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New similarity solutions of the thermocline equations with vertical variations of diffusion

by Simon Hood[1]

ABSTRACT

Three new classes of exact solution of the thermocline equations are obtained through use of an ansatz based method (Clarkson and Kruskal, 1989), which is related to the older Classical Lie Group Method used by Salmon and Hollerbach (1991) to obtain exact solutions. The newer method has not previously been applied to oceanographic problems. Our results are more general than those of Salmon and Hollerbach in two distinct ways: we obtain new classes of solution not obtainable by the older method and, in addition, we determine solutions in which the vertical temperature diffusion profile is an arbitrary function of depth. Application of the solutions to understand the control of the main thermocline is considered in a companion paper.

1. Introduction

In this paper we look for new classes of exact, analytic solutions of the *thermocline equations* (e.g. Pedlosky, 1986, Section 6.21). The paper builds on work done by Salmon and Hollerbach (1991) (here after referred to as SH). To determine such solutions we make use of *symmetry reductions*—which for the purposes of this paper are considered to be a special kind of *similarity reduction,*[2] which we obtain by means of the Direct Method (Clarkson and Kruskal, 1989).

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2. There is some ambiguity in the literature regarding exactly what is meant by a similarity reduction or solution. For the purposes of this paper we consider a similarity solution (here these are obtained by first finding similarity reductions) of a PDE to be one obtained by *a priori* assuming a specific form of solution. The method used here makes use of the symmetries of a differential equation and we suppose symmetry-based solutions are a special type of similarity solution.

Our results are more general than those obtained by SH for two distinct reasons. First, using the Direct Method we find new classes of solutions which cannot be found using the method used by SH. Second, SH assumed diffisuion, κ, to be spatially uniform, while we consider reductions and solutions in which diffusion varies with depth, i.e., $\kappa = \kappa(z)$. We compute solutions in such a way that the diffusion profile may be prescribed *a posteriori.*

This paper is the first of two: here we concentrate on determining symmetry reductions of the thermocline equations, while in the second (Hood and Williams, 1996, hereafter 'Paper II') we consider how the associated solutions relate to the thermocline; velocity, pressure and temperature fields are computed, and boundary conditions imposed. These boundary conditions are satisified by placing conditions on arbitrary functions found in reductions determined here.

2. Background

The *thermocline equations* consist of the geostrophic and hydrostatic balances, continuity and a thermodynamic equation, and in nondimensional form are,

$$fu = -\phi_y, \quad fv = \phi_x, \quad \theta = \phi_z, \tag{2.1i}$$

$$u_x + v_y + w_z = 0, \tag{2.1ii}$$

$$u\theta_x + v\theta_y + w\theta_z = (\kappa\theta_z)_z, \tag{2.1iii}$$

where u, v and w are the velocity components, ϕ is pressure and θ is temperature. These equations are used to model gyre-scale ocean flow and investigate the dominant process or processes which are responsible for the ubiquitous temperature profile found in many of the world's oceans—the so called *thermocline problem.* (A detailed description of the thermocline problem is given by Pedlosky, 1986 (Section 6.21) and 1987; see also Paper II).

The model assumes steady motion, and inertial, friction and horizontal temperature diffusion are neglected. As such it is assumed to be an adequate description of time-averaged, large-scale ocean circulation away from boundaries. Small-scale motion and surface forcing are parameterized here by $\kappa(z)$. Clearly this is a crude parameterization, but it will allow us to examine the effect on solutions of the diffusion profile. No doubt more realistic choices for the r.h.s. of (2.1iii) would give further behavior.

a. Early work

Several authors have considered exact, analytic solutions of the thermocline equations, including Robinson and Stommel (1959), Needler (1967), Welander (1959, 1971) and Killworth (1987). A review of the early work is given by Veronis

(1969). Robinson and Stommel consider a diffusive ocean and look for solutions recoverable from the similarity transformation $\tilde{z} = zx^k$, $\theta = x^{3k+1}\tilde{\theta}(\tilde{z})$, $w = x^k\tilde{w}(\tilde{z})$, where $\tilde{z}$, $\tilde{\theta}$ and $\tilde{w}$ are new variables. Welander (1959) considers the purely advective case and derives an equation similar to (2.2) and then looks for solutions to this equation of the form $M(x,y,z) = P(x,y)Q(y,z)$, where $P(x,y)$ and $Q(y,z)$ are functions to be determined. It turns out that $Q(y,z)$ simplifies so that Welander's solutions are of the form $M(x,y,z) = M_0(x,y)\exp(z)$. Needler (1967) also considers the purely advective case. A significant weakness of all these solutions is that they assume the same vertical structure throughout the domain of solution.

Welander (1971) takes a different approach. He again considers the purely advective limit (i.e., $\kappa = 0$ and the flow is adiabatic) and shows that in this case potential vorticity, q, density, ρ and the Bernoulli functional, B, must be functionally related, i.e., $q = F(\rho,B)$, where F is to be determined. This represents a first integral of the thermocline equations. Solutions of the form $q = F(a\rho + bB + c)$, where a, b and c are constants, are sought. As with the solutions of Robinson and Stommel, Welander (1959) and Needler (1967), problems exist with matching of realistic boundary conditions. Killworth (1987) obtains three exact solutions of the thermocline equations, again in the advective limit, by making use of Welander's functional relationship between q, ρ and B. These solutions are a significant advance over those mentioned above since density and Ekman pumping can be chosen as "fairly arbitrary" functions at the surface.

The results of the work mentioned above illustrate two limiting cases: firstly, an advective case ($\kappa = 0$) in which the effects of diffusion are assumed negligible and the thermocline is supposed to be an advective phenomenon, and secondly, the diffusive limit, in which upwelling of cold water balances downward diffusion of heat. Salmon and Hollerbach (1991) use an approach based on symmetry reductions of the thermocline equations in which one is not required to make strong assumptions *a priori* about the form of solution (cf. Welander, 1959 or Needler, 1967), nor is it necessary to choose between a purely advective or a strongly diffusive limit. In addition, solutions contain arbitrary functions and constants which can be used to match realistic boundary conditions. We use an approach similar to, though more general than, that used by SH.

b. *Previous work—Salmon and Hollerbach*

We outline the pertinent methodology and results from Salmon and Hollerbach (1991):

In the case when κ is a constant, the thermocline system (2.1) may be written with no loss of generality as

$$M_xM_{zzz} + y(M_{xx}M_{yzz} - M_{yz}M_{xzz}) - \kappa y^2 M_{zzzz} = 0, \qquad (2.2)$$

where

$$u = -\frac{1}{y}M_{yz} \quad v = \frac{1}{y}M_{xz} \quad w = \frac{1}{y^2}M_x \tag{2.3i}$$

$$\phi = M_z \quad \theta = M_{zz}. \tag{2.3ii}$$

Salmon and Hollerbach (1991) reformulated the thermocline system in this way and used the *Classical Lie Group Method* (see below) to compute *symmetries* of the thermocline equation (2.2) and hence find transformations which *reduce* (2.2) to a partial differential equation (PDE) in just two independent variables. Sixteen such reductions were found. Simple solutions of the reduced equations were considered for some reductions and the associated velocity, pressure and temperature fields found using (2.3). (We explain the terms in italics below.)

