Journal of Marine Research, 49, 249–280, 1991

Similarity solutions of the thermocline equations

by Rick Salmon[1] and Rainer Hollerbach[1,2]

ABSTRACT

We apply symmetry group methods to find the group of transformations of the dependent and independent variables that leave the thermocline equations unchanged. These transformations lead to an optimal subset of sixteen forms of similarity solution. Each form obeys an equation with one fewer dependent variable than the original thermocline equations. Previously obtained similarity solutions, which are based solely upon scaling symmetries, are special cases of just three of these forms. Two of the sixteen forms lead to linear, two-dimensional, advection-diffusion equations for the temperature, Bernoulli functional or potential vorticity. Simple exact solutions contain "internal boundary layers" that resemble the thermocline in subtropical gyres.

1. Introduction

The thermocline equations (2.1) or (2.3) govern geostrophic, hydrostatic flow that advects its own mass density. Despite the severity of the approximations they embody, the thermocline equations are mathematically quite formidable, and are sometimes considered an adequate description of the large-scale, time-average ocean circulation outside frictional or inertial boundary layers. Whether adequate or not, it seems clear that the solutions of still more complicated model equations cannot be understood without a better physical understanding of the solutions to the thermocline equations.

In this paper we record all the *point symmetries* of the thermocline equations; that is, we find all transformations of the general form (3.5) that leave the thermocline equations (2.1) or (2.3) unchanged. These transformations can be used to transform solutions into other solutions by the explicit formulae given in column 3 of Table 1. More importantly, the transformations form the basis for a classification of similarity solutions to the thermocline equations. Each similarity solution obeys an equation in which the number of independent variables has been reduced from three to two. As illustrated by the calculations of Salmon (1990), a reduction from three to two space dimensions permits high-resolution numerical experiments that can be used to study

1. Scripps Institution of Oceanography A-025, La Jolla, California 92093-0225, U.S.A.
2. Present address: Department of Mathematics and Statistics, The University, Newcastle upon Tyne, NE1 7RU, England.

fronts, "internal boundary layers" in which temperature diffusion is important and flow properties change rapidly.

Our basic result is that all two-dimensional similarity solutions of the thermocline equations (of a general class defined below) can be obtained by selecting one of the sixteen similarity forms given in Table 3, where G is a function to be determined by substitution back into the thermocline equation (2.3), and then applying an arbitrary combination of the transformations given in column 3 of Table 1. In two of the sixteen cases, the resulting equations for the undetermined function G are linear, and can therefore be thoroughly analyzed. These two cases lead to simple exact solutions of the thermocline equations in which the temperature or potential vorticity changes rapidly across a front corresponding to the main thermocline in a subtropical gyre.

The previously known similarity solutions of the thermocline equations (2.3) are based upon *scaling symmetries*. These lead to the familiar type of similarity solution, in which the scalar potential $M(x, y, z)$ depends on ratios of x, y, z raised to various powers. However, symmetry group theory allows us to find all the invariant transformations of the thermocline equations, and to construct the general family of similarity solutions based upon them. In fact, only three of the sixteen similarity forms listed in Table 3 are based solely on scaling symmetries, and the similarity solutions discussed in Sections 6–8 below are not based upon scaling symmetries at all. The solutions in Sections 6–8 rely on one previously unnoticed symmetry property of the thermocline equations, and on the gauge symmetry of the M-equation (2.3).

This paper is organized as follows. In Section 2, we introduce the thermocline equations, and establish our notation. In Sections 3 and 4, we apply symmetry group methods to the thermocline equations to obtain the results summarized in Tables 1 and 3. These two sections offer a gentle introduction to symmetry group methods in terms that should appeal to readers with a background in fluid mechanics. However, readers who have no interest in group theory can skim ahead to Section 5, which begins with a summary of the results.

