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Energetics of linear geostrophic adjustment in stratified rotating fluids

by Roger H. J. Grimshaw[1], Andrew J. Willmott[2] and Peter D. Killworth[3]

ABSTRACT

The energy conversion ratio, γ, is shown to be bounded below by 0 and above by 1/2 in the two-dimensional linear geostrophic adjustment of a continuously stably stratified, incompressible, inviscid non-Boussinesq fluid. "Two-dimensional" refers to problems in which the initial isopycnal displacement field is an arbitrary function of the vertical (parallel to the rotation axis) and a single horizontal coordinate. By using Fourier analysis techniques, the paper also identified classes of initial isopycnal displacement profiles for which the adjustment process leads to $\gamma > 1/3$. Finally, an expression for γ is derived when the initial isopycnal displacement profile is three dimensional.

1. Introduction

The problem of how a fluid, initially not in geostrophic balance, adjusts to that balance is a fundamental problem in the theory of rotating fluids. Blumen (1972) gave the first review in the area, and discussed many concepts which have been used since, including potential vorticity conservation and minimum energy principles.

In this paper, upper and lower bounds are derived for the energy conversion ratio in the geostrophic adjustment problem for a uniformly rotating stratified fluid. A feature of all geostrophic adjustment problems is that only a fraction of the potential energy released, ΔPE, is converted into kinetic energy, ΔKE, of the final geostrophically adjusted state. The energetics of the geostrophic adjustment problem are usually expressed in terms of the energy conversion ratio $\gamma = \Delta KE/\Delta PE$.

In the literature, following Blumen's work, the "classical" linear geostrophic adjustment problem refers to the adjustment of a horizontally unbounded, uniformly rotating, barotropic fluid which initially is at rest (with respect to the rotating frame of reference) with a step in the fluid surface which is maintained by a vertical barrier. Upon removal of the barrier the fluid adjusts to a steady geostrophic state by the propagation of Poincaré waves. In the adjusted state Gill (1976, 1982) shows that $\gamma = 1/3$; the radiation of Poincaré waves implying a loss of energy from any finite region.

1. Department of Mathematics and Statistics, Monash University, Clayton, Victoria 3168, Australia.
2. Department of Mathematics, Keele University, Keele, Staffordshire, ST5 5BG, United Kingdom.
3. Southampton Oceanography Centre, European Way, Southampton SO14 3ZH, United Kingdom.

In this paper we focus on the small amplitude (linearized) geostrophic adjustment problem. Within the context of this linearized theory, what is the value of γ for an arbitrary initial fluid surface displacement when the fluid is initially at rest? This question is partially answered by Middleton (1987) for an initial surface displacement which depends only upon one horizontal spatial variable. Middleton (1987) demonstrates that $\gamma \leq \frac{1}{2}$ for this case. When the initial surface displacement is sinusoidal with wavelength, λ, Killworth (1986) and Middleton (1987) demonstrate that γ smoothly varies from 0 to $\frac{1}{2}$ as λ increases from 0 to infinity. Both authors, therefore, stress the importance of the horizontal characteristic length scale of the initial fluid surface displacement in determining the value of γ.

Problems in which the initial surface displacement is of square wave form with a horizontal extent given by L lead to an adjusted state in which $0 \leq \gamma \leq \frac{1}{3}$, as shown by Killworth (1986). As $L \to \infty$, $\gamma \to \frac{1}{3}$, because the adjustment process at each end of the initial square wave profile occurs independently of the other. On the other hand, $\gamma \to 0$ as $L \to 0$ because the effects of rotation are then negligible in the adjustment process and hence $\Delta KE \to 0$ as $L \to 0$.

The nonlinear version of the problem considered by Gill (1976, 1982) is discussed by Boss and Thompson (1995). To calculate the geostrophically adjusted state in the nonlinear problem, conservation of mass and potential vorticity (PV) are required. Boss and Thompson (1995) show that $\gamma = \frac{1}{3}$, although ΔKE is slightly smaller than its linear counterpart. The authors also consider the nonlinear geostrophic adjustment for a two-layer fluid with a rigid lid and show that $\gamma = \frac{1}{3}$ in this case too. However, both ΔKE and ΔPE in the nonlinear adjusted state are less than the equivalent values for the linear problem. This difference is greatest when the mean depth of the interface is close to the upper or lower boundary, and increases with the height of the initial step in the interfacial displacement.

In a model of tidal mixing, van Heijst (1985) calculates the nonlinear geostrophically adjusted state which results after the removal of a barrier separating a stably stratified two-layer fluid from a homogeneous fluid of intermediate density. In essence this study is similar to that of Boss and Thompson (1995) except that the number of fluid layers is increased. van Heijst (1985) shows that $\gamma = \frac{1}{3}$, irrespective of the value of the initial depth ratio of the stratified region.

Nonlinear geostrophic adjustment of a continuously stratified inviscid, incompressible Boussinesq fluid is considered by Ou (1984, 1986), Blumen and Wu (1995) and Wu and Blumen (1995). All these studies consider two-dimensional motion in a fluid which is unbounded horizontally and bounded in the vertical by two rigid planes. Ou considers a fluid that initially is at rest, with a density field that is homogeneous in the vertical, but in which the density varies smoothly from one uniform value to another in the horizontal. Consequently, the initial value of the PV is zero, and is conserved throughout the adjustment. Blumen and Wu (1995) refer to the Ou problem as a "zero PV flow." Ou (1986) and Blumen and Wu (1995) establish that $\gamma = \frac{1}{2}$ for the zero PV adjustment problem provided that fronts do not form during the adjustment. Since a two-layer fluid is an asymptotic limit of a continuously stratified fluid, Ou (1986) deduces that γ must decrease smoothly from ½ to ⅓ in the parameter regime where fronts form. Blumen and Wu (1995) also extend the Ou study to the case when the fluid is initially at rest with a basic state density field which is made up from the sum of two parts; a field which is linear in depth and one which depends only upon a horizontal spatial coordinate. These initial conditions lead to a uniform value for the *PV* which is conserved throughout the flow, and Blumen and Wu (1995) show that $\gamma \leq 1/2$ for particular distributions of the initial density field. The value of γ depends upon the horizontal length scale of the initial density field.

Wu and Blumen (1995) extend the Ou (1984, 1986) studies to include an initial geostrophic velocity, but still maintain the zero PV assumption. Although the initial velocity field can alter the distributions of the density and velocity fields in the adjusted state compared with the Ou problem, it is found that $\gamma = 1/2$.

Kuo (1997) examines the temporal evolution of a radially symmetric disturbance with initially unbalanced velocity or pressure, for general profiles, but does not discuss the energy of the final system from the perspective of γ.

It seems that an overlooked aspect of the geostrophic adjustment problem relates to the linear adjustment of a continuously stratified, inviscid, incompressible fluid bounded above by a free surface, and below by a rigid horizontal plane, unbounded in the horizontal, with an initial PV distribution which is an arbitrary function of depth and one horizontal spatial coordinate. Under these assumptions, the motion is two-dimensional and the goal of this paper is to derive upper and lower bounds for γ.