Two reductions are of particular interest; these are

$$M(x, y, z) = x\alpha(y) + xz\beta(y) + G(y, z), \tag{2.4}$$

$$M(x, y, z) = x\alpha(y) + xz^2\beta(y) + G(y, z), \tag{2.5}$$

that is, similarity forms S_{12} and S_{13}—in the notation of SH, respectively. Based on these similarity forms, SH considered solutions of the form

$$M(x, y, z) = xy^2\left(\frac{z^2 - z_0 z}{1 - z_0}\right)w_E(y) + G(y, z), \tag{2.6i}$$

where $G(y, z)$ satisfies

$$\frac{w_E(y)}{h_0}\left[(z^2 - z_0 z)\,G''' + y(2z - z_0)G_y'' - 2yG_y'\right] = \kappa G^{\mathrm{IV}}, \tag{2.6ii}$$

$w_0(y) = \alpha(y)/y^2$ and $W(y) = -\beta(y)/y^2$. By finding solutions to (2.6ii), exact analytical solutions of (2.2) were found and these solutions suggest that the thermocline is a front. This idea is supported by separate numerical experiments by Salmon (1990).

c. The approach—Symmetry reductions

Finding the general solution of the Thermocline System (2.1) is impossible, at least with current techniques, so we look for physically significant special solutions. A method which often proves useful for this is that of *symmetry reductions.* We contend that such reductions and their associated solutions are important. First, one can infer information about the physical system represented by the PDE, using such solutions. SH do this when they argue that the thermocline is a front. Second, the solutions may be directly used: for example, Cates, 1990 and Cates and Crighton, 1990, examine such solutions for ocean acoustics. In addition, general solutions of PDEs often asymptote to symmetry solutions.

Before going any further, we provide the unfamiliar reader with some background regarding symmetries of PDEs. Consider the general fourth order PDE in three independent variables,

$$\Delta(x, y, z, M, M_x, M_y, \ldots, M_{zzzz}) = 0. \tag{2.7}$$

Applying the transformation

$$\tilde{x} = \tilde{x}(x, y, z, M), \quad \tilde{y} = \tilde{y}(x, y, z, M), \quad \tilde{z} = \tilde{z}(x, y, z, M), \quad \tilde{M} = \tilde{M}(x, y, z, M), \tag{2.8}$$

to (2.7) yields

$$\tilde{\Delta}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{M}, \tilde{M}_x, \tilde{M}_y, \ldots, \tilde{M}_{zzzz}) = 0. \tag{2.9}$$

If (2.7) and (2.9) are identical then (2.8) is a *symmetry* of the differential equation (2.7) which is *invariant* under that transformation. If the symmetry is a group and depends continuously on a single parameter, ϵ say, then it may be used to reduce the number of independent variables in any PDE invariant under the transformation; the transformation is referred to as a *one-parameter Lie group.* It is often the case that the resulting *reduced equation* will be easier to solve than the original, whether analytically or numerically.

Note that the general solution of the reduced equation does not yield the general solution of the original—in general applying symmetry reductions enables one to find only special solutions of PDEs; general boundary conditions cannot be met. Nevertheless, it is often the case that sufficiently general boundary conditions for physical significance may be satisfied.

The question remains, how to determine such transformations and how to use them? The standard method is usually referred to as the *Classical Lie Group Method,* which is based on a group-theoretic approach and this is the method used by SH to obtain their results. A good outline of the method is given in Section 3 of their work and we do not repeat the description here. In addition, Hill (1992) provides an excellent *practical* introduction to the subject. (Another accessible introduction by Stephani (1990) and Olver (1992) reviews the entire subject of exact solutions of nonlinear partial differential equations; many further references are given in the latter.)

The Classical Lie Group Method does not in general find all point symmetries—and consequently reductions—of a given PDE. Certainly it does not for the thermocline equations and so we use a more general method, the so called *Direct Method* to compute new families of solution. This method is based on an alternative ansatz-based approach, where one makes an informed "guess" for a general form of reduction.

d. Generalization of previous work

We build on the results of Salmon and Hollerbach in two distinct ways. First, by use of the Direct Method for finding symmetry reductions, which is ansatz based, i.e., one assumes a general form of reduction and proceeds to find conditions on that form from the given PDE. (The method is outlined in Section 3.) Second, by considering reductions in which diffusion varies with depth. The computations by SH are under the simplifying assumption that diffusion is spatially uniform, i.e., κ is constant. In the computation of symmetry reductions of the thermocline equations this is neither necessary nor desirable: diffusion is not spatially uniform in the ocean, but rather it is larger nearer the surface or at boundaries. We address this point more fully in our second paper.

In accord with this, we look for reductions of the thermocline equations in which diffusion is an arbitrary function of z, so that we may prescribe the diffusion profile *a posteriori.* With $\kappa = \kappa(z)$ then the thermocline equation becomes

$$M_xM_{zzz}+y[M_{xz}M_{yzz} - M_{yz}M_{xzz}] - y^2(\kappa(z)M_{zzz})_z = 0 \tag{2.10}$$

(cf. (2.2), which has a simpler final term).

The remaining sections of this paper are organized as follows: in Section 3 the method used to determine solutions of the thermocline equations is introduced and then in Section 4 calculations and results are presented; finally, in Section 5 we compare our results to those of Salmon and Hollerbach, and include some discussion on the rôle of similarity solutions in physical oceanography. Note that much of Sections 3 and 4 is mathematical, so the reader whose interest lies more in the results than the method may care to skim these. (The results are summarized in Table 1.)

3. The Direct Method

We describe the fundamental features of the method here, although, as it is not fully algorithmic, it is best understood by considering the examples in the next section. A detailed description of the method is given by Clarkson and Kruskal, 1989.

Given a partial differential equation in the three independent variables x, y and z (cf. 2.10), then inspired by the reductions which one can obtain by means of the Classical Lie Method one seeks a reduction to a PDE in just two independent variables in the form,

$$M(x, y, z) = F(x, y, z, G(\xi(x, y, z), \zeta(x, y, z))), \tag{3.1}$$

where the functions $F(x, y, z, G)$, $\xi(x, y, z)$ and $\zeta(x, y, z)$ are to be determined. This is the most general transformation which is not implicit, i.e., F and G are independent of M. In fact it is usually sufficient to consider

$$M(x, y, z) = \beta(x, y, z)G(\xi(x, y, z), \zeta(x, y)) + \alpha(x, y, z), \tag{3.2}$$

which is a considerable simplification (and it is often straightforward to prove this).

First, M is now linear in the new dependent variable, $G(\xi, \zeta)$ and second, only one of the new independent variables depends on all of the old independent variables. If it is not sufficient to consider (3.2) then this usually indicates that the given PDE can be written in a simpler form by a suitable transformation. One can also consider transformations which reduce the given PDE directly to an ordinary differential equation by supposing

$$M(x, y, z) = F(x, y, z, G(\xi(x, y, z))), \tag{3.3}$$

where, $F(x, y, z, \xi)$, $G(\xi)$ and $\xi(x, y, z)$ are to be determined. As before, it is usually the case that F is linear in G, i.e.,

$$M(x, y, z) = \beta(x, y, z)G(\xi) + \alpha(x, y, z). \tag{3.4}$$

Assuming that the linear ansätze are sufficient, the fundamental idea is to substitute (3.2) or (3.4) into the given PDE and *require* that the result be a differential equation in fewer independent variables. Thus the ratios of different products and powers of G and its derivatives must be functions of ξ and ζ only. This condition yields an over-determined, nonlinear system of *determining equations* for $\alpha(x, y, z)$, $\beta(x, y, z)$ and the new independent variable(s).

It is convenient to make use of three freedoms in the ansatz (3.2) to make the method productive. Since (3.2) is linear in $G(\xi, \zeta)$ then we may translate and rescale $G(\xi, \zeta)$ as convenient:

Freedom (i): Given that $\alpha(x, y, z)$ has the form, $\alpha(x, y, z) = \alpha_0(x, y, z) + \beta(x, y, z)\Omega(\xi, \zeta)$, then we may take $\Omega(\xi, \zeta) \equiv 0$ by the translation $G(\xi, \zeta) \to G(\xi, \zeta) - \Omega(\xi, \zeta)$.

