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A Discrete Morse Approach for Computing Homotopy Types: An Exploration of the Morse, Generalized Morse, Matching, and Independence Complexes

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An Exploration of the Morse, Generalized Morse, Matching, and Independence Complexes

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Abstract

In this thesis, we study possible homotopy types of four families of simplicial complexes–the Morse complex, the generalized Morse complex, the matching complex, and the independence complex–using discrete Morse theory. Given a simplicial complex, K, we can construct its Morse complex, $\mathcal{M}(K)$, from all possible discrete gradient vector fields on K. A similar construction will allow us to build the generalized Morse complex, $\mathcal{GM}(K)$, while considering edges and vertices will allow us to construct the matching complex, $\mathrm{M}(K)$, and independence complex, I_K.

In Chapter 3, we use the Cluster Lemma and the notion of star clusters to apply matchings to families of Morse, generalized Morse, and matching complexes, computing their homotopy types. Notably, we show that the Morse complex of a subset of extended star graphs is homotopy equivalent to a wedge of spheres (Theorem 17) and the matching complex of a Dutch windmill graph is homotopy equivalent to a point, sphere, or wedge of spheres (Theorem 29). In Chapter 4, we use a degenerate Hasse diagram, along with strong collapses to compute the homotopy type of many families of Morse complexes. Recognizably, we provide computations showing wedged complexes as suspensions (Corollary 56, Proposition 58) and provide a sufficient condition for strongly collapsible Morse complexes (Theorem 61). Lastly, in Chapter 5 we study chord diagrams–a largely unexplored topic–and provide insight into the possible homotopy types of the independence complex of intersection graphs of chord diagrams. We realize spheres and wedges of spheres as possible homotopy types (Corollaries 68 and 70) and begin to explore what families of intersection graphs can be represented as a chord diagram. From here, most interesting, we show that ladder graphs can be represented as chord diagrams, and the independence complex of a ladder graph has the homotopy type of a sphere (Proposition 78).

KEYWORDS: Morse complex, matchings, strong collapses, homotopy type, chord diagrams

Dedicated to my sisters: Erin, Kylae, and Regan.
May your daydreams explode into your realities.

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Chapter 1

Introduction

The gradient vector field was developed by Robin Forman in 1998 [11, 12] as a powerful tool to study a simplicial complex K. A gradient vector field, which can represent a sequence of collapses on a complex, has evolved to become an extremely helpful tool in computing the number of critical simplices a complex has [11, Section 8] and thus helping determine the homotopy type of K. For example, one can find a discrete Morse function on a path with only one critical 0-simplex. This discrete Morse function can be represented by any maximum gradient vector field showing this sequence of collapses. This one critical 0-simplex tells us that the path has the homotopy type of a point.

Nearly a decade later, in 2005, Chari and Joswig [7] introduced a new complex that would become the main complex of interest in this paper. It is known as the Morse complex, denoted $\mathcal{M}(K)$, and it has a unique construction based on the compatibility of different gradient vector fields on the simplicial complex, K. Up to isomorphism, K can be reconstructed using $\mathcal{M}(K)$ as its guide because of the unique set of gradient vector fields that is associated to it [5]. Although this is a powerful and interesting topological usage, the same authors showed that $\mathcal{M}(K)$ does not uniquely determine the simple homotopy type of K.

In 2013, Jonathan Ariel Barmak [4] introduced the subcomplex of K known as the star cluster of a simplex in K. He proved that if K is a flag complex (also known as clique), then the star cluster is contractible. For Barmak, this provided an incredibly useful tool for studying the topology of a family of complexes determined by independence sets, known as independence complexes. One notable result he proved is that if there is an independence complex of a triangle-free graph, then that independence complex has the homotopy type of a suspension. Further, he used the idea of a star cluster to show that the independence complex of a forest is either contractible or homotopy equivalent to a sphere. Some other notable uses of the star cluster are constructing a matching tree for the independence complex of square grids with cyclic identification [15] and useing star clusters to compute the homotopy type of the independence complexes of the generalized Mycielskian of complete graphs [13].

A tool that is used hand-in-hand with star clusters in this thesis is the Cluster Lemma. The Cluster Lemma, independently developed by both Jonsson [16, Lemma 4.2] and Hersh [14, Lemma 4.1], is a tool to apply an acyclic matching to a simplicial complex by applying many small acyclic matchings and then gluing them all together. Throughout Section 3.3, we are able to show a large portion of a complex is collapsible by considering a star cluster and then applying small acyclic matchings to the rest of the complex to form a larger matching using the Cluster Lemma.

In Chapter 3, our goal will be to utilize both the Cluster Lemma and star clusters to determine the homotopy type of a variety of families of simplicial complexes: Morse complexes, generalized Morse complexes, and matching complexes. Each holds unique combinatorial information relating back to an original simplicial complex, and thus, studying the topological structure of these objects is an interesting and engaging pursuit. Although the Morse complex, $\mathcal{M}(K)$ is not a flag complex and star clusters are only relevant in flag complexes, we are able to show that the Morse complex is a flag complex when $K = T$ is a tree. In this special case, we use star clusters and the Cluster lemma to show that $\mathcal{M}(T)$ has the homotopy type of a suspension (Proposition 13). We also compute the homotopy type of the Morse complex on what we define as the "extended star graph" (Theorem 17) and provide an alternate computation of the homotopy type of the Morse complex of a path (Proposition 14), as originally computed by D. Kozlov in [18]. We next study the homotopy type of the generalized Morse complex, as first defined in [22]. This complex is slightly easier to study using this technique because it is always a flag complex, and so we show that the Morse complex and Generalized Morse complex of a cycle with a single leaf have the same homotopy type. Additionally, we utilize the Cluster Lemma and star clusters to compute the homotopy type of the matching complex of certain complexes. Again, this complex has the advantage of always being a flag complex. If we consider a graph G, the matching complex, denoted $\mathrm{M}(G)$, is the complex constructed from all independent edge sets on G. We provide alternate proofs for the homotopy type of the matching complexes of paths and cycles (Proposition 26, Proposition 27), as originally computed by Kozlov [18], and provide a new, more involved, computation of the homotopy type of Dutch windmill graphs (Theorem 29). One of the more relevant past results regarding the matching complex was made in [6], where the authors relate the matching complex to the Morse complex. They prove that there is a one-to-one correspondence between elements in the generalized Morse complex of a graph G and matchings on the barycentric subdivision of G. So, it immediately follows that $\mathcal{GM}(G) \cong \mathrm{M}(\mathrm{sd}(G))$ and recalling that the Morse complex of a tree is flag, we have the relationship $\mathcal{M}(T) \cong \mathcal{GM}(T) \cong \mathrm{M}(\mathrm{sd}(T))$. This acts as a nice bridge between our results regarding the generalized Morse complex and our results of the matching complex.

We choose this as a focus for Chapter 3 because the Cluster Lemma is a powerful tool in that it adds greater simplicity to choosing an ideal gradient vector field on our simplicial complex–by ideal, we mean a gradient vector field that allows us to determine the homotopy type. Due to a result by Robin Forman [10, Corollary 3.5], if our gradient vector field on a simplicial complex satisfies certain conditions regarding the critical simplices that are produced, then we can uniquely determine the homotopy type of that complex. In all of our work, we will satisfy Forman's result.

In Chapter 4, our focus will change to using strong collapses as our main tool for computing homotopy types, along with what we call the "degenerate Hasse diagram." This will allow us to prove some notable results–one of the most significant being Theorem 61, where we provide a sufficient condition for when we can guarantee that a Morse complex is strongly collapsible. We also extend a result of Ayala et. al. [2] to show that the pure Morse complex–the Morse complex generated by the maximum gradient vector fields on a simplicial complex–is strongly collapsible. Additionally, we are able to use our tools to compute the homotopy type of the Morse complex for cycles with a leaf (Theorem 46), centipede graphs (Corollary 48), and the strong collapse sequence for paths with a leaf (Proposition 52). Further, we provide a handful of results in Section 4.3 that allow us to more easily identify when a variety of different Morse complexes are suspensions.

We choose this as a focus for Chapter 4 because with the development of the degenerate Hasse diagram, it becomes much easier to interpret the effect strong collapses have in the Hasse diagram. We can easily model these strong collapses in the Hasse diagram using simple drawings, leading to quick computations of strongly collapsible complexes, and complexes that strongly collapse in a notable way (such as collapsing into a suspension). Although this method is fairly restricted by recognition of the degenerate Hasse diagram, it is a quick way to achieve topological information.

In Chapter 5, we explore an area of research that has been little studied–the independence complex of the intersection graph of a chord diagram. Given a chord diagram $C(n)$, we can associate an intersection graph, $\Gamma(C(n))$, based on which chords intersect each other. From there, we can construct the independence complex of $\Gamma(C(n))$, denoted $I_{\Gamma(C(n))}$, and study this complex. It is an open question whether $I_{\Gamma(C(n))}$ is always homotopy equivalent to a point, a sphere, or a wedge of spheres, so we explore that question. We first prove that the homotopy types of sphere and wedge of spheres can be realized by $I_{\Gamma(C(n))}$ and describe chord diagrams that satisfy this realization (Corollaries 68 and 70). After, we work to construct families of chord diagrams that realize certain families of graphs as the intersection graph. Namely, we define cycle chord diagrams, path chord diagrams, ladder chord diagrams, complete bipartite chord diagrams, and centipede chord diagrams, and compute the homotopy type of the independence complex of the intersection graph of these complexes (Propositions 74, 75, 78, 79, and 82).

Lastly, we would like to note that many results in this undergraduate thesis have been previously published ([8], [9]). If a result is not cited in this thesis, it is either our previously published work, or new results altogether. Note that these sources may be cited in some cases as an attempt to give credit to the other contributing authors.

Chapter 2

Overview

A reoccurring goal in algebraic topology is to use a variety of topological tools to transform a large, high-dimensional object into its smallest equivalent. The intuition behind this is the following question: is there any way we can bend, shrink, or expand an object into another while maintaining some overall "shape"? If so, we can consider two objects to be homotopy equivalent. It is important to note that homotopy equivalence is a related but distinct idea from homeomorphism. A single point is homotopy equivalent to a solid disk because the solid disk can be continuously squeezed down to a point. However, there is no bijection between a point and a solid disk, and thus they cannot be homeomorphic.

Ideally, we can use topological tools to determine the homotopy type of a simplicial complex, providing us with a complete understanding of the complex's topological properties. One of the hopes for this work is to provide new understanding to the homotopy type of certain objects. When objects start to live in arbitrarily large dimensions, it becomes extremely difficult to pinpoint its homotopy type, relating it to a family of bigger and smaller objects. So, through this work, we hope to provide a stronger basis for studying the homotopy type of a variety of simplicial complexes.

However, when it is impractical to compute the homotopy type because of the complexity of a simplicial complex, information can still be gained by resorting to properties such as topological invariants, connectivity, and homology. These properties will not be covered in this work, as we will take a deep dive into the world of homotopy types.

We refer to [21] for much of our foundational knowledge of simplicial complexes and discrete Morse theory.

2.1 Simplicial Complexes

Conveniently, higher-dimensional spaces can be easily modeled with an object called a **simplicial complex**. Recall that a graph G can be communicated by a vertex set, $V(G)$, and an edge set $E(G)$. Two vertices are **adjacent** if they are joined together by an edge. Additionally, if G is an acyclic graph, then we call G a **tree**. The number of edges a vertex is incident to is the **degree** of the vertex. A **leaf** is any vertex of degree 1. Now, an easy way to think of a simplicial complex is to think of a graph without limitations to its dimension. So, in addition to a vertex set and an edge set, there can exist higher dimensional faces communicating that more than two vertices are related.

Definition 2.1.1. Let $n \geq 0$ be an integer and $[v_n] := \{v_0, v_1, \ldots, v_n\}$ be a collection of $n+1$ symbols. An **(abstract) simplicial complex** K **on** $[v_n]$, or a **complex**, is a collection of subsets of $[v_n]$, excluding $\emptyset$, such that

- if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$;
- $\{v_i\} \in K$ for every $v_i \in [v_n]$.

The set $[v_n]$ is called the **vertex set** of K and the elements $\{v_i\} \in [v_n]$ are called **vertices**, or **0-simplices**. Similarly to graphs, it is also commonplace to denote the vertex set of K as $V(K)$.

The elements of K are called **faces**. A set σ with cardinality $n+1$ is called an n**-dimensional simplex** or n**-simplex** for short. Thus, a vertex is a 0-simplex, an edge is a 1-simplex, a triangular face is a 2-simplex, a tetrahedron is a 3-simplex, and this continues on. If a simplex, σ, of K cannot be contained within a larger simplex in K, then σ is called a **facet** of K.

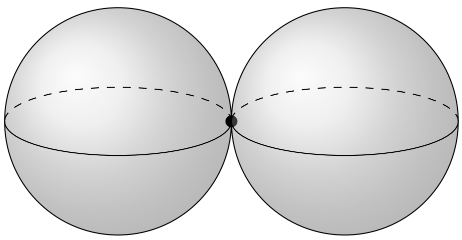
Additionally, some terms that will be incredibly beneficial in our work, especially in computing homotopy types of very large simplicial complexes, are the **star** and **link** of a vertex.

Definition 2.1.2. The **star** of a vertex v in a simplicial complex K is $\text{star}_K(v) = \{\sigma \in K : v \in \sigma\}$ and can be denoted $\text{st}(v)$ when the context is clear.

Definition 2.1.3. The **link** of a vertex v in a simplicial complex K is $\text{link}_K(v) = \{\sigma \in K : v \notin \sigma, \{v\} \cup \sigma \in K\}$ and can be denoted $\text{lk}(v)$ when the context is clear.

Throughout our work, we reference the **wedge sum** of two simplicial complexes quite often, denoted "$\vee$." A wedge of spheres is a common homotopy type, and we frequently wedge paths onto other families of simplicial complexes as a means of studying new families. By the **wedge sum**, we mean the one-point union of any number of simplicial complexes. We can think of this as the gluing of simplicial complexes together at a point.

Example 1. We can illustrate the wedge of two $2-$spheres; i.e. two three-dimensional spheres.



We have sufficiently introduced simplicial complexes as objects of study. We now move on to the tools that we will use to study these complicated objects.

2.2 Discrete Morse Theory

As referenced at the beginning of the overview, mathematicians are frequently interested in different notions of "sameness." There are more than one ways to define what being "the same" is, but for our purposes, we will be interested in both the **simple** and **strong homotopy type** of a complex. The intuition of simple homotopy type follows easily from the idea of squeezing a complex down to its roots step-by step, performing moves known as **elementary collapses** and **elementary expansions**. **Strong homotopy type** strengthens this idea, with **strong collapses** allowing us to perform many elementary collapses at once.

Before we give a foundation for discrete Morse theory, we can first ask, what is it that discrete Morse theory allows us to do? There must be a reason that it has proven fruitful in our studies. So, what is the intuition behind discrete Morse theory? First, it will help us to detect and declare the number of holes that exist in a simplicial complex. Tools of discrete Morse theory, namely collapses and expansions, allow us to pinpoint which simplices within a complex are causing holes–a very powerful and unique way to find a property of topological spaces. At this point, it is not clear what a "hole" is but basic intuition can help us think of what a hole could be. Secondly, discrete Morse theory allows us to replace simplicial complexes with a topologically equivalent one that is smaller. Since homotopy type is an equivalence relation it is incredibly useful to be able to replace a large, inconceivable complex with a smaller one holding the same properties. This can save mathematicians a lot of time in their studies. It is not clear what replacing a complex by a smaller, equivalent, one means, but after we introduce some necessary background, this idea should become comprehensible.

2.2.1 Simple Homotopy Type

Let K be a simplicial complex such that there is a pair of simplices $\{\sigma^{(p-1)}, \tau^{(p)}\}$ in K such that σ is a face of τ and σ has no other cofaces. Then $K - \{\sigma, \tau\}$ is a simplicial complex called an **elementary collapse** of K. The action of an elementary collapse can be denoted by the symbol $\searrow$. Thus, we write $K \searrow K - \{\sigma, \tau\}$.

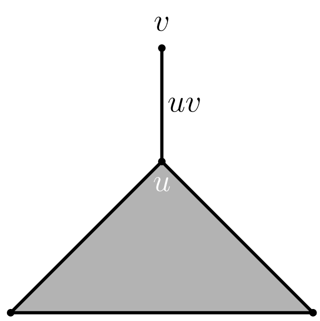
Conversely, suppose $\{\sigma^{(p-1)}, \tau^{(p)}\}$ is a pair of simplicies not in K where σ is a face of τ and all other faces of τ are in K. Then, we can perform a **elementary expansion** of K, adding $\{\sigma^{(p-1)}, \tau^{(p)}\}$ to K, denoted $K \nearrow K \cup \{\sigma, \tau\}$.

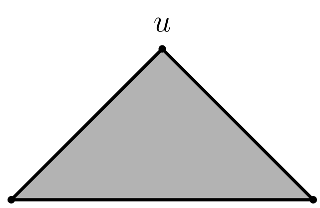
For either an elementary collapse or expansion, our pair of simplicies $\{\sigma, \tau\}$ is called a **free pair**.