In Sections 6, 7, and 8 we investigate the similarity solutions that result from two of the sixteen cases in Table 3. These cases correspond to three-dimensional flows in which the temperature, Bernoulli functional, or potential vorticity obey linear, two-dimensional, advection-diffusion equations in the yz-plane. These equations are exactly hyperbolic when the temperature diffusivity κ vanishes, and exactly parabolic when $\kappa \neq 0$. (The general thermocline equations have defied any such classification.) When $\kappa = 0$, the solutions can be written in a general explicit form involving several arbitrary functions. When $\kappa \neq 0$, fronts appear where the flow along characteristics converges.

Readers who want only a sampling of results can skip from Section 2 all the way to Section 8. Section 9 summarizes our results.

2. The thermocline equations

In nondimensional form, the *thermocline equations* governing large-scale ocean flow are

$$\begin{aligned} -fv &= -\phi_x \\ +fu &= -\phi_y \\ \theta &= \phi_z \\ u_x + v_y + w_z &= 0 \\ u\theta_x + v\theta_y + w\theta_z &= \kappa\theta_{zz}. \end{aligned} \tag{2.1}$$

Here, $\mathbf{u} = (u, v, w)$ is the velocity in the (east, north, vertical) direction with coordinates (x, y, z), $f = y$ is the Coriolis parameter, ϕ is the pressure, θ the temperature, and κ is a coefficient of vertical temperature diffusivity. The equations (2.1) assume steady flow, and neglect inertia, friction, and temperature diffusion in horizontal directions. Nevertheless, (2.1) are mathematically quite formidable, and are sometimes considered an adequate description of the time-average, large-scale ocean circulation away from boundaries. For a more complete discussion of the thermocline equations, including a review of previous solutions, refer to Veronis (1969, 1973) and Pedlosky (1987). For a discussion of the possible roles played by boundary layers, see Salmon (1990).

In this paper, we set aside questions about boundaries and matching conditions, and seek solutions to (2.1) of whatever form. These solutions will contain adjustable constants and arbitrary functions that can be used to satisfy boundary and matching conditions, as appropriate. To be sure, our prejudices about boundary conditions will determine which of the similarity solutions are ultimately the most interesting. However, it does no harm to defer such matters, and to study (2.1) by themselves.

With no loss in generality, the representation

$$u = -\frac{1}{y}M_{zy}, \qquad v = \frac{1}{y}M_{zx}, \qquad w = \frac{1}{y^2}M_x, \qquad \phi = M_z, \qquad \theta = M_{zz} \tag{2.2}$$

satisfies (2.1a–d). Here $M(x, y, z)$ is a function to be determined by substitution into (2.1e). There results:

$$y[-M_{zy}M_{zzx} + M_{zx}M_{zzy}] + M_xM_{zz} = y^2\kappa M_{zzzz}. \tag{2.3a}$$

Thus (2.2) and (2.3a) are equivalent to (2.1). When $\kappa = 0$, (2.3a) reduces to the *ideal thermocline equation,*

$$y[-M_{zy}M_{zzx} + M_{zx}M_{zzy}] + M_xM_{zz} = 0. \tag{2.3b}$$

For future reference, we note that the thermocline equations (2.1) imply

$$\mathbf{u} \cdot \nabla B = z\kappa\left[\frac{B_z}{z}\right]_z \tag{2.4}$$

and

$$\mathbf{u} \cdot \nabla q = \kappa q_{zz} \tag{2.5}$$

where

$$B \equiv \phi - z\theta = M_z - zM_{zz} \tag{2.6}$$

is the Bernoulli functional, and

$$q \equiv y\theta_z = yM_{zzz} \tag{2.7}$$

is the potential vorticity.

3. Symmetry groups

Symmetry group methods are attractive because they apply to general nonlinear equations. Good basic references include the books by Bluman and Cole (1974), Olver (1986), and Bluman and Kumei (1989). In this paper, we apply symmetry group methods to obtain a general family of similarity solutions to the thermocline equation. Our explanation of the methods will be very brief, and is designed to appeal to readers with a background in fluid dynamics. For a complete and rigorous explanation, the reader should consult Chapters 2 and 3 of Olver's book. Readers who are uninterested in the methods can skim ahead to Section 5, which begins with a summary of the results.