Freedom (ii): Given that $\beta(x, t)$ has the form, $\beta(x, y, z) = \beta_0(x, y, z)\Omega(\xi, \zeta)$, then we may take $\Omega(\xi, \zeta) \equiv 1$, by the rescaling $G(\xi, \zeta) \to G(\xi, \zeta)/\Omega(\xi, \zeta)$.

We may also redefine ξ:

Freedom (iii): Given that $\xi(x, y, z)$ is given by an equation of the form, $\Omega(\xi, \zeta) = \xi_0(x, y, z)$, where $\Omega(\xi)$ is assumed to be an invertible function, then we may take $\xi = \xi_0(x, y, z)$ by transforming $\xi \to \Omega^{-1}(\xi)$. We have of course the same freedom in ζ.

Each of Freedoms (i)–(iii) may be used once with no loss of generality.

The Direct Method is very different from that due to Lie in both theory and practice. We mentioned above that the Direct Method is more general; equally important is the method's flexibility: for the full ansatz, (3.1), the complexity of the thermocline equation, (2.10), yields a determining system which (at least presently) appears intractable (recall that it is nonlinear). However, we may choose to consider simplified ansätze and therefore a simpler determining system. In this way we are able to find new reductions and corresponding exact solutions. In addition, the sheer quantity of algebra involved in obtaining and solving the determining system is an order of magnitude less than that necessary for Lie's method—while the use of computer algebra systems for group-theoretic methods is a virtual necessity, it is not for this method.

4. Application of the Direct Method to the thermocline equation

We do not attempt to compute the full set of reductions of the thermocline equation, (2.10), obtainable by the Direct Method here (the full problem appears intractable). Instead we consider three simplified ansätze, each of which leads to a new class of reductions. These in turn lead to new solutions which we believe are physically significant.

First we consider a simplified form of ansatz (3.4) in which we set $\beta = 1$, i.e.,

$$M(x, y, z) = G(\xi(x, y, z)) + \alpha(x, y, z), \tag{4.1}$$

where $G(\xi)$, $\xi(x, y, z)$ and $\alpha(x, y, z)$ are to be determined. The analysis in this case is tractable and while these reductions do not admit an arbitrary diffusion profile, they do illustrate how the Direct Method is more general than that due to Lie. We obtain a set of solutions which may not be found by using the Classical Lie Method.

Secondly, we are particularly interested in reductions in which diffusion remains an arbitrary function of z. To admit such an arbitrary function, one of the "new" independent variables must be z itself. We therefore consider reductions obtainable from the two ansätze

$$M(x, y, z) = \beta(x, y)G(\xi(x, y), z) + \alpha(x, y, z), \tag{4.2}$$

$$M(x, y, z) = \beta(x, y)G(z) + \alpha(x, y, z), \tag{4.3}$$

where the functions $\beta(x, y)$, $G(\xi, z)$ (or $G(z)$), $\xi(x, y)$ and $\alpha(x, y, z)$ once more remain to be determined.

Unless otherwise stated, the following notation is used: Γ_i, $i = 1, 2 \ldots$, are functions of the given argument, which are obtained upon derivation of the determining equations for a reduction, and are to be determined; often such functions are polynomial in ζ and we write $\Gamma_i(\xi) = \gamma_{i0} + \gamma_{i1}\xi + \ldots$; if the function is constant we write simply γ_i; λ_1, $i = 1, 2 \ldots$, are constants introduced during computation of reductions, for example constants of integration, or upon separation of variables. Partial derivatives are denoted by subscripts; we reserve primes, f', to indicate differentiation with respect to z, $\partial/\partial z$.

a. *Reductions admitting only uniform diffusion: $M = G(\xi) + \alpha(x, y, z)$*

In this section we consider a class of reductions which lead to solutions in which the diffusion is uniform. These serve to show that the set of reductions obtained by SH is incomplete and to illustrate the power of the Direct Method.

Substituting (4.1) into (2.2) we find

$$\begin{aligned}
&- \kappa y^2\xi_z^4 G^{\mathrm{IV}} + \{\xi_x\xi_z^3 + y\xi_z^2[\xi_{xz}\xi_y - \xi_x\xi_{yz}]\}G'G''' + 2y\xi_z^2[\xi_x\xi_{yz} - \xi_{xz}\xi_y](G'')^2 \\
&+ \{3\xi_x\xi_z\xi_{zz} + y[(\xi_{xz}\xi_y - \xi_x\xi_{yz})\xi_{zz} + (\xi_x\xi_{yzz} - \xi_{xzz}\xi_y)\xi_z]\}G'G'' \\
&+ \{\xi_x\xi_{zzz} + y[\xi_{xz}\xi_{yzz} - \xi_{xzz}\xi_{yz}]\}(G')^2 \\
&+ \{\alpha_x\xi_z^3 + y\xi_z^2[\alpha_{xz}\xi_y - \alpha_{yz}\xi_x] - 6\kappa y^2\xi_z^2\xi_{zz}\}G''' \\
&+ \{3\alpha_x\xi_z\xi_{zz} + y[\alpha_{xz}(\xi_y\xi_{zz} + 2\xi_{yz}\xi_z) - \alpha_{yz}(\xi_x\xi_{zz} + 2\xi_{xz}\xi_z) \\
&\quad - \alpha_{xzz}\xi_y\xi_z + \alpha_{yzz}\xi_x\xi_z] - \kappa y^2(4\xi_z\xi_{zzz} + 3\xi_{zz}^2)\}G'' \\
&+ \{\alpha_{zzz}\xi_x + \alpha_x\xi_{zzz} + y[\alpha_{xz}\xi_{yzz} - \alpha_{xzz}\xi_{yz} - \alpha_{yz}\xi_{xzz} + \alpha_{yzz}\xi_{xz}] - \kappa y^2\xi_{zzzz}\}G' \\
&+ \alpha_x\alpha_{zzz} + y[\alpha_{xz}\alpha_{yzz} - \alpha_{xzz}\alpha_{yz}] - \kappa y^2\alpha_{zzzz} = 0.
\end{aligned} \tag{4.4}$$

First order partial derivatives of ξ with respect to both x and z occur in (4.4); partial derivatives with respect to y occur only in mixed derivatives. This leads to the existence of four distinct cases to consider: (i), $\xi_x\xi_z \neq 0$, (ii), $\xi_x = 0$ with $\xi_z \neq 0$, (iii), $\xi_x \neq 0$ with $\xi_z = 0$ and (iv) $\xi_x = \xi_z = 0$. Since the main thrust of this paper is to compute reductions which admit an arbitrary diffusion profile, we consider only Case (i) (the most interesting) here.

Normalizing the coefficients of $G'G'''$ and $(G'')^2$ against that of G^{IV}, respectively, we find

$$y^2\xi_z^2\Gamma_a(\xi) = 2y(\xi_x\xi_{yz} - \xi_{xz}\xi_y), \tag{4.5a}$$

$$y^2\xi_z^2\Gamma_b(\xi) = \xi_x\xi_z + y(\xi_{xz}\xi_y - \xi_x\xi_{yz}). \tag{4.5b}$$

Adding (4.5a) to twice (4.5b) yields

$$y^2\Gamma(\xi)_z - \xi_x = 0, \tag{4.6i}$$

where

$$\Gamma(\xi) = \tfrac{1}{2}(\Gamma_a(\xi) + 2\Gamma_b(\xi)). \tag{4.6ii}$$

Note that $\Gamma \neq 0$ since, by assumption, $\xi_x \neq 0$ in this case. Integrating (4.6) we obtain the general solution

$$z + xy^2\Gamma(\xi) = \mathscr{F}(\xi), \tag{4.7}$$

where $\mathscr{F}(\xi)$ is a function of integration and is to be determined.

We note that from (4.7) it is readily deduced that

$$2x\xi_x = y\xi_y, \tag{4.8a}$$

$$2x\xi_{xz} = y\xi_{yz} \tag{4.8b}$$

and then from (4.7), after making use of (4.8) and using (4.5) that $\Gamma_a = 0$ and $\Gamma_b = \Gamma$.

