows

We now define a "move" in the game which consists of removing a coin from the array that has two empty cells just below it and replacing it with two coins in those empty cells. For example, here is a game with two possible starting moves as shown in Figure 2.

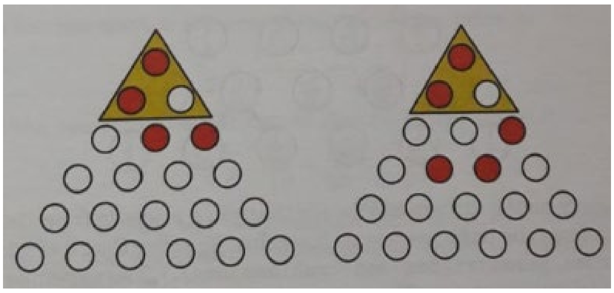

Figure 2. Two Possible Starting Positions of the Game

The objective of this game is to empty the three red coins (inside the yellow triangle at the Apex forming first two rows) of the triangular array with suitably replaced coins at other rows. Can any player accomplish this task? The answer to this question was explained in this paper.

Leibniz's Harmonic Triangle

We first consider the well-known Pascal's triangle which is a triangular array of numbers which is symmetrical about the central vertical line. The numbers of the Pascal's triangle are viewed as binomial coefficients. Pascal's triangle has innumerably many properties and have huge occurrence in many combinatorial and probability problems. In our case, we use Pascal's triangle to construct a new triangle called Leibniz Harmonic Triangle as shown in Figure 3.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1
\end{array}
\qquad
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad \frac{1}{2} \quad 1 \\
1 \quad \frac{1}{3} \quad \frac{1}{3} \quad 1 \\
1 \quad \frac{1}{4} \quad \frac{1}{6} \quad \frac{1}{4} \quad 1 \\
1 \quad \frac{1}{5} \quad \frac{1}{10} \quad \frac{1}{10} \quad \frac{1}{5} \quad 1
\end{array}
\qquad
\begin{array}{c}
1 \\
\frac{1}{2} \quad \frac{1}{2} \\
\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3} \\
\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4} \\
\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{20} \quad \frac{1}{5} \\
\frac{1}{6} \quad \frac{1}{30} \quad \frac{1}{60} \quad \frac{1}{60} \quad \frac{1}{30} \quad \frac{1}{6}
\end{array}
$$

Figure 3. Construction of Leibniz Triangle from Pascal Triangle

The leftmost picture is simply the Pascal's Triangle. Take the reciprocals of all numbers in every row of Pascal's triangle to obtain the triangle in middle picture. Now divide first row number by 1, second row numbers by 2, third row numbers by 3, fourth row numbers by 4 and so on to obtain the triangle in the rightmost picture. This resulting triangle is shown in Figure 4.

We notice that each number apart from 1 in the top row is a unit fraction. Further the numbers in the leftmost (or at the rightmost) leading diagonals are the harmonic numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots$. These numbers are precisely the reciprocals of the natural numbers. The German Polymath Genius Gotfried Wilhelm Leibniz provided some mathematical properties of the numbers in the triangle of Figure 4. Hence the triangle got the name Leibniz's Harmonic Triangle.

$$
\begin{array}{c}
1 \\
\frac{1}{2} \quad \frac{1}{2} \\
\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3} \\
\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4} \\
\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{20} \quad \frac{1}{5} \\
\frac{1}{6} \quad \frac{1}{30} \quad \frac{1}{60} \quad \frac{1}{60} \quad \frac{1}{30} \quad \frac{1}{6}
\end{array}
$$

Figure 4. Leibniz's Harmonic Triangle

Though the construction of the triangle is explained through numbers of the associated

Pascal's triangle we also see the following generating rule for constructing Leibniz's Harmonic Triangle of Figure 4. Beginning with 1 at the top row, we always consider $\frac{1}{n}$ at the beginning and end of row n for $n = 1, 2, 3, 4, \ldots$ To construct the middle numbers from third and other subsequent rows, let us consider the generation rule given in Figure 5.