Given (2.3) in the abstract form

$$\Delta(x, y, z, M, M_x, M_y, \ldots, M_{zzzz}) = 0 \tag{3.1}$$

we seek solutions

$$\Gamma(x, y, z, M) = 0. \tag{3.2}$$

Here Δ and Γ are ordinary functions of their respective arguments. The situation is that Δ is a given function and Γ must be found.

From a more geometric viewpoint, (3.1) is a hypersurface in a high-dimensional *jet space* with coordinates

$$x, y, z, M, M_x, M_y, \ldots, M_{zzzz}. \tag{3.3}$$

A solution (3.2) is a 3-dimensional hypersurface in the 4-dimensional *base space* with coordinates

$$x, y, z, M. \tag{3.4}$$

The strategy is to find transformations of the dependent and independent variables for which the thermocline equation is unchanged. We therefore consider transformations of the variables from "old coordinates" (x, y, z, M) to "new coordinates" (x', y', z', M'). Under certain assumptions such transformations form a group. If the group depends continuously on a parameter s, then it is called a *Lie group:*

$$\begin{aligned} x' &= f(x, y, z, M; s) \\ y' &= g(x, y, z, M; s) \\ z' &= h(x, y, z, M; s) \\ M' &= j(x, y, z, M; s). \end{aligned} \tag{3.5}$$

It is conventional to let $s = 0$ correspond to the identity element of the group. Thus

$$\begin{aligned} x &= f(x, y, z, M; 0) \\ y &= g(x, y, z, M; 0) \\ z &= h(x, y, z, M; 0) \\ M &= j(x, y, z, M; 0). \end{aligned} \tag{3.6}$$

One way to generate such a group is as the solution to equations of the form

$$\begin{aligned} \frac{dx'}{ds} &= \xi^x(x', y', z', M'), \qquad x'(0) = x \\ \frac{dy'}{ds} &= \xi^y(x', y', z', M'), \qquad y'(0) = y \\ \frac{dz'}{ds} &= \xi^z(x', y', z', M'), \qquad z'(0) = z \\ \frac{dM'}{ds} &= \xi^M(x', y', z', M'), \qquad M'(0) = M. \end{aligned} \tag{3.7}$$

It is then useful to think of (x', y', z', M') as the "location" at "time" s of a particle initially at (x, y, z, M) that moves always with the "velocity"

$$\mathbf{v} = [\xi^x(x, y, z, M), \xi^y(x, y, z, M), \xi^z(x, y, z, M), \xi^M(x, y, z, M)]. \tag{3.8}$$

The simplifying feature is that the "velocity field" (3.8) is "steady," i.e. s-independent. Thus $\mathbf{v}$ is everywhere tangent to the base-space trajectories that define the transformation.

The "velocity field" **v** determines a corresponding "velocity field"

$$pr\,\mathbf{v} = \left[\xi^x, \xi^y, \xi^z, \xi^M, \frac{d(M_x)}{ds}, \frac{d(M_y)}{ds}, \ldots, \frac{d(M_{zzzz})}{ds}\right] \tag{3.9}$$

in the jet space with coordinates (3.3). (The notation $pr\,\mathbf{v}$ stands for "prolongation of **v**," a good terminology.) The first four components of (3.9) are the same as (3.8). The remaining components of (3.9) can be expressed in terms of ξ^x, ξ^y, ξ^z, ξ^M and their derivatives. It is obvious that such expressions must exist, from the fact that the formula for the transformation of a function implicitly determines formulas for the transformations of all its derivatives. As an example, we calculate $d(M_x)/ds$.