2. Formulation of the problem

Consider an incompressible, stably stratified, inviscid fluid rotating with uniform angular speed $f/2$ about a vertical axis, where f is the Coriolis parameter in geophysical applications. The governing equations for small amplitude motions in the hydrostatic approximation about an equilibrium state in which the density is $\rho_0(z)$ are

$$u_t - fv + \rho_0^{-1} p_x = 0, \tag{2.1}$$

$$v_t + fu + \rho_0^{-1} p_y = 0, \tag{2.2}$$

$$p_z = -g\rho, \tag{2.3}$$

$$u_x + v_y + w_z = 0, \tag{2.4}$$

$$\rho_t - \rho_0 N^2 g^{-1} w = 0, \tag{2.5}$$

where

$$N^2 = -g(d\rho_0/dz)\rho_0^{-1},$$

is the buoyancy frequency. The system (2.1) to (2.5) is referred to a right-handed Cartesian coordinate frame $0xyz$, with $0z$ directed vertically upward; t is time; u, v, w denote the velocity components along the $0x$, $0y$ and $0z$ axes, respectively; p is the perturbation pressure; ρ is the perturbation density; g is the gravitational acceleration. If in (2.1) to (2.5) the Boussinesq approximation is made, then $\rho_0(z)$ becomes a constant, ρ^*, but $d\rho_0/dz$ remains unchanged.

For a fluid of uniform depth h, with the undisturbed free surface at $z = 0$, the boundary conditions for (2.1) to (2.5) are

$$w = 0 \qquad \text{at} \qquad z = -h \tag{2.6}$$

$$\left.\begin{aligned} w &= \eta_t \\ p - g\rho_0\eta &= 0 \end{aligned}\right\} \qquad \text{at} \qquad z = 0. \tag{2.7}$$

The first of (2.7) states that a fluid particle in the free surface will remain in it (the kinematic boundary condition), while the second of (2.7) is a statement of continuity of pressure at the fluid surface (see LeBlond and Mysak, 1978, Ch. 2, Section 9). Here η is the free-surface displacement.

It will be convenient to recast the equations in terms of the vertical isopycnal displacement field $\zeta(x, y, z, t)$. Consider an isopycnal surface $\rho_0 + \rho =$ constant, which in the rest state coincides with $z = z_0$. Let $\zeta(x, y, z_0, t)$ denote the vertical displacement of this surface from $z = z_0$:

$$z - z_0 = \zeta(x, y, z_0, t).$$

In fully nonlinear motion, $\rho_0 + \rho$ is conserved following a fluid particle and therefore

$$\rho_0 + \rho = \rho_0(z - \hat{\zeta}), \tag{2.8}$$

where

$$\hat{\zeta}(x, y, z, t) \equiv \zeta(x, y, z_0, t) = \zeta(x, y, z - \zeta, t). \tag{2.9}$$

Then to the leading linear order it is readily seen that,

$$\hat{\zeta} = \zeta(x, y, z, t) + \ldots, \tag{2.10}$$

and

$$\rho = -\rho_{0z}\zeta + \ldots \tag{2.11}$$

Holliday and McIntyre (1981) derive similar expressions to higher order in amplitude in a paper which considers expressions for the potential energy density in a continuously stratified incompressible fluid when isopycnal displacements are large. Note that (2.11) shows that to this leading linear order,

$$w(x, y, z, t) = \zeta_t. \tag{2.12}$$

Thus (2.12) in effect replaces (2.5), and to this leading linear order, the hydrostatic balance, (2.3), becomes,

$$p_z + \rho_0 N^2\zeta = 0. \tag{2.13}$$

Finally, the two boundary conditions (2.7) are replaced by the single boundary condition,

$$p - g\rho_0\zeta = 0 \qquad \text{at} \qquad z = 0. \tag{2.14}$$

To summarize, the governing equations are (2.1), (2.2), (2.4), (2.11), (2.12), and (2.13). In a discussion of the governing equations for a nonrotating continuously stratified incompressible fluid, Gill (1982, Ch. 6) also introduces an isopycnal displacement function which is equivalent to ζ. An energy equation for this system is straightforward to derive and takes the form

$$\frac{\partial}{\partial t}(K + P) + \nabla \cdot (p\mathbf{u}) + (pw)_z = 0, \tag{2.15}$$

where the kinetic and potential energy densities are given by

$$K = \frac{1}{2}\rho_0(u^2 + v^2),$$

$$P = \frac{1}{2}\rho_0 N^2\zeta^2,$$

respectively, $\mathbf{u} = (u, v)$ and ∇ denotes the two-dimensional horizontal gradient operator. The kinetic and potential energy densities per unit area are then obtained by integrating these expressions over the depth, so that,

$$\frac{\partial}{\partial t}(KE + PE) + \nabla \cdot \mathbf{F} = 0 \tag{2.16}$$

where the expressions KE, PE and $\mathbf{F}$ are given by,

$$\left.\begin{aligned} KE &= \int_{-h}^{0} \frac{1}{2}\rho_0(u^2 + v^2)\, dz, \\ PE &= \int_{-h}^{0} \frac{1}{2}\rho_0 N^2\zeta^2\, dz + \frac{1}{2}\rho_0(z = 0)g\zeta^2(z = 0). \\ \mathbf{F} &= \int_{-h}^{0}(p\mathbf{u})\, dz. \end{aligned}\right\} \tag{2.17}$$

Initially the fluid is at rest with a specified density anomaly:

$$\mathbf{u} = 0,\, w = 0,\, \zeta = \zeta_0(x, y, z) \qquad \text{at} \qquad t = 0. \tag{2.18}$$

From (2.1), (2.2), (2.4) the perturbation PV, Π, which is a conserved quantity following the motion, is given by

$$\Pi = v_x - u_y - f\zeta_z = -f\zeta_{0z} \tag{2.19}$$

and is therefore spatially nonuniform.

3. Normal mode decomposition

Next all dependent variables are written in terms of the modal expansion (e.g., Gill (1982) though with a different notation):

$$\mathbf{u} \equiv (u, v) = \sum_{n=0}^{\infty} \mathbf{B}_n(x, y, t) \frac{d\phi_n}{dz}, \tag{3.1}$$

$$p = \sum_{n=0}^{\infty} \rho_0 D_n(x, y, t) \frac{d\phi_n}{dz}, \tag{3.2}$$

$$\zeta = \sum_{n=0}^{\infty} A_n(x, y, t)\phi_n(z), \tag{3.3}$$

where the vertical modal functions $\phi_n(z)$ satisfy

$$\frac{d}{dz}\left[\rho_0 \frac{d\phi}{dz}\right] + \frac{\rho_0 N^2}{c^2} \phi = 0, \tag{3.4a}$$

subject to

$$\phi = 0 \qquad \text{at} \qquad z = -h, \tag{3.4b}$$

$$\frac{d\phi}{dz} = \frac{g}{c^2} \phi \qquad \text{at} \qquad z = 0. \tag{3.4c}$$