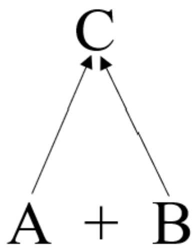

Figure 5. Generation Rule for Leibniz's Harmonic Triangle

Thus for example, the middle number in third row is $\frac{1}{6}$ because, according to generating rule of Figure 5, we have $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$ and so $\frac{1}{6}=\frac{1}{2}-\frac{1}{3}$. Similarly, the two central numbers of fourth row is obtained by the generating rule of Figure 5, since $\frac{1}{4}+\frac{1}{12}=\frac{1}{3}$ we get, $\frac{1}{12}=\frac{1}{3}-\frac{1}{4}$. Similarly, since $\frac{1}{12}+\frac{1}{12}=\frac{1}{6}$ we get, $\frac{1}{6}-\frac{1}{12}=\frac{1}{12}$

Modified Number Triangle

We now construct a modified number triangle similar to that of Leibniz's Harmonic Triangle with the same generation rule as given in Figure 5. Instead of considering the harmonic numbers along the leading leftmost and rightmost diagonals if we consider the reciprocals of powers of 2 of the form $\frac{1}{2^n}$ for $n = 0, 1, 2, 3, 4, \ldots$ we get a triangle which we call Modified Number Triangle shown in Figure 6.

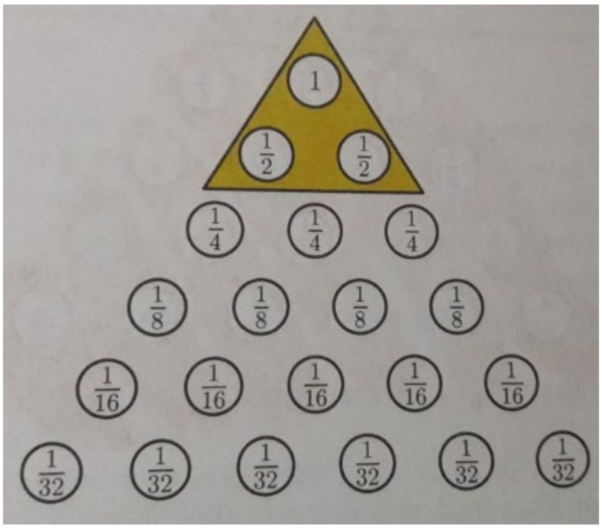

Figure 6. Modified Number Triangle

In constructing as per the given instructions following the same generating rule as in Leibniz's Harmonic Triangle, we see that the modified number triangle in Figure 6 is such that it possess numbers of the form $\frac{1}{2^n}$ for $n = 0, 1, 2, 3, 4, \ldots$ in row n of the triangle.

Using the modified number triangle in Figure 6, we now try to determine the solution to the coin game provided in section 2.

Theorem 1

As the game is played, the sum of the values of all the coins at any stage of the game is always 2.

Proof: The game begins with three coins occupying the first two rows in little yellow triangle of values $1, \frac{1}{2}, \frac{1}{2}$. The sum of these three values is $1+\frac{1}{2}+\frac{1}{2}=2$. We notice that any move of the game removes a coin and replace it two coins just below it. But if we make these replacements of two new coins then their sum would be equal to the value of the single coin considered before replacing because of the generating rule of Leibniz's Harmonic Triangle which is same as that for Modified Number Triangle of Figure 6.

Thus, for example, if we make the move of first coin of value $\frac{1}{2}$ in the second row, with two numbers of values $\frac{1}{4},\frac{1}{4}$ in the third row, then by the generating rule in Figure 5, we see that $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$. The same phenomenon continues for every move in the modified number triangle of Figure 6.

In general, we notice that by replacing a coin of value $\frac{1}{2^n}$ in nth row of the modified number triangle with two coins of values $\frac{1}{2^{n+1}},\frac{1}{2^{n+1}}$ in the $(n+1)$th row, then we see that $\frac{1}{2^{n+1}}+\frac{1}{2^{n+1}}=\frac{1}{2^n}$.

These ideas ensures us the total value of the coins after each move at any stage of the game never changes and will be equal to the values of the first three coins considered initially which is $1+\frac{1}{2}+\frac{1}{2}=2$. This completes the proof.