Definition 2.2.1. Let K and L be simplicial complexes. We say that K and L have the same **simple homotopy type**, denoted $K \simeq L$, if there is a sequence of elementary collapses and expansions from K to L.

In the case where $L = \{v\} = *$, we say that K has the **simple homotopy type of a point**. If $K \simeq *$ is achieved through *only* using elementary collapses, we call K **collapsible**.

Example 2. We can illustrate an elementary collapse. A visualization can be very useful in understanding what it means for a pair of simplices to be a free pair.





Notice how v is a face of uv, and v has no other cofaces. Hence, $\{v, uv\}$ is our free pair, making for a clear elementary collapse on our simplicial complex.

Since simple homotopy is a type of equivalence relation, reflexivity holds, and thus if $K \simeq H$ and $H \simeq L$, then $K \simeq L$. This is an important property of simple homotopy because it allows us to form families of complexes with alike topological structure.

Definition 2.2.2. Let K and L be two simplicial complexes with no vertices in common. We define the **join** of K and L, denoted $K * L$, by

$$K * L := \{\sigma, \tau, \sigma \cup \tau : \sigma \in K, \tau \in L\}.$$

When $L = \{v, w\}$ for vertices $v, w \notin K$, then $K * L$ is the **suspension** of K denoted ΣK.

We consider another special case of the join called the **cone**, denoted $C(K)$. This is the join of a simplicial complex K with a single vertex. The following is an incredibly useful proposition in computing simple homotopy types.

Proposition 3. The cone $C(K)$ over any simplicial complex K is collapsible.

2.2.2 Strong Homotopy Type

Next, we introduce a stronger equivalence between two complexes, hence the term "strong homotopy type." Note that there is a foundation centered around simplicial maps (functions that map one simplicial complex to another) if we choose to introduce this concept starting from its roots. However, we choose to omit the notion of simplicial maps in favor of a very intuitive definition of strong homotopy type. The hope is that this allows for a clearer understanding of the actions we perform to achieve strong homotopy type. The first necessary concept that we must introduce is that of a **dominating vertex**.

Definition 2.2.3. Let K be a simplicial complex. A vertex v is said to **dominate** v' if every maximal simplex (facet) of v' also contains v.

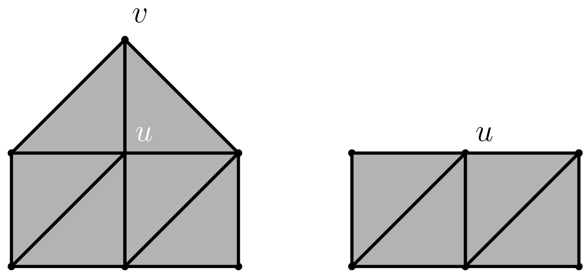
Using the fairly explicit notion of a dominating vertex, we are able to formally define **strong homotopy type**.

If v dominates v' in a simplicial complex K, the removal of v' from K is called an **elementary strong collapse** and is denoted by $K \searrow\searrow K - \{v'\}$. Conversely, the addition of a dominated vertex is an **elementary strong expansion** and is denoted by $\nearrow\nearrow$. A sequence of elementary strong collapses and expansions is called a strong collapse or strong expansion, respectively, and is denoted in the same manner.

Definition 2.2.4. Let K and L be simplicial complexes. If there is a sequence of strong collapses and expansions from K to L, then K and L are said to have the same **strong homotopy type**.

In the case where $L = *$, then K is said to have the **strong homotopy type of a point**. If there is a sequence of *only* strong collapses from K to a point, K is **strongly collapsible**.

Example 4. We can illustrate a strong collapse. Note how a strong collapse acts as many elementary collapses happening simultaneously.



Starting with the simplicial complex to the left, it is evident that u dominates v, as every facet of v also contains u. So we can make a strong collapse, leaving the reduced simplicial complex on the right.

This definition brings to light the question of whether collapsibility and strong collapsibility are the same thing. By definition, strongly collapsible implies collapsible. However, the converse is false, and can be shown so by counterexample. A famous counterexample is the Argentinian complex, as it is clearly collapsible, but has no dominating vertices, and thus cannot be strongly collapsible.

2.2.3 Discrete Gradient Vector Fields

Consider a simplicial complex K. A way to represent elementary collapses on K is by a type of induced function known as a **gradient vector field**. Each arrow in a gradient vector field represents an elementary collapse that can be performed. If done strategically, this can slowly squeeze K down to its simple homotopy type. However, there are some rules that we must satisfy when choosing our gradient vector field on K, and so we provide a formal definition and basis of this rich tool.

Definition 2.2.5. Let K be a simplicial complex. A **discrete vector field** V on K is defined by

$$V := \{(\sigma^{(p)}, \tau^{(p+1)}) : \sigma < \tau, \text{ each simplex of } K \text{ in at most one pair}\}.$$

If $(\sigma, \tau) \in V$, (σ, τ) is called a **vector**, an **arrow**, or a **matching**. All three terms are commonplace. The element σ is a **tail** while τ is a **head**.

Any pair $(\sigma, \tau) \in V$ is called a **regular pair**, and σ, τ are called **regular simplices** or just **regular**. If $(\sigma^{(p)}, \tau^{(p+1)}) \in V$, we say that $p + 1$ is the **index** of the regular pair. Any simplex in K which is not in V is called **critical**.

The formal definition of a discrete vector field puts implicit conditions on the types of vectors allowed. For any discrete vector field on a simplicial complex K, exactly one of the following holds:

1. σ is the tail of exactly one arrow.

2. σ is the head of exactly one arrow.

3. σ is neither the head nor the tail of an arrow. In other words, σ is **critical**.

Although these conditions are clear and easy to implement, they do not always correspond to a **gradient vector field** on our simplicial complex, K. There is one more condition we must satisfy: namely, we need to solve the problem of "closed paths."

Definition 2.2.6. Let V be a discrete vector field on a simplicial complex K. A V**-path** or **gradient path** is a sequence of simplices

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)} \dots, \beta_{k-1}^{(p+1)}, \alpha_k^{(p)}$$

of K such that $(\alpha_i^{(p)}, \beta_i^{(p+1)}) \in V$ and $\beta_i^{(p+1)} > \alpha_{i+1}^{(p)} \neq \alpha_i^{(p)}$ for $0 \leq i \leq k-1$.

If $k \neq 0$, then the V-path is called **non-trivial.** A V-path is said to be **closed** if $\alpha_k^{(p)} = \alpha_0^{(p)}$. A discrete vector field V which contains no non-trivial closed V-paths is called a **gradient vector field**. We sometimes use f to denote a gradient vector field.

We will refer to gradient vector fields consisting of only a single matching as a **primitive** gradient vector field. Taking multiple primitive gradient vector fields, we are allowed to combine them to form new, larger gradient vector fields.

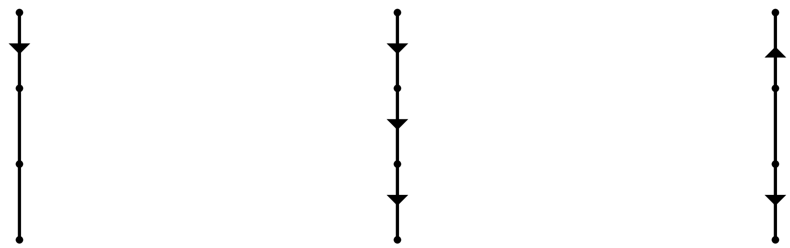
If f, g are two gradient vector fields on K, we can write $g \leq f$ whenever the regular pairs of g are also regular pairs of f. This denotes that g is a sub-gradient vector field of f. Generally, we call a collection of primitive gradient vector fields $f_0, f_1, \dots, f_n$ **compatible** if there exists a gradient vector field f such that $f_i \leq f$ for all $0 \leq i \leq n$.

It is important to note a distinction between two useful types of gradient vector fields when attempting to compute homotopy types: **maximal** and **maximum** gradient vector fields.

Definition 2.2.7. A **maximal** gradient vector field on a simplicial complex, K, is one that cannot be properly contained within any other gradient vector field on K.

Let K be a simplicial complex. A gradient vector field on K is **maximum** if as many simplices are matched as possible.

Example 5. Here, we can illustrate the distinction between primitive gradient vector fields (left), maximum gradient vector fields (center), and maximal gradient vector fields (right).



Note that maximum gradient vector fields are always maximal, but the converse is not necessarily true. Further, maximum and maximal gradient vector fields are beneficial in understanding the construction of the Morse complex, which will be explained in Chapter 3.

Chapter 3

The Cluster Lemma and Computing Homotopy Types

Our first area of results is in using what is called the "Cluster Lemma" [14, 16] to compute homotopy types of large simplicial complexes. In this chapter, we investigate the homotopy type of families of the Morse complex, the Generalized Morse complex, and the Matching complex. This chapter focuses mainly on computing simple homotopy type. However, other useful results describing the topology of these intricate simplicial complexes may arise, such as determining whether a complex is a suspension of another.

3.1 Motivation

Homotopy type is of great interest in topology and the study of different spaces. Similarities and differences of higher-dimensional spaces are a frequent focus in the discipline and homotopy invariants can be found more easily, computing the exact homotopy type takes patience and a variety of approaches. In [4], Barmak introduced the **star cluster** of a simplex in K and proved that if K is a **flag** (or a **clique**) complex, then the star cluster is collapsible. This proved to be a useful tool in studying the topology of a family of complexes known as independence complexes. So, our goal here is to expand this tool for independence complexes to other families of complexes.

The Morse, generalized Morse, and matching complexes store a variety of combinatorial information of a simplicial complex K. Thus, determining the homotopy types of the complexes is an interesting topological question with potential applications in topological data analysis and determining similarities between data sets.

3.2 Star Clusters and The Cluster Lemma

There are several famous types of complexes–one being the **flag complex**. A result by Jonathan Barmak [4], declaring that the "star cluster of a simplex of a flag complex is collapsible" will be crucial in our computations of homotopy types. The previous phrase is not currently clear, so let us introduce the necessary prerequisites.

Definition 3.2.1. A simplicial complex K is a **flag complex** if for each non-empty set of vertices σ such that $\{v_i, v_j\} \in K$ for every $v_i, v_j \in \sigma$, we have that $\sigma \in K$.

The flag complex of a graph G is the smallest simplicial complex that has G as a 1-skeleton.

Definition 3.2.2. [4, Definition 3.1] Let σ be a simplex of a simplicial complex, K. We define the **star cluster** of σ in K as the subcomplex

$$\mathrm{SC}_K(\sigma) = \bigcup_{v\in\sigma} \mathrm{st}_K(v) + \mathrm{lk}_K(v)$$

We denote the star cluster of σ by $\mathrm{SC}(\sigma)$ when the context is clear.

We will now formally introduce Barmak's result which will act as a fundamental tool in our work.

Proposition 6. [4, Lemma 3.2] The star cluster of a simplex in a flag complex is collapsible.

Next, we must properly introduce the "Cluster Lemma." This result was independently formulated by both Jonsson [16, Lemma 4.2] and Hersh [14, Lemma 4.1]. It is a simple way to construct a gradient vector field on a complex by applying several of smaller gradient vector fields disjointly and gluing them together. This result, along with Proposition 6 will simplify the process of computing homotopy types on complexes that are flag.

Lemma 7. ([16, Lemma 4.2] and [14, Lemma 4.1]) [Cluster Lemma] Let Δ be a simplicial complex which decomposes into collections Δ_σ of simplices, indexed by the elements σ in a partial order P which has a unique minimal element $\sigma_0 = \Delta_0$, Furthermore, assume that this decomposition is as follows:

1. Each simplex belongs to exactly one Δ_σ.
2. For each $\sigma \in P$, $\bigcup_{\tau\leq\sigma} \Delta_\tau$ is a subsimplicial complex of Δ.

For each $\sigma \in P$, let M_σ be an acyclic matching in Δ_σ. Then $\bigcup_{\sigma\in P} M_\sigma$ is an acyclic matching on K.

In addition to the Cluster Lemma [Lemma 7] allowing us to apply an acyclic gradient vector field on a whole complex by mending together smaller acyclic matchings, what is most important for us to take note of the simplices that go unmatched. If our gradient vector field is picked strategically enough, in some cases, specific collections of critical simplices can uniquely determine the homotopy type of the original complex–Morse, generalized Morse, and matching complex alike; of which, we will introduce later in this chapter. This unique determination is due to the following well-known result by Robin Forman:

Theorem 8. [10, Corollary 3.5] Let K be a simplicial complex and M an acyclic matching on K with m_i critical simplices of dimension i. Then K has the homotopy type of a CW complex with exactly m_i cells of dimension i. In particular, if $m_0 = 1, m_n = k$, and $m_j = 0$ for all $j \neq 0, n$, then K has the homotopy type of a k-fold wedge of S^n.

In our work, all of our computations will satisfy Theorem 8 and allow us to uniquely determine the homotopy type of the specific complex. We choose not to include the proof of Forman's result, as it is outside the realm of this thesis. However, we can note that a CW complex is a more general concept than a simplicial complex. Thus, simplicial complexes are encapsulated in the definition of a CW complex.

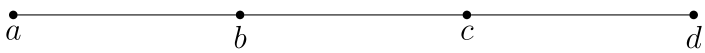
3.3 Computing Homotopy Types using Star Clusters and the Cluster Lemma

3.3.1 The Morse Complex

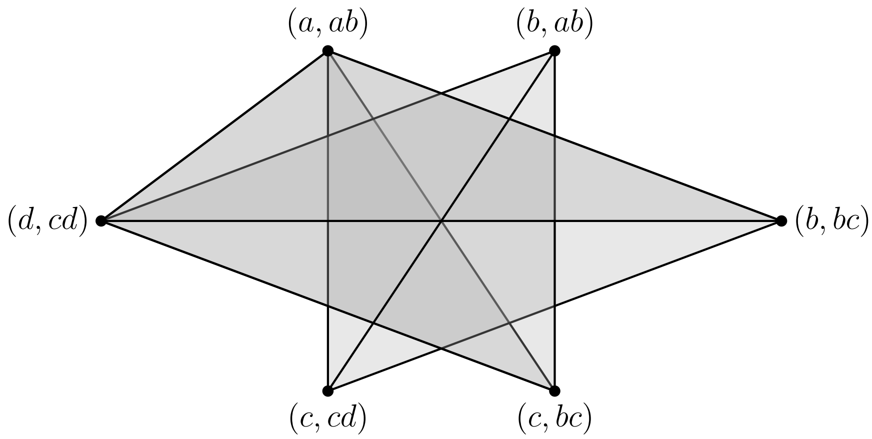
We continue onward to our first large classification of complex–the Morse complex. Informally, the **Morse complex** of a simplicial complex, K, is the complex containing information of all possible gradient vector fields on K, and is denoted $\mathcal{M}(K)$. Although the Morse complex is not a flag complex in general, it is when $K = T$, T a tree. In this special case, we are able to determine that $\mathcal{M}(T)$ is a suspension using star clusters and the Cluster Lemma [Lemma 7]. Further, we are able to use these same ideas to determine the homotopy type of a variety of different families of Morse complexes.

Definition 3.3.1. The **Morse complex** of K, denoted $\mathcal{M}(K)$, is the simplicial complex whose vertices are given by primitive gradient vector fields and whose n-simplices are given by gradient vector fields with $n+1$ regular pairs. A gradient vector field f is then associated with all primitive gradient vector fields $f := \{f_0, \ldots, f_n\}$ with $f_i \leq f$ for all $0 \leq i \leq n$.

Example 9. As a simple example, we can find the More complex of the following complex K:



Although the simplicial complex K is fairly simple, the Morse complex still requires some work to create. First, we will consider the six primitive gradient vector fields on K. These are $(a, ab), (b, ab), (b, bc), (c, bc), (c, cd)$, and (d, cd). Further, we will consider the five compatibilities $V_1 = \{(a, ab), (b, bc), (c, cd)\}, V_2 = \{(a, ab), (b, bc), (d, cd)\}, V_3 = \{(a, ab), (c, bc), (d, cd)\}, V_4 = \{(b, ab), (c, bc), (d, cd)\}$, and $V_5 = \{(b, ab), (c, cd)\}$. From these, we can construct our Morse Complex, $\mathcal{M}(K)$:



We find that V_1, V_2, V_3, and V_4 form triangle faces in $\mathcal{M}(K)$, while V_5 only forms an edge between (b, ab) and (c, cd). We can now draw a connection back to maximal and maximum gradient vector fields. It is clear that $V_1, \ldots, V_4$ are maximum gradient vector fields on K, while V_5 is only maximal.

Remark 10. If (a, ab) is a primitive gradient vector field, we sometimes denote this as $(a)b$, and if (ac, acb) is a primitive gradient vector field, we sometimes denote this as $(ac)b$. This notation is less cumbersome to work with, and thus we will use it moving forward.

Lemma 11. [9] The Morse complex $\mathcal{M}(K)$ is a flag complex if and only if K is a tree.

Proof. Let T be a tree and $\mathcal{M}(T)$ the Morse complex of T. For $\mathcal{M}(T)$ to be a flag complex, each non-empty set of mutually compatible vertices needs to be all together compatible. In other words, for each non-empty set of vertices σ such that $\{v, w\} \subseteq \mathcal{M}(T)$ for every $v, w \in \sigma$, we have that $\sigma \in \mathcal{M}(T)$. Now the only case when a collection of pairwise compatible primitive gradient vector fields may not be compatible is when they form a cycle. But since trees are acyclic, a collection of pairwise compatible primitive gradient vector fields can never form a cycle so that $\mathcal{M}(T)$ is a flag complex.