The modal functions have dimensions of length, while c is the linear long-wave phase speed. Both are indexed such that $n = 0$ denotes the barotropic mode. The orthogonality condition is

$$\int_{-h}^{0} \rho_0 \phi_{nz} \phi_{mz}\, dz = \delta_{nm} I_n, \tag{3.5a}$$

which defines I_n when $n = m$, and upon integrating by parts it becomes

$$\int_{-h}^{0} \rho_0 N^2 \phi_n \phi_m\, dz + (\rho_0 g \phi_n \phi_m)\Big|_{z=0} = c_n^2 \delta_{nm} I_n, \tag{3.5b}$$

upon using (3.4). In (3.5), δ_{nm} denotes the Kronecker delta. Multiplying (3.1) and (3.2) by $\rho_0 \phi_{mz}$, vertically integrating and applying (3.5a) gives

$$I_n \mathbf{B}_n = \int_{-h}^{0} \rho_0 \phi_{nz} \mathbf{u}\, dz, \qquad I_n D_n = \int_{-h}^{0} p \phi_{nz}\, dz.$$

Similarly, multiplying (3.3) by $\rho_0 N^2 \phi_m$, vertically integrating and applying (3.5b) yields

$$I_n c_n^2 A_n = \int_{-h}^{0} \rho_0 N^2 \phi_n \zeta\, dz + (\rho_0 g \phi_n \zeta)\Big|_{z=0}.$$

The hydrostatic balance (2.13) can be written as

$$c_n^2 I_n A_n = -\int_{-h}^{0} p_z \phi_n \, dz + (p\phi_n)\Big|_{z=0}$$

$$= \int_{-h}^{0} p\phi_{nz} \, dz = I_n D_n$$

which gives

$$D_n = c_n^2 A_n. \tag{3.6}$$

To obtain a single equation for A_n, two approaches are possible. First, we may either proceed from the full linearized time-dependent equations, which yields after some algebra

$$A_{ntt} + f^2(A_n - A_{n0}) = c_n^2 \nabla^2 A_n, \tag{3.7}$$

which is a two-dimensional Klein-Gordon equation and demonstrates that the adjustment process involves Poincaré wave propagation for each mode. In (3.7), $A_{n0}(x, y)$ denotes the Fourier coefficients of ζ_0. Since (3.7) is simply a re-statement of the conservation of the *PV* equation (2.19), we can also proceed by expressing that conservation between initial and final states. Now, in the final steady state, for each vertical mode, the flow is geostrophic, so that

$$f\mathbf{k} \times \mathbf{B}_n = -\nabla D_n = -c_n^2 \nabla A_n, \tag{3.8}$$

after use of (3.6). Then (2.19) gives for each vertical mode (i.e., multiply (2.19) by $\rho_0 \phi_{nz}$ and integrate over z)

$$\frac{\nabla^2 D_n}{f} - fA_n = -fA_{n0},$$

where we have used (3.8). Then using (3.6) we get

$$f^2(A_n - A_{n0}) = c_n^2 \nabla^2 A_n, \tag{3.9}$$

which is merely the steady state version of (3.7).

Hereafter, we will consider a two-dimensional adjustment problem in which $\zeta_0 \equiv \zeta_0(x, z)$, is otherwise arbitrary. In the final geostrophically adjusted steady state let

$$A_n = A_{n0}(x) + \alpha_n(x),$$

where

$$\alpha_n'' - R_n^{-2} \alpha_n = -A_{n0}'', \tag{3.10}$$

from (3.9), $R_n = c_n/f$ is the Rossby radius of deformation associated with the n^{th} vertical mode and the prime denotes a derivative with respect to x. The solution of (3.10) which satisfies $\alpha_n \to 0$ as $|x| \to \infty$ is given by

$$\alpha_n = \frac{R_n}{2}\int_{-\infty}^{\infty} \exp\,[-R_n^{-1}|s - x|]A''_{n0}(s)ds, \tag{3.11}$$

which upon integration by parts twice becomes

$$\alpha_n = -A_{n0} + \frac{1}{2R_n}\int_{-\infty}^{\infty} \exp\,[-R_n^{-1}|s - x|]A_{n0}(s)ds,$$

and hence

$$A_n = \frac{1}{2R_n}\int_{-\infty}^{\infty} \exp\,[-R_n^{-1}|s - x|]A_{n0}(s)ds. \tag{3.12}$$

Eq. (3.12), in slightly different form, was derived by Blumen (1972). The Green's function $\exp[-R_n{}^{-1}|s - x|]$ demonstrates that initial conditions spread a distance $O(R_n)$ during the adjustment process. It also implies that in the geostrophically adjusted state, $\zeta(x, z)$ has a characteristic horizontal length scale bounded above by R_1.

4. Energy conversion ratio

From (2.17) the kinetic energy of the adjusted state is given by

$$\Delta KE = \int_{-\infty}^{\infty}\int_{-h}^{0} \frac{1}{2}\,\rho_0(u^2 + v^2)\;dz\;dx,$$

$$= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}\,I_n|\mathbf{B}_n|^2\;dx$$

where (3.5a) has been used. Using (3.8), this gives

$$\Delta KE = \sum_{n=0}^{\infty}\frac{c_n^4}{f^2}\,\frac{1}{2}\,I_n\int_{-\infty}^{\infty}(A_n')^2\;dx. \tag{4.1}$$

Hereafter, the range of the summation index will be omitted, and unless stated otherwise, Σ denotes an infinite sum in which the index n of the n^{th} term satisfies $n \geq 0$.

From (2.17) the potential energy released during the adjustment is given by

$$\begin{aligned}\Delta PE = &\int_{-\infty}^{\infty}\int_{-h}^{0} \frac{1}{2}\,\rho_0 N^2(\zeta_0^2 - \zeta^2)\;dzdx\\ &+ \int_{-\infty}^{\infty} \frac{1}{2}\,\rho_0(0)\;(\zeta_0^2 - \zeta^2)\Big|_{z=0}\;dx,\end{aligned} \tag{4.2}$$

where the second term on the right-hand side of (4.2) represents the free surface contribution. With the aid of (3.3) and (3.5b), (4.2) can be written as

$$\Delta PE = \frac{1}{2}\sum\int_{-\infty}^{\infty}[(A_{n0})^2 - (A_n)^2]\,dx\left\{\int_{-h}^{0}\rho_0 N^2\phi_n^2\,dz + g\rho_0(0)\phi_n^2(0)\right\},$$

$$= \frac{1}{2}\sum c_n^2 I_n\int_{-\infty}^{\infty}[(A_{n0})^2 - (A_n)^2]\,dx.$$

Therefore, the energy conversion ratio

$$\gamma = \frac{\Delta KE}{\Delta PE} = \frac{\sum_{n=0}^{\infty} c_n^4 I_n\int_{-\infty}^{\infty}(A_n')^2 dx}{\sum_{n=0}^{\infty} c_n^2 I_n f^2\int_{-\infty}^{\infty}(A_{n0}^2 - A_n^2)\,dx}. \tag{4.3}$$