In order to determine $\xi(x, y, z)$ it remains to consider the coefficients of $G'G''$ and

$(G')^2$; the remaining coefficients yield an over-determined system for $\alpha(x, y, z)$. Normalizing the coefficient of $G'G''$ against that of $G'G'''$ and using (4.8) we find

$$\xi_z^2\Gamma_c(\xi) = \xi_{zz}, \tag{4.9}$$

then integrating with respect to z, exponentiating and integrating again we obtain

$$\int^{\xi} \exp\left[-\int^{\xi_1} \Gamma_c(\xi_2)d\xi_2\right] d\xi_1 = z\theta(x, y) + \phi(x, y), \tag{4.10}$$

where $\theta(x, y)$ and $\phi(x, y)$ are functions of integration, to be determined. Next, applying Freedom (iii) we may write

$$\xi(x, y, t) = z\theta(x, y) + \phi(x, y), \tag{4.11}$$

without loss of generality and consequently $\Gamma_c = 0$ (cf. 4.9). Using (4.8b) it is easy to see that the coefficient of $(G')^2$ vanishes.

Finally, we return to (4.6): since ξ_x is linear in z and ξ_z is independent of z (cf. (4.11)), then it is clear that

$$\Gamma(\xi) + \gamma_{01}\xi + \gamma_{00} = 0, \tag{4.12}$$

and using this result in (4.7) we find

$$\mathscr{F}(\xi) = \lambda_1\xi + \lambda_0; \tag{4.13}$$

hence eliminating ξ in favor of z in (4.7), using (4.11) and equating coefficients of like powers of z yields

$$\lambda_1\theta = 1 - \xi_{01}xy^2\theta, \tag{4.14a}$$

$$\lambda_1\phi + \lambda_0 = -xy^2(\gamma_{01}\phi + \gamma_{00}), \tag{4.14b}$$

from which it is straightforward to obtain $\theta(x, y)$ and $\phi(x, y)$.

Reduction 4.a.1. We have therefore computed the complete set of reductions for the case $\xi_x\xi_z \neq 0$ and these are given by

$$M(x, y, z) = G(\xi) + \alpha(x, y, z), \tag{4.15a}$$

$$\xi(x, y, z) = \frac{z - \gamma_{00}xy^2 - \lambda_0}{\gamma_{01}xy^2 + \lambda_1}, \tag{4.15b}$$

where $\alpha(x, y, z)$ satisfies the system of PDEs,

$$y^2\theta^2\Gamma_1(\xi) = \alpha_x\theta + y[\alpha_{xz}(z\theta_y + \phi_y) - \alpha_{yz}(z\theta_x + \phi_x)], \tag{I}$$

$$y\theta^3\Gamma_2(\xi) = 2\alpha_{xz}\theta_y - 2\alpha_{yz}\theta_x - \alpha_{xzz}(z\theta_y + \phi_y) + \alpha_{yzz}(z\theta_x - \phi_x), \tag{II}$$

$$y^2\theta^4\Gamma_3(\xi) = \alpha_{zzz}(z\theta_x + \phi_x) + y[\alpha_{yzz}\theta_x - \alpha_{xzz}\theta_y], \tag{III}$$

$$y^2\theta^4\Gamma_4(\xi) = \alpha_x\alpha_{zzz} + y[\alpha_{xz}\alpha_{yzz} - \alpha_{xzz}\alpha_{yz}] - \kappa y^2\alpha_{zzzz}, \tag{IV}$$

and $G(\xi)$ satisfies

$$\kappa G^{\mathrm{IV}} + (\gamma_{01}\xi + \gamma_{00})G'G''' - \Gamma_1(\xi)G''' - \Gamma_2(\xi)G'' - \Gamma_3(\xi)G' - \Gamma_4(\xi) = 0. \quad (4.16)$$

Notice that (III) is a *first order linear* PDE for α_{zz}; if $\alpha_{zz} = 0$ then (III) vanishes and (II) is a first order linear PDE for α_z. Hence there are three cases to consider: (a), $\alpha_z = 0$, (b), α linear in z and (c), $\alpha_{zz} \neq 0$. (Notice also that in the case $\gamma_{01} = \gamma_{00} = 0$ (4.16) is a linear ODE.) Hence Reduction 4.a.1 is in fact a *set* of reductions and gives rise to many solutions of the thermocline equations, (2.1). Each of these solutions is new, so we have illustrated that the set of reductions obtained by SH is incomplete. Recall that these solutions are for uniform diffusion, the case considered by SH—that extra solutions are found is attributable solely to the method used. Note also that Cases (ii), (iii) and (iv) will yield yet more new solutions.

b. Reductions admitting on arbitrary diffusion profile, I

In this section we consider the first of our two ansätze which yield reductions admitting an arbitrary diffusion profile, namely (4.2).

Substituting (4.2) into the thermocline equation (2.10) we find

$$\begin{gathered} -y^2\beta\kappa(z)G^{\mathrm{IV}} + \beta\beta_x GG''' + \beta^2\xi_x G_\xi G''' \\ + \mathscr{G}(G_\xi' G'', G'G_\xi'', G_\xi G'', GG_\xi'', G_\xi G', GG_\xi', GG'', GG_\xi, GG', G^2, \\ G_\xi'', G_\xi', G_\xi, G''', G'', G', G; \alpha, \beta, \xi, \kappa) = 0, \end{gathered} \quad (4.17)$$

where $\mathscr{G}$ is a known function which depends linearly on its arguments, $G_\xi' G'', \ldots, G$, and on the derivatives of its parameters, $\alpha, \ldots, \kappa$. We begin by comparing the coefficients of G^{IV}, GG''' and $G_\xi G'''$. It turns out that $\xi_x \neq 0 \Rightarrow \beta_x = 0$, so there are three cases to consider: $\xi_x \neq 0$, $\xi_x = 0$ with $\beta_x \neq 0$ and $\xi_x = 0$ with $\beta_x = 0$.

Case 1: $\xi_x \neq 0$. Normalizing the coefficients of $G_\xi G'''$ and GG''' in (4.17) against that of G^{IV} we obtain

$$\beta\xi_x = \Gamma_a(\xi, z)y^2, \quad (4.18a)$$

$$\beta_x = \Gamma_b(\xi, z)y^2, \quad (4.18b)$$

and we may set $\Gamma_a = 1$ without loss of generality through Freedom (ii); hence differentiating (4.18a) with respect to x yields $\beta_x\xi_x + \beta\xi_{xx} = 0$, and if $\beta_x \neq 0$ we may eliminate β_x between this result and (4.18b) yielding

$$\frac{\xi_{xx}}{\xi_x} + \Gamma_b(\xi, z)\xi_x = 0. \quad (4.19)$$

Further, since ξ is independent of z then $\Gamma_{b,z} = 0$; hence, integrating (4.19) with respect to x twice we obtain

$$\int^\xi \exp\left[\int^{\xi_1} \Gamma_a(\xi_2)d\xi_2\right] d\xi_1 = x\theta(y) + \phi(y), \quad (4.20)$$

where $\theta(y)$ and $\phi(y)$ are functions of integration, to be determined, and using Freedom (iii) we may write

$$\xi(x, y) = x\theta(y) + \phi(y), \tag{4.21}$$

without loss of generality; consequently $\beta(y) = y^2\theta^{-1}(y)$, i.e., $\beta_x = \beta_z = 0$.