Taylor expansions of (3.7) yield

$$\begin{aligned} x &= x' - s\xi^x(x', y', z', M') + \cdots \\ y &= y' - s\xi^y(x', y', z', M') + \cdots \\ z &= z' - s\xi^z(x', y', z', M') + \cdots \\ M' &= M + s\xi^M(x, y, z, M) + \cdots. \end{aligned} \tag{3.10}$$

Thus

$$\begin{aligned} \frac{\partial M'}{\partial x'} &= \left(\frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x'}\frac{\partial}{\partial z}\right)[M + s\xi^M(x, y, z, M)] + O(s^2) \\ &= \frac{\partial M}{\partial x} + s\{D_x\xi^M - M_xD_x\xi^x - M_yD_x\xi^y - M_zD_x\xi^z\} + O(s^2) \end{aligned} \tag{3.11}$$

where

$$D_x \equiv \frac{\partial}{\partial x} + M_x\frac{\partial}{\partial M}. \tag{3.12}$$

Therefore

$$\frac{d(M_x)}{ds} = D_x\xi^M - M_xD_x\xi^x - M_yD_x\xi^y - M_zD_x\xi^z. \tag{3.13}$$

Returning to the thermocline equation (2.3), we seek transformations of the form (3.7) for which the thermocline equation takes the same form in new variables as in old, i.e.

$$\Delta(x', y', z', M', M'_{x'}, M'_{y'}, \ldots, M'_{z'z'z'z'}) = 0 \tag{3.14}$$

where $\Delta(,,,,)$ is the same function of its arguments in (3.14) as it is in (3.1). We expand (3.14) in s,

$$\Delta(x + s\xi^x + \cdots, y + s\xi^y + \cdots, \ldots\ldots\ldots) = 0 \tag{3.15}$$

and subtract (3.1) from (3.15). Letting $s \to 0$, we obtain

$$\xi^x(x, y, z, M)\frac{\partial \Delta}{\partial x} + \xi^y(x, y, z, M)\frac{\partial \Delta}{\partial y} + \cdots + \frac{d(M_{zzzz})}{ds}\frac{\partial \Delta}{\partial M_{zzzz}} = 0. \tag{3.16}$$

By changing the definitions (3.8) and (3.9) slightly to

$$\mathbf{v} \equiv \xi^x(x, y, z, M)\frac{\partial}{\partial x} + \xi^y(x, y, z, M)\frac{\partial}{\partial y} + \xi^z(x, y, z, M)\frac{\partial}{\partial z} + \xi^M(x, y, z, M)\frac{\partial}{\partial M} \tag{3.17}$$

and

$$pr\,\mathbf{v} \equiv \mathbf{v} + \frac{d(M_x)}{ds}\frac{\partial}{\partial M_x} + \cdots + \frac{d(M_{zzzz})}{ds}\frac{\partial}{\partial M_{zzzz}} \tag{3.18}$$

we can rewrite (3.1) and (3.16) compactly as

$$(pr\,\mathbf{v})\Delta = 0 \tag{3.19a}$$

on

$$\Delta = 0. \tag{3.19b}$$

The *tangent vectors* (3.17) and (3.18) are simply the "advective derivatives" associated with the "velocity fields" (3.8) and (3.9). Eq. (3.19) just states that the jet-space "trajectory" must lie in the hypersurface (3.1) corresponding to the thermocline equation.

We have solved (3.19) to determine the components of the "velocity field" that defines the transformation. This involves equating the coefficients of monomials in the derivatives of M to zero in (3.19a) after using (3.19b) to remove one of the derivatives. There results a very large set of coupled differential equations in the functions ξ^x, ξ^y, ξ^z, $\xi^M(x, y, z, M)$. These equations, which are called the *determining system* of the transformation, are *linear* (cf. Eq. 3.13), even though the thermocline equation (2.3) is nonlinear. This is what makes the symmetry group method so attractive. For the thermocline equation, the determining system contains over one hundred differential equations. Fortunately, there now exist symbolic manipulation programs that do nearly all the work of setting up and solving these systems. We have used the program SPDE developed by Fritz Schwarz (see Schwarz, 1988) in REDUCE 3.3.

