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Vorticity balance of boundary currents

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ABSTRACT

Friction at the seafloor acts as a source of potential vorticity (PV) for individual isopycnic layers of a boundary current. The rate of PV transport (flux times layer thickness) equals, to a good approximation, the divergence of alongstream shear stress in the bottom boundary layer at the seafloor, which in turn equals the alongstream gradient of Montgomery potential. Mean PV transport is continuous along isopycnals between the bottom boundary layer and a boundary current in statistically steady state. Within the boundary current, Reynolds flux of vorticity transports PV. The divergence of this transport balances planetary vorticity advection and other terms in the vorticity equation. PV transport is equivalent to horizontal shear force, and its continuity from the seafloor to the interior of the boundary current implies that the total shear force exerted by the seafloor over the broad footprint of an isopycnic layer acts as much increased shear over the shallow depth of the same layer offshore.

A drag law of the bottom boundary layer connects shear stress at the seafloor to velocity outside the boundary layer, a similarity argument yields the functional form of the shear stress gradient-friction velocity relationship, and hence the boundary condition on PV transport from the seafloor. This is neither free-slip nor no-slip, but closer to the latter.

1. Introduction

The idea that individual isopycnic layers of the ocean move more or less independently of one another is a theoretical distillation of much observational evidence to the effect that overlying water masses of different properties move in contrasting ways (see e.g. Worthington, 1976; Reid, 1981). Rossby (1936) and Montgomery

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(1938) have already exploited the idea in discussing the circulation of the North Atlantic; recent work by Rhines and Young (1982), McDowell *et al.* (1982), and others, gave it renewed emphasis. For western boundary currents the idea implies that, to enable them to penetrate higher or lower latitudes, a mechanism must operate to counteract the advection of planetary vorticity, in each individual isopycnic layer separately. Our purpose here is to elucidate that mechanism.

First a word about "isopycnic" terminology, prompted by a referee. As we understand recent evolution of this field, "isopycnal surface" means a surface of constant potential density. The conjunction is often shortened by dropping the noun, to "isopycnal," the adjective turned noun. "Isopycnic" means something involving isopycnals, such as an isopycnic coordinate system (one using isopycnals for coordinate surfaces) or an isopycnic layer (between two isopycnals), isopycnic remaining an adjective. "Diapycnal" (-mixing, for example) denotes direction, and means across-isopycnal, "epipycnal" along-isopycnal, both always adjectives. While epipycnal is not in wide use, it seems preferable to the clumsy alternative, along-isopycnal.

Haynes and McIntyre (1987, 1990) have crystallized key ideas, contained in the classical theorems on potential vorticity of Ertel (1942) and others, in an "impermeability theorem" for isopycnal surfaces, and what might be called an "indestructibility theorem" for isopycnic potential vorticity (PV). According to these theorems, the total stock of PV within an isopycnic layer of the ocean remains constant, except for inward or outward directed flux at the intersection of the layer with the free surface or the seafloor. This makes it possible to treat vorticity balance of a single isopycnic layer independently of other layers. Boundary flux of PV into or out of a layer arises from tangential force or from diabatic heating or freshening. Exploiting Haynes and McIntyre's analysis, Marshall and Nurser (1992) have recently discussed PV fluxes into the surface outcropping of isopycnic layers of the main oceanic thermocline. They have reviewed the theorems mentioned, and summarized their physical content. Here we further exploit these theorems to explore the vorticity balance of isopycnic layers intersecting the continental slope along an ocean boundary. Boundary flux of PV into such layers comes mainly from tangential force, density flux across the seafloor being negligible, diapycnal mixing feeble.

At the intersection of an isopycnal surface and the continental slope the tangential boundary force is the divergence (vertical gradient) of bottom stress. Because velocity and acceleration vanish at a solid boundary (including Coriolis acceleration), the stress gradient must be balanced by a pressure gradient, a constraint familiar from boundary layer theory. In isopycnic layers the Montgomery potential gradient plays the role of the pressure gradient. Transport of PV into a layer intersecting the seafloor thus comes to equal the alongstream gradient of the Montgomery potential. Away from the boundary intersection, turbulent shear stress decays, and PV flux must be handled in some other way.

As satellite images vividly illustrate, western boundary currents behave very irregularly, develop large meanders, cast off and reabsorb eddies, subduct surface water, and expose water from depth. Eddy-resolving numerical models also suggest that the irregular motions, collectively known as geostrophic turbulence, transfer vorticity. Observations of the epipycnal distribution of various scalar properties (e.g. oxygen concentration, Bower *et al.*, 1985) show anomalies spreading across the flow in the manner of eddy mixing, implying Reynolds fluxes and fairly large epipycnal eddy diffusivity. It is therefore possible that Reynolds flux of PV takes over from the boundary force, and distributes the PV transport over a broader region of the boundary current. Implicit in such an idea is the postulate of statistically steady state, an isopycnic layer moving about a fixed stochastic mean position, in practice mean over a period long compared to the lifetime of meanders. The mean isopycnals intersect the seafloor in a fixed location, excluding the possibility of mean Ekman drift in the bottom mixed layer.

