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On the parameterization of eddy transfer Part I. Theory

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ABSTRACT

This is the first of three linked papers which develop an eddy parameterization scheme for mean flows which are wide compared with a deformation radius. The scheme is partly based on the behavior of potential vorticity and thickness fluxes in linear instability, where the former are downgradient (apart from a turning matrix, not present in channel models) and the latter are not precisely downgradient, except on an f-plane. The scheme leads to a diffusivity which varies quite strongly with depth and is smallest at surface and floor. Intrinsic delta-function fluxes also occur at surface and floor, and these are worked out in detail. It is shown that all such parameterization schemes (whether linked to linear instability or not) must satisfy a necessary consistency condition, in the form of a vertical integral. A uniform diffusivity does not satisfy this requirement unless it is defined to vanish at surface and floor. Two methods to compute approximate diffusivities efficiently are given, and their results compare well with exact results from instability theory.

1. Introduction

The most well-known difficulty in numerical ocean modeling has been present since the earliest runs of eddy-permitting models: the inherent conflict between the need for simulations of centuries to examine climate issues, and the enormous computer resources necessary to undertake such integrations while retaining at least some degree of eddy activity in the model. For the foreseeable future, climate models will have to continue to include parameterizations of the eddy field from a purely pragmatic perspective. It is also frustrating to need a model delineating every eddy in order to understand broader-scale climate issues, so that a physically meaningful parameterization is useful for comprehension also.

Until the last five years, oceanographic representations of eddy effects were largely crude, save for early work by Welander (1973) and Marshall (1981), and homogeneous turbulence studies by, e.g., Haidvogel and Held (1980) and Panetta and Held (1988). The representation used simple downgradient diffusion terms for momentum and tracers. With the centered differences used in numerical ocean models, such terms were necessary to maintain numerical stability, rather than as representations of eddy effects *per se*. Solomon (1971) and Redi (1982) had shown how to rotate the mixing tensor to account for

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predominantly isopycnic mixing, but horizontal lateral mixing terms remained for stability; Griffies *et al.* (1997) have shown how to avoid the lateral terms and maintain stability.

Advances in the meteorological literature (e.g., Plumb and Mahlman, 1987) proceeded independently on the two main — and distinct — oceanographic approaches. (Only more recently, in discussion of transformed Eulerian means, are the two approaches converging.) One approach, by Gent and McWilliams (1990), sought to parameterize the effect on tracers by time- and space-varying eddy fields. The other, by Eby and Holloway (1994), parameterized the effects on momentum of the presence of a statistical equilibrium field of eddies. These two approaches both appear to yield improved large-scale oceanographic representations, despite their differing approaches. (Although appealing to different equations, terms in a tracer equation can be maneuvred into the momentum equation and vice versa; cf. Lee and Leach, 1996, for example.) We shall restrict attention here to conserved tracers such as temperature and salinity.

These tracers are moved both by time- and space-mean flow, but also by eddy transport terms which can be thought of as an additional 'bolus' advective velocity (the effect has many names and a long history; cf. Gent *et al.*, 1995 for a discussion). Attempts stemming from Gent and McWilliams' (1990) work have largely concentrated on prescriptions for the bolus velocity. They assume that eddy motions represent a release of mean energy through instability mechanisms. Since most parts of the mean ocean circulation have length scales large compared with a deformation radius, much of this release is assumed to occur through a baroclinic, rather than a barotropic, mechanism, following the early suggestions of Green (1970). Underlying these attempts, then, is the concept of reduction of mean available potential energy by eddy fluxes, as well as motions preferentially on density surfaces (we follow other authors here and do not consider difficulties relating to neutral surfaces, etc.; cf. McDougall, 1987 for details). Such concepts cannot, by their nature, deal with long-distance advection of tracers within sub-mesoscale coherent vortices like Meddies (Armi *et al.*, 1989), or many other aspects of ocean flow: barotropic instability, spatial growth and decay, etc.

Gent *et al.* (1995) give an excellent summary of the requirements of a parameterization based on tracers, and discuss two versions of their parameterization (Eqs. (18) and (22)), involving downgradient mixing of isopycnic layer thickness in a manner similar to that in use in isopycnic co-ordinate models (Bleck *et al.*, 1992). Treguier *et al.* (1997) investigated, *inter alia,* constraints on such parameterizations caused by horizontal boundaries; they also noted that potential vorticity, rather than layer thickness, was the relevant conserved quantity, and discussed some of the changes using this quantity would produce. Visbeck *et al.* (1997) examined four potential parameterizations in the light of eddy-resolving initial value problems. Lee *et al.* (1997) have examined how tracers are spread in a statistically steady eddy-resolving channel model, also drawing attention to the importance of potential vorticity conservation.

This paper looks at a possible form of parameterization. Taking a suggestion of Treguier *et al.* (1997), it examines the slowly varying (locally vertical) baroclinic instability problem. This is a return to the concepts of Green (1970), together with results of how properties could be mixed by eddies (Killworth, 1981). A form for the eddy diffusion of layer thickness and potential vorticity is produced (Section 2) which turns out to involve intrinsically a vertical variation of diffusivity. Because of the variation of Coriolis parameter, pure downgradient diffusion of layer thickness does not occur (but downgradient diffusion of potential vorticity — ignoring the contribution from relative vorticity — does occur), and an extra term is predicted which is clearly visible in Lee *et al.*'s (1997) eddy-resolving runs. The use of a local linear solution is then largely relaxed for the rest of the paper, with discussion predominantly about the form of the parameterization. Section 3 gives a form for the bolus velocity, and Section 4 discusses boundary conditions. It is shown in Section 5 that all parameterizations similar to the form developed here must satisfy a vertical integral consistency condition related to the implied delta-functions at surface and floor. (These delta-functions are responsible for the cross-channel movement of light and dense fluid in the Eady problem, for example.) Sections 6 and 7 create two different approximate forms for the diffusivity which take this consistency condition into account. Calculations of fastest growth rate in simple problems discussed in Section 8 show that these forms give reasonable answers in such cases.

Part II of this paper compares the results of this parameterization with those of Gent and McWilliams (1990) in an eddy-resolving three-dimensional channel model of a front. Part III examines the parameterization of tracers, and also the modifications necessary when there is a region of zero vertical density gradient present (because of a surface mixed layer, or because of deep convection).

2. Slowly varying linear theory

As a preliminary guide to possible eddy effects, we consider linear perturbation theory applied to a slowly varying mean flow. Linear theory is not a good descriptor of fully developed geostrophic turbulence, particularly for momentum fluxes (e.g., Simmons and Hoskins, 1978). Nonetheless, linear theory has the virtue that its motions are solutions of the relevant equations, and fluxes of buoyancy — which are better described by linear theory — do behave in a physically reasonable way. Green (1970) used a similar approach to that described here for parallel shear flow and has deduced forms of many of the results of this paper.

We assume the background (mean) flow is slowly varying, in the sense that its length scales are large compared with a deformation radius. For coarse ocean climate models this holds automatically on resolution grounds. This slow variation thus precludes barotropic instability, and mitigates against areas of the ocean which have smaller length scales, such as western boundary layers (which are poorly represented in coarse models anyway). There are cases when eddies can remove this convenient scale separation.

Density co-ordinates are used, to facilitate the expansion. Accordingly density is assumed a linear function of temperature and salinity in what follows, and the effects of more complicated equations of state (cf. McDougall, 1987) are ignored. The beta effect is included; it can act to stabilize or destabilize the flow. Conservation of momentum, mass, and the hydrostatic relation become, if there is no diapycnal mixing,

$$u_t + \mathbf{u} \cdot \nabla u - fv = -\frac{B_x}{\rho_0} \tag{1}$$

$$v_t + \mathbf{u} \cdot \nabla v + fu = -\frac{B_y}{\rho_0} \tag{2}$$

$$z_{\rho t} + \nabla \cdot (\mathbf{u} z_\rho) = 0 \tag{3}$$

$$B_\rho = gz \tag{4}$$

where the axes are (x, y) oriented east and north, z is directed upward, with zero at the surface, ρ is the density (ρ_0 is a reference density), B represents the linear Bernoulli or Montgomery function $p + \rho gz$, g is the acceleration due to gravity, f the Coriolis parameter (with northward variation β), and t the time. The gradient and divergences are taken in the horizontal directions only.

From these equations we can derive conservation of potential vorticity $q = (f + v_x - u_y)/z_\rho$,

$$q_t + \mathbf{u} \cdot \nabla q = 0. \tag{5}$$

Small perturbations are taken against a mean background (denoted by an overbar) which is in steady (or, more formally, slowly varying) geostrophic and mass balance. The expansion procedure will be terminated at the zeroth order, representing a local vertical problem. The first use of such a slowly-varying expansion appears to be Simmons (1974) for a quasi-geostrophic parallel jet shear flow; Robinson and McWilliams (1974) wrote a description in terms of fast and slow space and time variables, but terminated this expression at the leading term. Killworth (1980) extended Simmons' (1974) approach to a general quasi-geostrophic parallel shear flow, and recently Killworth *et al.* (1997) have used a similar theory for primitive equation ring instability.