Table 1. The symmetry generators of the thermocline Eq. (2.3a). The second column contains the corresponding transformation that leaves (2.3a) unchanged. The third column gives the corresponding rule for transforming an arbitrary solution $M = F(x, y, z)$ of (2.3a) into another solution. The functions α, β and γ are arbitrary, and c is an arbitrary constant.

Generator	Finite transformation	Transformation of solutions
$\mathbf{v}_1[\alpha(y)] = \alpha(y)\partial_x$	$(x, y, z, M) \rightarrow (x + \alpha(y), y, z, M)$	$M = F(x + \alpha(y), y, z)$
$\mathbf{v}_2 = 2x\partial_x - y\partial_y$	$(x, y, z, M) \rightarrow (c^2x, y/c, z, M)$	$M = F(c^2x, y/c, z)$
$\mathbf{v}_3 = \partial_z$	$(x, y, z, M) \rightarrow (x, y, z + c, M)$	$M = F(x, y, z + c)$
$\mathbf{v}_4 = x\partial_x + z\partial_z$	$(x, y, z, M) \rightarrow (cx, y, cz, M)$	$M = F(cx, y, cz)$
$\mathbf{v}_5 = x\partial_x + M\partial_M$	$(x, y, z, M) \rightarrow (cx, y, z, cM)$	$M = F(cx, y, z)/c$
$\mathbf{v}_6 = z\partial_M$	$(x, y, z, M) \rightarrow (x, y, z, M + cz)$	$M = F(x, y, z) + cz$
$\mathbf{v}_7 = z^2\partial_M$	$(x, y, z, M) \rightarrow (x, y, z, M + cz^2)$	$M = F(x, y, z) + cz^2$
$\mathbf{v}_8[\beta(y)] = \beta(y)\partial_M$	$(x, y, z, M) \rightarrow (x, y, z, M + \beta(y))$	$M = F(x, y, z) + \beta(y)$

For the ideal thermocline equation (2.3b), the above table is unchanged, except that the first row is replaced by

$\mathbf{v}_1[\gamma(x, y)] = \gamma(x, y)\partial_x$	$(x, y, z, M) \rightarrow (\alpha(x, y), y, z, M)$	$M = F(\alpha(x, y), y, z)$

For the thermocline equation (2.3a) with $\kappa \neq 0$, the general solution of the determining system turns out to be

$$\begin{aligned} \xi^x(x, y, z, M) &= \alpha(y) + 2c_2x + c_4x + c_5x \\ \xi^y(x, y, z, M) &= -c_2y \\ \xi^z(x, y, z, M) &= c_3 + c_4z \\ \xi^M(x, y, z, M) &= c_5M + c_6z + c_7z^2 + \beta(y) \end{aligned} \tag{3.20}$$

where the c_i are arbitrary constants, and $\alpha(y)$ and $\beta(y)$ are arbitrary functions.[3] Equivalently, we can say that the thermocline equation is invariant to the transformations corresponding to an arbitrary linear combination of the 8 generators listed in the first column of Table 1. Two of these generators, $\mathbf{v}_1[\alpha]$ and $\mathbf{v}_8[\beta]$, actually correspond to infinite-dimensional vector fields, because $\alpha(y)$ and $\beta(y)$ are arbitrary functions. For each generator, Table 1 also shows the corresponding finite transformation obtained by solving (3.7), and the corresponding rule for transforming an arbitrary solution,

$$M = F(x, y, z) \tag{3.21}$$

of the thermocline equation into another solution.

3. Throughout this paper, we use α, β, and γ to denote arbitrary functions of x, y or (x, y). The arbitrary function α in any particular equation is generally unrelated to the arbitrary function α in any other equation, and similarly for β and γ.

In the case of the ideal thermocline equation (2.3b), the generators are the same as for (2.3a), *except* that the arbitrary function $\alpha(y)$ in $\mathbf{v}_1[\alpha(y)]$ is replaced by an arbitrary function $\alpha(x, y)$ of both horizontal coordinates. Thus if $\kappa = 0$, Table 1 is unchanged, except that the first row is replaced by the row set off at the bottom of Table 1.