The "handover" relation, PV transport through the seafloor equals PV transport via Reynolds flux outside the bottom boundary layer, then serves as a boundary condition for vorticity balance over the interior portion of an isopycnic layer. Integrated across a western boundary current, boundary input of PV balances planetary vorticity advection, as well as other terms in the vorticity equation.

Continuity of PV transport between the bottom boundary layer and the interior of the boundary current implies continuity of an equivalent shear force, and absence of diapycnal Ekman transport, as already pointed out. Thus the isopycnic bottom boundary layer differs from the atmospheric ("planetary") Ekman boundary layer, in that "Ekman veering" is absent. The shear stress divergence at the seafloor should nevertheless depend on the key parameters of Ekman layers, friction velocity u^* and Coriolis parameter f. This, and a conventional drag law, then links PV transport to the velocity outside the bottom boundary layer.

To proceed from vorticity balance to velocity distribution, a gradient transport model of PV diffusion is plausible, with eddy diffusivity derived from observations of scalar property spread. The resulting equation for PV transport and diffusion contains the layer depth, however, the cross-stream distribution of which cannot be determined from the vorticity equation alone. To illustrate the consequences of PV transport continuity nevertheless, we explore the somewhat overidealized case of constant isopycnic layer depth and constant velocity of inflow from gyre interior into a western boundary current. This is equivalent to barotropic western boundary current models of constant depth oceans, discussed by Moore (1963), Ierley (1987), Cessi *et al.* (1990) and others. The boundary condition is, however, different, neither free slip nor no slip. The solutions are qualitatively similar to the no slip ones, their main message being that the classical results apply with little change to individual isopycnic layers.

2. Potential vorticity conservationa.

a. The flux form of Ertel's theorem. Conservation of potential vorticity may be expressed in the following "flux" form (Haynes and McIntyre, 1987):

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{1}$$

with the definitions:

$$q = \nabla\theta \cdot (f\mathbf{k} + \omega)$$

$$\omega = \nabla \times \mathbf{u}$$

$$\mathbf{J} = \mathbf{u}q + \nabla\theta \times \mathbf{F} + \dot{\theta}(f\mathbf{k} + \omega).$$

Here q is potential vorticity (PV), θ negative potential density anomaly (buoyancy, for short), $\dot{\theta} = d\theta/dt$, $\rho = \rho_0(1 - \theta)$, ρ_0 a deep reference density, f Coriolis parameter, **u** velocity vector, **k** vertical unit vector. **J** is PV flux vector, containing advective flux and contributions from the nonconservative force vector **F** (in kinematic units, i.e. divided by the reference density) and from density tendency $\dot{\theta}$. Our discussion here centers on the magnitude and properties of the PV flux vector on an isopycnal surface, at and near the intersection of such a surface with a solid boundary.

b. The theorems of Haynes and McIntyre. Haynes and McIntyre (1987) summarized the physical content of Eq. 1, having in mind atmospheric applications, in the following statements:

(1) There can be no net transport of PV across any isentropic surface ("impermeability theorem").

(2) PV can neither be created nor destroyed, within a layer bounded by two isentropic surfaces ("indestructibility theorem").

Haynes and McIntyre (1990) discussed these theorems further, especially the continuity of PV "transport" (flux times layer depth). Marshall and Nurser (1992) reviewed the ideas involved in an oceanographic context, isopycnals replacing isentropes. Our analysis relies on those ideas, primarily on the continuity of PV transport.

3. Isopycnic layer in contact with the continental slope

a. PV transport through the seafloor. Figure 1 illustrates the intersection with the seafloor of two adjacent isopycnal surfaces enclosing an isopycnic layer. Locally, the seafloor is a plane of small inclination, parallel to the y-axis, the x-axis pointing offshore. There is no density flux across the seafloor, so that the isopycnals intersect the seafloor at right angles. Outside a bottom boundary layer a boundary current flows toward positive or negative y, causing the isopycnals to slope up or down in the

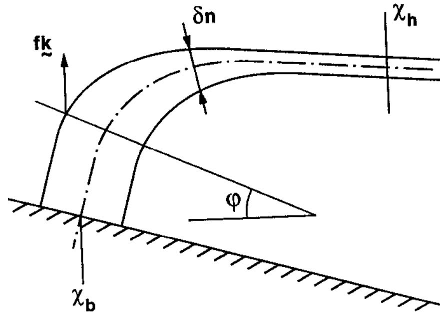

Figure 1. Stylized sketch of an isopycnic layer where it meets the seafloor. Isopycnals intersect the seafloor at right angles, merging into their interior quasi-horizontal position outside the bottom boundary layer.

x-direction, at a small angle to the horizontal. The exact shape of the isopycnals in between is unimportant for our analysis; the figure shows a smooth connection. Recent articles dealing with bottom boundary layers over sloping bottom contain similar illustrations (Garrett, 1990; McCready and Rhines, 1993).