Now suppose $\mathcal{M}(K)$ is a flag complex. Clearly neither K nor the 1-skeleton of K can contain a cycle since otherwise there would exist a collection of mutually compatible vertices on $\mathcal{M}(K)$ that are not all together compatible. Thus K must be a tree.

□

Although the flag condition greatly reduces the kind of Morse complexes that we can study directly using star clusters, the following result of Barmak will allow us to make a blanket statement about the Morse complex of all trees. Specifically, as previously stated, that the Morse complex of all trees is a suspension.

Lemma 12. [4, Lemma 3.4] Let K be a simplicial complex and K_1, K_2 be two collapsible subcomplexes such that $K = K_1 \cup K_2$. Then K is homotopy equivalent to $\Sigma(K_1 \cap K_2)$.

Proposition 13. [9] Let T be a tree. Then $\mathcal{M}(T)$ has the homotopy type of a suspension.

Proof. We apply Lemma 12 by constructing two collapsible subcomplexes of $\mathcal{M}(T)$ whose union is all of $\mathcal{M}(T)$. Pick any leaf $\{v_0, v_0v_1\}$ of T and consider the maximum gradient vector field σ_0 rooted in v_0 and the maximum gradient vector field rooted in v_1 [20, Proposition 3.3]. These correspond to simplices $\sigma_0, \sigma_1 \in \mathcal{M}(T)$, respectfully. Define $\mathcal{M}_1(T) = \mathrm{SC}_{\mathcal{M}(T)}(\sigma_0)$ and $\mathcal{M}_2(T) = \mathrm{SC}_{\mathcal{M}(T)}(\sigma_1)$. Then $\mathcal{M}_1(T)$ and $\mathcal{M}_2(T)$ are collapsible subcomplexes of $\mathcal{M}(T)$ by Lemma 11. Furthermore, it is easy to see that $\mathcal{M}(T) = \mathcal{M}_1(T) \cup \mathcal{M}_2(T)$. Thus $\mathcal{M}(T) \simeq \Sigma(\mathcal{M}_1(T) \cap \mathcal{M}_2(T))$.

□

Hence, we are limited with making claims about the homotopy types when directly using star clusters and the Cluster Lemma [Lemma 7]. However, we can still use this technique to compute the homotopy type of the Morse complex of specific classes of trees. Our first computation is the homotopy type of the Morse complex of a path. This has been shown by Kozlov [18], so we are providing an alternate proof technique utilizing our method of collapsing star clusters and the Cluster Lemma [Lemma 7].

First, recall that a path graph on t vertices, denoted P_t, is a tree with two vertices of degree one and the other $t-2$ vertices having degree two. It is a convenient starting point for an illustration of our technique because it is a simple family of trees where all vertices and edges can be drawn in a straight line.

Proposition 14. [9] Let P_t be the path on t vertices, $t \geq 3$. Then

$$\mathcal{M}(P_t) \simeq \begin{cases} * & \text{if } t = 3n \\ \mathbb{S}^{2n-1} & \text{if } t = 3n+1 \\ \mathbb{S}^{2n} & \text{if } t = 3n+2 \end{cases}$$

Proof. We apply the Cluster Lemma. In order to do so, we decompose $\mathcal{M}(P_t)$ into collections Δ_k. First, we construct collections of sub-simplices σ_i for $i = 0, \ldots n$. We construct collections as follows:

1. Let $\sigma_0 := \mathrm{SC}((v_0)v_1, (v_1)v_2, \ldots, (v_{t-3})v_{t-2}, (v_{t-2})v_{t-1})$
2. For $1 \leq j \leq n$, we define the following:
   (a) When $j = 2k-1$, let $\sigma_j := \mathrm{st}((v_{t-(3k-1)})v_{t-3k})$
   (b) When $j = 2k$, let $\sigma_j := \mathrm{st}((v_{3k})v_{3k-1})$
3. Let $\sigma_{n+1} := \mathcal{M}(P_t) - \bigcup_{i=0}^{n} \sigma_i$

Now define $\Delta_0 := \sigma_0$ and $\Delta_k := \sigma_k - \bigcup_{j=0}^{k-1} \sigma_j$, and observe that $\bigcup_{k=0}^{n+1} \Delta_k = \mathcal{M}(P_t)$. Define an acyclic matching on each Δ_j as follows:

We know that Δ_0 is collapsible by Proposition 6 and Lemma 11 so Δ_0 has an acylcic matching with a single unmatched 0-simplex.

Let $j = 2k-1$, $1 \leq j \leq n$. Any simplex $V \in \Delta_j$ by definition contains $(v_{t-(3k-1)})v_{t-1}$. Match V with $V \cup \{(v_{t-(3k-2)})v_{t-(3k-1)}\}$ (or $V - \{(v_{t-(3k-2)})v_{t-(3k-1)}\}$ if V already contains this vector). In this way, all simplices in Δ_{2k-1} are matched with no unmatched simplices. Furthermore, since this matching is a subset of the matching on the cone on $(v_{t-(3k-1)})v_{t-1}$, it is acyclic.

Let $j = 2k$, $2 \leq j \leq n$. Any simplex $V \in \Delta_j$ by definition contains $(v_{3k})v_{3k-1}$. Match V with $V \cup \{(v_{3k-1})v_{3k-2}\}$ (or V with this vector removed, as above). In this way, all simplices in Δ_{2k} are matched with no unmatched simplices. Again, this matching is a subset of the matching on a cone so it is acyclic.

Now consider Δ_{n+1}. We have three cases:

(Note: When considering $n = 1$ in cases 2 and 3, disregard matchings containing vertices with negative indices e.g. v_{-1})

Case 1 Let $t = 3n$. Then $\Delta_{n+1} = \emptyset$, and thus $\mathcal{M}(P_t) \simeq *$.

Case 2 Let $t = 3n+1$. Then Δ_{n+1} contains a single simplex V of dimension $(2n-1)$ satisfying

$$(v_{3\lfloor \frac{n}{2} \rfloor})v_{3\lfloor \frac{n}{2} \rfloor -1}, (v_{3\lfloor \frac{n}{2} \rfloor +2})v_{3\lfloor \frac{n}{2} \rfloor +1} \notin V.$$

Thus by Theorem 8, $\mathcal{M}(P_t) \simeq \mathbb{S}^{2n-1}$.

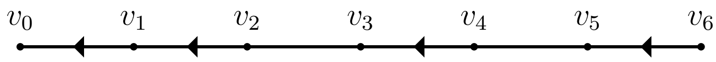
Case 3 Let $t = 3n + 2$. Then Δ_{n+1} contains a single simplex V of dimension $2n$ satisfying

$$(v_{3\lfloor \frac{n}{2} \rfloor})v_{3\lfloor \frac{n}{2} \rfloor - 1}, (v_{3\lfloor \frac{n}{2} \rfloor + 3})v_{3\lfloor \frac{n}{2} \rfloor + 2} \notin V.$$

Thus by Theorem 8, $\mathcal{M}(P_t) \simeq \mathbb{S}^{2n}$.

$\square$

Example 15. This computation gives us an algorithm to determine which simplex in the $\mathcal{M}(P_t)$ are critical. We can look at a small yet helpful example:



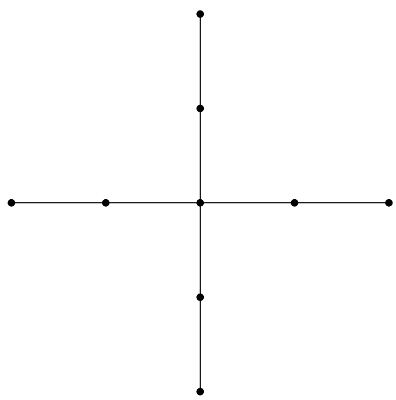
For $t = 7 = 3(2) + 1$, Case 2 of the proof of Theorem 14 implies that $V \in \Delta_3$ will result in the gradient vector field (critical simplex) in $\mathcal{M}(P_7)$ pictured above.

Next, recall that the **star graph**, S_n, on $n+1$ vertices is the complete bipartite graph $K_{1,n}$. Alternatively, we may view S_n as the result of taking n paths of length 1 and gluing them to a common endpoint (the so-called wedge). We generalize S_n in the following definition:

Definition 3.3.2. An **extended star graph**, denoted S_{v_1,v_2,v_3}, is the graph obtained by starting with v_1 paths of length 1, v_2 paths of length 2, and v_3 paths of lengths 3 and identifying an endpoint of each path with a fixed vertex c called the **center**. By an **extended leaf of length** k, we mean a path of length k from the center vertex, c, to a vertex, v_k, of degree 1.

Clearly $S_k = S_{k,0,0}$ recovers the star graph. It was shown in [8, Proposition 3.5] that not only is $\mathcal{M}(S_n)$ (strongly) collapsible for $n \geq 2$, but that any complex with at least two leaves sharing a common vertex is strongly collapsible. Hence, we let $v_0 = 0$ in our computation below.

Example 16. Here we illustrate a possible extended star graph: $S_{0,4,0}$.



By Corollary [19], we see that $\mathcal{M}(S_{0,4}) \simeq \mathbb{S}^4 \vee \mathbb{S}^4 \vee \mathbb{S}^4$.

Theorem 17. [9] Let $S_{0,n,m}$ be an extended star graph. Then,

$$\mathcal{M}(S_{0,n,m}) \simeq \vee^{n-1} \mathbb{S}^{2m+n}$$

Proof. Define a collection of subsimplices σ_i for $i = 0, \dots n$ on $\mathcal{M}(S_{0,n,m})$ as follows:

Let c be the center vertex of $S_{0,n,m}$ and let $\{v_{a_i}v_{b_i}, v_{b_i}\}$ be the leaf of each extended leaf of length 2, $i = 1, 2, \dots, n$, and $\{v_{\alpha_j}v_{\beta_j}, v_{\beta_j}\}$ the leaf of each extended leaf of length 3, $j = 1, 2, \dots, m$ with $v_{\gamma_j} \neq v_{\beta_j}$ the other neighbor of v_{α_j}.

1. Let σ_0 be the star cluster of the gradient vector field rooted in c. Such a gradient vector field exists and is unique by [20, Proposition 3.3].

2. Let $\sigma_1 := \cup_{i=1}^{m} \mathrm{st}(\{(c)v_{\gamma_i}\})$

Now define $\Delta_0 := \sigma_0, \Delta_1 := \sigma_1 - \sigma_0$, and $\Delta_2 := \mathcal{M}(S_{0,n,m}) - (\sigma_0 \cup \sigma_1)$. Clearly $\Delta_0 \cup \Delta_1 \cup \Delta_2 = \mathcal{M}(S_{0,n,m})$ so we can apply the Cluster Lemma. We define an acyclic matching on each Δ_j as follows:

First, Δ_0 is collapsible by Proposition 6 and Lemma 11 so there is an acyclic matching on Δ_0 with a single critical 0-simplex.

To construct a matching on Δ_1, we first observe that a typical element of Δ_1 is of the form $(c)v_{\gamma_i}$ along with other arrows pointing away from the center vertex c. Furthermore, because σ_0 contains all gradient vector fields with any arrow pointing towards c, all elements of Δ_1 are not compatible with any arrow pointing towards c. Upon inspection, there are exactly $2m$ such gradient vector fields. Match the $(2m + n - 1)$-simplex of Δ_1 containing $(c)v_{\gamma_i}$ but not containing $(v_{\gamma_i})v_{\alpha_i}$ to the corresponding $(2m + n)$-simplex containing both $(c)v_{\gamma_i}$ and $(v_{\gamma_i})v_{\alpha_i}$. This produces an acyclic matching on all elements in Δ_1.

Lastly, observe that Δ_2 contains $n+1$ elements. We will create a single matching, leaving $n-1$ unmatched $(2m+n)$-simplices and hence critical. A typical element of Δ_2 is of the form $(\bigcup_{i=1}^{n}(v_{a_i})v_{b_i}) \cup (\bigcup_{i=1}^{m}(v_{\gamma_i})v_{\alpha_i}) \cup (\bigcup_{i=1}^{m}(v_{\alpha_i})v_{\beta_1})$ along with possibly one of $(c)v_{a_i}$. Match the $(2m + n - 1)$-simplex of Δ_2 containing none of the $(c)v_{a_i}$ with the $(2m + n)$-simplex containing $(c)v_{a_1}$.

If $n > 1$ then, there are $n - 1$ unmatched $(2m + n)$-simplices τ_i, where each τ_i contains $(c)v_{a_i}$ for $i = 2, 3, \dots, n - 1$. Thus $\mathcal{M}(S_{0,n,m}) \simeq \vee^{n-1}\mathbb{S}^{2m+n}$.

□

We can quickly obtain several special cases for which we list as corollaries below.

Corollary 18. Let $S_{0,1,n}$ be an extended star graph. Then,

$$\mathcal{M}(S_{0,1,n}) \simeq *$$

Corollary 19. Let $S_{0,n}$ be an extended star graph. Then,

$$\mathcal{M}(S_{0,n}) \simeq \vee^{n-1}\mathbb{S}^{n}$$

Corollary 20. Let $S_{0,0,n}$ be an extended star graph. Then,

$$\mathcal{M}(S_{0,0,n}) \simeq \mathbb{S}^{2n-1}$$

As one can quickly see, using star clusters in conjunction with the Cluster Lemma 7 can be a simple yet powerful tool to computing homotopy types. We will continue to use this method throughout Chapter 2, and it will next be done with the **generalized Morse complex**.

3.3.2 The Generalized Morse Complex

The **generalized Morse complex** was defined in [22] in order to estimate the connectivity of the Morse complex. However, we see merit in using our technique to learn more about the topology of the object. Specifically, the homotopy type, and how it can possibly be related back to the Morse complex.

Definition 3.3.3. The **generalized Morse complex** $\mathcal{GM}(K)$ of a simplicial complex, K, is the simplicial complex whose vertices are the primitive gradient vector fields on K, with a finite collection of vertices spanning a simplex whenever the primitive gradient vector fields are pairwise compatible. Reworded, the simplices of $\mathcal{GM}(K)$ are the discrete vector fields on K, with face relation given by inclusion.

Note that $\mathcal{GM}(K)$ is a flag complex since it allows closed V-paths on K. We now compute the homotopy type of the generalized Morse complex for cycles. First, a definition.

Definition 3.3.4. Let C_t be a cycle, $t \geq 3$, with vertices $v_0, \ldots, v_{t-1}$. Let $V_k := \{(v_{i+1})v_i : k \leq i \leq t-1\}$. We define $\mathrm{st}_{\mathrm{mod}}(V_k) := \{\sigma \in \mathrm{st}(V_k) : (v_k)v_{k-1} \notin \sigma\}$.

Theorem 21. [9] Let C_t be the cycle on t vertices, $t > 3$. Then

$$\mathcal{GM}(C_t) \simeq \begin{cases} \mathbb{S}^{2n-1} \vee \mathbb{S}^{2n-1} & \text{if } t = 3n \\ \mathbb{S}^{2n} & \text{if } t = 3n+1 \\ \mathbb{S}^{2n} & \text{if } t = 3n+2 \end{cases}$$

Proof. We decompose $\mathcal{GM}(C_t)$ into collections Δ_k. We begin by constructing the following collections:

1. Let $\sigma_0 := \mathrm{SC}(\{(v_0)v_1, (v_1)v_2, \ldots, (v_{t-1})v_0\})$
2. For $1 \leq j \leq t-2$, let $\sigma_j := \mathrm{st}_{\mathrm{mod}}(\{(v_j)v_{j-1}(v_{j+2})v_{j+1}\})$

Define $\Delta_0 := \sigma_0$ and $\Delta_k := \sigma_k - \bigcup_{j=0}^{k-1} \sigma_j$. Then $\bigcup_{k=0}^{t-2} \Delta_k = \mathcal{GM}(C_t)$. Clearly Δ_0 is collapsible. Now match k-simplex of the form $\{(v_j)v_{j-1}(v_{j+2})v_{j+1} \ldots\}$ with the $(k+1)$-simplex of the form $\{(v_j)v_{j-1}(v_{j+1})v_j(v_{j+2})v_{j+1} \ldots\}$. There are three cases to consider.

Case 1 Let $t = 3n$. Then there will be two critical $(2n-1)$-simplices, both of which were excluded from every σ_j by the definition of $\mathrm{st}_{\mathrm{mod}}$. However, all other simplices have been matched. Thus $\mathcal{GM}(C_{3n}) \simeq \mathbb{S}^{2n-1} \vee \mathbb{S}^{2n-1}$.

Case 2 Let $t = 3n+1$. Then there will be one critical $(2n)$-simplex while all other simplices have been matched. Thus $\mathcal{GM}(C_{3n+1}) \simeq \mathbb{S}^{2n}$.

Case 3 Let $t = 3n+2$. There is a single critical $(2n)$-simplex which was excluded from every σ_j by the definition of $\mathrm{st}_{\mathrm{mod}}$ with all other simplices matched. Thus $\mathcal{GM}(C_{3n+2}) \simeq \mathbb{S}^{2n}$.