To exemplify the way Table 1 has been constructed, consider the transformation generated by $\mathbf{v}_3 = \partial_z$. By (3.17) and (3.7), it is:

$$x' = x, \qquad y' = y, \qquad z' = z + s, \qquad M' = M. \tag{3.22}$$

Now let (3.21) be a particular solution of the thermocline equation. By (3.22), the $\mathbf{v}_3$-transform of (3.21) is

$$M' = F(x', y', z' - s). \tag{3.23}$$

But we know that, by design, the primed variables also satisfy the thermocline equation. We therefore conclude that if $F(x, y, z)$ is a solution, then so must be $F(x, y, z + c)$, for any constant c. Repeating this logic for all the generators of the thermocline equation, we obtain the results given in column 3 of Table 1.

We emphasize that Table 1 contains the complete results of using the computer program SPDE to solve the huge determining system resulting from (3.19), and that the results in Table 1 can be easily checked; it is a trivial matter to verify that each of the transformations given in column 2 of Table 1 leaves the thermocline equation unchanged. In fact, most of the transformations given in column 2 of Table 1 could be guessed from an inspection of (2.3). However, the transformation corresponding to the generator $\mathbf{v}_1$ is an important exception that proves the value of the symbolic manipulation software. For the case $\kappa = 0$, this generator leads to the important result that, if (3.21) is any solution to the ideal thermocline equation (2.3b), then

$$M = F(\alpha(x, y), y, z) \tag{3.24}$$

is also a solution, where $\alpha(x, y)$ is an *arbitrary* function of the horizontal coordinates. Again, this statement can be verified by a direct substitution into (2.3b), and the ensuing cancellations seem almost miraculous. If $\kappa \neq 0$, the corresponding statement is much weaker: If (3.21) is a solution of (2.3a), then so is

$$M = F(x + \alpha(y), y, z). \tag{3.25}$$

However, even the less general transformation (3.25) could be used to make an arbitrary solution (3.21) of the thermocline equations satisfy the condition of no-normal-depth-averaged flow at a longitudinal boundary, a strategy we follow in Section 8. These general properties of (2.3a, b) have apparently not been previously noticed.

As the preceding paragraph suggests, the ability to transform solutions into solutions is sometimes useful by itself; the transformation of even trivial solutions can yield nontrivial results. However, it is in the determination and classification of similarity solutions that symmetry group methods show their full power.

4. Similarity solutions

In Section 3 we drew an analogy between a transformation group of a differential equation and the particle trajectories in a steady flow. The generators of the group are analogous to the velocity field of the flow. Knowing the generators is equivalent to knowing the transformation group, but, as the fluid mechanical analogy would suggest, it is usually much easier to deal with the generators than it is to deal with the group.

The generators form a *Lie algebra* with mathematical properties that reflect the underlying group. The most interesting of these properties is this: If $\mathbf{v}_a$ and $\mathbf{v}_b$ are any two generators, then their *Lie bracket,* defined as

$$[\mathbf{v}_a, \mathbf{v}_b] = \mathbf{v}_a\mathbf{v}_b - \mathbf{v}_b\mathbf{v}_a \tag{4.1}$$

is a linear combination of all the generators. For example, consulting Table 1, we find that

$$[\mathbf{v}_3, \mathbf{v}_7] = 2z\partial_M = 2\mathbf{v}_6. \tag{4.2}$$

This closure property of the Lie algebra is a consequence of the correspondence between generators and transformation groups.[4]

We now turn to invariant (similarity) solutions. Let $\mathbf{v}$ be a particular generator and recall that $(pr\,\mathbf{v})$ is, by hypothesis, tangent to the "equation surface" (3.1), corresponding to the thermocline equation, in the high-dimensional jet-space. However, $\mathbf{v}$ is not necessarily tangent to an arbitrarily chosen "solution surface" (3.2). That is, $(pr\,\mathbf{v}) \cdot \Delta = 0$ but $\mathbf{v}\Gamma \neq 0$. In fact, it is the "flow" across solution surfaces that carries solutions into other solutions, as in column 3 of Table 1.