Advective PV flux across the seafloor is nonexistent, θ due only to diapycnal mixing, contributing $f\theta$ to cross-bottom PV flux. Subject to later justification, we neglect this contribution, and suppose that $\nabla\theta \times \mathbf{F}$ provides most of the cross-bottom PV flux. The bottom buoyancy gradient $\nabla\theta$ is parallel to the seafloor, hence nearly horizontal, so that the cross-bottom PV flux is to a good approximation:

$$J_b = \frac{\partial\theta}{\partial x}\frac{\partial\tau}{\partial z}. \tag{2}$$

In a "left-bounded" current (e.g. in boundary currents of a northern subtropical gyre) both gradients in this expression are negative, in the coordinate system chosen, hence PV flux into the isopycnic layer of such a current is positive. This is a direct consequence of stable stratification and retardation of the current by friction, and is true for any layer of a left-bounded boundary current in contact with the seafloor, along the eastern, western or any other ocean margin.

PV transport (flux times width of an isopycnic layer between isopycnals θ and θ + δθ) into the isopycnic layer is now, putting $-\partial x/\partial\theta$, a positive quantity in the present coordinates, for layer width:

$$-J_b\frac{\partial x}{\partial\theta} = -\frac{\partial\tau}{\partial z}. \tag{2a}$$

At a solid boundary, where all accelerations vanish, (including the Coriolis acceleration) alongstream pressure gradient must balance the divergence of alongstream shear stress. In an isopycnic layer on the continental slope, a gradual alongstream rise (or fall) of the isopycnals makes it necessary to take into account the force of gravity, and the force balancing shear stress comes to be the gradient of the Montgomery potential. This may be shown from first principles, or from writing down equation of motion in generalized coordinates, see e.g. Dutton (1976). The result is:

$$\frac{\partial \tau}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\phi}{\rho}\right) \tag{3}$$

with ϕ the Montgomery potential, $\phi = p + \rho g z$.

b. Handover of PV flux. When an isopycnal surface moves up and down over the continental slope with the meanders of the boundary current, its bottom trace (intersection with the seafloor) moves across-stream. Boundary currents may be taken to exist, however, in a statistically steady state, with the stochastic mean position of the bottom trace in a fixed position at some $x = \bar{x}_b(\theta, y)$. This is an idealization of a quasi-steady long-term mean state, long-term standing for long compared to meander period. The mean cross-stream fluid velocity then vanishes at the seafloor and in the immediately adjacent portion of the isopycnic layer. We will take Figure 1 to show the mean position of isopycnal surfaces near the seafloor in an xz cross-section.

Consider now mean PV inputs and outputs to a control volume of the isopycnic layer indicated in Figure 1, bounded by the seafloor, two adjacent isopycnals θ and $\theta + \delta\theta$, vertical xz planes at y and $y + \delta y$, and a "handover section" or vertical yz plane at x_h, far enough seaward so that at this section both isopycnals enclosing the isopycnic layer are outside the boundary layer, and nearly horizontal. The mean PV transport into the layer through the seafloor is as in Eq. 2a, but with overbars added. The divergence of the mean PV flux $\bar{\mathbf{J}}$ vanishes according to Eq. 1, so that by Gauss's theorem mean PV fluxes across the boundaries of the control volume integrate to zero. In virtue of the impermeability theorem there is no PV flux across the isopycnal portions of the boundaries. The balance of other mean PV transports out of the control volume is:

$$-\overline{J_b \frac{\partial x}{\partial \theta}}\,\delta\theta = \overline{J_h \frac{\partial z}{\partial \theta}}\,\delta\theta + \overline{Q_y} \tag{4}$$

where J_h is PV flux through the handover section, and Q_y is the divergence of integrated alongstream PV transport:

$$Q_y = \frac{\partial}{\partial y}\int vq\,\delta n\,ds$$

with the integration extending over the curved centerline of the isopycnic layer in the xz-plane sections. The velocity v is small near the seafloor, but q may not be. From the definition of q, approximately (neglecting $\partial u/\partial y$ and $\partial w/\partial y$):

$$q = \frac{\partial \theta}{\partial n}\left(f \sin \varphi - \frac{\partial v}{\partial z}\cos \varphi + \frac{\partial v}{\partial x}\sin \varphi\right)$$

where φ is the angle of the normal to the centerline, so that $\sin \varphi = dx/ds$, $\cos \varphi = dz/dx$, and the velocity gradients exactly cancel. This leads to:

$$Q_y = \delta\theta \frac{\partial}{\partial y}\int fv\, ds.$$

Substituting into Eq. 4, also using Eq. 2a, we find for PV transport through the handover section:

$$\overline{\sigma J_h} = -\frac{\partial \overline{\tau}}{\partial z} - \frac{\partial}{\partial y}\int \overline{fv}\, ds \tag{5}$$

where we put $\sigma = dz/d\theta$ for layer thickness at the handover section. Velocities near the seafloor are small, and vary on a long alongstream scale, so that the contribution of the integral on the right is small. Instead of dropping it, however, it is more illuminating to include it on the left-hand side, by increasing J_h slightly. This may be thought of as extending the nearly-horizontal portion of the isopycnals backward toward the bottom intersection, and applying the full PV transport, $-\partial\tau/\partial z$, there.