Collecting results (4.17) becomes

$$\begin{aligned} &y^4\theta^{-1}\{G_\xi G''' - \kappa(z)G^{IV}\} + y^3(y^2\theta^{-1})_y\{G_\xi' G'' - G'G_\xi''\} + y^2\theta^{-1}(\alpha_x - y^2\kappa')G''' \\ &\quad + y^3\theta^{-1}[(x\theta_y + \phi_y)\alpha_x' - \alpha_y'\theta]G_\xi'' + y^3\theta^{-1}[\alpha_y''\theta - (x\theta_y + \phi_y)\alpha_x'']G_\xi' \\ &\quad + y\alpha_x(y^2\theta^{-1})_y G'' + y^2\alpha'''G_\xi - y\alpha_x''(y^2\theta^{-1})_y G' \\ &\quad + \alpha_x\alpha_{zzz} + y[\alpha_x'\alpha_y'' - \alpha_y'\alpha_x''] - y^2\kappa(z)\alpha^{IV} - y^2\kappa'\alpha''' = 0. \end{aligned} \tag{4.22}$$

Normalizing the coefficient of $G_\xi' G''$ and $G'G_\xi''$ against that of $G_\xi G'''$ and G^{IV} we obtain $2 - y\theta^{-1}\theta_y = \Gamma_0(\xi, z)$, and since the l.h.s. is independent of both x and z then Γ_0 must be constant; hence we trivially integrate to obtain $\theta(y) = \lambda_1 y^{2-\gamma_0}$, where λ_1 is the nonzero constant of integration and consequently $\beta(y) = \lambda_1^{-1}y^{\gamma_0}$.

Next, normalizing the coefficient of w''' in the same way we find $\alpha_x - y^2\kappa' = \Gamma(\xi, z)y^2$ and integrating with respect to x we obtain

$$\alpha(x, y, z) = \lambda_1^{-1}y^{\gamma_0}\int^\xi \Gamma(\xi_1, z)\, d\xi_1 + xy^2\kappa' + \alpha_0(y, z), \tag{4.23i}$$

$$= \beta\left\{\int^\xi \Gamma(\xi_1, z)\, d\xi_1 + \kappa'\xi\right\} - \frac{y^2\phi\kappa'}{\theta} + \alpha_0(y, z), \tag{4.23ii}$$

where $\alpha_0(y, z)$ is a function of integration, to be determined. We may set the term in braces equal to zero through Freedom (i); hence $\alpha_x = 0$, i.e., dropping the subscript-zero and absorbing the second term in (4.23ii) into the third, $\alpha = \alpha(y, z)$. Consequently (4.22) simplifies further and it is now straightforward to find the most general form of $\alpha(y, z)$ for which we obtain a reduction to a PDE in $G(\xi, z)$. We obtain the following set of reductions:

Reduction 4.b.1.

$$M(x, y, z) = \lambda_1^{-1}y^{\gamma_0}G(\xi, z) + \left(\frac{\gamma_2}{2\lambda_1}z^2 - \frac{\lambda^2}{\lambda_1}z\right)\int y^{\gamma_0 - 1}\, dy + \alpha_0(y), \tag{4.24a}$$

$$\xi = \lambda_1 x y^{2-\gamma_0} + \phi(y), \tag{4.24b}$$

where $\alpha_0(y)$, $\phi(y)$ and γ_0 remain arbitrary, and $G(\xi, z)$ satisfies

$$G_\xi G''' - \kappa(z)G^{IV} - \kappa'G''' + \gamma_0(G_\xi' G'' - G'G_\xi'') + \left(\frac{\lambda_2 - z}{\lambda_1}\right)G_\xi'' + \gamma_2 G_\xi' = 0. \tag{4.25}$$

Reduction 4.b.1 represents a second *set* of reductions not obtainable by the Classical Lie Method and hence not obtained by SH, and hence a second set of new solutions.

Case 2: $\xi_x = 0$. Since $\xi_x = 0$ we may set $\xi = y$ without loss of generality (cf. Freedom (iii)), i.e., we assume solutions of the form

$$M(x, y, z) = \beta(x, y)G(y, z) + \alpha(x, y, z). \tag{4.26}$$

Substituting (4.26) into (2.10) we find

$$\begin{aligned} &- y^2\kappa(z)\beta G^{\mathrm{IV}} + \beta\beta_x\{GG''' + yG'G_y'' - yG_y'G''\} + y\alpha_x\beta G_y'' \\ &\quad + (\alpha_x\beta - y^2\kappa'\beta)G''' - y\alpha_x''\beta G_y' + (\alpha_x'\beta_y - \alpha_y'\beta_x)yG'' + (\alpha_y''\beta_x - \alpha_x''\beta_y)yG' \\ &\quad + \alpha'''\beta_x G + \alpha_x\alpha''' + y[\alpha_x'\alpha_y'' - \alpha_y'\alpha_x''] - y^2\kappa'\alpha''' - y^2\kappa\alpha^{\mathrm{IV}} = 0. \end{aligned} \tag{4.27}$$

There are two sub-cases to consider: (a), $\beta_x = 0$ and (b), $\beta_x = 0$ (cf. the coefficient of the terms in braces):

Case 2a: $\beta_x \neq 0$. Normalizing the first term in (4.27) against the terms in braces we find $\beta_x = \Gamma_a(y)$ ($\Gamma_a' = 0$ necessarily as $\beta' = 0$) and integrating with respect to x we obtain

$$\beta(x, y) = x\Gamma_a(y) + \beta_0(y), \tag{4.28}$$

where $\beta_0(y)$ is a function of integration, to be determined. We may write $\beta = \Gamma_a(x + \hat{\beta}_0)$, or $\beta = \beta_0(x\hat{\Gamma}_a + 1)$, where $\hat{\beta}_0$ and $\hat{\Gamma}_a$ are defined in the obvious way; hence we may set exactly one of the functions of y in the r.h.s. of (4.28) equal to 1 without loss of generality (cf. Freedom (i)), i.e., there are two further cases to consider:

Case (i): $\beta = x + \beta_0(y)$. Normalizing the coefficient of G''' we find $\alpha_x - y^2\kappa' = \Gamma_b(y, z)$ and integrating with repect to x we obtain

$$\alpha(x, y, z) = x[y^2\kappa' + \Gamma_b(y, z)] + \alpha_0(y, z), \tag{4.29}$$

where $\alpha_0(y, z)$ is a function of integration, to be determined. We may use Freedom (i) to set $\Gamma_b(y, z) = -y^2\kappa'$, i.e., $\alpha_x = 0$ without loss of generality: translating $\alpha_0(y, z) \to \alpha_0(y, z) - \beta_0(y)(y^2\kappa' + \Gamma_b(y, z))$ we may write

$$M(x, y, z) = (x + \beta_0(y))[G(y, z) - y^2\kappa' - \Gamma_b(y, z)] + \alpha_0(y, z), \tag{4.30}$$

and the result follows immediately.

Collecting results, i.e., $\beta_x = 1$ and $\alpha_x = 0$ and then (4.27) simplifies further, and normalizing the remaining coefficients and equating coefficients of like powers of x it is now straightforward to obtain the following reduction:

Reduction 4.b.2.

$$M(x,y,z) = (x + \beta_0(y))G(y,z) + \lambda_2 z^2 + \lambda_1 z + \alpha_{00}(y), \tag{4.31a}$$

where $\alpha_{00}(y)$, $\beta_0(y)$, λ_1 and λ_2 remain arbitrary, and $G(y,z)$ satisfies

$$GG''' - y^2\kappa G^{IV} + y[G'G_y'' - G_y'G''] - y^2\kappa' G''' = 0. \tag{4.31b}$$

Case (ii): $\beta = x\Gamma_a(y) + 1$. As for Case (i), normalizing the coefficient of G''' we obtain (4.29); hence, translating $\alpha_0(y,z) \rightarrow \alpha_0(y,z) - \Gamma_a^{-1}(y)[y^2\kappa' + \Gamma_b(y,z)]$ we may write

$$M(x,y,z) = (x\Gamma_a(y) + 1)\{G(y,z) - \Gamma_a^{-1}(y)[y^2\kappa' + \Gamma_b(y,z)]\} + \alpha_0(y,z) \tag{4.32}$$

so again it follows that we may set $\Gamma_b(y,z) = -y^2\kappa$, i.e., $\alpha_x = 0$, without loss of generality through Freedom (i). Once more, normalizing the remaining coefficients and equating coefficients of like powers of x it is straightforward to complete the calculation. We obtain the following reduction:

Reduction 4.b.3.

$$M(x,y,z) = (x\Gamma_a(y) + 1)G(y,z) + \lambda_2 z^2 + \lambda_1 z + \alpha_{00}(y), \tag{4.33a}$$

where $\alpha_{00}(y)$, $\Gamma_a(y)$, λ_1 and λ_2 remain arbitrary, and $G(y,z)$ satisfies

$$\Gamma_a(y)\{GG''' + y[G'G_y'' - G_y'G'']\} - y^2\kappa' G''' - y^2\kappa G^{IV} = 0. \tag{4.33b}$$

Case 2b: $\beta_x = 0$. In this case $\beta_x = \beta_z = 0$, i.e., $\beta = \beta(y)$; hence rescaling $G(y,z)$ (cf. Freedom (ii)) we may set $\beta = 1$ without loss of generality. So (4.27) becomes

$$\begin{aligned} -y^2\kappa G^{IV} + (\alpha_x - y^2\kappa')G''' + y\alpha_x' G_y'' - y\alpha_x'' G_y' \\ + \alpha_x\alpha''' + y[\alpha_x'\alpha_y'' - \alpha_y'\alpha_x''] - y^2\kappa'\alpha''' - y^2\kappa\alpha^{IV} = 0. \end{aligned} \tag{4.34}$$

Normalizing the coefficient of G''' against that of G^{IV} we find $\alpha_x - y^2\kappa' = \Gamma_a(y,z)$; hence integrating with respect to x we obtain

$$\alpha(x,y\,z) = x[y^2\kappa' + \Gamma_a(y,z)] + \alpha_0(y,z), \tag{4.35}$$

where $\alpha_0(y,z)$ is a function of integration, to be determined. Translating $G(y,z) \rightarrow G(y,z) - \alpha_0(y,z)$ (cf. Freedom (i)) we may set $\alpha_0 = 0$ without loss of generality.

Given (4.35), the coefficients of G_y'' and G_y' are functions of y and z only; hence it remains to consider just the terms not involving G in (4.34). Normalizing against the other coefficients, using (4.35) (with $\alpha_0 = 0$) and equating coefficients of like powers of x we find

$$\begin{aligned} &y^2\kappa^{IV}\Gamma_a + \Gamma_a\Gamma_a''' - y^4\kappa\kappa^{V} - y^2\kappa\Gamma_a^{IV} \\ &\qquad + y[y^2\kappa''\Gamma_{a,y}'' + 2y\kappa'''\Gamma_a' + \Gamma_a'\Gamma_{a,y}'' - 2y\kappa''\Gamma_a'' - y^2\kappa'''\Gamma_{a,y}' - \Gamma_{a,y}'\Gamma_a''] = 0 \end{aligned} \tag{4.36}$$

which determines $\Gamma_a(y, z)$.

Reduction 4.b.4. We have obtained the set of reductions given by

$$M(x, y, z) = G(y, z) + x[y^2\kappa' + \Gamma_a(y, z)], \tag{4.37a}$$

where $G(y, z)$ satisfies

$$-y^2\kappa G^{IV} + \Gamma_a(y, z)G''' + y(y^2\kappa'' + \Gamma_a')G_y'' - y(y^2\kappa''' + \Gamma_a'')G_y'\ 0, \tag{4.37b}$$

and $\Gamma_a(y, z)$ is determined by (4.36).

Reductions 4.b.2, 4.b.3 and 4.b.4 are not obtainable by using the Classical Lie Method except in special cases. In fact, the reductions used by SH to explore the structure of the thermocline (cf. Sections 6, 7 and 8, Salmon and Hollerbach, 1991) are special cases of Reduction 4.b.4.

c. Reductions admitting an arbitrary diffusion profile, II

In this section we consider the second of our two ansätze which yield reductions admitting an arbitrary diffusion profile, namely (4.3), i.e., we assume

$$M(x, y, z) = \beta(x, y)G(z) + \alpha(x, y, z). \tag{4.38}$$

None of the reductions and corresponding solutions of the thermocline system, (2.1) were found by SH. Recall that Salmon and Hollerbach computed only reductions from an equation in three independent variables, (2.2), to an equation in two independent variables. Reductions to an equation in just one independent variable, i.e., and ordinary differential equation, were not considered.

Substituting (4.38) and (2.10) we find

$$\begin{aligned} &\beta\beta_x GG''' - y^2\beta\kappa(z)G^{IV} + \beta(\alpha_x - y^2\kappa')G''' \\ &\qquad + y(\alpha_x'\beta_y - \alpha_y'\beta_x)G'' + y(\alpha_y''\beta_x - \alpha_x''\beta_y)G' + \alpha'''\beta_x G \\ &\qquad\qquad + \alpha_x\alpha''' + y(\alpha_x'\alpha_y'' - \alpha_y'\alpha_x'') - y^2\kappa'\alpha''' - y^2\kappa(z)\alpha^{IV} = 0. \end{aligned} \tag{4.39}$$

From the coefficient of GG''', we see that there are two distinct cases to consider: (1), $\beta_x = 0$ and (2), $\beta_x \neq 0$. (From the first two terms in (4.39) it is obvious that setting $\beta_y = 0 \Rightarrow \beta_x = 0$ and we then have a special case of (1).)

Case 1: $\beta_x = 0$. Normalizing against the coeffiecent of G^{IV}, we obtain the following system of determining equations:

$$-y^2\Gamma_1(z) = \alpha_x - y^2\kappa', \tag{I}$$

$$-y\beta\Gamma_2(z) = \alpha'_x\beta_y, \tag{II}$$

$$y\beta\Gamma_3(z) = \alpha''_x\beta_y, \tag{III}$$

$$-y^2\beta\Gamma_4(z) = \alpha_x\alpha''' + y(\alpha'_x\alpha''_y - a'_y\alpha''_x) - y^2\kappa'\alpha''' - y^2\kappa(z)\alpha^{IV}, \tag{IV}$$

and $G(\xi)$ satisfies

$$\kappa(z)G^{IV} + \Gamma_1(z)G''' + \Gamma_2(z)G'' + \Gamma_3(z)G' + \Gamma_4(z) = 0. \tag{4.40}$$

There are two sub-cases to consider: (a), $\beta_y = 0$ and (b), $\beta_y \neq 0$ (cf. (II) and (III)).

Case 1a: $\beta_y = 0$. In this case we may set $\beta = 1$ without loss of generality (cf. Freedom (ii)); $\Gamma_2 = \Gamma_3 = 0$, and (II) and (III) vanish. Integrating (I) with respect to x we obtain

$$\alpha(x, y, z) = xy^2(\kappa' + \Gamma_1(z)) + \alpha_0(y, z), \tag{4.41}$$

where $\alpha_0(y, z)$ is a function of integration, to be determined.

It remains to consider (IV). Substituting (4.41) into (IV) and equating coefficients of like powers of x yields the equations

$$(\kappa^{IV} + \Gamma'''_1)\Gamma_1 - (\kappa^{V} + \Gamma^{IV}_1)\kappa = 0, \tag{4.42}$$

$$(\kappa' + \Gamma_1)\alpha'''_0 + y\{(\kappa'' + \Gamma'_1)\alpha''_{0,y} - (\kappa''' + \Gamma''_1)\alpha'_{0,y}\} - \kappa'\alpha'''_0 - \kappa(z)\alpha^{IV}_0 = 0. \tag{4.43}$$

The former is an ODE for $\Gamma_1(z)$, given $\kappa(z)$ and the latter is a linear PDE for $\alpha_0(y, z)$, given $\kappa(z)$ and $\Gamma_1(z)$.