We now consider the special solutions for which $\mathbf{v}\Gamma = 0$. The generator $\mathbf{v}_3$ offers a trivial but prototypical example. $\mathbf{v}_3$ has a component in the z-direction, but no components in the x-, y-, or M-directions. Thus $\mathbf{v}_3\Gamma = 0$ only if $\Gamma = \Gamma(x, y, M)$. That is, solutions invariant to the transformation generated by $\mathbf{v}_3$ must take the form $M = G(x, y)$ and are independent of z.

For an arbitrary generator $\mathbf{v}$, the invariant solutions are found by a method that amounts to finding the special coordinates in (x, y, z, M)-space for which $\mathbf{v}$ takes the *canonical form* (3.22). This is most easily done by the method of characteristics: one

4. Briefly, the commutator (4.1) is the generator of the composite transformation consisting of an infinitesimal displacement along the trajectory corresponding to $\mathbf{v}_a$, followed by an infinitesimal displacement in the direction of $\mathbf{v}_b$, followed by backwards displacements in the directions of $\mathbf{v}_a$, and then $\mathbf{v}_b$. This composite transformation is certainly a member of the general group of transformations, and therefore its generator, (4.1), is some linear combination of the basis vectors $\mathbf{v}_i$.

determines the functions $\mu_1(x, y, z, M)$, $\mu_2(x, y, z, M)$, $\mu_3(x, y, z, M)$, called *differential invariants,* whose iso-surfaces intersect to form the trajectories of **v**. The similarity solution then takes the form $\mu_1 = G(\mu_2, \mu_3)$ where G is a function to be determined by substitution into the original differential equation.

As an example, we calculate the similarity solutions corresponding to the generator $\mathbf{v}_3 + c\mathbf{v}_6$ of the thermocline equation, with c an arbitrary constant. The characteristic equations

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{1} = \frac{dM}{cz} \tag{4.3}$$

yield the differential invariants

$$\mu_1 = x, \qquad \mu_2 = y, \qquad \mu_3 = M - \tfrac{1}{2}cz^2 \tag{4.4}$$

so that the similarity solution takes the form

$$M = G(x, y) + \tfrac{1}{2}cz^2 \tag{4.5}$$

with $G(x, y)$ left to be determined by substitution in the thermocline equation.

To study the most general similarity solution of the thermocline equation (2.3a), we must use the general generator

$$\mathbf{w} = \mathbf{v}_1[\gamma] + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 + c_6\mathbf{v}_6 + c_7\mathbf{v}_7 + \mathbf{v}_8[\mu] \tag{4.6}$$

where c_i are arbitrary constants and $\gamma(y)$, $\mu(y)$ are arbitrary functions. Unfortunately, the characteristic equations corresponding to (4.6) are very difficult to solve. However, this task can be circumvented by a procedure that forms the most powerful part of the theory.

The essential idea is very well illustrated by the example (4.3–5). To obtain (4.5) we can use a combination of $\mathbf{v}_3$ and $\mathbf{v}_6$ as above; or we can use $\mathbf{v}_3$ *by itself* to obtain (4.5) with $c = 0$, and then use the transformation property, obtained from $\mathbf{v}_7$ in the last column of Table 1, that a constant multiple of z^2 can be added to any solution. That is, we can use a more restricted generator to obtain our similarity solutions if we combine the results with the rules for transforming solutions into solutions. It can be shown, in fact, that if $c_3 \neq 0$ in (4.6), then we can take $c_6 = 0$ with no loss in generality.