Suppose that A_{n0} is everywhere bounded, with $A_{n0} \to A_{n\infty}^{+}$ (a constant) as $x \to \infty$ and $A_{n0} \to A_{n\infty}^{-}$ (a constant) as $x \to -\infty$. Clearly, as $|x| \to \infty$, A_n has identical bounds to A_{n0}. In the adjusted state, (3.7) becomes

$$c_n^2 A_n'' = f^2(A_n - A_{n0}). \tag{4.4}$$

By integrating by parts and using (4.4) it can be shown that

$$\frac{c_n^2}{f^2}\int_{-\infty}^{\infty}(A_n')^2\,dx = \int_{-\infty}^{\infty}A_n(A_{n0} - A_n)\,dx,$$

$$= \int_{-\infty}^{\infty}(A_{n0}^2 - A_n^2)\,dx + \int_{-\infty}^{\infty}A_{n0}(A_n - A_{n0})\,dx.$$

Substituting into (4.3) we obtain

$$\gamma = 1 - \frac{\sum c_n^2 I_n\int_{-\infty}^{\infty}A_{n0}(A_n - A_{n0})\,dx}{\sum_{n=0}^{\infty} c_n^2 I_n\int_{-\infty}^{\infty}(A_{n0}^2 - A_n^2)\,dx}, \tag{4.5a}$$

$$= \frac{\sum c_n^2 I_n\int_{-\infty}^{\infty}A_n(A_{n0} - A_n)\,dx}{\sum c_n^2 I_n\int_{-\infty}^{\infty}(A_{n0}^2 - A_n^2)\,dx}. \tag{4.5b}$$

Finally, in terms of α_n, (4.5a) takes the form

$$\gamma = 1 - \frac{\sum c_n^2 I_n \int_{-\infty}^{\infty} A_{n0}\alpha_n dx}{\sum c_n^2 I_n \int_{-\infty}^{\infty} \alpha_n(\alpha_n + 2A_{n0}) dx}. \tag{4.6}$$

The results (4.5) and (4.6) have not appeared elsewhere as far as we can determine. They are quite general in that they enable the energy conversion ratio to be calculated for two-dimensional linear geostrophic adjustment of a non-Boussinesq continuously stratified incompressible fluid, given an initial two-dimensional isopycnal displacement field. Furthermore, it will be shown in the next section that (4.6) can be bounded above and below.

5. Bounds on the energy conversion ratio

A useful simplification of the formula (4.6) is obtained if we introduce a new variable, β_n, where

$$\beta_n' = \alpha_n, \qquad \text{and} \qquad \beta_n(-\infty) = 0. \tag{5.1}$$

Substitution into (3.10), and one integration yields

$$\beta_n'' - R_n^{-2}\beta_n = -A_{n0}', \tag{5.2}$$

which also serves to establish that $\beta_n(\infty) = 0$. Then it follows that, using successive integrations by parts,

$$\begin{aligned} \int_{-\infty}^{\infty} A_{n0}\alpha_n \, dx = \int_{-\infty}^{\infty} A_{n0}\beta_n' \, dx &= -\int_{-\infty}^{\infty} \beta_n A_{n0}' \, dx \\ &= \int_{-\infty}^{\infty} \beta_n(\beta_n'' - R_n^{-2}\beta_n) \, dx \\ &= -\int_{-\infty}^{\infty} (\beta_n'^2 + R_n^{-2}\beta_n^2) \, dx < 0. \end{aligned} \tag{5.3}$$

Thus the numerator in the expression for $1 - \gamma$ in (4.6) is negative. Next, consider the denominator,

$$\begin{aligned} \int_{-\infty}^{\infty} \alpha_n(\alpha_n + 2A_{n0}) \, dx &= \int_{-\infty}^{\infty} \beta_n'^2 dx + 2\int_{-\infty}^{\infty} \alpha_n A_{n0} \, dx, \\ &= -\int_{-\infty}^{\infty} (\beta_n'^2 + 2R_n^{-2} \beta_n^2) \, dx < 0, \end{aligned} \tag{5.4}$$

on using (5.3). Thus the denominator is also negative, confirming that $\gamma < 1$ as expected. Further, on substituting (5.3) and (5.4) into (4.6) we find

$$\gamma = \frac{\sum I_n \int_{-\infty}^{\infty} \beta_n^2 \, dx}{\sum I_n \int_{-\infty}^{\infty} (R_n^2 \beta_n'^2 + 2\beta_n^2) \, dx}. \tag{5.5}$$

It follows immediately that

$$0 < \gamma < \frac{1}{2}. \tag{5.6}$$

Further, it is clear that if the scale of β_n is much larger than R_n, then γ becomes arbitrarily close to ½. Alternatively, if the scale of β_n is much smaller than R_n, then γ becomes arbitrarily close to 0. It is clear that for general flows these bounds are the tightest possible.

In our stratified example, the bound of ½ is achieved for initial conditions with very large length scales. However, for initial conditions with any finite (but small) length scale, sufficiently high-order vertical modes will have a smaller deformation radius than this length scale, so that some of these initial conditions must lead to adjusted states with γ greater than, but close to, zero. For a single vertical mode, Middleton (1987) and Killworth (1986) give examples which achieve these bounds.

Let us consider some examples. The simplest is a pure sinusoidal initial condition (as in the Middleton (1987) and Killworth (1986) examples above). If only one mode (n, say) is present, $A_{n0} = \sin kx$ implies $\alpha_n = -k^2R_n^2 \sin kx/(1 + k^2R_n^2)$ and $\gamma = 1/(2 + k^2R_n^2)$. Suggestively, this takes the value 1/3 when $kR_n = 1$; i.e., when the length scale of the initial condition is exactly R_n[4].

This leads us to consider whether an improved lower bound can be obtained for γ under restrictions corresponding to physically sensible conditions. From previous studies, and the above discussion, we might expect $\gamma > ⅓$. With the idea in mind of finding under what circumstances $\gamma > ⅓$, we consider

$$\gamma - \frac{1}{3} = \frac{\sum I_n \int_{-\infty}^{\infty} (\beta_n^2 - R_n^2 \beta_n'^2) \, dx}{3 \sum I_n \int_{-\infty}^{\infty} (R_n^2 \beta_n'^2 + 2\beta_n^2) \, dx} \equiv \frac{NUM}{DEN}, \tag{5.7}$$

4. Note that strictly A_{n0} and A_n do not satisfy the required boundary conditions at infinity, and are essentially Fourier transforms (see also the Appendix). However, such simple sinusoidal expressions can be used here provided we replace the infinite integrals above by integrals over one wavelength, $2\pi/k$.

on using (5.5). Thus $\gamma > 1/3$ when $NUM > 0$. Clearly, we can anticipate that this will be the case when the scale of β_n is much larger than R_n. To illustrate this, suppose we choose

$$\beta_n = b_n \exp(-x^2/L_n^2). \tag{5.8}$$

where $\{b_n\}$ are a set of constants with dimensions of length. Of course, this is an inverse problem as we should first choose A_{n0} and then determine β_n from (5.2). However this inverse procedure is simpler to implement, and substitution into (5.2) readily shows that

$$A_{n0} = \frac{1}{2}(A_{n0}^+ + A_{n0}^-) + \frac{2b_n x}{L_n^2}\exp(-x^2/L_n^2) + \frac{1}{2}(A_{n0}^+ - A_{n0}^-)\,\mathrm{erf}\,(x/L_n) \tag{5.9}$$

where the constants A_{n0}^+ and A_{n0}^- denote the value of A_{n0} as $x \to \infty, -\infty$ respectively, and $A_{n0}^+ - A_{n0}^- = -b_n L_n \sqrt{\pi}/R_n^2$. Consider the case when $A_{n0}^+ = 1 = -A_{n0}^-$ and nondimensionalize β_n and x by R_n. Then (5.9) becomes

$$A_{n0} = \mathrm{erf}\,(x/\lambda_n) + \frac{2x}{\lambda_n^2}\exp(-x^2/\lambda_n^2),$$

where $\lambda_n = L_n/R_n$. This is plotted in Figure 1 with λ_n as a parameter.