At the handover section the friction force vanishes by hypothesis, leaving only advective PV transport, $\sigma J_h = \sigma u q = u(f + \zeta)$, with $q = (f + \zeta)/\sigma$, ζ the isopycnic relative vorticity,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

The fact that layer thickness cancels in the product σq greatly simplifies PV *transport* balances. The mean value of the PV transport at the handover section is $\overline{\sigma J_h} = \overline{u}(f + \overline{\zeta}) + \overline{u'\zeta'}$. The mean velocity $\overline{u}$ at the handover section is vanishingly small under the supposed statistically steady conditions. In this respect, the seafloor terminus of an isopycnic layer behaves like a vertical wall. It allows, however, fluctuating velocities and vorticities, hence PV transport. One must then suppose that PV transport coming from the seafloor is carried into the interior of a boundary current ("handed over" at x_h) by the Reynolds flux of vorticity, $\overline{u'\zeta'}$. Hydrodynamic instability of the boundary current is responsible for the eddy motions that make such vorticity transport possible.

Putting our results so far together, neglecting small quantities, we have for the handover transport of PV:

$$\overline{\sigma J_h} = -\frac{\partial \overline{\tau}}{\partial z} = -\overline{\frac{\partial}{\partial y}\left(\frac{\phi}{\rho}\right)} = (\overline{u'\zeta'})_h. \tag{6}$$

The result implies a horizontal force exerted on the isopycnic layer at x_h: the handover transport $\overline{J_h(\partial z/\partial\theta)}$ is equivalent to a shearing force in yz planes, or shear stress divergence,

$$-\frac{\partial\tau}{\partial x} = \overline{J_h \frac{\partial z}{\partial\theta}}.$$

It is interesting to note that while the equivalent shear force $\partial\tau/\partial x$ equals the bottom shear stress force $\partial\tau/\partial z$, the equivalent shear *stress* τ in yz planes is much larger, because the x-scale of variation is the boundary current width, a distance some 3 orders of magnitude greater than the depth of the bottom boundary layer on the seafloor. In physical terms, the broad footprint of an isopycnic layer experiences a braking force from the seafloor that, distributed over the shallow depth of the layer offshore, acts as huge shear stress. Continuity of PV transport establishes a connection between a realistically small bottom stress on the footprint and high equivalent shear stress offshore.

c. Boundary current vorticity balance. Seaward of the handover section, continuity of PV transport yields:

$$\frac{\partial}{\partial x}[\bar{u}(f+\bar{\zeta})] + \frac{\partial}{\partial y}[\bar{v}(f+\bar{\zeta})] = -\frac{\partial}{\partial x}(\overline{u'\zeta'}) - \frac{\partial}{\partial y}(\overline{v'\zeta'}) \qquad (7)$$

showing a simple balance between mean advection and Reynolds flux of vorticity ζ'. To express the bulk PV balance of a boundary current, we integrate this equation between x_h and another vertical yz plane at x_i, at the outer edge of the boundary current. There we postulate vanishing vorticity ζ and streamwise velocity v, no friction force $\boldsymbol{F}$, a small mean inflow velocity $-\bar{u}_i$, and negligible fluctuations of velocity and vorticity. The resulting PV balance is made simple again, here and in the last equation, by the cancellation of the layer depth in PV transport:

$$(\overline{u'\zeta'})_h = f\bar{u}_i + \frac{\partial}{\partial y}\int_{X_h}^{X_i}\{\bar{v}(f+\bar{\zeta}) + (\overline{v'\zeta'})\}dx. \qquad (8)$$

The terms under the integral sign include planetary vorticity advection, $\beta\int\bar{v}\,dx$. In a left-bounded boundary current the positive handover flux $(\overline{u'\zeta'})_h$ helps balance planetary vorticity advection at the western boundary of a northern subtropical gyre, while the two combine to add positive PV at an eastern boundary. Replacing $(\overline{u'\zeta'})_h$ by the alongstream pressure gradient, from Eq. 6, yields an equation that can also be derived directly from the x-component isopycnic equation of motion, integrating with respect to x across the boundary current, and differentiating with respect to y. Appropriate streamwise pressure drop, and the balancing shear force at the seafloor, are necessary corollaries of planetary vorticity advection by a boundary current in statistically steady state.

The remarkable aspect of the above results is that they apply to any isopycnic layer in contact with the seafloor, independently of any other layer. They also imply that mechanical turbulence at the seafloor maintains the stress gradient at the required strength, while geostrophic turbulence is vigorous enough to transmit PV input from the seafloor to the body of the boundary current. Two kinds of instability and chaotic motion are apparently key determinants of boundary current behavior. How exactly these two kinds of turbulent motions discharge the functions imputed to them, or what conditions or consequences their role entails, remains to be further investigated.