Reduction 4.c.1. Solution of (4.42) and (4.43) for $\Gamma_1(z)$ and $\alpha_0(y, z)$, respectively, yields a set of reductions where

$$M(x, y, z) = G(z) + xy^2(\kappa' + \Gamma_1(z)) + \alpha_0(y, z), \tag{4.44}$$

and

$$\kappa(z)G^{IV} + \Gamma_1(z)G''' = 0, \tag{4.45}$$

which is a first order linear ODE for G'''.

Case 1b: $\beta_y \neq 0$. We may set $\Gamma_2 = 1$ through Freedom (ii); hence differentiating (I) through with respect to z and eliminating α_{xz} between the result and (II) we find $-\beta = y\beta_y(\kappa'' - \Gamma'_1)$; separating variables yields

$$\kappa'' - \Gamma'_1 = \lambda_1, \qquad -\beta = \lambda_1 y\beta_y, \tag{4.46}$$

and integrating we obtain

$$\Gamma_1(z) = \kappa' - \lambda_1 z - \lambda_2, \qquad \beta(y) = \lambda_3 y^{-1/\lambda_1}. \tag{4.47}$$

Differentiating (I) with respect to z twice and eliminating α_{xzz} between the result and (III) we find

$$\beta\Gamma_3(z) = y\beta_y(\kappa''' - \Gamma_1''), \tag{4.48}$$

and using (4.46) we see that the r.h.s. is zero; hence the equation vanishes and $\Gamma_3 = 0$.

Next, using (4.47) in (I) we find $y^2(\lambda_1 z + \lambda_2) = \alpha_x$ and integrating with respect to x we obtain

$$\alpha(x, y, z) = xy^2(\lambda_1 z + \lambda_2) + \alpha_0(y, z), \tag{4.49}$$

where $\alpha_0(y, z)$ is a function of integration, to be determined.

It remains to consider (IV). We may set $\Gamma_4 = 0$ without loss of generality through freedom of translation in G. Then substituting (4.49) into (IV) we find

$$(\lambda_1 z + \lambda_2)\alpha_0''' + \lambda_1 y\alpha_{0,y}'' - (\kappa(z)\alpha_0''')' = 0, \tag{4.50}$$

which is a second order linear PDE for α_0''. We have obtained the following set of reductions:

Reduction 4.c.2.

$$M(x, y, z) = \lambda_3 y^{-1/\lambda_1}G(z) + xy^2(\lambda_1 z + \lambda_2) + \alpha_0(y, z), \tag{4.51}$$

where $\alpha_0(y, z)$ is determined by (4.50) and $G(z)$ satisfies

$$\kappa(z)G^{IV} + (\kappa(z) - \lambda_1 z - \lambda_2)G''' + G'' = 0, \tag{4.52}$$

which is a second order linear ODE in G''; $\kappa(z)$ remains arbitrary.

Case 2: $\beta_x \neq 0$. Normalizing against the coefficient of w^{IV}, then from (4.39) we obtain the following system of determining equations:

$$-y^2\Gamma_1(z) = \beta_x, \tag{I}$$

$$-y^2\Gamma_2(z) = \alpha_x - y^2\kappa', \tag{II}$$

$$-y^2\beta\Gamma_3(z) = \alpha'''\beta_x, \tag{III}$$

$$-y\beta\Gamma_4(z) = \alpha_x'\beta_y - \alpha_y'\beta_x, \tag{IV}$$

$$-y\beta\Gamma_5(z) = \alpha_y'\beta_x - \alpha_x'\beta_y, \tag{V}$$

$$-y^2\beta\Gamma_6(z) = \alpha_x\alpha''' + y(\alpha_x'\alpha_y'' - \alpha_y'\alpha_x'') - y^2\kappa'\alpha''' - y^2\kappa(z)\alpha^{IV}, \tag{VI}$$

and $G(\xi)$ satisfies

$$\kappa(z)G^{\text{IV}} + \Gamma_1(z)GG''' + \Gamma_2(z)G''' + \Gamma_4(z)G'' + \Gamma_5(z)G' + \Gamma_3(z)G + \Gamma_6(z) = 0. \quad (4.53)$$

Since $\beta_z = 0$ then Γ_1 is necessarily constant; we are free to rescale $G(z)$ (cf. Freedom (ii)) and so we may set $\Gamma_1 = 1$ without loss of generality. Hence integrating (I) with respect to x we obtain

$$\beta(x, y) = \beta_0(y) - xy^2, \quad (4.54)$$

where $\beta_0(y)$ is a function of integration, which remains arbitrary.

Next, integrating (II) with respect to x we obtain

$$\alpha(x, y, z) = \alpha_0(y, z) + xy^2(\kappa' - \Gamma_2(z)), \quad (4.55)$$

where $\alpha_0(y, z)$ is a function of integration, to be determined. We may use Freedom (i) to set $\alpha_x = 0$, i.e., $\Gamma_2 = \kappa'$, without loss of generality: translating $\alpha_0(y, z) \to \alpha_0(y, z) - \beta_0(y)(\Gamma_2(z) - \kappa')$ we may write

$$M(x, y, z) = (\beta_0(y) - xy^2)(G(z) + \Gamma_2(z) - \kappa') + \alpha_0(y, z), \quad (4.56)$$

and the results follows immediately.

Substituting (4.54) and (4.55), with $\Gamma_2 = \kappa'$, into (III) and equating coefficients of like powers of x we find $\Gamma_3 = 0$ and consequently $\alpha_0''' = 0$. Hence, integrating we obtain

$$\alpha_0(y, z) = \alpha_{02}(y)z^2 + \alpha_{01}(y)z + \alpha_{00}(y), \quad (4.57)$$

where $\alpha_{02}(y)$, $\alpha_{01}(y)$ and $\alpha_{00}(y)$ are functions of integration, to be determined.

It remains to consider (IV), (V) and (VI). Collecting results, then from (IV) we find $\Gamma_4 = 0$ and consequently $2z\alpha_{02,y} + \alpha_{01,y} = 0$. We deduce that $\alpha_{02,y} = \alpha_{01,y} = 0$ and it is then straightforward to show that (V) and (VI) yield only $\Gamma_5 = \Gamma_6 = 0$. We have computed one more reduction:

Reduction 4.c.3.

$$M(x, y, z) = (\beta_0(y) - xy^2)G(z) + \lambda_2 z^2 + \lambda_1 z + \alpha_{00}(y), \quad (4.58)$$

where $\beta_0(y)$ and $\alpha_{00}(y)$ remain arbitrary; $\kappa(z)$ also remains arbitrary and $G(z)$ satisfies

$$\kappa(z)G^{\text{IV}} + GG''' = 0. \quad (4.59)$$

5. Discussion

In this section we compare our results to those of Salmon and Hollerbach; the results are summarized in Table 1, for convenience. We also include some general

Table 1. Summary of results from Section 3. All parameters or functions are arbitrary unless indicated otherwise. Reduction 4.a.1 is for uniform diffusion, all others admit an arbitrary diffusion profile. The reduction used by SH to yield frontal solutions is a special case of Reduction 4.b.4.