Next it is readily shown from (5.5) that

$$\gamma = \frac{\sum I_n b_n^2 L_n}{\sum I_n b_n^2 L_n (2 + R_n^2/L_n^2)}. \tag{5.10}$$

For a single mode (e.g., $b_n = 0$ for all $n \neq N$), clearly $\gamma \gtrless 1/3$ according as $R_N \lessgtr L_N$. If the horizontal length scale associated with β_n is independent of n (L, say) we can also infer that that $\gamma > 1/3$ if $L \geq R_1$, since $R_n \to 0$ as $n \to \infty$, with $R_1 > R_2 > R_3 > \ldots$.

Further progress is mathematically technical and can be found in the Appendix. The results are summarized in Section 7.

6. Examples

In this section we consider two forms for the initial isopycnal displacement field, namely

$$\zeta_0(x, z) = \zeta_0^* \frac{z}{h}\left(1 + \frac{z}{h}\right) e^{-\mu|x|}, \tag{6.1}$$

and

$$\zeta_0(x, z) = \zeta_0^* \frac{z}{h}\left(1 + \frac{z}{h}\right) F(x), \tag{6.2}$$

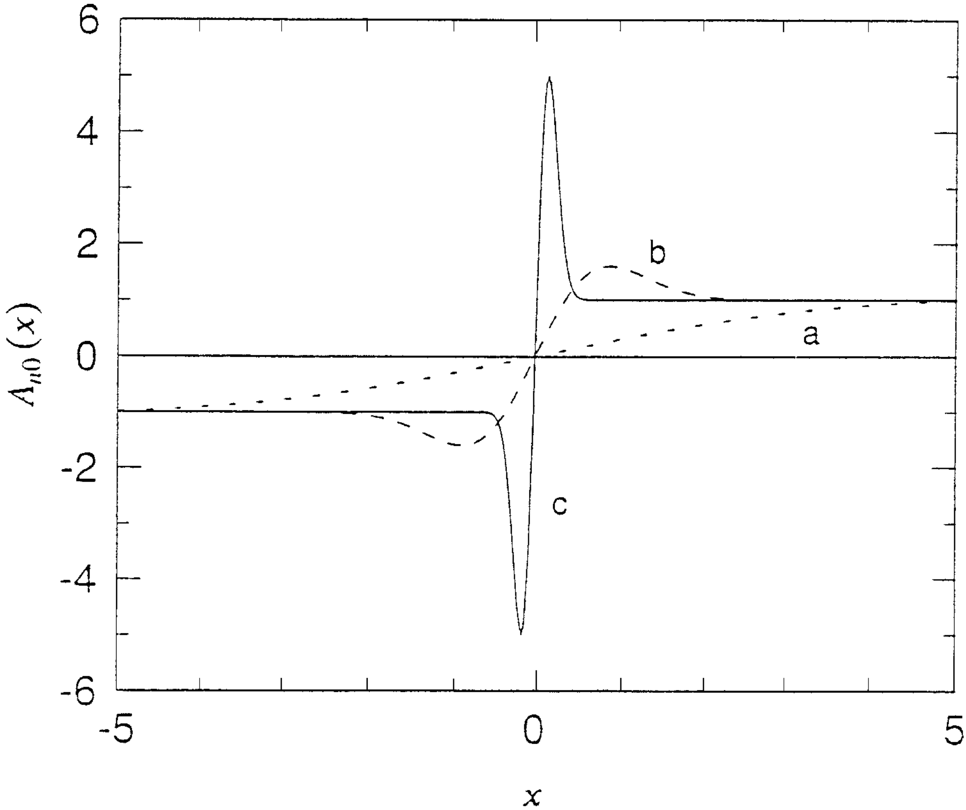

Figure 1. Plot of $A_{n0}(x) = \mathrm{erf}(x/\lambda_n) + 2x\exp(-x^2/\lambda_n^2)/\lambda_n^2$ for (a) $\lambda_n = 0.2$; (b) $\lambda_n = 1.0$; (c) $\lambda_n = 5.0$.

where

$$F(x) = \begin{cases} 1, & x > b \\ \dfrac{x}{b}, & |x| \le b \\ -1, & x < -b. \end{cases}$$

In (6.1) and (6.2), μ and b are positive constants, while ζ_0^* is a constant, with dimensions of length, which controls the amplitude of the isopycnal displacement field. The exact value of γ will be calculated for (6.1) and (6.2). Specific conditions relating to the structure of ζ_0 are derived in the Appendix which lead to $1/3 < \gamma < 1/2$, and their utility will be illustrated using (6.1) and (6.2).

For convenience a rigid lid is imposed at $z = 0$, thereby filtering out the barotropic mode. In addition, the Boussinesq approximation is made throughout this section, in which case the eigenvalues and eigenfunctions become $c_n = N_0h/(n\pi)$ and $\phi_n(z) = \zeta_0^* \sin(n\pi z/h)$, respectively, for a fluid with constant buoyancy frequency N_0. Now the Fourier coefficients of (6.1) are simply

$$A_{n0}(x) = a_n e^{-\mu|x|}, \tag{6.3}$$

while for (6.2) they are

$$A_{n0}(x) = a_n F(x), \tag{6.4}$$

where

$$a_n = \frac{8}{\pi^3(2n+1)^3}, \qquad n = 0, 1, 2, \ldots$$

Consider the initial isopycnal displacement field (6.1). After adjustment

$$A_n = \frac{a_n}{K_n^2 - 1}\left(K_n e^{-|x|/R_n} - e^{-\mu|x|}\right),$$

where $K_n = \mu R_n$. Figure 2 shows a plot of the initial and adjusted isopycnal displacements at selected depths. Twenty baroclinic modes are used to calculate the adjusted isopycnal displacement field, and indeed the value of γ, for both examples in this section. Using (4.3) we find that

$$\gamma = \frac{1}{2}\frac{\sum_{n=0}^{\infty}(2n+1)^{-6}\frac{K_n^2}{(K_n^2-1)^2}\left[1 + K_n - \frac{4K_n}{1+K_n}\right]}{\sum_{n=0}^{\infty}(2n+1)^{-6}\frac{K_n^2}{(K_n^2-1)^2}\left[\frac{K_n^2}{2} - \frac{K_n}{2} - 1 + \frac{2}{1+K_n}\right]}.$$

Table 1 shows the values of γ for various values of μ, using the parameter values $N_0^2 = 10^{-5}\ \text{s}^{-2}$, $h = 4000$ m.