4. The bottom boundary layer

A pardonable mistake would be to believe that the bottom boundary layer at the seafloor-isopycnal intersection is an "ordinary" turbulent Ekman layer associated with the bottom shear stress, similar to the planetary boundary layer of the atmosphere. That this cannot be so, however, is at once clear from the vanishing mean cross-stream velocity outside the boundary layer. In a turbulent Ekman layer over an infinite plane the cross-shear component of velocity increases from zero to its "geostrophic departure" value above the boundary layer. Cross-shear Ekman transport in the boundary layer balances the boundary shear stress.

In an isopycnic layer in contact with the seafloor, by contrast, continuity of PV transport implies continuity of shear force, as we already discussed. The shear force-pressure gradient balance varies only slowly with distance from the seafloor within the layer, at first up and then seaward, as dictated by the divergence of alongstream PV transport. This bottom boundary layer behavior shows some resemblance to what MacCready and Rhines (1991, 1993) have found in a boundary layer over sloping bottom, in a stratified fluid, after the "shutdown time": diapycnal Ekman transport decays to insignificance, and boundary layer thickness increases, as without rotation. An alongstream pressure gradient, however, makes the boundary current case quite different: in MacCready and Rhines' analysis the shear stress divergence vanishes at the seafloor, alongstream pressure gradient being absent.

Although conditions imposed at the top of the isopycnic bottom boundary layer differ from those of classical turbulent Ekman layer theory, much the same similarity arguments should apply. The shear stress may be written as a drag coefficient c_d times squared velocity at the top of the isopycnic bottom boundary layer, which we take to be the alongstream velocity at the handover section, v_h. In the classical turbulent Ekman layer the "geostrophic departure" is $u_d = -Cu^*$, where $u^{*2} = c_d v_g^2$, and C a universal constant (Csanady, 1967). The shear stress force at the boundary equals fu_d, and thus equals $\partial\tau/\partial z = -fC\sqrt{c_d}v_g$. Allowing for the differences between the two cases, we take the same law to apply to the isopycnic bottom boundary layer, with v_h in place of v_g, and the constant adjusted on the basis of observation. In physical terms, we take the shear stress divergence to depend only on u^* and f, the same key parameters that determine the properties of the turbulent Ekman layer. The friction velocity derives from a drag law, and connects to velocity above the boundary layer. The latter is sustained by the geostrophic turbulence of the boundary current at a value required for steady state, i.e. for closing the vorticity balance.

Weatherly (1972, 1977) reported observations in the bottom boundary layer of the Florida Current. This boundary layer should by typical of what one finds under swift boundary currents. The velocity above the boundary layer, and the bottom shear stress, varied with strong tidal currents, and with the upward-downward movements of the isopycnal surfaces, manifested by temperature changes at the observation site. Averaged over longer periods, a mean (kinematic) stress $\tau_0/\rho = u^{*2}$ of 2×10^{-5} m^{2}s^{-2} was associated with a mean velocity above the boundary layer of $v = 0.15$ m s^{-1}, a stress 3.2 times higher with a velocity 2.6 times greater. These observations yield drag coefficients of $10^3 c_d = 0.9$ and 0.5, but do not directly shed light on the shear stress divergence, a quantity rarely reported in similar studies.

According to Eq. 3, the shear stress divergence equals the alongshore Montgomery potential gradient. An approximation to the latter, for the isopycnals in the main thermocline, should be the alongshore pressure gradient acting on the continental shelf of the South Atlantic Bight, known to be about 2.10^{-6} m s^{-1} (Lee *et al.*, 1984). For the combined constant $C\sqrt{c_d}$ this estimate, and Weatherly's observations, yield the values of 0.18 and 0.075, with C of 6.0 and 3.3 respectively. If, as seems likely, the pressure gradient is an overestimate for the isopycnals to which Weatherly's observations apply, C should be reduced.

Putting our results here together with Eq. 6, we express the handover transport of PV:

$$(\overline{u'\zeta'})_h = fC\sqrt{c_d}v_h. \tag{9}$$

Viewing this as a condition imposed at the effective lateral boundary of the isopycnic layer, it is a boundary condition somewhere between "free slip" and "no slip."

5. Boundary current structure

To explore the structure of a boundary current within an individual isopycnal layer, i.e. to solve Eq. 7, we have to connect PV transport to other flow variables. This requires further assumptions which impair the realism of our model. The results nevertheless add some insight into the role of an isopycnic layer's footprint in boundary current dynamics.

a. Gradient transport of PV. The conservation property of PV justifies a mixing length argument for two-dimensional epipycnal eddy motions, similar to what G.I. Taylor introduced many years ago for the turbulent boundary layer (1915, see Goldstein 1938). If local fluctuations of PV occur on account of a mean PV gradient $d\bar{q}/dx$, then they are of magnitude $l'd\bar{q}/dx$, with l' the random cross-stream displacement of a parcel on an isopycnal surface. The resulting Reynolds flux of PV is:

$$\overline{u'q'} = -K\frac{\partial \bar{q}}{\partial x} \tag{10}$$

where $K = \overline{u'l'}$ is an eddy mixing coefficient. Because irregular horizontal excursions of an isopycnal over the seafloor are essentially unrestricted (unlike at a vertical wall) the mixing coefficient should behave more as in free turbulent flow, than as in a wall layer: it should not vary much across the boundary current.