Theorem 2

It is impossible to empty the three red coins at the Apex in a triangular array with finitely many rows.

Proof: We first notice from Figure 2, that the game is presented in a triangular array board with six rows in which the first two rows contains three coins at the Apex. As described in section 2, the objective of the game is to have no coins at the top three positions of the array (inside the yellow triangle). To achieve this, we have to produce a configuration of coins covering only cells of value $\frac{1}{4}$ or less.

From Figure 6, we see that there are three positions of value $\frac{1}{4}$, four positions of value $\frac{1}{8}$, five positions of value $\frac{1}{16}$, six positions of value $\frac{1}{32}$ and so on.

We now try to compute the total values of these coins assuming that we extend the triangular array board to infinitely many rows (not just with six rows as in Figure 2). That is, we now try to find the value of

$$\left(3\times\frac{1}{4}\right)+\left(4\times\frac{1}{8}\right)+\left(5\times\frac{1}{16}\right)+\left(6\times\frac{1}{32}\right)+\cdots \quad (1)$$

Using General Binomial Theorem we get

$$(1-x)^{-2}=1+2x+3x^2+4x^3+5x^4+6x^5+\cdots \quad (2)$$

The infinite series in (2) converges if $-1<x<1$. Hence considering $x=\frac{1}{2}$ in (2), we get

$$\left(1-\frac{1}{2}\right)^{-2}=1+2\left(\frac{1}{2}\right)+\left[\left(3\times\frac{1}{4}\right)+\left(4\times\frac{1}{8}\right)+\left(5\times\frac{1}{16}\right)+\left(6\times\frac{1}{32}\right)+\cdots\right]$$

Thus,

$$\left(3\times\frac{1}{4}\right)+\left(4\times\frac{1}{8}\right)+\left(5\times\frac{1}{16}\right)+\left(6\times\frac{1}{32}\right)+\cdots=2 \quad (3)$$

The calculation from equation (3) reveals the fact that if we place a coin on each and every cell of value $\frac{1}{4}$ or less, then the total value of that configuration will be 2, which is the sum of the values of initial three coins at the Apex in the little yellow triangle. Thus if we leave any cells uncovered, then the total value of the coins placed on each subsequent move will be less than 2.

But we know from Theorem 1, that 2 is the total value at any stage of playing the game. Also, at any stage of playing the game there will be only finitely many coins on the triangular array board. By Theorem 1, the sum of values covered by the coins at any stage of play must be 2. To get a sum of values equal to 2 away from the top two rows at the Apex of the triangular array (marked in little yellow triangle), by equation (3), we must have an infinite number of coins to play. But our original triangular array contains only finitely many (in fact six as in Figure 2) rows and hence finitely many coins can occupy at any stage of the game. Hence with finitely many coins at any stage it is impossible to get the sum of values of the coins as 2. This completes the proof.

We finally try to answer the question that if you have special super power to possess an board with infinitely many rows and assuming that you can play for as long as you please, will you be able to remove the three red coins at the Apex and complete the task of the game? This is answered in the following theorem.

Theorem 3

It is impossible to empty the three red coins at the Apex in a triangular array with infinitely many rows.

Proof: From theorem 1, we know that sum of the values of all the coins at any stage of the game is always 2. Suppose if we have a very special triangular array board with infinitely many rows and if we can have enough energy to play the game as long as we can, then we know that we have to produce a configuration of coins covering only cells of value $\frac{1}{4}$ or less. Assuming that you make very clever moves at each stage of the game, you will notice that after certain finite number of moves either not all the cells below the first two rows are covered. Hence by (3) of Theorem 2, the sum of the values of the coins replaced at any stage will be less than 2. Thus, even with infinitely many rows, it is impossible to get the total value of the coins as 2. This completes the proof.

Conclusion

By considering a simple combinatorial game with three coins and relating it to Leibniz's Harmonic Triangle through which we obtained Modified Number Triangle we deduced that the objective of the game cannot be accomplished for any triangular array board either with finitely many rows or even with infinitely many rows. These facts were established through three theorems in the paper presented in section 5. It is wonderful to see how a simple game with easy to describe move pattern generate such a wide range of connections with seemingly unrelated concepts, which is often the case with mathematical research.

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