The characteristic horizontal length scale of A_{n0} (and hence ζ_0) is μ^{-1}, and when $\mu^{-1} \gg R_1$ we expect $\gamma > 1/3$ (see the Appendix). Table 1 shows that when $\mu^{-1} \geq 100\ \text{km} > R_1 \simeq 40$ km, the value of γ exceeds 0.4. Following the notation adopted in the Appendix, we find that

$$G_n(k) = \frac{4a_n^2\mu^2k^2}{(\mu^2 + k^2)^2},$$

which has local maxima at $k = \pm\mu$. As μ decreases, the symmetric function G_n becomes concentrated around the origin, and this example falls into category (ii) of Section 6.

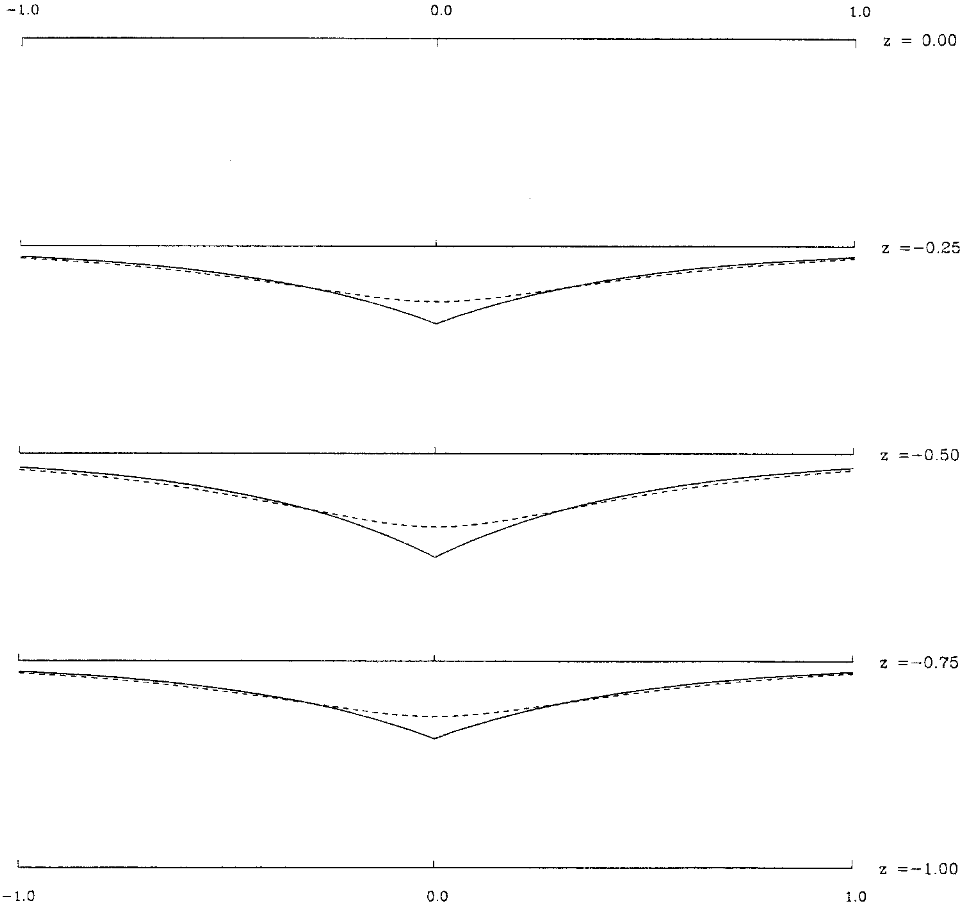

Figure 2. Plots of the initial (solid lines) and adjusted (broken lines) isopycnal displacement fields at $z/h = -0.25$, -0.5 and -0.75, using the initial condition (6.1) with $\mu = 10^{-5}$ m^{-1}. On the horizontal axis, one unit corresponds to $2/\mu$ meters. $\zeta_0(0, -0.5\,h) = -0.25\zeta_0^*$ can be used to relate the plot displacements to dimensional units.

Table 1. The energy conversion ratio for various values of the parameter μ when $A_{n0} = a_n \exp(-\mu|x|)$, where the coefficients a_n are given in Section 6.

μ (m^{-1})	γ
10^{-3}	2.372×10^{-2}
10^{-4}	0.166
10^{-5}	0.416
10^{-6}	0.49

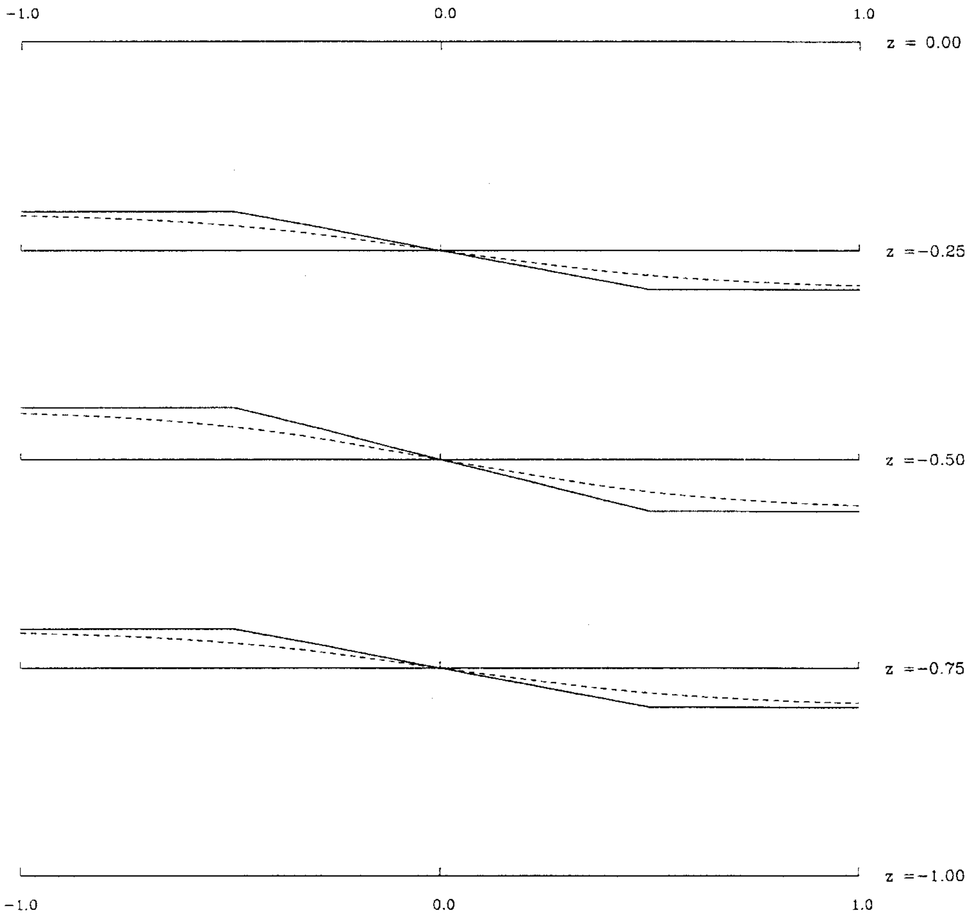

Figure 3. As in Figure 2, but for the initial condition (6.2). One unit on the horizontal axis corresponds to $2b$ km. To relate the plot displacement to dimensional units, note that $\zeta_0(x, -0.5h) = -0.25\zeta_0^*$ for $x > b$.