Alongstream velocity fluctuations in a boundary current are of the same order as cross-stream ones, but the alongstream scale of variations in $\bar{q}$ is much larger. On this boundary layer approximation we neglect $\overline{v'q'}$ in comparison with $\overline{u'q'}$. To connect mean PV, $\bar{q}$, to mean vorticity $\bar{\zeta}$, PV fluctuation to ζ', we find from the definition of q:

$$\bar{\zeta} = \bar{\sigma}\,\bar{q} - f + \overline{\sigma'q'}$$

$$\zeta' = \bar{\sigma}\,q' + \sigma'\bar{q}$$

$$\overline{u'\zeta'} = \bar{\sigma}\,\overline{u'q'} + \bar{q}\,\overline{u'\sigma'}$$

The correlation $\overline{\sigma'q'}$ is difficult to determine by observation, requiring the simultaneous recording of velocity- and density-gradient. Keyser and Rotunno (1990) analyze the physical implications of such a correlation, and suggest that $\overline{\sigma'q'}$ becomes significant only in the presence of diapycnal mass transfer. We neglect this term, and take mean PV to be $(f + \bar{\zeta})/\bar{\sigma}$. As argued elsewhere, the Reynolds volume flux $\overline{u'\sigma'}$ may play an important role in energy dissipation (Csanady, 1989). It also transports mean PV, as substitution of $\overline{u'\zeta'}$ into Eq. 7 reveals, but only from one part of the boundary current to another, the divergence of the Reynolds volume flux vanishing. We also neglect this term.

The above expression for $\overline{u'\zeta'}$, using the flux-gradient relationship of Eq. 10, now yields:

$$\overline{u'\zeta'} = -\bar{\sigma}K\frac{\partial}{\partial x}\left(\frac{f + \bar{\zeta}}{\bar{\sigma}}\right) \tag{11}$$

to be substituted into Eq. 7.

b. Quasi-barotropic boundary current model. Unfortunately, the mean depth distribution, $\bar{\sigma}(x)$, on an isopycnal layer cannot be determined independently of other layers.

Thus in the vorticity balance this remains an externally impressed variable. Hydrographic sections of the Gulf Stream do not show major cross-stream variations of isopycnic layer depth, suggesting the simple model, $\bar{\sigma}$ = constant. Such an assumption is of course overidealized, leading to a quasi-barotropic model. Advance over classical barotropic ocean circulation models lies in the more realistic treatment of boundary conditions at an isopycnal-seafloor intersection.

Given constant layer depth, and absence of diapycnal mixing, the divergence of the mean velocity vanishes, so that a streamfunction may be introduced, $v = \partial\psi/\partial x$, $u = -\partial\psi/\partial y$, $\zeta = \nabla^2\psi$, having deleted the overbars. Putting also $\partial f/\partial y = \beta$, $\partial f/\partial x = 0$, for the western boundary current of a (northern) subtropical gyre, the vorticity equation becomes:

$$K\psi_{xxxx} + \psi_y\psi_{xxx} - \psi_x\psi_{yxx} - \beta\psi_x = 0. \tag{12}$$

This equation, and a similarity solution of the form:

$$\psi = u_i y\, F(x/L)$$

where $L = (K/\beta)^{1/3}$ is a constant length scale, have been discussed in the literature on several occasions, in connection with constant-depth ocean circulation models (Moore, 1963; Ierley, 1987; Cessi *et al.*, 1990). Boundary conditions are, at infinity $u = -u_i$, or $F = 1$, all derivatives vanishing, at the coast $F = 0$, plus usually a no-slip, sometimes a free-slip condition. The conditions at infinity yield:

$$-F'''(0) = -RF'^2(0) + 1.$$

The left-hand side is nondimensional vorticity input, the right-hand side the cross-stream integral of nonlinear terms and planetary vorticity advection. $F'(0)$ is nondimensional velocity at the handover section. The new feature of our model is the boundary condition expressed by Eq. 9, replacing free slip or no slip. For the similarity solution this translates into:

$$C\sqrt{c_d}F'(0) = -E\,F'''(0)$$

where $E = K/fL^2$, Ekman number. This is a second equation connecting $F'(0)$ and $F'''(0)$, and together with the previous one leads to:

$$Ro\,P^2 + C\sqrt{c_d}\,P - E = 0$$

where $P = F'(0)$, and $Ro = u_i/fL$ is a Rossby number. Both E and Ro being small, the positive root of the last equation is approximately $P = E/C\sqrt{c_d}$. This supplies a second condition at the handover section.

A first integral of Eq. (12) is, observing conditions at infinity:

$$F''' = RF'^2 - RFF'' + F - 1 \tag{13}$$

where primes denote differentiation with respect to the argument x/L, and $R = u_i L/K$, an eddy Reynolds number.