□

We next investigate the generalized Morse complex of a cycle with a leaf attached. We use the notation $C_t \vee \ell$ to denote the cycle of length t with a leaf ℓ joined to some vertex of C_t.

Theorem 22. [9] Let C_t be the cycle on t vertices, $t > 3$. Then

$$\mathcal{GM}(C_t \vee \ell) \simeq \begin{cases} * & \text{if } t = 3n \\ \mathbb{S}^{2n} & \text{if } t = 3n+1 \\ \mathbb{S}^{2n+1} & \text{if } t = 3n+2 \end{cases}$$

Proof. Let $\{v_1, v_0v_1\}$ be the leaf attached to $v_1 \in C_t$.

To apply the Cluster lemma, we first construct collections as follows:

1. Let $\sigma_0 := \mathrm{SC}(\{(v_0)v_1(v_1)v_2(v_2)v_3 \ldots (v_n)v_1\})$
2. For $1 \leq j \leq n$,
   (a) Let $j = 2k-1$ and define
   $$\sigma_j := \mathrm{st}(\{(v_{1+3(k-1)})v_{0+3(k-1)}(v_{3+3(k-1)})v_{2+3(k-1)}\})$$
   (b) Let $j = 2k$ and define
   $$\sigma_j := \mathrm{st}(\{(v_{t-3(k-1)})v_{(t-1)-3(k-1)}(v_{(t-2)-3(k-1)})v_{(t-3)-3(k-1)}\})$$
3. For $j = n+1$,
   (a) if $n+1 = 2k-1$, then
   $$\sigma_{n+1} := \mathrm{st}(\{(v_{t-3(k-1)})v_{(t-1)-3(k-1)}(v_{(t-1)-3(k-1)})v_{(t-2)-3(k-1)}\})$$
   (b) if $n+1 = 2k$, then
   $$\sigma_{n+1} := \mathrm{st}(\{(v_{1+3(k-1)})v_{0+3(k-1)}(v_{2+3(k-1)})v_{1+3(k-1)}\})$$

Let $\Delta_0 := \sigma_0$ and $\Delta_k := \sigma_k - \bigcup_{j=0}^{k-1} \sigma_j$. Then $\bigcup_{k=0}^{n+1} \Delta_k = \mathcal{GM}(C_t \vee l)$. Clearly Δ_0 is collapsible. Now match each Δ_j for $1 < j < n$ by the following:

If $j = 2k-1$, match each m-simplex of the form

$$\{(v_{1+3(k-1)})v_{0+3(k-1)}(v_{3+3(k-1)})v_{2+3(k-1)} \ldots\}$$

to the corresponding $m+1$-simplex of the form

$$\{(v_{1+3(k-1)})v_{0+3(k-1)}(v_{2+3(k-1)})v_{1+3(k-1)}(v_{3+3(k-1)})v_{2+3(k-1)} \ldots\}$$

If $j = 2k$, match each m-simplex of the form

$$\{(v_{t-3(k-1)})v_{(t-1)-3(k-1)}(v_{(t-2)-3(k-1)})v_{(t-3)-3(k-1)} \ldots\}$$

to the corresponding $m+1$-simplex of the form

$$\{(v_{t-3(k-1)})v_{(t-1)-3(k-1)}(v_{(t-1)-3(k-1)})v_{(t-2)-3(k-1)}(v_{(t-2)-3(k-1)})v_{(t-3)-3(k-1)} \ldots\}$$

Thus all simplicies in Δ_j for $1 < j < n$ have been matched.

Now we must match simplices in Δ_{n+1}. We consider three cases:

Case 1 Let $t = 3n$. Then $\Delta_{n+1} = \emptyset$, and thus $\mathcal{GM}(C_t \vee l) \simeq *$.

Case 2 Let $t = 3n + 1$. Then Δ_{n+1} only contains one $2n$-simplex. Thus $\mathcal{GM}(C_t \vee l) \simeq \mathbb{S}^{2n}$.

Case 3 Let $t = 3n + 2$. Then Δ_{n+1} only contains one $2n + 1$-simplex. Thus $\mathcal{GM}(C_t \vee l) \simeq \mathbb{S}^{2n+1}$.

$\square$

The homotopy type of the Morse complex of $C_t \vee \ell$ is computed in Theorem 46. It turns out to be the same as the homotopy type of the Generalized Morse complex of $C_t \vee \ell$. We thus have

Corollary 23. Let $C_t \vee \ell$ be a cycle with a leaf. Then,

$$\mathcal{GM}(C_t \vee \ell) \simeq \mathcal{M}(C_t \vee \ell).$$

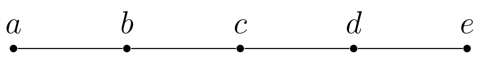
A collapse of $\mathcal{GM}(C_t \vee \ell)$ onto $\mathcal{M}(C_t \vee \ell)$ can be seen by considering the closed V-paths in $\mathcal{GM}(C_t \vee \ell)$ that are added to $\mathcal{M}(C_t \vee \ell)$. We see that there are four such V-paths: a clockwise cycle, a counterclockwise cycle, a clockwise cycle with an inward facing arrow on the leaf, and a counterclockwise cycle with an inward facing arrow on the leaf. By matching the clockwise cycle with the clockwise cycle with an inward facing arrow on the leaf and also matching the counterclockwise cycle to the counterclockwise cycle with an inward facing arrow on the leaf, we have collapsed $\mathcal{GM}(C_t \vee \ell)$ back into $\mathcal{M}(C_t \vee \ell)$, showing a homotopy equivalence.

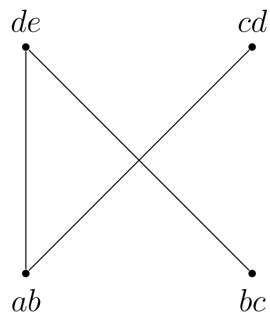
3.3.3 The Matching Complex

Our last complex of interest is the **matching complex**. The matching complex is a very well-known complex that can be associated to a graph, and it is fairly well-studied. However, there is still a lot of discovery to be done when thinking about the homotopy type of the matching complex.

Definition 3.3.5. Let the **matching complex** of a graph, G, denoted $\mathrm{M}(G)$, be the simplicial complex with vertices given by edges of G and faces given by matchings of G, where a matching is a subset of edges $H \subseteq E(G)$ such that any vertex $v \in V(H)$ has degree at most 1.

Example 24. We can begin with the simplicial complex, P_5 (left), and using our definition of the matching complex, we can construct $\mathrm{M}(P_5)$ (right).





The homotopy types of the matching complexes of the path and cycle were computed in [18]. As in the case of the Morse complex for a path, we are able to provide alternative proof methods using star clusters and the Cluster Lemma. Additionally, this method proves fruitful as we are able to provide a new result, computing the homotopy type of the matching complex for Dutch windmill graphs.

First, a simple yet powerful observation that we can make is $\mathcal{GM}(G) \cong \mathrm{M}(\mathrm{sd}(G))$ for any graph G [6]. Thus the results in Section 3.3.2 hold for the matching complex on the barycentric subdivision of the graph in question. By barycentric subdivision of the graph, we mean that $\mathrm{sd}(G)$ is the graph that results from dividing every edge in G into two by adding a vertex.

It was furthermore proved in [8, Proposition 3.5] (and is formally stated as Proposition 59) that if a graph G has two leaves sharing a common vertex, then the Morse complex is contractible. The same result holds for the generalized Morse complex. We thus have the following:

Corollary 25. [9] If a graph G has two leaves sharing a common vertex, then $\mathrm{M}(\mathrm{sd}(G))$ is contractible.

Proposition 26. [9] Let P_t be a path on $t \geq 3$ vertices. Then

$$\mathrm{M}(P_t) \simeq \begin{cases} \mathbb{S}^{n-1} & \text{if } t = 3n \\ \mathbb{S}^{n-1} & \text{if } t = 3n+1 \\ * & \text{if } t = 3n+2 \end{cases}$$

Proof. We apply the Cluster Lemma. In order to do so, we decompose $\mathrm{M}(P_t)$ into collections Δ_k. First, we construct collections of sub-simplices σ_i. We construct collections as follows:

1. Let $\sigma_0 := \mathrm{SC}(\{\bigcup_{i=0}^{k}(v_{3i}v_{3i+1})\})$, $k \leq n$
2. Let $\sigma_1 := \mathrm{st}\{(v_1v_2)\}$

Let $\Delta_0 := \sigma_0$ and $\Delta_1 := \sigma_1 - \sigma_0$. Now any maximal matching of P_t contains either v_0v_1 or v_1v_2. If it contains v_0, then it is in Δ_0. If it contains v_1v_2, then it is in Δ_1. Hence $\Delta_0 \cup \Delta_1 = \mathrm{M}(P_t)$ so that we define an acyclic matching on Δ_0, Δ_1 and apply the Cluster Lemma.

Now Δ_0 is flag so it is collapsible by Proposition 6 and Lemma 11. To construct a matching on Δ_1, we consider three cases:

Case 1 Let $t = 3n$. Then Δ_1 is a single simplex given by $\{\bigcup_{i=0}^{n-1}(v_{3i+1}v_{3i+2})\}$. Hence this corresponds to an $(n-1)$-simplex in the Morse complex and thus is critical so that $\mathrm{M}(P_{3n}) \simeq \mathbb{S}^{n-1}$.

Case 2 Let $t = 3n+1$. As in Case 1, Δ_1 is a single matching given by $\{\bigcup_{i=0}^{n-1}(v_{3i+1}v_{3i+2})\}$. This matching corresponds to a critical $(n-1)$-simplex in the Morse complex and thus $\mathrm{M}(P_{3n+1}) \simeq \mathbb{S}^{n-1}$.

Case 3 Let $t = 3n+2$. Then $\Delta_1 = \emptyset$. Thus $\mathrm{M}(P_{3n+2}) \simeq *$.

□

We can also provide an alternate proof for computing the homotopy type of the matching complex of the cycle using the same technique and a similar matching.

Proposition 27. [9] Let C_t be a cycle on $t \geq 3$ vertices. Then

$$\mathrm{M}(C_t) \simeq \begin{cases} \mathbb{S}^{n-1} \vee \mathbb{S}^{n-1} & \text{if } t = 3n \\ \mathbb{S}^{n-1} & \text{if } t = 3n+1 \\ \mathbb{S}^{n} & \text{if } t = 3n+2 \end{cases}$$

Proof. As usual, we apply the Cluster Lemma by first constructing collections of subsimplices σ_i.

1. Let $\sigma_0 := \mathrm{SC}(\{\bigcup_{i=0}^{k}(v_{3i}v_{3i+1})\})$, $k \leq n$ ($k \leq n-1$ when $t = 3n+1$)
2. Let $\sigma_1 := \mathrm{st}\{(v_{(t-1)}v_0)\}$
3. Let $\sigma_2 := \mathrm{st}\{(v_1v_2)\}$

Define $\Delta_0 := \sigma_0, \Delta_1 := \sigma_1 - \sigma_0$, and $\Delta_2 := \sigma_2 - (\sigma_0 \cup \sigma_1)$. Since every matching of C_t is in one of the σ_i, it follows that $\Delta_0 \cup \Delta_1 \cup \Delta_2 = \mathrm{M}(C_t)$. To define an acyclic matching on the Δ_i, we first observe that Δ_0 is collapsible.

The matchings on both Δ_1 and Δ_2 are considered in three cases:

Case 1 Let $t = 3n$. In Δ_1, there exists one $(n-1)$-simplex, $\{\bigcup_{i=0}^{k}(v_{2+3i}v_{3+3i})\}$. Thus it cannot be matched so it it critical. In Δ_2, there exists one $(n-1)$-simplex of the form $\{\bigcup_{i=0}^{k}(v_{1+3i}v_{2+3i})\}$ which also cannot be matched. Thus $\mathrm{M}(C_{3n}) \simeq \mathbb{S}^{n-1} \vee \mathbb{S}^{n-1}$.

Case 2 Let $t = 3n+1$. Any $(n-1)$-simplex V in Δ_1 does not contain $\{(v_{t-3}v_{t-2})\}$ so we match V with $V \cup \{(v_{t-3}v_{t-2})\}$. This yields a perfect acyclic matching on Δ_1. Now there is only one simplex in Δ_2; namely, the $(n-1)$-simplex $\{\bigcup_{i=0}^{n-1}(v_{3i+1}v_{3i+2})\}$. This $(n-1)$-simplex is critical, hence $\mathrm{M}(C_{3n+1}) \simeq \mathbb{S}^{n-1}$.

Case 3 Let $t = 3n+2$. For each $(n-1)$-simplex V of Δ_1, there is exactly one k, $0 \leq k \leq n-1$, such that both $v_{3k+1}v_{3k+2}$ and $v_{3k+2}v_{3k+3}$ are not in V. Match this V with $V \cup \{v_{3k+2}v_{3k+3}\}$. Then there is one n-simplex left unmatched, namely, $\{\bigcup_{i=0}^{n}(v_{3i+1}v_{3i+2})\}$. Observe that Δ_2 is empty, and thus $\mathrm{M}(C_{3n+2}) \simeq \mathbb{S}^{n}$.

□

Next, we can use the same technique to compute the homotopy type for some new families of graphs. The first being the **centipede graph**, and the second being the **Dutch windmill graph**.

Definition 3.3.6. A **centipede graph**, $\mathcal{C}_t$ is a graph obtained by adding a leaf to each vertex on a path P_t. If $v_0, \ldots, v_{t-1}$ are the vertices of P_t, denote the vertex of the leaf added to v_i by v_i'.

Proposition 28. [9] Let $\mathcal{C}_t$ be a centipede graph. Then

$$\mathrm{M}(\mathcal{C}_t) \simeq \begin{cases} \mathbb{S}^{n-1} & \text{if } t = 2n \\ * & \text{if } t = 2n+1 \end{cases}$$

Proof. Let $\mathcal{C}_t$ be a centipede graph. We apply the Cluster Lemma and construct collections as follows:

1. Let $\sigma_0 := \mathrm{SC}(\{\bigcup_{i=0}^{t-1}(v_i v_i')\})$
2. Let $\sigma_1 := \mathrm{st}((v_0 v_1))$

Define $\Delta_0 := \sigma_0$ and $\Delta_1 := \sigma_1 - \sigma_0$ so that $\Delta_0 \cup \Delta_1 = \mathrm{M}(\mathcal{C}_t)$. Define an acyclic matching on each Δ_i as follows:

We know Δ_0 is collapsible by Proposition 6 and Lemma 11.

For Δ_1, we have two cases:

Case 1 Let $t = 2n$. Then the only element in Δ_1 is $\{\bigcup_{j=0}^{n-1}(v_{2j}v_{2j+1})\}$, an $(n-1)$-simplex. Hence $\mathrm{M}(\mathcal{C}_t) \simeq \mathbb{S}^{n-1}$.

Case 2 Let $t = 2n+1$. Then $\Delta_1 = \emptyset$. Thus $\mathrm{M}(\mathcal{C}_t) \simeq *$.

□

Definition 3.3.7. Let D_m^n be a **Dutch windmill graph**. D_m^n is obtained by taking n copies of the cycle C_m and joining them at a common vertex.

Theorem 29. [9] Let D_m^n be a Dutch windmill graph. Then

$$\mathrm{M}(D_m^n) \simeq \begin{cases} * & \text{if } m = 3k \\ \mathbb{S}^{nk-1} & \text{if } m = 3k+1 \\ \bigvee^{2n-1}\mathbb{S}^{nk} & \text{if } m = 3k+2 \end{cases}$$

Proof. Let D_m^n be the Dutch windmill graph with center vertex v_0 and for each of the n cycles C_m, let $v_{(j)_i}$ denote vertex j of cycle i, $0 \le j \le m-1$ and $1 \le i \le n$. We apply the Cluster lemma by defining the following collections:

1. Let $\sigma_0 := \mathrm{SC}\{\bigcup_{i=1}^{n}(\bigcup_{j=0}^{k-1}(v_{(3j+1)_i}v_{(3j+2)_i}))\}$
2. For $1 \le \nu \le k-1$, let $\sigma_\nu := \bigcup_{i=1}^{n}(\mathrm{st}(\bigcup_{j=1}^{k-\nu}(v_{(3j)_i}v_{(3j+1)_i})))$
3. Let $\sigma_k := \bigcup_{i=1}^{n}(\mathrm{st}(\bigcup_{j=0}^{k-1}(v_{(3j+2)_i}v_{(3j+3)_i})))$

Define $\Delta_0 := \sigma_0$ and $\Delta_\beta := \sigma_\beta - \bigcup_{\alpha=0}^{\beta-1}\sigma_\alpha$. Then $\bigcup_{\beta=0}^{k}\Delta_\beta = \mathrm{M}(D_m^n)$. We now define an acyclic matching on each Δ_β as follows:

We know Δ_0 is collapsible by Proposition 6 and Lemma 11. Observe that $\Delta_1, \ldots, \Delta_k = \emptyset$ for $m = 3k$, which implies that $\mathrm{M}(D_m^n) \simeq *$.