Now consider the initial isopycnal displacement field (6.2). After adjustment, A_n is the odd function given by

$$A_n(x) = a_n \begin{cases} \dfrac{x}{b} - \dfrac{R_n}{b} e^{-b/R_n} \sinh\left(\dfrac{x}{R_n}\right), & 0 \le x \le b \\ 1 - \dfrac{R_n}{b} e^{-x/R_n} \sinh\left(\dfrac{b}{R_n}\right), & x > b. \end{cases}$$

Figure 3 shows a plot of the initial and adjusted isopycnal displacement fields at various depths when $b = 50$ km.

Table 2. As in Table 1, except that $A_{n0} = a_n F(x)$ where F is an odd function with $F(x) = 1, x > b$ and $F(x) = x/b,\ 0 \le x \le b$.

b (km)	γ
1	0.337
10	0.366
100	0.468
1000	0.497

The energy conversion ratio is given by

$$\gamma = \frac{\sum_0^\infty (2n+1)^{-6}\lambda_n^{-2}\left[1 + \frac{1}{2}e^{-2\lambda_n} - \frac{3}{2}e^{-\lambda_n}\lambda_n^{-1}\sinh\lambda_n\right]}{\sum_0^\infty (2n+1)^{-6}\lambda_n^{-2}\left[2 + \frac{1}{2}e^{-2\lambda_n} - \frac{5}{2}e^{-\lambda_n}\lambda_n^{-1}\sinh\lambda_n\right]}$$

where $\lambda_n = b/R_n$. We also find that

$$G_n(k) = \frac{4a_n^2}{k^2}\sin^2(bk).$$

As $b \to 0$, the initial isopycnal displacement field becomes a step distribution and $\gamma \to 1/3$, from above. Table 2 shows the values of γ for various values of b. Referring to the Appendix, we observe that when $b = 1$, 10 and 100 km

$$G_1(R_1^{-1}) \ge \max_{k \ge k_1} G_1(k)$$

where k_1 is the smallest positive turning point, which for this example, is approximately π/b. When $b = 100$ km, $R_1 > k_1$. The first baroclinic mode is dominant in the adjusted field and in the calculation of γ.

7. Summary and discussion

The energy conversion ratio (γ) is shown to be bounded below by 0 and above by ½ in the two-dimensional linear geostrophic adjustment of an incompressible, continuously (stably) stratified, inviscid, non-Boussinesq uniformly rotating fluid. By "two-dimensional" we mean that the initial isopycnal displacement field is an arbitrary function of the vertical coordinate and a single horizontal coordinate. A vertical normal mode expansion technique is used to derive the upper bound for γ.

Whether the energy conversion ratio lies within the range (0, ½) depends on the length scale of the initial conditions. If this scale is *long*, the ratio tends to ½. If the length scale is *small*, less can be said since any finite length scale will still be much larger than the deformation radius for an infinite number of high-order modes (of course, such modes would be affected more strongly by nonideal fluid effects such as diffusion and viscosity).

We also address in this paper whether there is a class of initial isopycnal displacement profiles for which a nonzero lower bound for γ applies. To this end the initial two-dimensional isopycnal displacement field $\zeta_0(x, z)$ is decomposed into a sum of vertical normal modes of the form

$$\zeta_0(x, z) = \sum A_{n0}(x)\phi_n(z).$$

Loosely speaking, $\gamma > 1/3$ when $\hat{\chi}_n(k) \equiv \mathscr{F}\{dA_{n0}/dx\}$, where $\mathscr{F}$ denotes the Fourier transform operator, is concentrated around $k = 0$, but the precise conditions we have obtained are as follows:

(i) When $\hat{\chi}_n(k) = \mathscr{F}\{dA_{n0}/dx\}$ is a monotonically decreasing function for $k \geq 0$.

(ii) When A_{n0} is continuous and has a characteristic length scale $L \gg R_n$, the Rossby radius of deformation.

(iii) When A_{n0} is antisymmetric, with A_{n0} bounded as $|x| \to \infty$, provided that either $|\hat{\chi}_n|^2$ satisfies condition (i) or

$$G_n(R_n^{-1}) \geq \max_{k \geq k_1 > 0} G_n(k),$$

where $G \equiv |\hat{\chi}_n|^2$, k_1 is the smallest (positive) turning point of G_n and $R_n > k_1{}^{-1}$.

The utility of these precise conditions is shown in Section 6, where two particular initial isopycnal displacement profiles are examined. In each case, γ is calculated using (4.3) and is shown to lie in the interval $(1/3, 1/2)$ when at least one of the three conditions above is satisfied.

In the case when $A_{n0} = A_{n0}(x, y)$ the ensuing geostrophic adjustment is three-dimensional and it can be shown, using the methods discussed in Section 4, that

$$\gamma = \frac{\sum c_n^4 I_n \int\int |\nabla A_n|^2\, dS}{\sum c_n^4 I_n \int\int [2\,|\nabla A_n|^2 + R_n^2(\nabla^2 A_n)^2]\, dS}$$

where the domain of integration is the entire x–y plane. Once again $\gamma < 1/2$, and if A_n is purely large-scale, such that $|\nabla A_n|$ is small everywhere then $\gamma \to 1/2$. By choosing A_{n0} with "small horizontal scales" the solution for A_n will exhibit the same behavior and γ can be made arbitrarily close to zero.

Finally, it would be interesting to extend this investigation to the case in which localized topography is included. Willmott and Johnson (1995), and the references therein, discuss how the geostrophic adjustment problem is modified by the presence of an infinitely long step escarpment in both homogeneous and two-layer systems, and could provide a useful starting point for the analysis.