Eq. 13 can now be integrated, starting at $x = 0$ with a trial value of $F''(0)$, which is adjusted until the boundary conditions at infinity are satisfied, that is, F asymptotically approaches unity. Except for the presence of F (which is the scaled β-term), and different constants, Eq. 13 is the Falkner-Skan equation, familiar in boundary layer theory (see Schlichting, 1960, p. 143). Ierley (1987) discussed solutions of this equation for a range of the Reynolds number (parameter λ of Ierley). Our analysis here makes this and other classical results applicable to individual isopycnic layers of a boundary current, overidealized as our model may be (yet not as extremely as a flat bottom ocean with a vertical wall for a coast).

The remaining question is, how the boundary condition of Eq. 9 affects the solution of Eq. 13.

c. Calculated example. We have carried out calculations for a latitude where $\beta = 2 \times 10^{-11}$ m^{-1} s^{-1}, taking the inflow velocity to be $u_i = 0.02$ and 0.04 m s^{-1}. For the eddy diffusivity we have used $K = 250$ m^2 s^{-1}, the value estimated by Bower *et al.* (1985) for the epipycnal diffusivity of dissolved oxygen in the Gulf Stream. The same coefficient yields a rate of epipycnal spread for salinity, also in the Gulf Stream, in agreement with observation (Csanady and Hamilton, 1988). The length scale L is then 23.2 km, the Reynolds number R 1.86 and 3.71, at the two inflow velocities, both rather larger than the limit of $\lambda = 1$ to which Ierley's investigations extend. Empirical values for eddy diffusivity are responsible for the relatively high R. The least certain quantity is the value of $C\sqrt{c_d}$, 0.075 to 0.18 according to our order of magnitude estimates above. As an extreme lowest likely value, we have taken $C\sqrt{c_d} = 0.025$. The Ekman number with the chosen parameters is $E = 0.005$, the Rossby number $Ro = 0.01$, so that the highest likely value of the nondimensional velocity at the handover section is $F'(0) = 0.2$, the lowest value very close to zero.

Integration reveals the boundary value of nondimensional vorticity $F''(0)$ to be about 0.82 at $R = 1.86$, $F'(0) = 0$, about 0.58 at the same R, but $F'(0) = 0.2$. Figure 2 shows the profiles of streamfunction, velocity and vorticity across the boundary current for these two limiting cases. There is no qualitative difference between them, nor does the higher inflow velocity we have experimented with cause significant changes. The velocity profile for the no slip case (Fig. 2a) is similar to those shown by Cessi *et al.* (1990), obtained using different parameters.

6. Discussion

The argument for the continuity of PV transport from the seafloor to the interior of the boundary current rests on the idealization that the mean position of an isopycnal trace is fixed on the continental slope, implying that the mean cross-stream velocity vanishes immediately outside the bottom boundary layer as well as at the

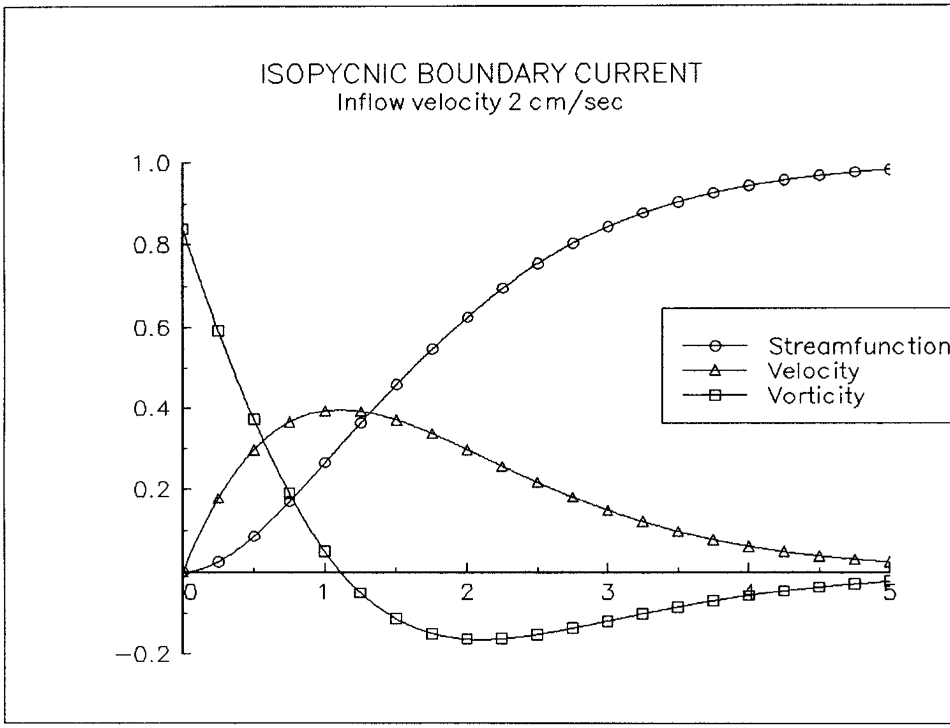

Figure 2. Distribution of streamfunction, velocity and vorticity in model isopycnic western boundary current: (*a*) $F'(0) = 0$; (*b*) $F'(0) = 0.2$.

seafloor. This comes from observation. One may then suppose that the eddy motions of the boundary current maintain the requisite eddy fluxes of PV and momentum to prevent cross-shear drift characteristic of Ekman layers. If they did not, such drift would move the mean isopycnal trace shoreward and upward, an action that might well increase the cross-stream Montgomery potential gradient, hence the speed of the current. A plausible speculation is that this would increase the eddy fluxes mentioned, and act as negative feedback.