The perturbations, denoted by primes, satisfy

$$u'_t + \bar{u}u'_x + u'\bar{u}_x + \bar{v}u'_y + \bar{u}_y v' - fv' = -\frac{B'_x}{\rho_0} \tag{6}$$

$$v'_t + \bar{u}v'_x + u'\bar{v}_x + \bar{v}v'_y + v'\bar{v}_y + fu' = -\frac{B'_y}{\rho_0} \tag{7}$$

$$z'_{\rho t} + \nabla \cdot (\bar{\mathbf{u}} z'_\rho + \mathbf{u}'\bar{z}_\rho) = 0 \tag{8}$$

$$B'_\rho = gz'. \tag{9}$$

As in Robinson and McWilliams (1974), we suppose that the mean flow is spatially slowly varying, i.e., that its length scale L is much larger than a, where a is the local deformation radius. Here $a \approx (gH\Delta\rho/\rho_0)^{1/2}/f$, where H is the depth scale, $\Delta\rho$ a density scale, so that the velocity scales on $gH(\Delta\rho/\rho_0)/fL$. We set

$$\frac{a}{L} \equiv \epsilon \tag{10}$$

which is assumed to be a small quantity. We also assume the mean flow varies temporally on a long time scale $T = \epsilon^{-2}f^{-1}$.

The perturbation quantities vary on more rapid scales: spatially, of order a, and temporally of order $\epsilon^{-1}f^{-1}$. Thus

$$B = \overline{B}(X, Y, \rho, T) + B'(x, y, \rho, t, X, Y, T) \tag{11}$$

is assumed, where (X, Y, T) represent the slow variables and (x, y, t) the fast. We now pose

$$B' = Re\,[B(\rho)\exp ik(x\cos\theta + y\sin\theta - ct)] \tag{12}$$

with similar expressions for (u', v', z'), plus smaller ageostrophic terms, where (k, θ) are the local wavenumber and direction of the perturbation and are functions of the slow variables. The phase velocity c will be complex in general. Choices for k and θ will be made later.

Substitution into (8) gives, dropping primes,

$$-ikcz_\rho + \bar{z}_\rho \nabla \cdot \mathbf{u} + u\bar{z}_{\rho x} + v\bar{z}_{\rho y} + ik\tilde{u}z_\rho = 0 \tag{13}$$

to leading order (the remaining term $z_\rho \nabla \cdot \bar{\mathbf{u}}$, involving gradients of the mean flow is $O(\epsilon)$ smaller). Here

$$\tilde{u} = \bar{u}\cos\theta + \bar{v}\sin\theta \tag{14}$$

is the projection of the mean flow in direction θ. The momentum equations become

$$-ikcu + ik\tilde{u}u - fv = -\frac{ik\cos\theta}{\rho_0}B + \text{small} \tag{15}$$

$$-ikcv + ik\tilde{u}v + fu = \frac{ik\sin\theta}{\rho_0}B + \text{small} \tag{16}$$

where we note that the non-Coriolis terms on the l.h.s. of (15, 16) are $O(\epsilon)$ smaller than the geostrophic balance

$$v = \frac{ik\cos\theta}{\rho_0 f}B, \qquad u = -\frac{ik\sin\theta}{\rho_0 f}B \tag{17}$$

but are necessary for the divergence terms in (13) since geostrophy is divergence-free. Cross-differentiating (15) and (16) to give $\nabla \cdot \mathbf{u}'$, and substitution into (13) then gives to leading order the purely vertical problem

$$(\tilde{u} - c)\left(B_{\rho\rho} + \frac{g}{f^2\rho_0} k^2\bar{z}_\rho B\right) - B\left(\frac{g}{f^2\rho_0} \beta\bar{z}_\rho \cos\theta + \tilde{u}_{\rho\rho}\right) = 0 \tag{18}$$

(the latter term is proportional to the gradient of potential vorticity in the direction of the wave vector) plus boundary conditions at top and bottom of vanishing vertical velocity

$$(\tilde{u} - c)B_\rho = \tilde{u}_\rho B, \qquad \rho = \rho_b, \rho_t. \tag{19}$$

Here $\bar{z}(x, y, \rho_b) = -H$, $\bar{z}(x, y, \rho_t) = 0$ define the bottom ($z = -H$) and top ($z = 0$) densities ρ_b, ρ_t respectively. The condition (19) holds for a flat bottom only. Since most oceanic instability problems are not controlled by floor conditions (cf. Gill *et al.*, 1974) this is probably a reasonably good first approximation for a parameterization.

Eq. (18) is simply the standard quasi-geostrophic problem (e.g., Pedlosky, 1987) cast into density co-ordinates; Williams (1974) has used the simplification when the last term in (18) vanishes to great advantage in extending the Eady (1949) solutions.

A full analysis would involve at least the next two orders in the expansion in ϵ. In the parallel or radial cases referred to above, the asymptotics become complicated. The main feature of note is that a second length scale $l = (aL)^{1/2}$ appears over which the instability has an effect on the mean flow; this scale is both *long* compared with the deformation scale, and *short* compared with the mean flow scale.

In two horizontal directions, little is known about the higher order terms (Robinson and McWilliams, 1974 do not discuss them). For our purposes here we assume as usual that the local problem above, at least in an ensemble sense, is the leading order to the linear instability problem, and follow the expansion no further.

We now examine fluxes of perturbed quantities.

a. *Thickness fluxes*

Perturbation thickness fluxes, averaged over many eddy space scales,[2] can be estimated in the usual way to give

$$\overline{u'z'_\rho} = \frac{1}{2} Re\,(uz^*_\rho), \qquad \overline{v'z'_\rho} = \frac{1}{2} Re\,(vz^*_\rho) \tag{20}$$

where an asterisk denotes a complex conjugate. From (18),

$$z_\rho = \frac{1}{g} B_{\rho\rho} = -\frac{k^2\bar{z}_\rho}{f^2\rho_0} B + \frac{B}{\tilde{u} - c}\left(\frac{\beta\bar{z}_\rho}{f^2\rho_0} \cos\theta + \frac{\tilde{u}_{\rho\rho}}{g}\right). \tag{21}$$

2. Recall that these averages are on density surfaces.

Substitution into (20), and use of the fact that both u and v are proportional to iB, means that the terms involving the first term in (21) vanish, leaving

$$\overline{u'z'_\rho} = \frac{1}{2} Re \left\{ \frac{ik \sin\theta}{f\rho_0} B \cdot \frac{B^*}{|\tilde{u} - c|^2} (\tilde{u} - c) \left(\frac{\beta\cos\theta}{f^2\rho_0} \bar{z}_\rho + \frac{\tilde{u}_{\rho\rho}}{g} \right) \right\}$$
$$= -\frac{1}{2} \frac{kc_i \sin\theta}{f\rho_0} \left| \frac{B}{\tilde{u} - c} \right|^2 \left(\frac{\beta\cos\theta}{f^2\rho_0} \bar{z}_\rho + \frac{\tilde{u}_{\rho\rho}}{g} \right) \quad (22)$$

and similarly

$$\overline{v'z'_\rho} = +\frac{1}{2} \frac{kc_i \cos\theta}{f\rho_0 g} \left| \frac{B}{\tilde{u} - c} \right|^2 \left(\frac{\beta\cos\theta}{f^2\rho_0} \bar{z}_\rho + \frac{\tilde{u}_{\rho\rho}}{g} \right). \quad (23)$$

where c_i is the imaginary part of c. The modulus term in (22, 23) is proportional to the average of the square of particle movement (Green, 1970; Pedlosky, 1987).

Thermal wind applied to the geostrophic mean flow implies

$$\bar{u}_\rho = -\frac{g\bar{z}_y}{f\rho_0}, \qquad \bar{v}_\rho = \frac{g\bar{z}_x}{f\rho_0} \quad (24)$$

so that $\tilde{u}_{\rho\rho} = (g/f\rho_0)(-\bar{z}_{\rho y}\cos\theta + \bar{z}_{\rho x}\sin\theta)$. Thus

$$\begin{pmatrix} \overline{u'z'_\rho} \\ \overline{v'z'_\rho} \end{pmatrix} = -\frac{kc_i}{2f^2\rho_0^2} \left| \frac{B}{\tilde{u} - c} \right|^2 \mathbf{A} \begin{pmatrix} \bar{z}_{\rho x} \\ \bar{z}_{\rho y} \end{pmatrix} + \frac{kc_i\beta\cos\theta}{2f^3\rho_0^2} \left| \frac{B}{\tilde{u} - c} \right|^2 \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \bar{z}_\rho$$
$$\equiv -\kappa \left[\mathbf{A} \cdot \nabla \bar{z}_\rho - \frac{\beta}{f} \bar{z}_\rho \mathbf{A}_2 \right] = -\kappa f \mathbf{A} \cdot \nabla \left(\frac{\bar{z}_\rho}{f} \right) \quad (25)$$

where

$$\mathbf{A} = \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix} \quad (26)$$

is a non-negative definite matrix (its eigenvalues are 1 and 0), $\mathbf{A}_2$ is its second column, and

$$\kappa(x, y, \rho) = \frac{kc_i}{2f^2\rho_0^2} \left| \frac{B}{\tilde{u} - c} \right|^2 \quad (27)$$

is a diffusivity, proportional to the mean-square particle excursions, and would normally be largest in the interior of the fluid.