Reduction	Transformation and Reduced Equation
4.a.1	$M = G(\xi) + \alpha(x,y,z), \qquad \xi = \dfrac{z - \gamma_{00}xy^2 - \lambda_0}{\gamma_{01}xy^2 + \lambda_1}$ $\kappa G^{IV} + (\gamma_{01}\xi + \gamma_{00})G'G''' - \Gamma_1(\xi)G''' - \Gamma_2(\xi)G'' - \Gamma_3(\xi)G' - \Gamma_4(\xi) = 0$
4.b.1	$M = \lambda_1^{-1}y^{\gamma_0}G(\xi, z) + \left(\dfrac{\gamma_2}{2\lambda_1}z^2 - \dfrac{\lambda_2}{\lambda_1}z\right)\int y^{\gamma_0 - 1}\,dy + \alpha_0(y)$ $\xi = \lambda_1 xy^{2-\gamma_0} + \phi(y)$ $G_\xi G''' - \kappa(z)G^{IV} - \kappa' G''' + \gamma_0(G'_\xi G'' - G'G''_\xi) + \left(\dfrac{\lambda_2 - z}{\lambda_1}\right)G''_\xi + \gamma_2 G'_\xi = 0$
4.b.2	$M = (x + \beta_0(y))G(y,z) + \lambda_2 z^2 + \lambda_1 z + \alpha_{00}(y)$ $GG''' - y^2\kappa G^{IV} - \kappa' G''' + y(G'G''_y - G'_yG'') = 0$
4.b.3	$M = (x\Gamma_a(y) + 1)G(y,z) + \lambda_2 z^2 + \lambda_1 z + \alpha_{00}(y)$ $\Gamma_a(y)\{GG''' - G'G''_y - G'_yG''\} - y^2\kappa' G''' - y^2\kappa' G^{IV} = 0$
4.b.4	$M = G(y,z) + x[y^2\kappa' + \Gamma_a(y,z)]$ $\Gamma_a(y,z)G''' - y^2\kappa g^{IV} + y(y^2\kappa'' + \Gamma'_a)G''_y - y(y^2\kappa''' + \gamma''_a)G'_y = 0$
4.c.1	$M = G(z) + xy^2(\kappa' + \Gamma_1(z)) + \alpha_0(y,z)$ $\kappa(z)G^{IV} + \Gamma_1(z)G''' = 0$ ($\alpha_0(y,z)$ satisfies (4.43).)
4.c.2	$M = \lambda_3 y^{-1/\lambda_1}G(z) + xy^2(\lambda_1 z + \lambda_2) + \alpha_0(y,z)$ $\kappa(z)G^{IV} + (\kappa(z) - \lambda_1 z - \lambda_2)G''' + G'' = 0$ ($\alpha_0(y,z)$ satisfies (4.50).)
4.c.3	$M = (\beta_0(y) - xy^2)G(z) + \lambda_2 z^2 + \lambda_1 z + \alpha_{00}(y)$ $\kappa(z)G^{IV} + GG''' = 0$

discussion of the rôle of similarity solutions in physical oceanography and speculate on some future work.

a. Comparison with the results of Salmon and Hollerbach

Salmon and Hollerbach (1991) use the Classical Lie Method to obtain a class of symmetries of the partial differential equation, (2.2), which with no loss of generality represents the thermocline system, (2.1), and hence determine special, exact solutions to this system. Their results include a similarity form discovered by Young and Ierley (1986), which is itself a generalization of one given by Robinson and Welander

(1963). We have generalized these results further and the results are summarized in Table 1. Firstly, we have used a more general method, the so called Direct Method, due to Clarkson and Kruskal (1989). As a demonstration of the power of the method, in Section 4.a we determined a new, large class of reductions of (2.2), viz. Reduction 4.a.1. These are for the case considered by SH, in which diffusion is spatially uniform. The reductions are to an *ordinary* differential equation and in the case $\gamma_0 = 0$ this ODE is linear, so the general solution is easily determined. Secondly we have considered classes of reductions to both partial and ordinary differential equations where diffusion is required to be an arbitrary function of z. SH considered only solutions in which diffusion is spatially uniform. Reduction 4.b.4 contains the similarity forms used by SH to infer a frontal structure of the thermocline (cf. Sections 6, 7 and 8, Salmon and Hollerbach, 1991) as special cases. In fact this reduction remains more general than that obtained by SH even after setting $\kappa' = 0$, i.e, assuming κ is uniform. The other reductions in Section 4.b, namely 4.b.1, 4.b.2 and 4.b.3, are completely new, as are those in Section 4.c.

In the companion paper (Paper II) we impose various diffusion and vertical velocity profiles which are consistent with Reduction 4.b.4 in order to carry out a set of analytic experiments. By this means we investigate the process of front formation in the density/temperature field when vertical diffusion is small. This investigation follows directly from that of SH. We also construct a ventilated model, by using Reduction 4.b.1, in which a diffusive surface layer overlies an adiabatic interior. The remaining reductions and their associated solutions are left for future analysis.

b. *The rôle of similarity solutions in physical oceanography*

We outlined previous work which obtained exact solutions of the thermocline equations in Section 2. Recently, with the exception of the work by Salmon and Hollerbach, a change of direction, away from the use of similarity solutions, seems to have taken place, led by the layered work of Luyten *et al.* (1983). However, it seems too early to dismiss the similarity approach as a useful method in physical oceanography. Pedlosky (1986, 1987) discusses similarity solutions to the thermocline equations found by Robinson and Stommel (1959), Needler (1967) and Welander (1971). His criticisms of these solutions are firstly, that the vertical structure of the solution is the same everywhere (as in the work of Robinson and Stommel, and of Welander); secondly, that the same form of solution is required to hold throughout the entire domain of flow, and also that physical understanding has not advanced very far as a result of using similarity solutions, in particular that similarity solutions do not allow us to ask the kind of physical questions we would like to. Our solutions address these criticisms. First, we do not make the assumption that the vertical structure is the same everywhere—indeed we explicitly use the fact that it is expected not to be by requiring that the diffusion, $\kappa(z)$, remains arbitrary in our reductions. Secondly, there is no reason why a similarity solution must have the same form throughout the ocean. As a simple example, consider Reduction 4.b.1: one can envisage, say, two different solutions, M_a and M_b, in which $\xi = \lambda_{1a}xy^{2-\gamma_{0a}} + \phi_a(y)$ and $\xi = \lambda_{1b}xy^{2-\gamma_{0b}} + \phi_b(y)$, respectively, holding in distinct parts of the ocean—the solution may evolve continuously from one to the other. Finally, we have explicitly constructed the method so that we *can* ask the kind of physical questions we would like to, namely, what effect do different diffusion profiles have on the thermocline (see Paper II)? In fact the *strength* of the similarity approach lies in this: one obtains local solutions and uses them to examine the physical processes in such a locality—there is no reason why a solution should be valid throughout the ocean in order to obtain useful information from it.

Some remarks on the boundary conditions that solutions found from our reductions can satisfy is in order. The method used here to determine solutions of the thermocline equations does not take account of boundary conditions in the usual way. Rather one hopes that at least one of the many solutions found will satisfy those conditions deemed necessary. In multidimensional problems, such as this, it is unlikely that solutions of this nature can satisfy the full set of conditions that one would wish to apply. However, if one wishes to investigate a particular physical process, rather than determine a global solution, then this need not be important.

c. Symmetries and conserved quantities

Finally, we speculate on some ideas for future analysis of the reductions determined here. The last 25 years have seen something of a renaissance in the study of special solutions, symmetries and conservation laws of differential equations and we might fruitfully apply these ideas.

It is possible to recover the symmetries of the thermocline equation, (2.10), to which the new solutions correspond and perhaps the conserved quantity associated with each. In addition, one might recover the conserved quantities associated with the solutions found by SH. We speculate that these quantities may have physical significance—recall that "conservation of potential vorticity" is a consequence of a symmetry of the governing equations (see Salmon, 1988, Section 4 and references contained therein)—and be useful in modeling. Salmon (1983, 1985), White (1987) and Ames *et al.* (1992) have all considered such ideas and their work indicates that symmetries can play a useful rôle in solving real problems. It is therefore reasonable to suppose that the conserved quantities of Eq. (2.10) can also play a fruitful rôle in the investigation of the properties of its solutions. Indeed, preliminary results from the author's current work on conserved quantities indicate that this is so.

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