Hence, suppose $m \neq 3k$. Let $1 \le \nu \le k-1$ and consider Δ_ν. For each $1 \le i \le n$, we match $\bigcup_{j=1}^{k-\nu}(v_{(3j)_i}v_{(3j+1)_i})$ with $(v_{(3(k-\nu)+2)_i}v_{(3(k-\nu)+3)_i}) \cup \bigcup_{j=1}^{k-\nu}(v_{(3j)_i}v_{(3j+1)_i})$. This produces an acyclic matching for all gradient vector fields in Δ_ν. It remains to put a matching on to Δ_k.

For Δ_k, we consider cases:

Case 1 Let $m = 3k+1$. Then Δ_k has one element, namely,

$$\bigcup_{i=1}^{n}\bigcup_{j=0}^{k-1}(v_{(2+3j)_i}v_{(3+3j)_i}).$$

This is an $(nk-1)$-unmatched simplex, so it is critical, and thus $\mathrm{M}(D_m^n) \simeq \mathbb{S}^{nk-1}$.

Case 2 Let $m = 3k+2$. Then Δ_k has $2n+1$ elements which are given by

$$\bigcup_{i=1}^{n}\bigcup_{j=0}^{k-1}(v_{(2+3j)_i}v_{(3+3j)_i})$$

For each $1 \le \ell \le n, (v_{(0)_\ell}v_{(1)_\ell}) \cup \bigcup_{i=1}^{n}\bigcup_{j=0}^{k-1}(v_{(2+3j)_i}v_{(3+3j)_i})$

For each $1 \le \ell \le n, (v_{(0)_\ell}v_{(m-1)_\ell}) \cup \bigcup_{i=1}^{n}\bigcup_{j=0}^{k-1}(v_{(2+3j)_i}v_{(3+3j)_i})$

We can only create one matching, namely, we match

$$\bigcup_{i=1}^{n}\bigcup_{j=0}^{k-1}(v_{(2+3j)_i}v_{(3+3j)_i}) \text{ with } (v_{(0)_1}v_{(1)_1}) \cup \bigcup_{i=1}^{n}\bigcup_{j=0}^{k-1}(v_{(2+3j)_i}v_{(3+3j)_i}).$$

This leaves $2n-1$ (nk)-simplices unmatched. Thus, $\mathrm{M}(D_m^n) \simeq \vee^{2n-1}\mathbb{S}^{nk}$.

□

Chapter 4

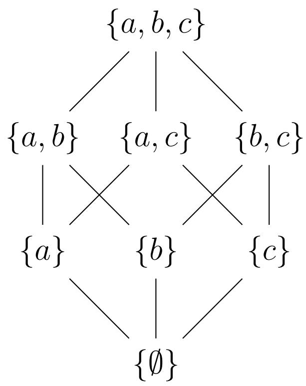
Computing Homotopy Types Through Strong Collapses and the Hasse Diagram

Through this chapter, we are further exploring homotopy types, so note that the same motivation of Chapter 3 holds. In this chapter, we hone our focus on the Morse complex, as we are particularly interested in the combinatorial information of gradient vector fields that it stores. Additionally, the **Hasse diagram** provides both a nice definition of the Morse complex, as well as another tool to study its topological structure.

4.1 Degenerate Hasse Diagrams and An Alternate Definition of the Morse Complex

Recall that a **partially ordered set**, or **poset** is a set P along with a reflexive, antisymmetric, and transitive relation, typically denoted by $\leq$. For example, if we consider a set X and the power set of X, $\mathcal{P}(X)$, we can show that $\mathcal{P}(X)$ is a poset under subset inclusion. If we have the set $X = \{a, b, c\}$, we can visualize the poset of $\mathcal{P}(X)$ by writing all the elements of $\mathcal{P}(X)$ out and drawing a line between a simplex and each codimension-1 face, while maintaining the cardinality in the same rows.

Example 30. We can illustrate the above example, as seen in [21]:



The above illustration is called a **Hasse diagram**, and will be a key part of an alternate definition of the Morse complex. We refer to each point in the Hasse diagram as a **node**, and we call each ascending row of the Hasse diagram the **level** i. In another lense, the level is used to denote the i-simplices of K in the Hasse Diagram.

Definition 4.1.1. Let K be a simplicial complex. The **Hasse diagram** of K, denoted $\mathcal{H}(K)$, is defined as the partially ordered set of simplices of K ordered by the face relations. Thus, $\mathcal{H}(K)$ is a graph such that there is a one-to-one correspondence between the nodes of $\mathcal{H}(K)$ and the simplices of K.

If we choose to abuse notation for easier explanation, if $\sigma \in K$, we can write $\sigma \in \mathcal{H}(K)$ for the corresponding node, and there is an edge between two simplices $\sigma, \tau \in \mathcal{H}(K)$ if and only if τ is a codimension-1 face of σ.

This can be visualized by a picture placing nodes in rows in $\mathcal{H}(K)$ such that every node in the same row corresponds to a simplex in K of the same dimension.

So, with the Hasse diagram formally defined, we are free to provide an alternate definition of the Morse complex that will be useful to keep in mind throughout Chapter 3:

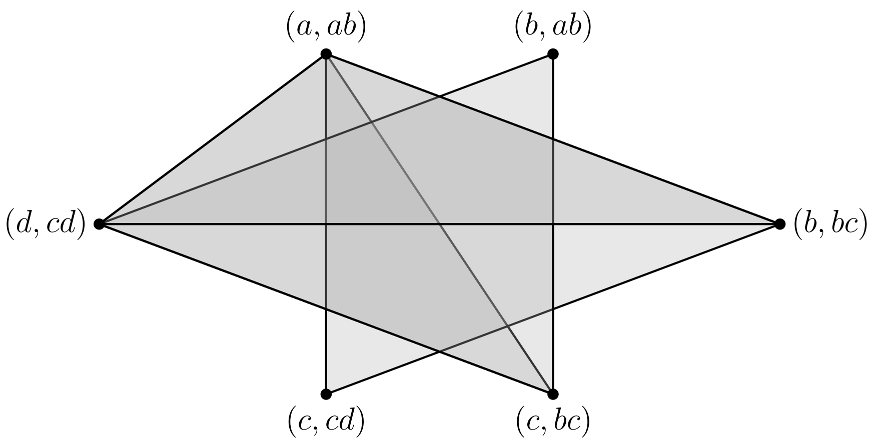
Definition 4.1.2. Let K be a simplicial complex. The **Morse Complex** of K, denoted by $\mathcal{M}(K)$, is the simplicial complex on the set of edges of $\mathcal{H}(K)$ defined as the set of subsets of edges of $\mathcal{H}(K)$ which form discrete Morse matchings (acyclic matchings), excluding the empty matching.

Definition 4.1.3. Let $\mathbb{P}$ be the set of all (finite) posets, and $\mathbb{K}$ be the set of all simplicial complexes. Define a function $f \colon \mathbb{P} \to \mathbb{K}$ as follows: for each $P \in \mathbb{P}$, construct a simplicial complex $f(P)$ whose vertex set is the edge set of P. Then let $\sigma = e_1 e_2 \cdots e_k$ be a simplex of $f(P)$ if and only if the edges $e_1, e_2, \cdots e_k$ oriented upward and all other edges oriented downward form an acyclic matching of P.

Remark 31. Take a moment to see that for any simplicial complex K, $\mathcal{M}(K) \simeq f(\mathcal{H}(K))$. Our definition 4.1.3 generalizes the notion of taking the Morse complex to degenerate Hasse diagrams. So, we will similarly call $f(P)$ the Morse complex of the poset P.

Lastly, we will now mention that the **pure Morse complex** of K, denoted $\mathcal{M}_P(K)$, is the subcomplex of $\mathcal{M}(K)$ generated by the maximum gradient vector fields on K.

Example 32. If we consider the Morse complex of P_4 from Example 9, we can make a slight tweak to to form the pure Morse complex, $\mathcal{M}_P(P_4)$:



Note that the only change when moving from $\mathcal{M}(P_4)$ to $\mathcal{M}_P(P_4)$ is we omit the edge from (c, cd) to (b, ab), as that is the only face that is maximal but not maximum.

We will state a general result regarding the strong collapsibility of pure Morse complex of trees, and so we see it fitting to include the pure Morse complex in this chapter, along with all other strong homotopy results.

4.2 The Morse Complex and Strong Homotopy Results

In this section, we use methods involving strong collapses and manipulation of the Hasse diagram to further explore the homotopy type of families of Morse complexes. One notable, and extremely applicable proposition that results from our results is that given two leafs wedged to the same vertex on a complex K, then $\mathcal{M}(K)$ is strongly collapsible. This fact gives rise to the question of what characterizes strong collapsibility. We leave that question to be pondered by the reader.

Our first result is a broad statement that completely characterizes the homotopy type of the pure Morse complex of trees, and extends a result by Ayala et. al. [2] showing that $\mathcal{M}_P(T)$ is collapsible where T is a tree. Our extension leads this result by Ayala et. al. to fall as an immediate corollary.

Proposition 33. Let T be a tree. Then $\mathcal{M}_P(T)$ is strongly collapsible.

Proof. Let T be a tree. By definition, T has at least one leaf, say $\{a, ab\}$. We will show that $(b)a$ is dominated and that after removing $(b)a$ from $\mathcal{M}_P(T)$, then $(a)b$ dominates all remaining primitive gradient vector fields, and hence $\mathcal{M}_P(T)$ is strongly collapsible.

Let bc be an edge incident with b, $c \neq a$. We claim that $(b)a$ is dominated by $(c)b$ in $\mathcal{M}_P(T)$. Suppose $\sigma \in \mathcal{M}_P(T)$ is a facet containing $(b)a$. Then σ is a maximal gradient vector of T, and since σ is in the pure Morse complex, σ is also maximum. Since $(b)a \in \sigma$ with a a leaf, σ is the only maximum gradient vector field containing $(b)a$ and thus is dominated by $(c)b$ (and in fact every primitive gradient vector field of σ). Since $(b)a$ is dominated, we may perform a strong elementary collapse and remove it from $\mathcal{M}_P(T)$.

Now we claim $(a)b$ dominates all remaining primitive gradient vector fields. Note that since $(b)a \notin \mathcal{M}_P(T) - \{(b)a\}$, $(a)b$ is compatible with $(\alpha)\beta \in \mathcal{M}_P(T) - \{(b)a\}$. Thus $(a)b$ dominates $(\alpha)\beta$ for all other $(\alpha)\beta \in \mathcal{M}_P(T) - \{(b)a\}$ so that $\mathcal{M}_P(T) - \{(b)a\}$ is a cone and thus strongly collapsible.

□

Thus, we can now formalize the corollary mentioned above:

Corollary 34. Let T be a tree. Then $\mathcal{M}_P(T)$ is collapsible.

Our next result strengthens a previous result by Kozlov [18, p. 119] computing the homotopy type of the Morse complex of the path, for which we provided an alternate proof in Chapter 1 14. We extend this result to state that a path on $3t$ vertices is strongly collapsible. However, first we state a lemma that will prove very useful throughout this chapter.

Lemma 35. *Let K be a simplicial complex with leaf $\{a, ab\}$ and c a neighbor of b not equal to a. Then $(b)c$ is dominated in $\mathcal{M}(K)$ by $(a)b$.*

Proof. Consider any facet of $(b)c$ in $\mathcal{M}(K)$. A facet of $\mathcal{M}(K)$ is a maximal gradient vector field on K, and since $(b)a$ is not compatible with $(b)c$ and $\{a, ab\}$ is a leaf, $(a)b$ must be in any maximal gradient vector field containing $(b)c$. Thus $(a)b$ dominates $(b)c$ in $\mathcal{M}(K)$.

□

Proposition 36. *Let P_{3n} be the path on $3n$ vertices, $n \geq 1$. Then $\mathcal{M}(P_t) \searrow\searrow *$*

Proof. By Lemma 35, $(v_1)v_2$ dominates $(v_2)v_3$. After removing $(v_2)v_3$, we see that $(v_3)v_2$ dominates $(v_4)v_3$, and so we remove $(v_4)v_3$. Continuing in this manner, we see that $(v_{3k-2})v_{3k-1}$ dominates $(v_{3k-1})v_{3k}$ for all $1 \leq k \leq n$, and $(v_{3k})v_{3k-1}$ dominates $(v_{3k+1})v_{3k}$ for for all $1 \leq k < n$. Hence we may remove each of these primitive gradient vector fields.

Now the last primitive gradient vector field removed is $(v_{3n-1})v_{3n}$ since it was dominated by $(v_{3n-2})v_{3n-1}$. We now claim that $(v_{3n})v_{3n-1}$ dominates every remaining vertex. To see this, observe that because $(v_{3n-1})v_{3n}$ has been removed, $(v_{3n})v_{3n-1}$ is compatible with all remaining vertices $(v_i)v_j$, and no $(v_i)v_j$ can exist in a facet of the remaining Morse complex without $(v_{3n})v_{3n-1}$. We remove all $(v_i)v_j$ until we are only left with $(v_{3n})v_{3n-1}$. Thus $\mathcal{M}(P_{3n-1})$ is strongly collapsible.

□

4.2.1 Morse Complex of the Disjoint Union

Now, before we give more results regarding strong homotopy, we must provide a couple useful results regarding the Morse complex of a disjoint union that we attribute to [8]. We will leave out the proofs, as they are quite involved.

Proposition 37. *Let K, L be connected simplicial complexes each with at least one edge. Then $\mathcal{M}(K \sqcup L) = \mathcal{M}(K) * \mathcal{M}(L)$.*

Corollary 38. *Let K be a simplicial complex. Then $\mathcal{M}(K \sqcup P_2) \simeq \Sigma\mathcal{M}(K)$.*

Additionally, given Remark 31 and Proposition 37, we also have the following:

Corollary 39. *Let A, B be posets. Then $f(\mathcal{H}(A) \sqcup \mathcal{H}(B)) \simeq f(\mathcal{H}(A)) * f(\mathcal{H}(B))$.*

If one of the factors in the join is strongly collapsible, then the join is strongly collapsible, due to the following result by Barmak:

Proposition 40. *[3, Proposition 5.1.16] Let K, L be simplicial complexes. Then $K * L$ is strongly collapsible if and only if K or L is strongly collapsible.*

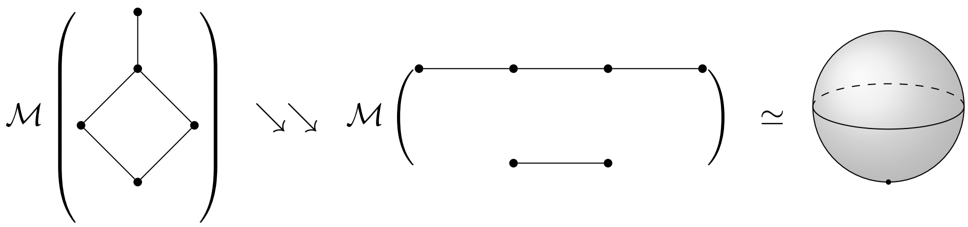
Now, we can use these results as tools for computing the homotopy type of a variety of families of simplicial complexes.

4.2.2 Cycles Wedge a Leaf

A useful observation is that for any simplicial complex, K, $\mathcal{M}(K)$ is minimal unless it contains at least one vertex, $v \in K$, with degree 1 [8].

Example 41. This is useful as it can tell us if strong collapses are possible. So, if we consider a cycle on 4 vertices, we can immediately know that the Morse complex of C_4 is not strongly collapsible. However, if we wedge a leaf to any vertex of the cycle, it is now possible to show that the resulting Morse complex strongly collapses down to the Morse complex of the disjoint union of P_2 and P_4, which ultimately results in the homotopy type of a sphere.

We can write the sequence of strong collapses in the following fashion:



We will be able to show that the Morse complex of a cycle wedged with a leaf strongly collapses to the Morse complex of a disjoint union of paths in Proposition 44 using this along with a handful of other results.

Remark 42. Conveniently, behavior of strong collapses can be studied by considering a modified version of the Hasse diagram–our degenerate Hasse diagram. In general, this is simply a poset and not necessarily a Hasse diagram by its formal definition. However, it still functions as a tool that allows us to predict the behavior of strong collapses in $\mathcal{M}(K)$ in many cases. We use $K \vee_v \ell$ to denote attaching a leaf ℓ to a vertex $v \in K$. We use $K \vee \ell$ when there is no need to make reference to the vertex.

Lemma 43. For any simplicial complex K and vertex $v \in V(K)$, the Morse complex $\mathcal{M}(K \vee_v \ell)$ strongly collapses to $f((\mathcal{H}(K) - v) \sqcup \mathcal{H}(\ell))$.

Proof. Write $\ell = vw$ for some vertex w and let $a_1, a_2, \ldots, a_k$ be the neighbors of v. By Lemma 35, the vertex (w, wv) dominates vertices $(v, va_1), (v, va_2), \ldots (v, va_k)$, leading to k strong collapses. In the Hasse diagram $\mathcal{H}(K \vee \ell)$, this corresponds to a removal of the edges connecting node v to nodes $va_1, va_2, \ldots, va_k$. As these are all the edges in K that v is connected to, the Hasse diagram now consists of $\mathcal{H}(K)$ with node v removed, together with a second component consisting of the Hasse diagram of the leaf vw. The entire Hasse diagram is $(\mathcal{H}(K) - v) \sqcup \mathcal{H}(\ell))$. Therefore $\mathcal{M}(K \vee_v \ell) \searrow\searrow f((\mathcal{H}(K) - v) \sqcup \mathcal{H}(\ell))$.