b. *Potential vorticity fluxes*

For small perturbations,

$$q' = \left(\frac{f + v_x - u_y}{z_\rho} \right)' = \frac{v'_x - u'_y}{\bar{z}_\rho} - \frac{f}{\bar{z}_\rho^2} z'_\rho = -\frac{k^2 B'}{\bar{z}_\rho} - \frac{f}{\bar{z}_\rho^2} \frac{B'_{\rho\rho}}{g}. \quad (28)$$

After substitution, we proceed as in (22). The relative vorticity term in (28) is uncorrelated with $\mathbf{u}'$, because the Laplacian of B is out of phase with the geostrophic velocity, so that

$$\overline{\mathbf{u}'q'} = -\frac{f}{\bar{z}_\rho^2}\overline{\mathbf{u}'z_\rho'} \tag{29}$$

for linear perturbations (similar to a result by Treguier *et al.*, 1997). The term in β becomes part of the mean q gradient, so that

$$\begin{pmatrix}\overline{u'q'}\\ \overline{v'q'}\end{pmatrix} = -\kappa\mathbf{A}\cdot\nabla\overline{q} \tag{30}$$

with no additional terms (because q, and not z_ρ, is a conserved quantity), where both $\mathbf{A}$ and κ are the same as in (25). In other words, through (29), a parameterization of thickness fluxes implies one for q, and vice versa. Eq. (30) approximately represents the well-known belief that potential vorticity is fluxed downgradient by eddies; see, e.g., McDougall (1995).

c. Discussion

For the next few sections we will explore parameterizations of fluxes of quantities, where the shape of the parameterization is that of (30) or (25), without reference to linear perturbation theory. The diffusivity κ is permitted to depend on all three spatial coordinates (and time) while the matrix $\mathbf{A}$ depends only on (x, y, t). Were $\mathbf{A}$ the identity matrix, (30) would correspond to purely downgradient (Fickian) transfer. Another example would be the special case of a channel flow in which there is no mean along-channel gradient ($\partial/\partial x = 0$), where the problem would reduce to two dimensions and $\mathbf{A}$ would become unity; (30) would then be completely downgradient. In a general flow which changes direction with height, q is not fluxed exactly downgradient everywhere by linear perturbations.

On an f-plane, layer thickness and large-scale potential vorticity are equivalent; the form (30) would apply in both cases. On a β-plane, this is not the case: only q is fluxed pseudo-downgradient — the 'pseudo' indicating the inclusion of the matrix $\mathbf{A}$ — while thickness has an extra term. For a channel flow, this extra term in (25) is $(\beta/f)\kappa\bar{z}_\rho$, and is of uniform sign; converted to actual layer thickness this yields a northward flux which would be present in the absence of mean thickness gradients. Such a flux is clearly visible in Lee *et al.*'s (1997, Fig. 4b) eddy-resolving channel experiments, where their middle layer is essentially of uniform thickness 500 m, and yet there is a uniform northward flux of about 0.2 $m^2\ s^{-1}$. Using the diffusivity they estimate (about 2000 $m^2\ s^{-1}$) a flux $(\beta/f)\kappa h$ of 0.1 $m^2\ s^{-1}$, which is clearly of the right order. Put another way, the term is equivalent to an advective velocity of $\beta\kappa/f$ which is typically of order 10^{-4} m s^{-1}. This is a very small velocity; but in the channel case is of uniform direction (northward) and, being persistent, can exert a strong influence in the steady state.

This β-term raises an apparent difficulty. Lee *et al.*'s (1997) example demonstrates that eddy fluxes of q, rather than z_ρ, lend themselves naturally to parameterization; indeed, Treguier *et al.* (1997) and Lee *et al.* (1997) call for such a parameterization. Greatbatch and Lamb (1990) also demonstrate that downgradient mixing of z_ρ leads to mixing of f/q rather than of q itself. However, most numerical models are not posed with q as a prognostic variable, so that unless some inversion procedure is undertaken — which would slow down the computation of a climate model drastically — knowledge of $\nabla \cdot (\overline{\mathbf{u}'q'})$ needs to be used indirectly. (If we represent the equation for q in the form $q_t + \mathbf{u} \cdot \nabla q = 0$, then divergence terms do not naturally enter. If the equation is written in (the equivalent) thickness-weighted form $(qz_\rho)_t + \nabla \cdot (\mathbf{u}qz_\rho) = 0$, which includes a divergence term, then this system merely predicts *the absolute* and not *potential* vorticity; recall that for linear theory the eddy fluxes of absolute vorticity are zero.)

It would be preferable to use the transformed Eulerian-mean system of momentum equations (Lee and Leach, 1996) which are driven directly by potential vorticity eddy flux terms, and substitute the parameterization there. This yields (Lee and Leach, 1996, Eq. (3.7)), among others, an extra forcing term in the u-momentum equation of form $\bar{z}_\rho \overline{v'q'}$. If this is parameterized by (30), with the matrix $\mathbf{A}$ set to the identity matrix (downgradient q transfer), this term becomes $-\kappa \bar{z}_\rho \bar{q}_y$. Part of this term is simply $-\beta\kappa$, a constant westward forcing. This term appears to have been first found by Welander (1973). Attempts to use this form of parameterization will be reported elsewhere.

The matrix $\mathbf{A}$ is symmetric, and so corresponds to a diffusion directly (though this can appear as an advection in the tracer equations); antisymmetric terms would imply advection by other pseudovelocities. The lack of antisymmetric terms comes from the slowly varying assumption which gave a local problem; had extra derivatives been included (cf. Plumb, 1979) then asymmetries would have appeared.

We note for future use two facts. First, it is easy to show from linear theory that

$$\kappa_\rho = \kappa_z = 0, \qquad \rho = \rho_b, \qquad \rho = \rho_s \tag{31}$$

although this may not hold for more general forms of the parameterization. Second, a parameterization of form (30) must satisfy a necessary vertical integral criterion to conserve mass, discussed in Section 5 (related to, but not the same as, one suggested by Treguier *et al.* 1997).

3. The form of the solution

If we define the 'bolus' velocity

$$\mathbf{u}^* = \frac{\overline{\mathbf{u}'z_\rho'}}{\bar{z}_\rho} \equiv -\frac{\overline{\mathbf{u}'q'}}{\bar{q}}; \qquad \mathbf{U} = \bar{\mathbf{u}} + \mathbf{u}^* \tag{32}$$

then

$$\bar{z}_{\rho t} + \nabla \cdot (\mathbf{U}\bar{z}_\rho) = 0 \tag{33}$$

so that

$$\bar{z}_{\rho t} + \nabla \cdot (\bar{\mathbf{u}}\,\bar{z}_\rho) = \nabla \cdot (\kappa \mathbf{A} \cdot \nabla \bar{z}_\rho) - \nabla \cdot \left(\beta \frac{\kappa \bar{z}_\rho}{f} \mathbf{A}_2\right) = \nabla \cdot \left\{\kappa f \mathbf{A} \cdot \nabla\left(\frac{\bar{z}_\rho}{f}\right)\right\}. \tag{34}$$

On an f-plane, since $\mathbf{A}$ is positive definite, solutions of (34) conserve integrals of layer thickness (and hence density itself), and the r.h.s. acts to decrease layer thickness variance. These are desirable properties, as noted by Gent and McWilliams (1990). If β is included, the additional effects have no such properties.

Conversion to (x, y, z) coordinates gives the horizontal part of $\mathbf{u}^*$ as

$$\mathbf{u}_H^* = \kappa \mathbf{A} \cdot \frac{\partial}{\partial z}\left(\frac{\nabla_H \bar{\rho}}{\bar{\rho}_z}\right) + \frac{\beta}{f} \kappa \mathbf{A}_2. \tag{35}$$

(The vertical component is obtained from the requirement of zero divergence, to which we shall return.) Note that the diffusivity κ is outside the derivative in (35), unlike the original shape proposed by Gent and McWilliams (1990). Gent *et al.* (1995) suggested that this form, without the β-term, permits a steady solution in which the isopycnals are flat (another desirable feature). The addition of the extra term appears to invalidate this; but recall that the diffusivity is that for a local instability problem and would vanish for flat isotherms. The large-scale density satisfies, in z-coordinates (Gent *et al.,* 1995)

$$\bar{\rho}_t + \mathbf{U} \cdot \nabla \bar{\rho} = 0. \tag{36}$$

4. Boundary conditions

An eddy parameterization must specify boundary conditions on the closed surfaces of the ocean domain. Because of the asymmetries in scales, the conditions on vertical and horizontal surfaces are not necessarily the same.

At vertical walls, conservation of density requires that

$$(\overline{\mathbf{u}' z_\rho'}) \cdot \mathbf{n} = 0 \tag{37}$$

where $\mathbf{n}$ is a unit vector normal to the wall. In a no-slip ocean model, both horizontal components of the large-scale flow would vanish on the walls, and so thermal wind would normally be small, so that (37) might already be reasonably satisfied by the large-scale flow. Thus its imposition is unlikely to cause difficulties in a model.