To justify our neglect of PV transport through the seafloor due to diapycnal mixing, we note our estimates of mass exchange between surface and thermocline layers, from the Florida Straits to Cape Hatteras: about 2 sverdrups compared to flows of 20 and 40 sverdrups in the two layers (Pelegri and Csanady, 1991). The buoyancy difference is of the order of 10^{-3}, and if all of the flow changed its density by this amount, estimating mean travel time at 2.10^6 s, $\dot{\theta}$ would amount to $5 \times 10^{-10}\ \mathrm{s}^{-1}$. Given that only between 5 and 10% of the fluid changes density, and most of that in the high speed core of the current (Pelegri and Csanady, 1994), a realistic estimate is $\dot{\theta} = 10^{-11}\ \mathrm{s}^{-1}$. Thus $f\dot{\theta}$ is about $10^{-15}\ \mathrm{s}^{-2}$, versus $\partial\theta/\partial x\ \partial\tau/\partial z = 2.10^{-14}\ \mathrm{s}^{-2}$, with our

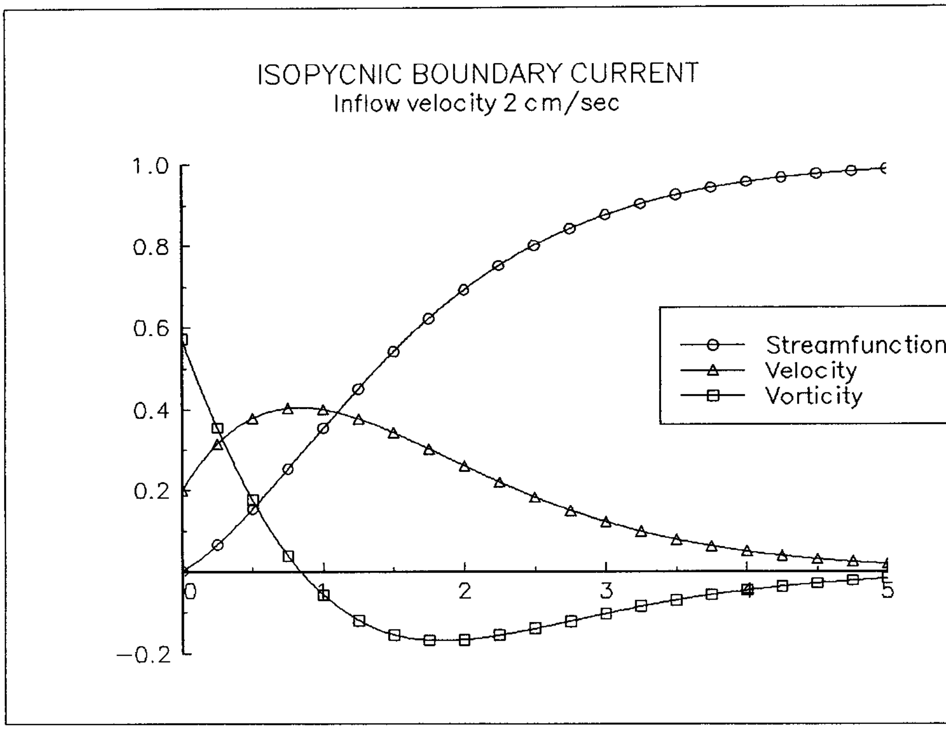

Fig. 2. (Continued)

previous estimate of $\partial\dot{\tau}/\partial z$, and $\partial\theta/\partial x = 10^{-8}\ \mathrm{m}^{-1}$, or the buoyancy difference used in the θ estimate distributed over a 100 km wide footprint.

Reviewers criticized our quasi-barotropic boundary current model on the grounds that constant layer depth is unrealistic and that the model is mathematically identical with nonlinear Munk models well explored by earlier workers. While the latter point is valid, except for the different boundary condition at the seafloor, the finding that such a model applies to an isopycnic layer makes the earlier studies that much more relevant to the real world. Constant depth is a blemish common to both models, although variations over the width of a boundary current are much greater in the depth of the entire water column than of an isopycnic layer. Also, if one layer's depth increases seaward, another one's must decrease. There must be some layers to which the constant depth model applies with tolerable approximation. In any case, some other distribution of layer depth, while undoubtedly influencing the details of velocity and vorticity distribution, does not at all affect various bulk relationships of forces and fluxes, and is therefore not likely to change the character of the boundary current within an isopycnic layer.

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