$\square$

Proposition 44. Let v be a vertex of C_n. Then $\mathcal{M}(C_n \vee_v \ell) \searrow\searrow \mathcal{M}(P_n \sqcup \ell)$.

Proof. By Lemma 43, we know that $\mathcal{M}(C_n \vee_v \ell) \searrow\searrow f((\mathcal{H}(C_n) - v) \sqcup \mathcal{H}(\ell))$. We also have $f((\mathcal{H}(C_n) - v) \sqcup \mathcal{H}(\ell)) \simeq f(\mathcal{H}(C_n) - v) * f(\mathcal{H}(\ell)) \simeq f(\mathcal{H}(C_n) - v) * \mathcal{M}(\ell)$. In addition, by Proposition 37 and Remark 31, $\mathcal{M}(P_n \sqcup \ell) \simeq \mathcal{M}(P_n) * \mathcal{M}(\ell) \simeq f(\mathcal{H}(P_n)) * \mathcal{M}(\ell)$. Observe that $\mathcal{H}(C_n) - v \simeq \mathcal{H}(P_n)$, thus $f(\mathcal{H}(P_n)) \simeq f(\mathcal{H}(C_n) - v)$. $\square$

We can know refer back to Proposition 14 or [18] to recall that

Proposition 45. Let P_n be a path on n vertices. Then

$$\mathcal{M}(P_n) \simeq \begin{cases} * & \text{if } n = 3k \\ \mathbb{S}^{2k-1} & \text{if } n = 3k+1 \\ \mathbb{S}^{2k} & \text{if } n = 3k+2. \end{cases}$$

Thus, the homotopy type of the Morse complex of a cycle wedge a leaf follows immediately by considering Propositions 44, 37, 45, 40, and Corollary 38:

Theorem 46. Let C_n be a cycle of length $n \geq 3$. Then

$$\mathcal{M}(C_n \vee \ell) \simeq \begin{cases} * & \text{if } n = 3k \\ \mathbb{S}^{2k} & \text{if } n = 3k+1 \\ \mathbb{S}^{2k+1} & \text{if } n = 3k+2. \end{cases}$$

4.2.3 Centipede Graphs

In this section, we will continue to use Proposition 37 to compute homotopy types. However, we will now return our focus to centipede graphs, as they were studied in Section 3.3.3.

Definition 4.2.1. Let K be a simplicial complex with vertex set $V(K)$. We denote the graph obtained by wedging a path of length n to each $v \in V(K)$ by $L_n(K)$.

It turns out that centipede graphs satisfy the following property:

Proposition 47. Let G be a connected graph with v vertices, and let $L_2(G)$ be the complex resulting from adding a leaf to each vertex of G. Then $\mathcal{M}(L_2(G)) \simeq \mathbb{S}^{v-1}$.

Proof. Let G be a connected graph, and we will denote the leaves we add to G to obtain $L_2(G)$ by $\ell_1, \ell_2, \dots, \ell_v$. Let $\{a, ab\}$ be any leaf of G. If c is any neighbor of b, $c \neq a$, then $(b)c$ is dominated in $\mathcal{M}(L_2(G))$ by Lemma 35. By adding a leaf to each vertex of G, every primitive gradient vector field on G is dominated, and thus can be removed from $\mathcal{M}(L_2(G))$. Since the Morse complex of a single leaf is $\mathbb{S}^0$, we have

$$\begin{aligned} \mathcal{M}(\ell_1 \sqcup \ell_2 \sqcup \ldots \sqcup \ell_v) &= \mathcal{M}(\ell_1) * \mathcal{M}(\ell_2) * \ldots \mathcal{M}(\ell_v) \\ &\simeq \mathbb{S}^{v-1} \end{aligned}$$

where the first equality is Proposition 37 and the second follows from the fact that $\Sigma\mathbb{S}^n \simeq \mathbb{S}^{n+1}$.

□

Thus, centipede graphs follow as an immediate corollary:

Corollary 48. Let $\mathcal{C}_v$ be a centipede graph. Then $\mathcal{M}(\mathcal{C}_v) \simeq S^{v-1}$.

Example 49. We can easily illustrate this property with an example. On the left, we start with $\mathcal{C}_3$. After performing the strong collapses that Lemma 35 allows, we then only have to take the Morse complex of the subcomplex on the right.

Proposition 50. Let G be a graph. Then $\mathcal{M}(L_{3i+2}(G)) \simeq \mathbb{S}^{v\cdot(2i+1)-1}$

Proof. Consider a connected graph, G, and $\mathcal{M}(L_{3i+2}(G))$. Starting at each leaf vertex, consider the sequence of strong collapses used in the proof of Proposition 36, along the wedged paths. If $\nu_i \in V(L_{3i+2}(G))$ is adjacent to each $v_i \in V(G)$, we see that $(\nu_i)v_i$ dominates all $(v_i)v_j$, v_j a neighbor of v_i not equal to ν_i. Thus, all primitive gradient vector fields from G have been strongly collapsed. Then, considering the Hasse diagram, we are left with $v \cdot (2i+1)$ copies of $f(H(\ell))$. Thus, using Then, using Proposition 37, Remark 31, and Lemma 43,

$$\begin{aligned}\mathcal{M}(L_{3i+2}(G)) &\simeq f(H(\ell_1)) * f(H(\ell_2)) * \ldots * f(H(\ell_{v\cdot(2i+1)})\\ &\simeq \mathcal{M}(\ell_1) * \mathcal{M}(\ell_2) * \ldots * \mathcal{M}(\ell_{v\cdot(2i+1)})\\ &\simeq \mathbb{S}^{v\cdot(2i+1)-1}\end{aligned}$$

□

4.2.4 Paths Wedge a Leaf

Next, we can once again use Proposition 37 to compute homotopy types. Now, we focus on paths wedge a leaf, and we obtain a general result that allows us to easily compute the exact homotopy type of a Morse complex given a path wedge a leaf.

Lemma 51. Let v_k be a vertex of P_t, $1 \le k \le t-2$ and $t \ge 2$. Then $\mathcal{M}(P_t \vee_{v_k} \ell) \searrow\searrow \mathcal{M}(P_{k+2} \sqcup P_{t-(k+2)} \sqcup \ell)$.

Proof. Write $\ell = v_k u$. By Lemma 35, $(u)v_k$ dominates $(v_k)v_{k+1}$ in $\mathcal{M}(P_t \vee_{v_k} \ell)$. In the corresponding Hasse diagram $\mathcal{H}(P_t \vee_{v_k} \ell)$, this corresponds to the removal of the edge between v_k and $v_k v_{k+1}$.

Furthermore, by Lemma 35, $(v_t)v_{t-1}$ dominates $(v_{t-1})v_{t-2}$. This corresponds to the removal of the edge between v_{t-1} and $v_{t-2}v_{t-1}$ on the Hasse diagram. Upon inspection, we see that this yields three components: the Hasse diagram of the path P_{k+2}, the Hasse diagram of the path $P_2 = \ell$, and an "upside-down" Hasse diagram of the path $\mathcal{H}(P_{t-(k+2)})$. Thus

$$\mathcal{M}(P_t \vee_{v_k} \ell) \searrow\searrow f(\mathcal{H}(P_{k+2}) \sqcup \mathcal{H}(P_{t-(k+2)}) \sqcup \mathcal{H}(\ell)).$$

By Proposition 37 and Remark 31, we have that

$$f(\mathcal{H}(P_{k+2}) \sqcup \mathcal{H}(P_{t-(k+2)}) \sqcup \mathcal{H}(\ell)) \simeq \mathcal{M}(P_{k+2} \sqcup P_{t-(k+2)} \sqcup \ell).$$

□

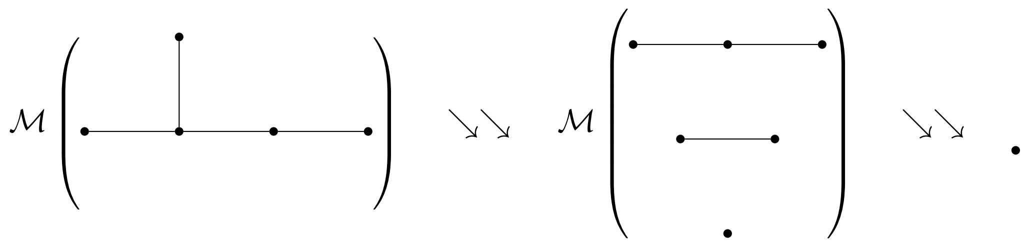
Combining Lemma 51 and Proposition 37, we have

Proposition 52. Let v_k be a vertex of P_t, $1 \leq k \leq t-2$. Then $\mathcal{M}(P_t \vee_{v_k} \ell) \simeq \mathcal{M}(P_{k+2}) * \mathcal{M}(P_{t-(k+2)}) * \mathcal{M}(\ell)$.

Considering Proposition 52, Proposition 40, and Proposition 36, we can finally achieve the following result:

Corollary 53. Let v_k be a vertex of P_t, $1 \leq k \leq t-2$. If $k+2 = 3j$ or $t-(k+2) = 3j$, then $\mathcal{M}(P_t \vee_{v_k} l) \searrow\searrow *$.

Example 54. We can illustrate the strong collapse sequence of $\mathcal{M}(P_4 \vee \ell)$ by the following diagram:



4.3 Strongly Collapsing to Suspensions

Although we showed earlier that the Morse complex of all trees are suspensions (Proposition 13), in general, it is unknown when a Morse complex is a suspension. Thus, it is an interesting question to ask when this may be true.

Here, we provide some convenient results for showing the Morse complex of many simplicial complexes are suspensions. Lemma 55 and Corollary 56 is for all simplicial complexes, while Proposition 58 is for cycles.

Lemma 55. Let K be a simplicial complex. Then, $\mathcal{M}(K \vee_v P_{3i+1}) \simeq \Sigma^{2i}\mathcal{M}(K)$

Proof. Suppose $v_0, v_1, \ldots, v_{3i} \in V(P_{3i+1})$ such that v_0 is our leaf vertex and v_{3i} is wedged with some $v \in V(K)$. Consider a sequence of strong collapses starting with vertex $(v_0)v_1$ dominating vertex $(v_1)v_2$ in the Morse complex. This can then be immediately followed by a strong collapse of $(v_2)v_1$ dominating $(v_3)v_2$ in the Morse complex. Upon inspection, each vertex $(v_{3t})v_{3t+1}$ dominates $(v_{3t+1})v_{3t+2}$ and each $(v_{3t+2})v_{3t+1}$ dominates $(v_{3(t+1)})v_{3t+2}$ along our wedged path. Note that this is the same sequence of strong collapses seen in Proposition 36.

These strong collapses correspond to the removal of the edges between each v_{3t+1} and $v_{3t+1}v_{3t+2}$ on the Hasse diagram, along with the edges between $v_{3(t+1)}$ and $v_{3(t+1)}v_{3t+2}$. We can quickly notice that this yields two components on our degenerate Hasse diagram per index i; all of which are Hasse diagrams of $P_2 = \ell$, either "right-side-up" or "upside-down."

Our v_{3i} is wedged to K, and those those $2i$ components are separated from $\mathcal{H}(K)$–it goes uneffected by the strong collapses. Thus,

$$\mathcal{M}(K \vee_v P_{3i+1}) \searrow\searrow f((H(K) \sqcup H(\ell_1) \sqcup \ldots \sqcup H(\ell_{2i}))$$

So, by Proposition 37 and Remark 31,

$$\begin{aligned} f((H(K) \sqcup H(\ell_1) \sqcup \ldots \sqcup H(\ell_{2i})) &\simeq f((H(K)) * f(H(\ell_1)) * \ldots * f(H(\ell_{2i})) \\ &\simeq \mathcal{M}(K) * \mathcal{M}(\ell_1) * \ldots * \mathcal{M}(\ell_{2i}) \\ &\simeq \Sigma^{2i}\mathcal{M}(K) \end{aligned}$$

□

Thus, we can immediately follow this with some corollaries.

Corollary 56. Let K be a simplicial complex. Then, for $v_1, v_2, \ldots, v_m \in V(K)$,

$$\mathcal{M}(K \vee_{v_1} P_{3n_1+1} \vee_{v_2} \ldots \vee_{v_m} P_{3n_m+1}) \simeq \Sigma^{2(n_1+\ldots+n_m)}\mathcal{M}(K).$$

Corollary 57. Let K be a simplicial complex on n vertices. Then, $\mathcal{M}(L_{3i+1}(K)) \simeq \Sigma^{2i(n)}\mathcal{M}(K)$.

Proposition 58. Let C_n be a cycle on n vertices. Then, $\mathcal{M}(C_n \vee_v P_{3i+2}) \simeq \Sigma^{2i+1}\mathcal{M}(P_{n-1})$

Proof. Firstly, it is clear to see that $\mathcal{M}(C_n \vee_v P_{3i+2}) = \mathcal{M}(C_n \vee_v \ell \vee_{v'} P_{3i+1})$ when the path is wedged onto the leaf vertex, v'.

Again, by Lemma 43, we know that for any simplicial complex, K, and vertex $v \in V(K)$, the Morse complex $\mathcal{M}(K \vee_v \ell)$ strongly collapses to $f((H(K) - v \sqcup H(\ell))$. It can be shown similarly, using the sequence of strong collapses seen in the proof of Proposition 36, that

$$\mathcal{M}(C_n \vee_v \ell \vee_{v'} P_{3i+1}) \searrow\searrow f((H(C_n) - v) \sqcup H(\ell_0) \sqcup H(\ell_1) \sqcup \ldots \sqcup H(\ell_{2i}))$$

Therefore, by Proposition 37 and Remark 31,

$$\begin{aligned} f((H(C_n) - v) \sqcup H(\ell_0) \sqcup H(\ell_1) \sqcup \ldots \sqcup H(\ell_{2i})) &\simeq f((H(C_n) - v) * f(H(\ell_0)) * \ldots * f(H(\ell_{2i})) \\ &\simeq \mathcal{M}(P_{n-1}) * \mathcal{M}(\ell_0) * \ldots * \mathcal{M}(\ell_{2i}) \\ &\simeq \Sigma^{2i+1}\mathcal{M}(P_{n-1}) \end{aligned}$$

□

4.4 Sufficient Condition for Strong Collapsibility

In [8], the following result is shown:

Proposition 59. [8, Proposition 3.5] If a simplicial complex, K, has two leaves sharing a vertex, then $\mathcal{M}(K)$ is strongly collapsible.

We also include the proof, as it will be useful to reference in the proof of Theorem 61.

Proof. Call the leaves $\{a, ab\}$ and $\{a, ac\}$ where $a, b, c \in V(K)$. These correspond to vertices $(a)b, (b)a, (a)c, (c)a \in V(\mathcal{M}(K))$. We claim that $(b)a$ dominates $(a)c$. Consider any facet σ of $(a)c$. The only vertex incompatible with $(b)a$ is $(a)b$, but since $(a)c$ and $(a)b$ are incompatible, $(a)b \notin \sigma$. Therefore we must have $(b)a \in \sigma$

since σ is maximal. Perform the strong collapse given by removing vertex $(a)c$. We claim that $(c)a$ dominates every vertex in the resulting complex. Consider an arbitrary $v \in V(\mathcal{M}(K)) - (a)c$ and a facet τ containing v. The only vertex that $(c)a$ is incompatible with in $V(\mathcal{M}(K))$ is $(a)c$. Since $(a)c \notin V(\mathcal{M}(K)) - (a)c$, we know that $(c)a$ is compatible with every vertex in τ, so $(c)a \in \tau$. Therefore $(c)a$ dominates v. We repeatedly apply the strong collapse removing each vertex v, strongly collapsing the Morse complex to $(c)a$.

□

One of the most significant results of this work is the following theorem. We are able to generalize Proposition 59 to include many pairs of paths wedged together at the same vertex. This is a strong sufficient condition for strong collapsibility, as the following is also shown in [8]:

Proposition 60. [8, Proposition 3.3] Let K be a simplicial complex. If all vertices $v \in V(K)$ have degree at least 2, then $\mathcal{M}(K)$ is minimal. In particular, $\mathcal{M}(K)$ is not strongly collapsible.

Thus, with the following result, we are on our way to answering when a Morse complex is strongly collapsible, which would be a very significant contribution of knowledge to the study of homotopy types, and more specifically, strong homotopy types.

Theorem 61. Suppose a simplicial complex, K, has two paths, of length $3n + 2$ and $3m + 2$ respectively, wedged at $v \in V(K)$, then $\mathcal{M}(K)$ is strongly collapsible.

Proof. We show by induction that this holds:

Consider a base case where both paths wedged at $v \in V(K)$ are of of length 2. Then two leaves are sharing a vertex, and so $\mathcal{M}(K) \searrow\searrow *$ by Proposition 59. Suppose this holds true for paths length 2 up to paths of length $3(n-1)+2$ and $3(m-1)+2$. Now, we will show that this holds for paths of length $3n+2$ and $3m+2$ respectively:

Consider the sequence of strong collapses used in Proposition 36 starting at the leaf vertex, v_1 of P_{3n+2}. Then, $(v_1)v_2$ dominates $(v_2)v_3$, $(v_3)v_2$ dominates $(v_4)v_3$, and then we are on to considering the remaining path of length $3(n-1)+2$. Similarly, we can consider the remaining path of length $3(m-1)+2$ for the other path. By assumption, $\mathcal{M}(K) \searrow\searrow *$ for paths of lengths $3(n-1)+2$ and $3(m-1)+2$.