The conditions at surface and floor are more subtle. At first sight, the equivalent of (37) is needed to provide boundary conditions on the vertical component of the effective vertical velocity, w^*. Indeed, Gent and McWilliams (1990) and Gent *et al.* (1995) use $w^* = 0$, and derive w^* from a vertical integral of the horizontal divergence of $\mathbf{u}^*$. They note that in general this causes a rapid decrease in the eddy flux terms, from some interior value to zero over the last grid point. Treguier *et al.* (1997) examine this behavior, partly because they also discuss the problems raised by surface mixed layers. (The treatment of regions which are unstratified in the vertical is discussed in Part III.) Treguier *et al.* (1997)

suggest, as do Gent *et al.* (1995), a ramping down of the eddy fluxes with some vertical structure. They argue that this implies strong horizontal fluxes to balance the divergence terms. Visbeck *et al.* (1997) do not discuss the surface boundary condition beyond that in Gent and McWilliams (1990).

Such a closure is somewhat *ad hoc,* and it is enlightening to examine what must occur at surface and floor in more detail. To begin with, Gent and McWilliams (1990) implicitly require

$$\int_{-H}^{0} \mathbf{u}^* dz = 0.$$

This is sufficient, but not necessary, to ensure w^* vanishes at top and bottom. [A sufficient condition would be that

$$\int_{-H}^{0} \mathbf{u}^* dz = \mathbf{k} \wedge \nabla \Psi$$

where $\Psi(x, y)$ is a streamfunction for the vertically integrated bolus velocity, which is therefore nondivergent. It is hard to see how an expression for Ψ could be determined. Below we show that for linear theory Ψ would indeed vanish.]

In density coordinates it is usual to think of outcropping lines as areas where density layers are flat and have no thickness (indeed, numerical isopycnic models adopt precisely this formulation). Thus the existence of nonzero thickness fluxes immediately adjacent to surface and floor implies the existence of delta-function changes, so that we must have

$$\mathbf{u}^* \bar{z}_\rho = \kappa[-\mathbf{A} \cdot \nabla \bar{z}_\rho + \beta \bar{z}_\rho \mathbf{A}_2 / f] - \kappa(\rho_s) \mathbf{A} \cdot \nabla \bar{z} \delta(\rho - \rho_s) + \kappa(\rho_b) \mathbf{A} \cdot \nabla \bar{z} \delta(\rho - \rho_b) \qquad (38)$$

where the signs are consistent with the downward increase of density, and we have assumed an immediate change from nonzero to zero slope in $\bar{z}$. The terms in $\nabla \bar{z}$ are evaluated just in the fluid interior. (The equivalent formulation for potential vorticity has the difficulty that q becomes infinite rather than zero; an expression using q^{-1} is very similar to (38), without the β-terms, and is not considered further.) Figure 1 shows a schematic of the situation at the surface.

The two delta functions in (38) are responsible for the intrusion at the surface of light water into the denser domain, and the intrusion at the bottom of dense water into the lighter domain, when baroclinic instability occurs. In the Eady (1949) problem, for example, density gradients are uniform across the channel, and there is identically no thickness flux in the fluid interior. The entire thickness flux occurs in the two delta functions at top and bottom.[3] Converted into z-coordinates, it is clear that the delta functions have the correct physical behavior:

$$\mathbf{u}^* = \kappa\left(\mathbf{A} \cdot \frac{\partial}{\partial z}\left(\frac{\nabla_H \bar{\rho}}{\bar{\rho}_z}\right) + \frac{\beta}{f} \mathbf{A}_2\right) - \kappa(\rho_s) \mathbf{A} \cdot \frac{\nabla_H \bar{\rho}}{\bar{\rho}_z} \delta(z) + \kappa(\rho_b) \mathbf{A} \cdot \frac{\nabla_H \bar{\rho}}{\bar{\rho}_z} \delta(z + H). \qquad (39)$$

3. The original use of delta functions at horizontal boundaries was made by Bretherton (1966) for potential vorticity.

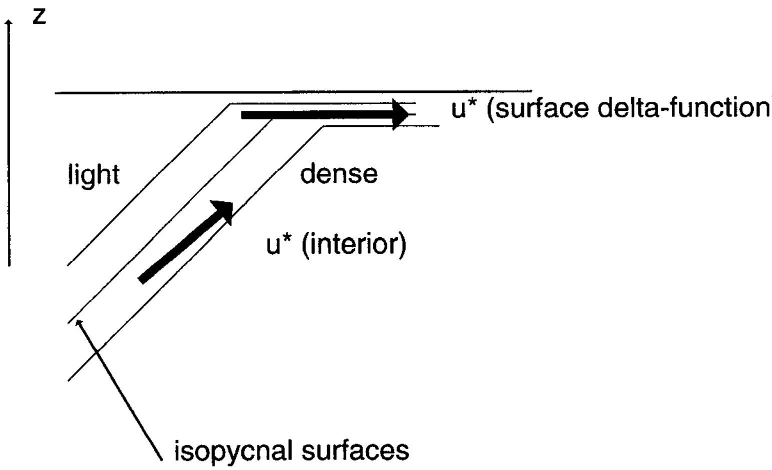

Figure 1. A schematic of the delta function structure near the surface as necessitated by a parameterization.

In the Gent and McWilliams (1990) formulation, the diffusivity κ was required to vanish at top and bottom. Here it becomes important that it not vanish, so that it may permit the correct additional fluxes of thickness near the boundaries. (The delta functions at top and bottom permit an elegant numerical treatment, discussed in Part II.)

In turn, however, the existence of the delta functions becomes part of a necessary condition which any parameterization of form (30) must satisfy; we now discuss this.

5. A necessary condition

Consider a parameterization of form (30). We assume that (38) holds (no net integral of the thickness fluxes); we shall return to this from linear theory below. We distinguish between surface density ρ_s and an interior value ρ_{s+} which is infinitesimally larger. A similar distinction occurs between ρ_b and ρ_{b-}. Then we must have

$$\begin{aligned} 0 &= \int_{-H}^{0} \mathbf{u}^*\, dz = \int_{\rho_b}^{\rho_s} \mathbf{u}^* \bar{z}_\rho\, d\rho = \left\{ \int_{\rho_{b-}}^{\rho_s+} + \int_{\rho_s+}^{\rho_s} + \int_{\rho_b}^{\rho_b-} \right\} \mathbf{u}^* \bar{z}_\rho\, d\rho \\ &= -[\kappa \mathbf{A} \cdot \nabla \bar{z}]_{\rho_{b-}}^{\rho_{s+}} + \int_{\rho_b}^{\rho_s} \kappa_\rho \mathbf{A} \cdot \nabla \bar{z}\, d\rho + \frac{\beta}{f} \int_{\rho_b}^{\rho_s} \kappa \mathbf{A}_2 \bar{z}_\rho\, d\rho \\ &\quad + \kappa(\rho_s) \mathbf{A} \cdot \nabla \bar{z}\big|_{\rho_s} - \kappa(\rho_b) \mathbf{A} \cdot \nabla \bar{z}\big|_{\rho_b} \end{aligned} \tag{40}$$

where the first two terms on the r.h.s. of (40) involve an integration by parts, and the last two terms on the r.h.s. are integrals of the delta functions in (38). All but the second and third terms cancel, leaving only

$$\int_{\rho_b}^{\rho_s} \kappa_\rho \mathbf{A} \cdot \nabla \bar{z}\, d\rho = -\frac{\beta}{f} \int_{\rho_b}^{\rho_s} \kappa \mathbf{A}_2 \bar{z}_\rho\, d\rho \tag{41}$$

where the integral is formally only between ρ_{b-} and ρ_{s+}, although this does not matter. The important point is that any formalism (30) places a constraint on the vertical form of the diffusivity through (41).[4] The equivalent in depth co-ordinates is

$$\int_{-H}^{0} \frac{\kappa_z}{\rho_z} \mathbf{A} \cdot \nabla \bar{\rho}\, dz = \frac{\beta}{f} \int_{-H}^{0} \kappa \mathbf{A}_2\, dz. \tag{41a}$$

If $\mathbf{A}$ is of full rank, we may separate out the two horizontal components and write

$$\begin{aligned} \int_{\rho_b}^{\rho_s} \kappa_\rho (\mathbf{A}_{11}\bar{z}_x + \mathbf{A}_{12}\bar{z}_y)\, d\rho + \frac{\beta}{f} \int_{\rho_b}^{\rho_s} \kappa \mathbf{A}_{12}\bar{z}_\rho\, d\rho = 0 \\ \int_{\rho_b}^{\rho_s} \kappa_\rho (\mathbf{A}_{21}\bar{z}_x + \mathbf{A}_{22}\bar{z}_y)\, d\rho + \frac{\beta}{f} \int_{\rho_b}^{\rho_s} \kappa \mathbf{A}_{22}\bar{z}_\rho\, d\rho = 0. \end{aligned} \tag{42}$$

However, in the linear instability problem above, $\mathbf{A}$ is not of full rank, and while there is a restriction of the form

$$\int_{\rho_b}^{\rho_s} \kappa_\rho (\sin\theta \bar{z}_x - \cos\theta \bar{z}_y)\, d\rho - \frac{\beta}{f} \cos\theta \int_{\rho_b}^{\rho_s} \kappa \bar{z}_\rho\, d\rho = 0$$

there is no information in the orthogonal direction, so that the projection of κ_ρ on $\cos\theta\bar{z}_x + \sin\theta\bar{z}_y$ is not restricted.