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APPENDIX

This Appendix discusses conditions under which $\gamma > \frac{1}{3}$. We recast *NUM* in Fourier space by invoking Parseval's theorem to obtain

$$NUM = \frac{1}{2\pi}\sum R_n^4 I_n \int_{-\infty}^{\infty} \frac{(1 - k^2R_n^2)}{(1 + k^2R_n^2)^2}\left|\hat{\chi}_n(k)\right|^2 dk, \tag{A1}$$

where

$$\hat{\chi}_n \equiv \mathscr{F}(A'_{n0}) = \int_{-\infty}^{\infty} A'_{n0}e^{-ikx}\,dx,$$

is the Fourier transform of A'_{n0}, and it follows from (5.2) that

$$\hat{\beta}_n \equiv \mathscr{F}(\beta_n) = \frac{R_n^2\hat{\chi}_n}{1 + k^2R_n^2}.$$

To determine the sign of *NUM*, we rewrite (A1) as

$$NUM = \frac{1}{\pi}\sum R_n^3 I_n \int_0^1 \frac{(1 - k^2)}{(1 + k^2)^2}\left[\left|\hat{\chi}_n\left(\frac{k}{R_n}\right)\right|^2 - \left|\hat{\chi}_n\left(\frac{1}{kR_n}\right)\right|^2\right] dk. \tag{A2}$$

It is illuminating at this stage to consider two specific examples which will facilitate the general treatment of (A2). Suppose that ζ_0 has a step discontinuity:

$$A_{n0} = A_{n0}^{+}H(x) + A_{n0}^{-}H(-x),$$

where ${A_{n0}}^{+/-}$ are constants and H denotes the Heaviside function. Now

$$A'_{n0} = \Delta_n\delta(x), \qquad \Delta_n = A_{n0}^{+} - A_{n0}^{-}.$$

and $\hat{\chi}_n = \Delta_n$. In this case (A2) shows that

$$NUM = 0.$$

and hence $\gamma = \frac{1}{3}$. However, two or more step discontinuities in A_{n0} will lead to $\hat{\chi}_n$ which is dependent upon k and $\gamma \neq \frac{1}{3}$.

If $A'_{n0} = (2L)^{-1} \exp(-|x|/L)$, we find that $\hat{\chi}_n = (1 + k^2L^2)^{-1}$, which is a monotonically decreasing function for $k \geq 0$, and (A1) becomes

$$NUM = \pi^{-1} \sum R_n^4 I_n \int_0^\infty \frac{(1 - k^2R_n^2)}{(1 + k^2R_n^2)^2} \frac{dk}{(1 + k^2L^2)^2},$$
$$= \pi^{-1} \sum R_n^3 I_n \int_0^\infty \frac{1 - k^2}{(1 + k^2)^2} \frac{dk}{(1 + a_n^2k^2)^2}, \tag{A3}$$

where $a_n = L/R_n$. To evaluate the integral in (A3) we note that for constants β and γ

$$\int_0^\infty \frac{x^2 dx}{(\beta + x^2)(\gamma + x^2)} = \frac{\pi}{2} (\sqrt{\beta} + \sqrt{\gamma})^{-1}, \tag{A4a}$$

and

$$\int_0^\infty \frac{dx}{(\beta + x^2)(\gamma + x^2)} = \frac{\pi}{2} (\beta\sqrt{\gamma} + \gamma\sqrt{\beta})^{-1}, \tag{A4b}$$

(see Gradshteyn and Ryzhik, 1980, p. 300), and by partially differentiating (A4) with respect to the constants β and γ and then setting $\beta = 1$, we obtain

$$\int_0^\infty \frac{(1 - x^2)}{(1 + x^2)^2(x^2 + \gamma)^2} dx = \frac{\pi}{4} \frac{(1 + 3\sqrt{\gamma})}{4\gamma^{3/2}(1 + \sqrt{\gamma})^3}.$$

Therefore (A3) becomes

$$NUM = \frac{1}{4} \sum R_n^3 I_n \frac{a_n(3 + a_n)}{(1 + a_n)^3} > 0,$$

and hence $1/3 < \gamma < 1/2$.

More generally if $\hat{\chi}_n(k_f/cn)$ is a monotonically decreasing function for $k \geq 0$ then

$$\left|\hat{\chi}_n\left(\frac{k}{R_n}\right)\right| > \left|\hat{\chi}_n\left(\frac{1}{kR_n}\right)\right|, \qquad 0 < k < 1,$$

and (A2) shows that $NUM > 0$, in which case $1/3 < \gamma < 1/2$. The second of the two examples above falls into this category.

We also see from (A1) or (A2) that if A_{n0} has a length scale $L \gg R_n$, then $\hat{\chi}_n$ will be narrow banded (i.e., essentially of nonzero support only on $(-k^*, k^*)$ where $k^* = L^{-1}$ and $NUM > 0$ (i.e., $\gamma > 1/3$). The exception to this rule is when the initial isopycnal profile has two or more step discontinuities in the horizontal. In this case $\gamma \to 1/3$ as $L \to \infty$, where L now denotes the minimum distance between any pair of adjacent step discontinuities in ζ_0. For example, when the initial profile for ζ_0 is of "top hat" type of width b, the generalized Fourier coefficients in (3.3) will take the form

$$A_{n0} = a_n[H(x) - H(x - b)],$$

where a_n and b are constants. Using (3.12) and (4.5b) it can be shown that

$$\gamma = \frac{\sum c_n^2 I_n [R_n(1 - e^{-\lambda_n}) - be^{-\lambda_n}]}{\sum c_n^2 I_n \left[\frac{7}{2} R_n(1 - e^{-\lambda_n}) - \frac{1}{2} R_n(1 - e^{-2\lambda_n}) - be^{-\lambda_n}\right]} \tag{A5}$$

where $\lambda_n = b/R_n$. As $b \to \infty$ (A5) shows that $\gamma \to 1/3$, as expected, because the adjustment at each end of the initial step profile takes place independently of the other end. Now suppose that the adjustment takes place in a single mode, $n = N$ say, in which case

$$\gamma = \frac{1 - e^{-\lambda_N} - \lambda_N e^{-\lambda_N}}{\frac{7}{2}(1 - e^{-\lambda_N}) - \frac{1}{2}(1 - e^{-2\lambda_N}) - \lambda_N e^{-\lambda_N}}. \tag{A6}$$

From (A6) it is readily seen that $\gamma \leq 1/3$ and that $\gamma \to 0$ as $b \to 0$.

Suppose that A_{n0} is antisymmetric with $A'_{n0} \geq 0$ and A_{n0} bounded as $|x| \to \infty$. Then, $\hat{\chi}_n(0) > 0$, $\hat{\chi}'_n(0) = 0$ and $\hat{\chi}''_n(0) < 0$, which implies that $G_n(k) \equiv |\hat{\chi}_n(k)|^2$ is symmetric with a local maximum at $k = 0$. Clearly the qualitative behavior of G_n falls into one of two categories. If G_n is a monotonically decreasing function for $k \geq 0$, then we have seen that $1/3 < \gamma < 1/2$. Otherwise, suppose that G_n has two or more turning points in $k > 0$, with the first occurring at $k = k_1 = L^{-1}$, say. Consider

$$H_n(k) \equiv G_n\left(\frac{k}{R_n}\right) - G_n\left(\frac{1}{kR_n}\right), \qquad 0 < k < 1,$$

which appears in the integrand of (A2). Now it is readily shown that if $R_n > L$ and

$$G_n(R_n^{-1}) \geq \max_{k \geq k_1} G_n(k)$$

then $H_n(k) \geq 0$, and so $NUM > 0$ (i.e. $\gamma > 1/3$).

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