It is clear to see that the sequence of strong collapses for paths of lengths $3(n-1)+2$ and $3(m-1)+2$ is not hindered by the remaining primitive gradient vector fields from v_0 to v_4, as each $(v_{3k-2})v_{3k-1}$ dominates $(v_{3k-1})v_{3k}$ for all $1 \leq k \leq n, m$, and each $(v_{3k})v_{3k-1}$ dominates $(v_{3k+1})v_{3k}$ for all $1 \leq k \leq n, m$.

So, using the same sequence of strong collapses we would for paths of lengths $3(n-1)+2$ and $3(m-1)+2$, $\mathcal{M}(K) \searrow\searrow *$ when paths of length $3n+2$ and $3m+2$ are both wedged at the same $v \in V(K)$.

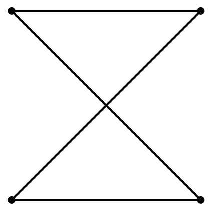
□

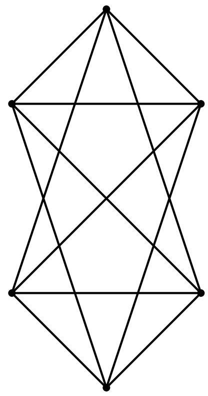
4.5 Realizing Cocktail Party Graphs Using Strong Collapses

As a fun final result of Chapter 4, we will talk about **cocktail party graphs**. This result is not as powerful or useful as some of the others in this thesis, but it still relates to the Morse complex and strong collapses. We choose to include it because we think it is a good display of a way that math can act as an artistic outlet, and a way of finding unusual hidden patterns.

Definition 4.5.1. The $n-$**cocktail party graph**, denoted $K_{n\times 2}$, is the complete n-partite graph where each partite set has size 2.

Example 62. On the left, we illustrate a $K_{2\times 2}$ cocktail party graph. On the right, we illustrate a $K_{3\times 2}$ cocktail party graph.





We call a simplicial complex, K, **minimal** if it contains no dominating vertices. The **core** of K is the minimal sub-complex $K_0 \subseteq K$ such that $K \searrow\searrow K_0$. Barmak [3, Theorem 5.1.10] proved that no matter the order of strong collapses, the core will be the same (the core is unique up to isomorphism).

The n-**skeleton of** K, denoted by $K^{(n)}$, is the sub-complex of K containing all simplices up to $n-$dimensional simplices.

For the following, let $\mathcal{M}_0(K)$ denote the core of the Morse complex of K.

Proposition 63. Let P_t be the path on $t \geq 4$ vertices. Then

$$\mathcal{M}_0(P_t)^{(1)} = \begin{cases} K_{2k\times 2} & \text{if } t = 3k+1 \\ K_{(2k+1)\times 2} & \text{if } t = 3k+2 \end{cases}$$

Proof. Proceed as in the proof of Proposition 36 by removing dominated vertices from $\mathcal{M}(P_t)$ starting with $(v_1)v_2$. The last vertex removed along the path differs depending on t.

Suppose $t = 3k+1$. Then the last vertex removed from the Morse complex is $(v_{3n})v_{3n-1}$ as it is dominated by $(v_{3n-1})v_{3n-2}$. We claim there are no more dominated vertices. We show that for any remaining primitive gradient vector field σ, there exists a unique primitive gradient vector field τ such that σ and τ are not compatible. If so, then σ is compatible with every other primitive gradient vector field, thus creating a maximal simplex of σ not containing τ. Observe that $(v_0)v_1$ and $(v_1)v_0$ are not compatible with each other, and that $(v_2)v_1$ is not compatible with $(v_2)v_3$. Additionally, each remaining primitive gradient vector field, $(v_{3n})v_{1+3n}$ is not compatible with the corresponding $(v_{1+3n})v_{3n}$, as well as each remaining $(v_{2+3n})v_{1+3n}$ is not compatible with the corresponding $(v_{2+3n})v_{3+3n}$. Other than these incompatibilities, every primitive gradient vector field is compatible with any other. Thus no primitive gradient vector field can dominate another and we have arrived at $\mathcal{M}_0(P_t)$.

We now determine the structure of the 1-skeleton of $\mathcal{M}_0(P_t)$. There were $6k$ primitive gradient vector fields on P_t, and we removed $2k$ of these above, yielding $4k$ vertices in $\mathcal{M}_0(P_t)$. As determined above, every vertex of $\mathcal{M}_0(P_t)$ is compatible every other vertex of $\mathcal{M}_0(P_t)$ other than a unique vertex. In other words, there is an edge between vertex v and every other vertex except a unique vertex v'. This is precisely the complete $2k$-partite graph with partite sets of size 2. Thus $\mathcal{M}_0(P_t))^{(1)} = K_{2k\times 2}$.

The $t = 3k + 2$ case is similar, and so we omit it.

□

Chapter 5

Chord Diagrams and Corresponding Independence Complexes

5.1 Chord Diagrams to Independence Complexes

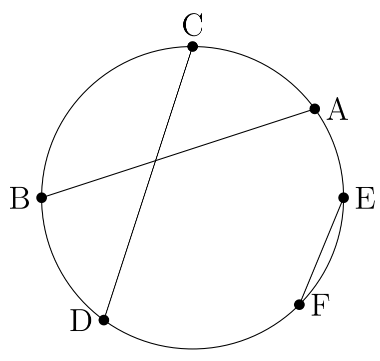
Definition 5.1.1. [1] A **chord diagram** of size n, denoted $C(n)$, or C when the context is clear, is a pairing of $2n$ given points on a circle with cyclic ordering and a one-to-one pairing on those points.

Given a chord diagram, we can associate an **intersection graph** which communicates which chords intersect.

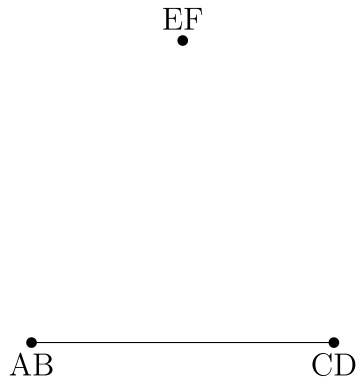
Definition 5.1.2. [1] For a chord diagram $C(n)$, we define the **intersection graph**, denoted $\Gamma(C(n))$, or $\Gamma(C)$, as follows: each chord in C becomes a vertex in $\Gamma(C)$, and two vertices are neighbors if and only if the corresponding chords cross each other in C.

From the intersection graph, $\Gamma(C)$, we can consider the **independence complex**, denoted $I_{\Gamma(C)}$. Recall the graph theoretic concept of independent sets in a graph G: a collection of vertices in G such that no two vertices in the set are neighbors. The **independence complex** of the intersection graph, $\Gamma(C)$, is a simplicial complex containing the independent sets of $\Gamma(C)$ as simplices. The independence complex is an interesting complex of study because of it's direct connection to the chromatic properties of graphs.

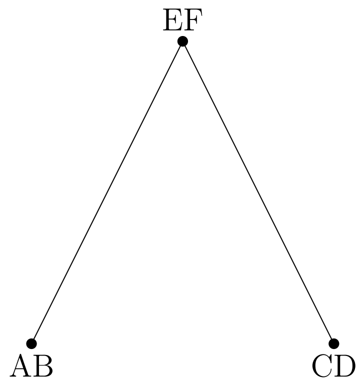
Example 64. We can run through one example of the construction for which we are interested in. Firstly, we can illustrate one possible $C(3)$:



From this specific $C(3)$, we can illustrate $\Gamma(C(3))$:



Lastly, we can construct $I_{\Gamma(C(3))}$, for which we will be interested in computing the homotopy type of.



From here, it is easy to apply a discrete gradient vector field and show that $I_{\Gamma(C(3))} \simeq *$.

It is important to note that the independence complex is also a clique complex [4], which will allow us to consider star clusters when computing homotopy types.

5.2 Homotopy Type of the Independence Complex of the Intersection Graph

5.2.1 Realizing Spheres and Wedges of Spheres

For simplicity in our upcoming results, we will introduce some easy terms for communicating properties of certain collections of chords.

Definition 5.2.1. Consider a chord diagram C. Define X**-chords** as a pair of crossing chords such that they are disjoint from all other chords in C.

Define $X(n)$**-chords** as n mutually crossing chords such that they are disjoint from all other chords in C.

Using this terminology, and the well-known fact that, if K and L are simplicial complexes, then $I_{K \sqcup L} \simeq I_K * I_L$, we can easily communicate some results regarding the homotopy type of $I_{\Gamma(C)}$.

Remark 65. [17, Observation 1.3]

1. Let K be contractible. Then, ΣK is also contractible.
2. $\Sigma(\vee \mathbb{S}^n) \simeq \vee \mathbb{S}^{n+1}$.

Lemma 66. Suppose C contains X-chords. Then $I_{\Gamma(C)}$ has the homotopy type of a suspension.

Proof. Consider a chord diagram C. Suppose we add X-chords to C. Call them a and b. Then, $\Gamma(C + \{a, b\})$ has two components: one for C and one for $\{a, b\}$. Additionally, our component for $\{a, b\}$ will resemble P_2. So,

$$\Gamma(C \sqcup \{a, b\}) = \Gamma(C) \sqcup \Gamma(P_2)$$

Thus,

$$\begin{aligned} I_{\Gamma(C) \sqcup \Gamma(P_2)} &\simeq I_{\Gamma(C)} * I_{P_2} \\ &\simeq \Sigma I_{\Gamma(C)}. \end{aligned}$$

□

Using Lemma 66, we can show that all spheres, and all wedges of spheres can be realized as the homotopy type of some $I_{\Gamma(C)}$.

Proposition 67. Suppose C is one pair of X-chords. Then, $I_{\Gamma(C)} \simeq \mathbb{S}^0$.

Proof. Suppose C is one pair of X-chords. Then $\Gamma(C) = P_2$, meaning $I_{\Gamma(C)}$ is two disjoint vertices, i.e. $I_{\Gamma(C)} \simeq \mathbb{S}^0$. □

Thus, considering Lemma 66 and Remark 65, if we continue to add X-chords to our chord diagram, we have

Corollary 68. Suppose C is one pair of X-chords. Then, $\Sigma^n I_{\Gamma(C)} \simeq \mathbb{S}^n$.

Proposition 69. Suppose C is a set of $X(n)$-chords. Then $I_{\Gamma(C)} \simeq \vee^{n-1} \mathbb{S}^0$.

Proof. Suppose C is a set of $X(n)$-chords. Then, $\Gamma(C)$ is a complete graph on n vertices. Thus, $I_{\Gamma(C)}$ is n disjoint vertices, i.e. $I_{\Gamma(C)} \simeq \vee^{n-1} \mathbb{S}^0$. □

Like Corollary 68, considering Lemma 66 and Remark 65, if we continue to add X-chords to our chord diagram, we have

Corollary 70. Suppose C is a set of $X(n)$-chords. Then $\Sigma^i I_{\Gamma(C)} \simeq \vee^{n-1} \mathbb{S}^i$.

5.2.2 Realizing Families as the Intersection Graphs of Chord Diagrams

Now, using previous results regarding independence complexes, we can say more about chord diagrams. Namely, if we can show that the intersection graph resembles a class of graphs for which we already know about the homotopy type of the independence complex, we can quickly achieve results, and develop classes of chord diagrams.

We first state two known results regarding the independence complex of paths and cycles.

Lemma 71. [18] Suppose P_t is a path on t vertices. Then,

$$I_{P_t} \simeq \begin{cases} * & \text{if } t = 3n \\ \mathbb{S}^n & \text{if } t = 3n+1 \\ \mathbb{S}^n & \text{if } t = 3n+2 \end{cases}$$

Lemma 72. [18] Suppose C_t is a cycle on t vertices. Then,

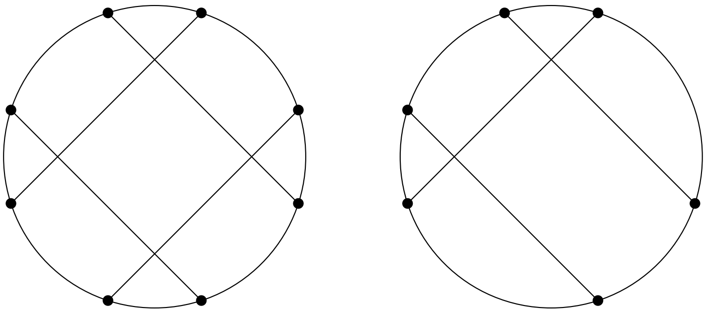
$$I_{C_t} \simeq \begin{cases} \mathbb{S}^{n-1} & \text{if } t = 3n \\ * & \text{if } t = 3n+1 \\ \mathbb{S}^n & \text{if } t = 3n+2 \end{cases}$$

Now, using Lemmas 71 and 72, we can define two families of chord diagrams and state results regarding the independence complex of their intersection graphs.

Definition 5.2.2. Suppose $C(n)$ contains n chords such that each crosses exactly two other chords. Then, $C(n)$ is a **cycle chord diagram**.

Definition 5.2.3. Suppose $P(n)$ contains n chords resembling a cycle chord diagram with exactly one chord removed. Then $P(n)$ is a **path chord diagram**.

Example 73. Here, we can illustrate a cycle chord diagram, $C(4)$, and a path chord diagram $P(3)$.



As one can see, a path chord diagram is as if we simply tear one of the chords out of our cycle chord diagram, which is an intuitive idea given the relationship between cycle and path graphs.

Proposition 74. Suppose $C(n)$ is a cycle chord diagram. Then

$$I_{\Gamma(C(n))} \simeq \begin{cases} \mathbb{S}^{k-1} & \text{if } n = 3k \\ * & \text{if } n = 3k+1 \\ \mathbb{S}^k & \text{if } n = 3k+2. \end{cases}$$

Proof. Suppose $C(n)$ is a cycle chord diagram. Then it is clear that $\Gamma(C(n))$ is a cycle graph with n vertices. Thus, by Lemma 72,

$$I_{\Gamma(C(n))} \simeq \begin{cases} \mathbb{S}^{k-1} & \text{if } n = 3k \\ * & \text{if } n = 3k+1 \\ \mathbb{S}^k & \text{if } n = 3k+2. \end{cases}$$

□

Proposition 75. Suppose $P(n)$ is a path chord diagram. Then,

$$I_{\Gamma(P(n))} \simeq \begin{cases} * & \text{if } n = 3k \\ \mathbb{S}^k & \text{if } n = 3k+1 \\ \mathbb{S}^k & \text{if } n = 3k+2. \end{cases}$$

Proof. Suppose $P(n)$ is a path chord diagram. Then it is clear that $\Gamma(P(n))$ is a path graph with n vertices. Thus, by Lemma 71,

$$I_{\Gamma(P(n))} \simeq \begin{cases} * & \text{if } n = 3k \\ \mathbb{S}^k & \text{if } n = 3k+1 \\ \mathbb{S}^k & \text{if } n = 3k+2. \end{cases}$$

□

Next, we provide a new computation for the homotopy type of the independence complex of ladder graphs.

Definition 5.2.4. We define a **square grid**, $SG_{r,s}$, such that $SG_{r,s} = P_r \times P_s$ where $\times$ denotes the Cartesian product. For a vertex $v \in V(SG_{r,s})$, we can denote $v_{\{i,j\}}$ such that $1 \leq i \leq r$ and $1 \leq j \leq s$.

Definition 5.2.5. We define a square grid, $SG_{2,s}$, to be a **ladder graph** for $s \geq 2$.

Lemma 76. Let $SG_{2,s}$ be a ladder graph. Then,

$$I_{SG_{2,s}} \simeq \begin{cases} \mathbb{S}^{\frac{s}{2}-1} & \text{if } s = 2k \\ \mathbb{S}^{\lfloor \frac{s}{2} \rfloor} & \text{if } s = 2k+1 \end{cases}$$

Notice that $SG_{2,s}$ is defined by two paths of length s. We will denote the vertices of one path by $v_i, 1 \leq i \leq s$, and the other by $u_i, 1 \leq i \leq s$, where the ordering of v and u has the same orientation, i.e. v_i and u_i are connected by a "ladder rung."