It is possible to use linear theory to derive the requirement of vanishing integral of the bolus velocities. In this case, direct evaluations shows that

$$\begin{aligned} \int_{\rho_{b-}}^{\rho_{s+}} \overline{u' z'_\rho}\, d\rho &= \frac{1}{2} Re \int_{\rho_b}^{\rho_s} \frac{-ik \sin\theta}{\rho_0 f} B \cdot \frac{B^*_{\rho\rho}}{g}\, d\rho \\ &= \frac{1}{2} Re \left[\frac{-ik\sin\theta}{\rho_0 f} B \cdot \frac{B^*_\rho}{g} \right]_{\rho_b}^{\rho_s} - \frac{1}{2} Re \int_{\rho_b}^{\rho_s} \frac{-ik\sin\theta}{\rho_0 f} B_\rho \cdot B^*_\rho\, d\rho \\ &= \frac{1}{2} Re\, [\overline{u'z'}]_{\rho_b}^{\rho_s} = -[\kappa \mathbf{A} \cdot \nabla \bar{z}]_{\rho_b -}^{\rho_s +}. \end{aligned}$$

This can be combined with the first line of (40) to give

$$\begin{aligned} \int_{\rho_b}^{\rho_s} \mathbf{u}^* \bar{z}_\rho\, d\rho &= \left\{ \int_{\rho_b-}^{\rho_s+} + \int_{\rho_s+}^{\rho_s} + \int_{\rho_b}^{\rho_b-} \right\} \mathbf{u}^* \bar{z}_\rho\, d\rho \\ &= -[\kappa \mathbf{A} \cdot \nabla \bar{z}]_{\rho_b-}^{\rho_s+} + \kappa(\rho_s) \mathbf{A} \cdot \nabla \bar{z}\big|_{\rho_s} - \kappa(\rho_b) \mathbf{A} \cdot \nabla \bar{z}\big|_{\rho_b} = 0 \end{aligned}$$

so that for linear theory there can be no net integral of the bolus velocity.

By rewriting the bolus velocity in terms of $\overline{\mathbf{u}'q'}$, it is straightforward to show that (41) is the condition that the eddies generate no vertically integrated momentum. A similar

4. Treguier *et al.* (1997) produce the quasi-geostrophic version of this condition.

condition was employed by Marshall (1981) in his channel model. Green (1970) found a similar, but not identical, expression for the integral requirement of no net zonal acceleration for a channel flow.

When $\mathbf{A}$ is of full rank, the consistency condition (41) places heavy constraints on the form of the diffusivity. Clearly that obtained above from linear instability automatically satisfies the (less stringent) constraint, and the above arguments confirm this. In particular, the trivial solution with κ independent of density (and thus independent of height), is not a solution satisfying the consistency condition, unless — as in Gent and McWilliams (1990) — the diffusivity is defined to vanish at surface and floor, and is inside the derivative in the equivalent of (35).

Whereas we know little about particle excursions in fully developed geostrophic turbulence, it is clear that under most circumstances we would anticipate that particles are more active, and thickness flux stronger, at mid-depth than at surface or floor. This can be confirmed for linear instability by direct solution of the equations (the Eady problem, as noted above, having no interior thickness flux, is not a good example in this respect). We would then prefer a diffusivity profile which reflected this belief.

If the diffusivity satisfies (41), then numerical computation of w^* from the horizontal divergence of $\mathbf{u}^*$, and imposition of the boundary conditions $w^* = 0, z = -H, 0$ will give a consistent response, provided that the delta functions are subsumed into additional horizontal fluxes in the uppermost and lowest grid points when the integration for w^* is made.

6. An approximate form using a small wavenumber expansion

A profile of diffusivity satisfying (41) is formally straightforward to obtain: one merely solves the local instability problem at each horizontal point and each timestep, chooses wavenumber and direction to maximize the local growth rate, and substitutes into (25) and (30). However, while this might be practical for the channel case (where direction is specified), the numerical loading would be too high for any practical use in three dimensions, as noted by Treguier *et al.* (1997). It is thus necessary to seek forms which satisfies (41) *approximately* as well as being physically motivated and rapid to calculate. This section and the next derive two such approximations, from a small wavenumber expansion and from an iterative procedure.

We consider first the small wavenumber approach.

An approximate profile can be created in three steps. First, seek an analytical solution to (18) valid for small wavenumber and small β. Second, choose a form for $\mathbf{A}$. Third, use the form thus obtained, but with a more accurate estimate for the wavenumber, to produce a profile which will approximately satisfy requirement (41).

a. *Small wavenumber expansion*

Approximate solutions to the local vertical problem (18) have been sought in many ways in the literature. The approach here was used in the zero β limit by Flierl (1975), and also in an unpublished report by Branscome (1983). Consider a wavenumber which is *small*, so that $ka \ll 1$. We shall also suppose that β is small, and so write

$$\beta \cos \theta = (ka)^2 \hat{\beta}$$

where $\hat{\beta}$ has the same units as β.

Since the quantity required is $|B/(\tilde{u} - c)|^2$, it is natural to pose

$$\phi = \frac{B}{\tilde{u} - c}, \tag{43}$$

a definition frequently made for the purpose of bounding instabilities (Pedlosky, 1987). The equation satisfied by ϕ is

$$[(\tilde{u} - c)^2 \phi_\rho]_\rho + \frac{g\bar{z}_\rho}{\rho_0 f} k^2 (\tilde{u} - c)^2 \phi - \frac{g\beta \cos \theta \bar{z}_\rho}{\rho_0 f^2} (\tilde{u} - c)\phi = 0 \tag{44}$$

$$\phi_\rho = 0, \qquad \rho = \rho_b, \rho_s. \tag{45}$$

Because for the moment we are interested only in the vertical *shape* of the diffusivity, we scale the problem by requiring $\phi = 1$, $\rho = \rho_b$; full scaling will be added later. Pose

$$\phi = \phi_0 + (ka)^2 \phi_1 + \cdots$$

$$c = c_0 + (ka)^2 c_1 + \cdots$$

To leading order, (44) becomes

$$[(\tilde{u} - c_0)^2 \phi_{0\rho}]_\rho = 0; \qquad \phi_{0\rho} = 0, \qquad \rho = \rho_b, \rho_s. \tag{46}$$

This has solution

$$\phi_0 \equiv 1. \tag{47}$$

To next order,

$$[(\tilde{u} - c_0)^2 \phi_{1\rho}]_\rho + \frac{g\bar{z}_\rho}{f^2 a^2 \rho_0} (\tilde{u} - c_0)^2 - \frac{g\hat{\beta}\bar{z}_\rho}{f^2 \rho_0} (\tilde{u} - c_0) = 0. \tag{48}$$

Integration of this, and use of the boundary conditions, gives immediately

$$\frac{1}{a^2} \int_{\rho_b}^{\rho_s} \bar{z}_\rho (\tilde{u} - c_0)^2 \, d\rho = \hat{\beta} \int_{\rho_b}^{\rho_s} \bar{z}_\rho (\tilde{u} - c_0) \, d\rho$$

$$\text{i.e., } \frac{1}{a^2} \int_{-H}^{0} (\tilde{u} - c_0)^2 \, dz = \hat{\beta} \int_{-H}^{0} (\tilde{u} - c_0) \, dz. \tag{49}$$

This has solution

$$c_0 = \bar{\tilde{u}} - \frac{\hat{\beta} a^2}{2} + i \sqrt{\left(\tilde{u}_s^2 - \frac{\hat{\beta}^2 a^4}{4}\right)} \tag{50}$$

where

$$\bar{\tilde{u}} = \frac{1}{H}\int_{-H}^{0} \tilde{u}\, dz; \qquad \tilde{u}_s = \left[\frac{1}{H}\int_{-H}^{0} \tilde{u}^2\, dz - \bar{\tilde{u}}^2\right]^{\frac{1}{2}} \tag{51}$$

are the vertical mean and standard deviation of $\tilde{u}$ respectively. In terms of the original β, (50) becomes

$$c_0 = \bar{\tilde{u}} - \frac{\beta \cos\theta}{2k^2} + i\sqrt{\left(\tilde{u}_s^2 - \frac{\beta^2 \cos^2\theta}{4k^4}\right)} \tag{52}$$

in which we recognize beta-corrections from the semi-circle theorems (Pedlosky, 1987). The solution is complex provided β is sufficiently small. Assuming, as we shall, that k is of order a^{-1}, the term under the square root in (52) is positive if (very roughly) the internal potential vorticity gradient $\beta - f^2(\tilde{u}_z/N^2)_z$ has a sign change. Thus the small wavenumber approximation gives instability approximately satisfying the internal of the three necessary conditions, and ignores the two at surface and floor. In practice, β is not usually large enough to prevent instability in (52).

The form of ϕ_1 is needed. Using (48), this implies that ϕ has the form

$$\phi \approx 1 - \frac{gk^2}{f^2\rho_0}\int_{\rho_b}^{\rho} \frac{d\rho'}{(\tilde{u}(\rho') - c_0)^2}\int_{\rho_b}^{\rho'} \bar{z}_\rho(\rho'') \cdot \left\{(\tilde{u}(\rho'') - c_0)^2 - \frac{\beta\cos\theta}{k^2}(\tilde{u}(\rho'') - c_0)\right\} d\rho'' \tag{53}$$

which can be written more conveniently in z-notation as

$$\phi \approx 1 + \frac{k^2}{f^2}\int_{-H}^{z} \frac{N^2(z')\, dz'}{(\tilde{u}(z') - c_0)^2}\int_{-H}^{z'} \left\{(\tilde{u}(z'') - c_0)^2 - \frac{\beta\cos\theta}{k^2}(\tilde{u}(z'') - c_0)\right\} dz''. \tag{54}$$

The value of $|\phi|^2$ is required for the parameterization, and this is simply

$$|\phi|^2 = 1 + 2\frac{k^2}{f^2} Re\left(\int_{-H}^{z} \frac{N^2(z')\, dz'}{(\tilde{u}(z') - c_0)^2}\int_{-H}^{z'} \cdot \left\{(\tilde{u}(z'') - c_0)^2 - \frac{\beta\cos\theta}{k^2}(\tilde{u}(z'') - c_0)\right\} dz''\right). \tag{55}$$

Suppose that $\mathbf{A}$ takes the linear instability form (26). Then if β is zero, both terms in (55) satisfy the integral constraint (41) on κ individually. They therefore do so in summation even when k is not small. When β is nonzero, the integral constraint remains satisfied to leading order provided merely that k is small. In both these cases, the *shape* (55) gives a good approximation to the required behavior. When the wavenumber and β terms are order

1 in the balance, however, the integral constraint is only approximately satisfied by this ansatz, with the approximation becoming better as the length scale of the flow, and hence the importance of β, decreases.

Provided β is sufficiently small, the approximation predicts instability at all points (save where $\mathbf{u}$ is vertically uniform, where κ should be set to zero to avoid divisions by zero). In the case where β is large, and c_i is predicted to vanish, the diffusivity should be set to zero.

b. *Choice of matrix* $\mathbf{A}$

A choice for $\mathbf{A}$ must be made. There are two simple choices possible:

i. Ensemble averaging. An obvious approach is to average over an ensemble of angles θ to obtain a pure downgradient flux

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

There are difficulties with this choice, apart from the fact that a total ensemble implicitly involves some measure of upgradient transfer. One problem is that it is necessary for the profile of diffusivity to satisfy both conditions (42). While this could be achieved with a uniform diffusivity in the vertical, this, as noted, does not satisfy the consistency condition except for zero β. An alternative approach would be to pose an expression for κ which was a linear combination of two arbitrary functions of z, and determine their amplitudes by applying (42). A second difficulty is that knowledge of the angle θ is needed to compute the phase speed c — and, indeed, the formulae above explicitly involve $\tilde{u}$ and B, both of which contain θ. There is thus no reason to believe that ensemble averaging would yield physically reasonable values (although an approach similar to (d) below could be employed).

ii. Maximal linear growth rate. Choose the angle θ which maximizes the linear growth rate. Again, this cannot be conveniently done for the full instability problem, but can be done for small wavenumber. To maximise c_{0i}, we note that from (51),

$$c_{0i}^2 = -\frac{\beta^2 \cos\theta}{4k^4} + u_s^2 \cos^2\theta + 2ru_s v_s \sin\theta\cos\theta + v_s^2 \sin^2\theta \tag{56}$$

where u_s, v_s are the (vertical) standard deviations of $\bar{u}$ and $\bar{v}$ respectively, and r is their vertical correlation

$$\left[H^{-1}\int_{-H}^{0} \overline{uv}\, dz - H^{-2}\left(\int_{-H}^{0} \bar{u}\, dz\right)\left(\int_{-H}^{0} \bar{v}\, dz\right)\right] \Big/ u_s v_s.$$

Thus

$$c_{0i}^2 = \frac{1}{2}\left(-\frac{\beta^2}{4k^4} + u_s^2 + v_s^2\right) + \left[\frac{1}{4}\left(-\frac{\beta^2}{4k^4} + u_s^2 - v_s^2\right)^2 + r^2u_s^2v_s^2\right]^{1/2} \cos(\gamma - 2\theta) \tag{57}$$

where

$$\tan\gamma = \frac{ru_sv_s}{\frac{1}{2}\left(-\frac{\beta^2}{4k^4} + u_s^2 - v_s^2\right)}. \tag{58}$$

The expression for c_{0i} is trivially maximized when

$$\theta = \gamma/2 \tag{59}$$

and is

$$c_{0i} = \left[\frac{1}{2}\left(-\frac{\beta^2}{4k^4} + u_s^2 + v_s^2\right) + \frac{1}{2}\left\{\left(-\frac{\beta^2}{4k^4} + u_s^2 - v_s^2\right)^2 + 4r^2u_s^2v_s^2\right\}^{1/2}\right]^{1/2} \tag{60}$$

Note that in the channel problem $\mathbf{A}$ is replaced by unity and θ by zero, so that this step is not necessary, though c_{0i} is still required for the size of the diffusivity.

c. A more accurate estimate of wavenumber

The analysis above was for small wavenumber. As indicated, we anticipate inverse wavenumbers around the deformation radius in size. Clearly, too large a wavenumber would induce difficulties in (55), whose second term can take both positive and negative values. A suitable compromise, and one which appears to work well, is to seek an approximation for the wavenumber of fastest growth, based on the Eady (1949) model.

For that model,

$$k_{max} = 1.606\,\frac{f}{NH} \tag{61}$$

(Gill, 1982; Pedlosky, 1987), where k_{max} is the wavenumber of maximum growth. To extend this to nonuniform stratification, we form an estimate of the internal wave speed by a WKBJ approximation to

$$w_{zz} + \frac{N^2}{C^2}w = 0; \qquad w = 0, \qquad z = -H, 0$$

(Gill, 1982) as

$$C \approx \frac{1}{\pi}\int_{-H}^{0} N(z)\,dz. \tag{62}$$

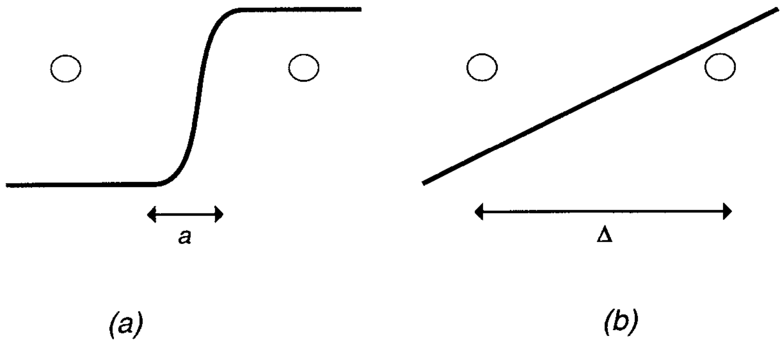

Figure 2. Schematics of the density field between two coarse gridpoints. In case (a), there is a front of width the deformation radius; in case (b), there is no front. Estimates of thermal wind will be inaccurate in case (b).

Using the Eady formula (61) then gives an estimate of the fastest growing wavenumber as

$$k = 0.51 \frac{f}{C}. \tag{63}$$

This would be expected to be accurate for mean flows "close" to the Eady problem and become progressively less so as the mean flow diverges from that problem. We test this formulation below in realistic circumstances and will see that it gives a very accurate estimate of the exact linear solution.

d. A scaling for the parameterization

We still know little about what controls the amplitude of mixing coefficients. Visbeck *et al.* (1997) produce a rationale which reduces to a diffusivity of order $\langle$velocity$\rangle \times \langle$length scale$\rangle$, where $\langle$velocity$\rangle$ is some measure of the vertical variability, and $\langle$length scale$\rangle$ is harder to define. They prefer a judicious mixture of deformation radius and the grid spacing (which is usually larger). Another possibility (e.g., Green, 1970) is to find a measure of how far the eddies are capable of mixing. Killworth (1981) suggests this distance would be $(La)^{1/2}$, where L is a length scale for the large-scale flow. Such a scale is notoriously difficult to define in all but the simplest circumstances.

Here, as a first guess, we follow Visbeck *et al.* (1997) and choose $\langle$length scale$\rangle$ = max (C/f, grid spacing).[5] Here, C/f is taken as an operant definition of a, the deformation radius. In a non-eddy-resolving model, no single choice for a length scale can be adequate. Figure 2 demonstrates two extreme cases. The coarse resolution model has the same values for density at its gridpoints in both cases. In case (a), the front is of width the deformation radius; in case (b), there is no front. In both cases, however, the velocity would be estimated from numerical thermal wind as the same (low) value, relevant only for case (b).