Proof. We apply the Cluster Lemma. In order to do so, we decompose $I_{SG_{2,s}}$ into collections Δ_k. First, we construct collections of sub-simplices as follows:

1. Let $\sigma_0 := \mathrm{SC}(\cup_{j=0}^{\lfloor \frac{s}{2} \rfloor} v_{2j+1} \cup_{j=1}^{\lfloor \frac{s}{2} \rfloor} u_{2j})$.
2. For $1 \leq i \leq \lfloor \frac{s}{2} \rfloor$, let $\sigma_i := \mathrm{st}(v_{2i})$.
3. Let $\sigma_{\lfloor \frac{s}{2} \rfloor + 1} := \begin{cases} (\cup_{j=0}^{\lfloor \frac{s}{2} \rfloor - 1} u_{2j+1}) & \text{for } s = 2k \\ (\cup_{j=0}^{\lfloor \frac{s}{2} \rfloor} u_{2j+1}) & \text{for } s = 2k+1. \end{cases}$

Now define $\Delta_0 := \sigma_0$ and $\Delta_i := \sigma_i - \cup_{j=0}^{i-1} \sigma_j$, and observe that $\bigcup_{k=0}^{\lfloor \frac{s}{2} \rfloor + 1} \Delta_k = I_{SG_{r,s}}$. Define an acyclic matching on each Δ_j as follows:

We know that Δ_0 is collapsible by Proposition 6 so Δ_0 has an acyclic matching with a single unmatched 0-simplex.

Now, consider Δ_i for $1 \leq i \leq \lfloor \frac{s}{2} \rfloor$. For each (n)-simplex, $\{\tau^{(n-1)} v_{2i}\} \in \Delta_i$, we will match it with the $(n+1)$-simplex of the form $\{\tau^{(n-1)} v_{2i} u_{2i-1}\} \in \Delta_i$. We can notice that considering only the collection of sub-simplices defined by Δ_1, u_1 dominates v_2. From there, it is easy to verify that as we progress through our Δ_i, each u_{2i-1}

dominates each v_{2i}, which provides a clear explanation for why our matching is valid. Further, we know each Δ_i matching is acyclic if we consider the nature of each individual pair, as they are all directed towards u_{2i-1}. Note that this matches all simplices in each Δ_i.

Then, this leaves us with $\Delta_{\lfloor \frac{s}{2} \rfloor+1}$, which is simply one element, and thus must be critical. So, we are left with one critical 0-simplex from Δ_0 and the critical simplex from $\Delta_{\lfloor \frac{s}{2} \rfloor+1}$. Thus,

$$I_{SG_{2,s}} \simeq \begin{cases} \mathbb{S}^{\frac{s}{2}-1} & \text{if } s = 2k \\ \mathbb{S}^{\lfloor \frac{s}{2} \rfloor} & \text{if } s = 2k+1. \end{cases}$$

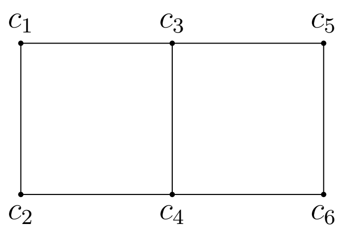
□

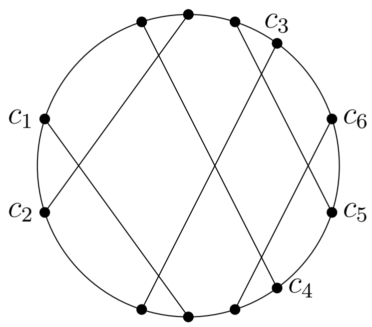
Definition 5.2.6. Denote $SG(2, s)$ a **ladder chord diagram** for $SG_{2,s}$.

Define $SG(2, 2)$ to be a cycle chord diagram on four chords ordered counterclockwise $c_1, \ldots, c_4$. Then, $SG(2, s+1) := SG(2, s) + \{c_i, c_j\}$ where i, j are ordered in general increasing order (i.e. $i = 5, j = 6$ for $SG(2, 3)$), and

- c_i intersects c_{i-2} and c_j;
- c_j intersects c_i and c_{j-2}.

Example 77. Here, we can illustrate $SG_{2,3}$ and the corresponding chord diagram, $SG(2, 3)$, such that $\Gamma(SG(2, 3)) = SG_{2,3}$





Thus, using 76, we can state the following proposition:

Proposition 78. Suppose $SG(2, s)$ is a ladder chord diagram. Then,

$$I_{\Gamma(SG(2,s))} \simeq \begin{cases} \mathbb{S}^{\frac{s}{2}-1} & \text{if } s = 2k \\ \mathbb{S}^{\lfloor \frac{s}{2} \rfloor} & \text{if } s = 2k+1. \end{cases}$$

Proof. Suppose $SG(2, s)$ is a ladder chord diagram. Then it is clear that $\Gamma(SG(2, s))$ is a ladder graph of length s. Thus, by Lemma 76,

$$I_{\Gamma(SG(2,s))} \simeq \begin{cases} \mathbb{S}^{\frac{s}{2}-1} & \text{if } s = 2k \\ \mathbb{S}^{\lfloor \frac{s}{2} \rfloor} & \text{if } s = 2k+1. \end{cases}$$

□

Definition 5.2.7. A chord diagram, $C(n)$, is **bipartite** if the n chords can be partitioned into two independent sets, A and B.

Definition 5.2.8. A chord diagram, $C(n)$, is **complete bipartite** if $C(n)$ is bipartite and all chords in A intersect all chords B.

Proposition 79. Suppose $C(n)$ is complete bipartite. Then $I_{\Gamma(C(n))} \simeq \mathbb{S}^0$.

Proof. Let $C(n)$ be complete bipartite. Then we can partition the chords of $C(n)$ into two independent sets, A and B. Suppose $|A| = n_1$ and $|B| = n_2$. Then, consider $I_{\Gamma(C(n))}$. If Δ^n is the n-simplex, it is clear that $I_{\Gamma(C(n))}$ will resemble $\Delta^{n_1-1} \sqcup \Delta^{n_2-1}$. We know that Δ^n is collapsible for all n, and thus $I_{\Gamma(C(n))} \simeq \Delta^{n_1-1} \sqcup \Delta^{n_2-1} \simeq \mathbb{S}^0$. □

We continue to explore families of intersection graphs that can be modeled by chord diagrams. Here, we give a short proof for the homotopy type of graphs, $L_2(G)$, defined in Proposition 47, and then define **centipede chord diagrams** and give the homotopy type of the resulting independence complex for their intersection graph.

Lemma 80. Let G be a connected graph with v vertices and let $L_2(G)$ be defined as in Proposition 47. Then, $I_{L_2(G)} \simeq *$.

Proof. Let G be a connected graph and denote the leaves we add to G to obtain $L_2(G)$ as $\ell_1, \ell_2, \ldots, \ell_v$. Consider the approach of the Cluster Lemma for computing homotopy types. Let us choose our first collection of sub-simplices, σ_0, to be defined as follows: $\sigma_0 := \mathrm{SC}(\cup_{j=1}^{v} \ell_j)$. Then, Δ_0 is collapsible by Proposition 6, and the only possible simplex left outside of Δ_0 would be the $(v-1)$-simplex defined by all v vertices of G. However, it is clear that not all v vertices are independent from each other, and thus, that simplex cannot exist in $I_{L_2(G)}$. Thus, $I_{L_2(G)} \simeq *$. □

Corollary 81. Let $\mathcal{C}_n$ be a centipede graph. Then, $I_{(\mathcal{C}_n)} \simeq *$.

Definition 5.2.9. Suppose we order the chords of $P(n)$ as c_i, $1 \leq i \leq n$. Then, define a **centipede chord diagram**, denoted $\mathcal{C}(n)$, as the chord diagram $P(n) \cup_{i=1}^{n} (c_i')$ such that each chord c_i' only intersects chord $c_i \in P(n)$.

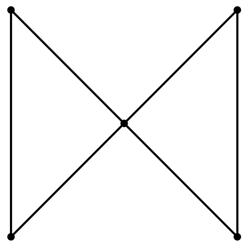
Proposition 82. Suppose $\mathcal{C}(n)$ is a centipede chord diagram. Then $I_{\Gamma(\mathcal{C}(n))} \simeq *$.

Proof. Suppose $\mathcal{C}(n)$ is a centipede chord diagram. Then, it is clear that $\Gamma(\mathcal{C}(n))$ corresponds to $\mathcal{C}_n$, and thus by Lemma 81, $I_{\Gamma(\mathcal{C}(n))} \simeq *$. □

Chapter 6

Future Directions

Open Question 1. One future direction that seems to hold a lot of potential for computing the homotopy type of the Morse complex relates the homotopy type of the Morse complex to the homotopy type of the generalized Morse complex. We argued using two elementary collapses that $\mathcal{GM}(C_t \vee l) \simeq \mathcal{M}(C_t \vee l)$. Because the generalized Morse complex is a flag complex, its homotopy type should theoretically be easier to compute, as we can use star clusters as a tool. So, consider another example:



Using the Cluster Lemma starting with a star collapse, we can apply a matching to $\mathcal{M}(D_3^2)$, finding that its homotopy type is collapsible. Slightly easier, we can use the Cluster Lemma starting with a star cluster collapse to apply a matching to $\mathcal{GM}(D_3^2)$ and compute the homotopy type of a point. So, we would be interested in pursuing the question of whether there is a way to use the generalized Morse complex as a tool for computing the homotopy type of the Morse complex for certain complexes.

Open Question 2. Another future direction that may prove interesting is studying what types of graphs can be modeled by the intersection graph of a chord diagram. For example, in this thesis, we were able to model ladder graphs as chord diagrams. However, we were not successful in modeling square grids at a larger scale. Right when we added a second row or squares in the grid, drawing a chord diagram with an intersection graph that realized the grid became, we think, impossible.

So, a question we might ask is what characteristics prevent an intersection graph from being modeled by a chord diagram? Can an intersection graph such that its independence complex has torsion be modeled by a chord diagram? This may shed more light on the largely unexplored area of chord diagrams, and bring more perspective as to whether all independence complexes of chord diagrams have the homotopy type of a point, sphere, or wedge of spheres.

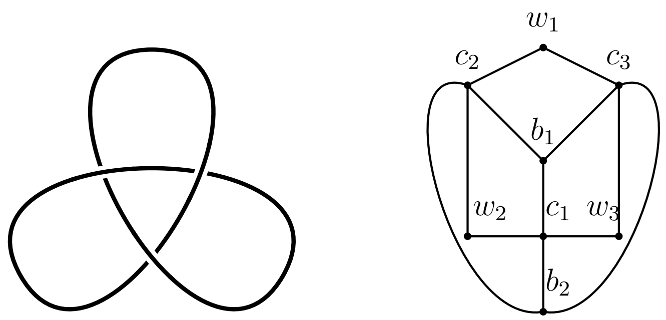
Open Question 3. One way to use the homotopy type of the generalized Morse complex to determine the homotopy type of the Morse complex is to show that the generalized Morse complex collapses to the Morse complex. This is not always possible, as we saw: the homotopy type of the Generalized Morse complex of a cycle computed in Theorem 21 does not agree with the homotopy type of the Morse complex of the cycle, as computed by Kozlov [18]. However, one can use the matching found in the proof of Theorem 21, throw out the closed V-paths in the matching, and obtain a matching on the Morse complex. In this case, Forman's theorem 8 is not satisfied because the critical cells occur in different dimensions,so the homotopy type is not uniquely determined. However, there may be special cases where the homotopy type can be recovered from knowledge of the critical cells and some other information. See, for example, [19, Theorem 2.2].

One complex of particular interest is the 3-simplex. The homotopy type of the Morse complex of the 3−simplex remains unknown. Chari and Joswig [7] showed that the 3−simplex satisfies $\{b_0 = 1, b_5 = 99\}$ using software. However, a matching on the 3-simplex has never been formed to yield these results. While star clusters are ineffective in collapsing the Morse complex of the n-simplex because it is not flag, can we create a matching on the generalized Morse complex and then remove the cyclic gradient vector fields from the matching? To find a matching, even just on the Morse complex of the 3-simplex, would be a major development in the study of the Morse complex.

Another object of particular interest, and one that would make for an interesting application, is the Tait graph of a knot–as well as the Morse complex of the Tait graph. We have studied the effect that Reidemeister moves have on the Tait graph, and we will provide proofs for both type I and type II Reidemeister moves. First, let us informally define the Tait graph of a knot as well as the Reidemeister moves.

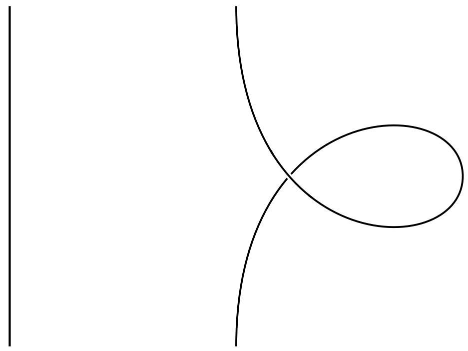
Definition 83. Let D be a knot diagram. Its corresponding **Tait graph**, denoted $\Gamma(D)$, is a graph corresponding to a checkerboard coloring of the knot diagram, where each region, as well as each crossing, corresponds to a vertex in $\Gamma(D)$. Then, each crossing has an edge connecting it to the vertices of its adjacent regions.

Example 84. Here we have the Trefoil knot, 3_1, and its corresponding Tait Graph, with crossings, and black and white regions, labeled.



Definition 85. Let D be a knot diagram, with strand s separating two regions. A **Type I Reidemeister Move (RI)** is the twisting of s, introducing a new crossing.

Example 86. We illustrate a Type I Reidemeister move. It is a "twist" in the strand, introducing a new crossing and a new region.



Definition 87. Let D be a knot diagram, with strands s_1 and s_2 such that they are not crossing. A **Type II Reidemeister Move (RII)** "pokes" s_1 either over or under s_2, introducing two new crossings.

Example 88. We illustrate a Type II Reidemeister move. It is a "poke" move, meaning we slide one strand under or over the other, introducing two new crossings.

So, our biggest interest regarding these Reidemeister moves is the effect, if any, that they have on the homotopy type of the Tait Graph and the Morse complex of the Tait graph. The Tait graph of knots gets large quick, and they do not tend to be trees. Thus, they become difficult objects to study, and it is likely that star clusters will not be a very useful tool.

We have pursued the effect that these Reidemeister moves have on the Tait graph–but not yet the effect on the homotopy type. So we will state those results. Note that these results are construction-oriented and so our results unfold in the proofs.

For any region, r_x, in knot diagram D, we denote its corresponding vertex in the $\Gamma(D)$ by v_{r_x}. Also, we denote an edge between two vertices, v_x, v_y in $\Gamma(D)$ by $v_x v_y$.

Lemma 89. Let D be a knot diagram with Tait Graph $\Gamma(D)$. Then performing a Type I Reidemeister move on D produces a new Tait Graph, $\Gamma(D_{RI}) = \Gamma(G_1)$.

Proof. Let D be a knot diagram with Tait Graph $\Gamma(D)$. Let $s \in D$ be a strand which separates regions $r_1, r_2 \in D$.

We can perform a RI on s by twisting the strand, protruding towards r_1, creating a new crossing, c, and a new region r_3. We can call knot projection D with a RI, $D_{RI} = G_1$, with Tait graph $\Gamma(D_{RI}) = \Gamma(G_1)$.

Performing the RI, crossing c and region r_3 are created.

Thus, $\Gamma(G_1)$ contains a new crossing vertex, v_c. From v_c, there will exist an edge connecting v_c to v_{r_3}, $v_c v_{r_3}$; an edge connecting v_c to v_{r_2}, $v_c v_{r_2}$; and two edges connecting v_c to v_{r_1}, $v_c v_{r_1}$ and $v_c v_{r_1}{}'$.

Tait Graphs are planar so these new edges exist such that they do not overlap with any other edges.

All other elements of $\Gamma(D)$ are unchanged in $\Gamma(G_1)$

□

Lemma 90. Let D be a knot diagram with Tait Graph $\Gamma(D)$. Then performing a Type II Reidemeister move on D produces a new Tait Graph, $\Gamma(D_{RII}) = \Gamma(G_2)$.

Proof. Let D be a knot diagram with Tait graph $\Gamma(D)$. Let there exist a strand $s_1 \in D$ between regions $r_1, r_2 \in D$ and a strand $s_2 \in D$ between regions $r_2, r_3 \in D$.

We can perform a RII by pushing s_1 across r_2, under/over (without loss of generality) s_2, and protruding into r_3. We can call this knot projection D with a RII, $D_{RII} = G_2$, with Tait graph $\Gamma(D_{RII}) = \Gamma(G_2)$.

Performing the RII, two new crossings, $c_1, c_2 \in D$, are created, and two new regions, $r_4, r_5 \in D$, are created.

Thus, $\Gamma(G_2)$ contains a new crossing vertex v_{c_1}. From v_{c_1}, there exists edge $v_{c_1}v_{r_1}$, $v_{c_1}v_{r_2}$, $v_{c_1}v_{r_3}$, and $v_{c_1}v_{r_4}$. $\Gamma(G_2)$ also contains a new crossing vertex v_{c_2}. From v_{c_2}, there exists edge $v_{c_2}v_{r_1}$, $v_{c_2}v_{r_2}$, $v_{c_2}v_{r_3}$, and $v_{c_1}v_{r_5}$.

Tait graphs are planar, these new edges exist such that they do not overlap with any other edges.

All other elements of $\Gamma(D)$ are unchanged in $\Gamma(G_2)$.

□

So, we can formally state a question regarding Tait graphs: what effect, if any, do Reidemeister moves have on the homotopy types of the Tait graph and the Morse complex of the Tait graph? Can the Cluster Lemma be a useful approach to this question? Can we create a matching on the generalized Morse complex and then remove the cyclic gradient vector fields from the matching?

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