5. Note that Killworth (1981), following Simmons (1974), finds a length scale for the influence of linear instability of order the geometric mean of the flow length scale (however defined) and the deformation radius.

In case (a), one would prefer to choose the grid spacing for a length scale rather than a simply to increase the estimated velocity to a correct value; in case (b), the velocity would be estimated correctly, and a would be more relevant. If we assume that in real oceanic regimes, fronts occur at widths of order a few deformation radii, then the velocity evaluated from thermal wind is of order (a/Δ) times the correct velocity. (Conversely, when grid spacings provide adequate resolution, computed velocities will be correct.) The tests in Part II are well resolved, so that a is used consistently; no tests have yet been made with the factor Δ.

For the velocity estimate it is natural to use c_i, which is here the standard deviation of the flow decomposed at angle θ. With a small grid spacing, the product gives the equivalent of $a^2 \cdot kc_i$ since k is chosen of order a^{-1}. (Here a is the deformation radius defined by $a = C/f$.) This product is thus directly a length scale squared times an inverse time scale (the growth rate).

Thus the final scaling is A max $(a, \Delta) \cdot c_i$, where A is some unknown scaling coefficient of order unity, to be tuned by numerical experiment. The above discussion then shows that poor resolution will underestimate velocities (and hence c_i) by a ratio a/Δ, so that the numerical product Δc_i becomes approximately the exact product ac_i as required.

e. Overall form for the parameterization

Combining the above, we have

$$\kappa = A \max(a, \Delta) c_i \left\{ 1 + 2\frac{k^2}{f^2} Re \left(\int_{-H}^{z} \frac{N^2(z')\, dz'}{(\tilde{u}(z') - c_0)^2} \int_{-H}^{z'} \cdot \left\{ (\tilde{u}(z'') - c_0)^2 - \frac{\beta \cos\theta}{k^2} (\tilde{u}(z'') - c_0) \right\} dz'' \right) \right\} \quad (64)$$

where A is of order unity, $a = C/f$, C and k are given by (62), (63) respectively, the angle θ, and hence the rotated velocity $\tilde{u}$, is given by (59), and c_i is given by (60). Finally, the implementation takes the form (25), (30), or (36), which are all equivalent, with the matrix $\mathbf{A}$ given by (26).

7. An approximate form using an iterative procedure

Another approach to finding a more accurate shape for κ, albeit with more computation, is to use an iterative version of the previous method. We use the same approximation (63) for wavenumber k of fastest growth. Pose an initial guess for ϕ as $\phi_0 \equiv 1$. The iterative procedure now assumes that we know ϕ_n, $n \geq 0$. To obtain c_{n+1}, integrate (44) from top to bottom. This gives

$$\int_{\rho_b}^{\rho_s} \bar{z}_\rho \phi_n \{k^2(\tilde{u} - c_{n+1})^2 - \beta \cos\theta(\tilde{u} - c_{n+1})\}\, d\rho = 0 \quad (65)$$

or, rewriting, a quadratic for c_{n+1}:

$$c_{n+1}^2\left[\int_{-H}^{0}\phi_n\,dz\right] + c_{n+1}\left[-2\int_{-H}^{0}\tilde{u}\phi_n\,dz + \frac{\beta\cos\theta}{k^2}\int_{-H}^{0}\phi_n\,dz\right]$$
$$+\left[\int_{-H}^{0}\tilde{u}^2\phi_n\,dz - \frac{\beta\cos\theta}{k^2}\int_{-h}^{0}\tilde{u}\phi_n\,dz\right] = 0. \quad (66)$$

The solution c_1 from (66) is precisely that in (52), of course. The next iterate ϕ_{n+1} is then obtained by integrating (44):

$$\phi_{n+1} = 1 + \frac{k^2}{f^2}\int_{-H}^{z}\frac{N^2(z')\,dz'}{(\tilde{u}(z') - c_{n+1})^2}\int_{-H}^{z}\phi_n$$
$$\cdot\left\{(\tilde{u}(z'') - c_{n+1})^2 - \frac{\beta\cos\theta}{k^2}(\tilde{u}(z'') - c_{n+1})\right\}dz \quad (67)$$

which is of form (54) when $n = 0$. This expression should replace that in curly brackets in (64) for the small wavenumber case; other aspects of the approximation remain unchanged. Note that the inner integral in (67) can itself be expressed as a quadratic in c_{n+1}, so that the integral can be tabulated while the solution of (66) is being prepared, saving computation time.

The small wavenumber solution is thus seen as the first term in the iterative scheme, although one would naturally use $|\phi_n|^2$ for the diffusivity and not the linear form (55). None of the individual functions ϕ_n satisfy the necessary condition (41) precisely, although when the solution is sufficiently converged, (41) will be well satisfied, but with a less than optimal wavenumber. In the three-dimensional case, the angle θ must also be chosen. Since θ enters (67) in a nonlinear fashion — the iterates ϕ_n depend intrinsically on θ — it is suggested that (59) be used as before as an approximation. This can easily be computed on the first pass of the iteration scheme.

For all cases tested, this iteration scheme converges rapidly (to the exact solution for the approximate k). Examples will be given in the next section. [We can ensure (41), or (41a), is satisfied exactly. Using either the approximate or iterated forms, merely compute the l.h. and r.h.s. of (41a), and subtract a value independent of z from κ which ensures (41a) is satisfied.]

8. Some comparisons with linear theory

The small wavenumber approach can be tested against maximum linear growth rate calculations in fairly realistic configurations. We consider the case when the vertical density structure and the horizontal velocity (here restricted to east-west only) are both exponentials in the vertical, but with differing scales

$$\rho = -\Delta\rho\exp az/H, \qquad u = u_0\exp\delta z/H.$$

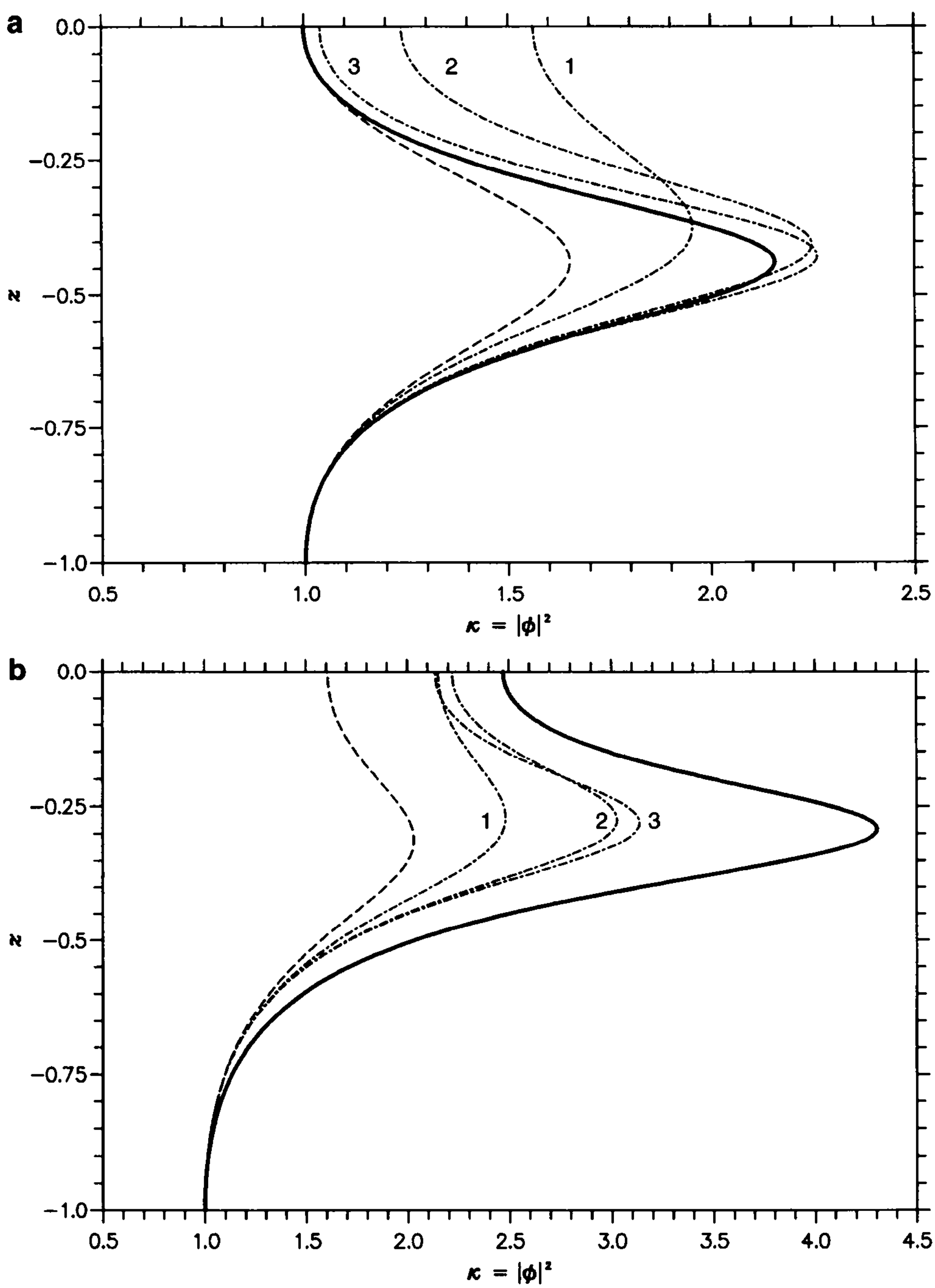
a
0.0
-0.25
-0.5
-0.75
-1.0
z
0.5
1.0
1.5
2.0
2.5
3
2
1
κ = |ϕ|²
b
0.0
-0.25
-0.5
-0.75
-1.0
z
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
1
2
3
κ = |ϕ|²

Figure 3

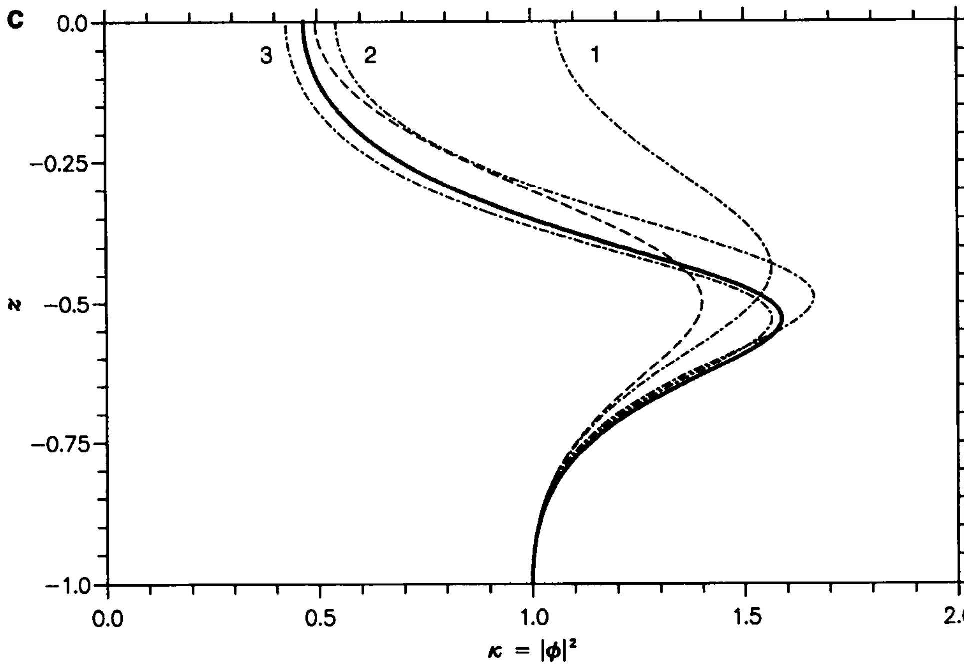

Figure 3. Comparison of the diffusivity from the fastest growing linear mode and from the approximate analytical form, for mean density and velocity varying exponentially in the vertical. The fastest mode diffusivity is the bold line; the small wavenumber form is the dashed line; and the three labeled dash-dotted lines are the first three iterated solutions. (a) shows $\alpha = \delta = 1$; $\beta = 0$; (b) shows $\alpha = 1$, $\delta = 2$, $\beta = 0.5$ with westward velocity; (c) shows $\alpha = 2$, $\delta = 1$; $\beta = 0.2$. The depth is nondimensionalized on H, and the diffusivity is normalized in both cases to be unity at the floor.

Here a and δ measure the nondimensional decay rates of density and velocity with depth, and $\Delta\rho$, u_0 are amplitudes of density and velocity respectively. A channel problem is chosen so that the angle θ does not enter the problem. Substitution into (44) permits numerical solution; that with maximum growth rate kc_i is chosen. These can be compared with the approximate solutions given by (55) or any of the iterates (67). Only the shape of the diffusivity in the vertical is examined here, with its value set to unity at the floor.

Results are presented nondimensionally, using u_0 as a scale for c, and the natural scales of $f/(g'H)^{1/2}$ for k, where $g' = g\Delta\rho/\rho_0$, and $u_0 f^2/g'H$ for β. Note that we can take u_0 of either sign when β is nonzero; when it is negative, this corresponds to westward flow.

Three cases are shown here, although fifteen have been run. In all cases, the diffusivity peaks at mid-depth, close to, but not always at, the position of the critical layer which would exist if c_i were zero. The diffusivity varies by a factor of 2–4 through the depth. The shape of this variation is in excellent agreement with values found empirically from eddy-resolving models (Treguier, 1997). The surface value of κ is larger than the bottom value when velocity decays slower than density in the vertical.

Figure 3a shows a simple case $a = \delta = 1$; $\beta = 0$. The exact fastest mode has $k_{max} = 2.03$, $c = 0.64 + 0.12i$. The approximate wavenumber is almost identical ($k = 2.04$). The small wavenumber approximation yields $c = 0.63 + 0.18i$. This behavior is typical of the cases studied; the real part of the phase speed is well approximated by (52), but the imaginary part is overestimated by about 50%. This means that the shape of κ in the small wavenumber approximation is underestimated near the (approximate) critical layer, since a division by $(\tilde{u} - c)^2$ is involved; this too is clearly visible in all cases studied.

The iterated solutions converge rapidly to a good approximation of the fastest growing mode. The first iterate corresponds to the small wavenumber solution, except that the full $|\phi_1|^2$ is being computed rather than the approximation (55). The difference between the two shows that $Im\ (\phi_1)$ must be large at the surface. By the second iteration, the solution is very close to the fastest growing mode.

Figure 3b shows the case $a = 1$, $\delta = 2$; $\beta = 0.5$, $u_0 < 0$. For typical ocean values, the value of β is rather high. This case demonstrates a larger variation in κ, with a peak near the effective critical layer, and is chosen to be one for which the approximate solutions are least accurate. The exact solution has $k = 2.28$, $c = -0.54 + 0.16i$. The approximate wavenumber is 2.04, an error of 10%. The small wavenumber solution for κ is now far from the exact solution, although the phase speed is $-0.49 + 0.23i$, again with a good estimate of $Re\ (c)$ but overestimating $Im\ (c)$. The iterated solutions converge to the exact solution for the incorrect k, so that the subsurface peak in κ is underestimated.

Finally, Figure 3c shows the case $a = 2$, $\delta = 1$; $\beta = 0.2$. This case has the subsurface maximum in κ below mid-depth, a feature reproduced by many of the approximate solutions. The exact solution has $k = 1.88$, $c = 0.59 + 0.12$, while the approximate wavenumber is 1.79. Both the small wavenumber solution and the second and higher iterates of the iterated solution give excellent agreement with the form of κ; the small wavenumber solution predicts $c = 0.60 + 0.18i$, again overestimating the imaginary part.

Other results (not shown) suggest that either the small wavenumber approximation, or two iterations of the iterated scheme, give good fits to the fastest growing mode diffusion profile. Each method requires only $O(n)$ computations at each fluid column containing n grid points; every iteration doubles the computational load. However, these are much smaller than the $O(n^3)$ operations needed to solve the local instability problem numerically (and this ignores the effort to select the correct eigenmode from the n returned from such a routine, which is nontrivial). Since there is little reason to require an exact fit to what is only a linear solution in the first place, it is not recommended that iterations beyond 2 are used in practice.

9. Conclusions

This paper has set of to achieve two parallel goals: to use linear perturbation theory to suggest a structure for a parameterization of baroclinic instability which is, to leading order at least, a solution of the equations of motion; and to deduce conditions on parameterizations which must be satisfied in general because of potential vorticity conservation.

Boundary conditions at surface and floor are considered carefully. This leads to a parameterization using either of two approximate forms as representations for the linear instability; both satisfy the necessary conditions approximately.

Although the scheme evaluates thickness mixing, it differs from Gent and McWilliams (1990) in many respects. First, the diffusion coefficient is predicted to vary with position, including a strong vertical structure to the mixing, similar to that deduced by Treguier (1997) from an eddy-resolving model. Second, the diffusion coefficient is outside the vertical derivative. Third, the mixing is *turned* by the matrix $\mathbf{A}$, to align itself along the pathways of the mixing of the fastest growing linear normal mode. Fourth, thickness itself is not mixed, but potential vorticity is. This does permit steady solutions with level isotherms, since the diffusivity is zero in such regions.

This paper has stressed various aspects of parameterization schemes, including the need to conserve potential vorticity rather than layer thickness and the variation of mixing with depth. It is not yet clear how important such details are. The northward flux computed by Lee *et al.* (1997) is clear evidence of potential vorticity, rather than thickness, mixing occurring in a layered model. The effective velocities computed from these fluxes are very weak, but monotonic. The necessary conditions deduced in this paper show that a vertically uniform diffusion coefficient cannot in general yield a nondivergent bolus velocity; yet under normal circumstances one could make a small correction to a vertically uniform coefficient which would satisfy the condition, without needing the strong vertical variation obtained, e.g., from linear theory. Stringent tests will be necessary to evaluate the relative importance of each of these features.

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