The special geometrical relationship between $\mathbf{v}_3$, $\mathbf{v}_6$ and $\mathbf{v}_7$ that allows this can be explained as follows: If the trajectories tangent to $\mathbf{v}_3$ are subjected to a coordinate transformation corresponding to $\mathbf{v}_7$, then the transformed trajectories are tangent to a linear combination of $\mathbf{v}_3$ and $\mathbf{v}_6$. We can regard the trajectories as material lines in a perfect fluid with velocity $\mathbf{v}_7$. The material lines are carried along by the fluid, and the tangents to these lines are *Lie dragged* by the velocity field $\mathbf{v}_7$ in the same way as the vorticity or magnetic field vectors in a perfect fluid. In complete analogy with the vorticity or induction equations, the evolution equation for the tangent, $\mathbf{v}$, is

$$\frac{d\mathbf{v}}{ds} = [\mathbf{v}_7, \mathbf{v}], \qquad \mathbf{v}(0) = \mathbf{v}_3. \tag{4.7}$$

The Taylor-series solution of (4.7) is

$$\begin{aligned}\mathbf{v}(s) &= \mathbf{v}_3 + s[\mathbf{v}_7, \mathbf{v}_3] + \tfrac{1}{2}s^2[\mathbf{v}_7, [\mathbf{v}_7, \mathbf{v}_3]] + \cdots \\ &= \mathbf{v}_3 + s(-2\mathbf{v}_6) + \tfrac{1}{2}s^2[\mathbf{v}_7, -2\mathbf{v}_6] + \cdots \\ &= \mathbf{v}_3 - 2s\mathbf{v}_6.\end{aligned} \tag{4.8}$$

Thus, a transformation ("advection") by $\mathbf{v}_7$ drags $\mathbf{v}_3$ into $\mathbf{v}_3 - 2s\mathbf{v}_6$.

More generally let $\mathbf{v}_j$ be any generator. Then a finite transformation corresponding to $\mathbf{v}_i$, by amount s, transforms the vector field $\mathbf{v}_j$ into

$$\mathbf{v}(s) = \mathbf{v}_j + s[\mathbf{v}_i, \mathbf{v}_j] + \tfrac{1}{2}s^2[\mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j]] \tag{4.9}$$

The right-hand side of (4.9) is called the adjoint operation of $\mathbf{v}_i$ on $\mathbf{v}_j$. The adjoint operations for all generators of the thermocline equation (2.3a) are given in Table 2.

To investigate all the similarity solutions obtainable from the general generator (4.6), we first operate to eliminate as many components of (4.6) as possible. Each elimination requires an assumption about the arbitrary constants in (4.6) (typically, that a particular c_i is nonzero), and the converse of each assumption must be separately examined. The final result is an *optimal subset* of generators, each very much simpler than (4.6). All the similarity solutions obtainable from (4.6) can then be obtained from this optimum subset (whose characteristic equations are much easier to integrate) *plus* the rules from Table 1 for transforming solutions into solutions.

We begin with (4.6) and operate first with $\mathbf{v}_8[\beta(y)]$. Consulting Table 2, we obtain

$$\mathbf{w} + s[c_2 y\beta'(y) + c_5\beta(y)]\partial_M. \tag{4.10}$$

So, unless $c_2 = c_5 = 0$ we can choose $\beta(y)$ to cancel the $\mu(y)\partial_M$ in $\mathbf{w}$. Then (4.6) reduces to

$$\mathbf{w}_1 = \mathbf{v}_1[\gamma(y)] + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 + c_6\mathbf{v}_6 + c_7\mathbf{v}_7. \tag{4.11}$$

We next operate on (4.11) with $\mathbf{v}_1[\alpha(y)]$ to obtain

$$\mathbf{w}_1 + s[(2\alpha + y\alpha')c_2 + (c_4 + c_5)\alpha]\partial_x. \tag{4.12}$$

So, unless $c_2 = c_4 + c_5 = 0$ we can choose $\alpha(y)$ to cancel the $\gamma(y)\partial_x$ in $\mathbf{w}_1$. Then (4.12) reduces to

$$\mathbf{w}_2 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 + c_6\mathbf{v}_6 + c_7\mathbf{v}_7. \tag{4.13}$$

Table 2. The adjoint table for the thermocline equation (2.3a). The table gives (4.9) with $\mathbf{v}_i$ the row element and $\mathbf{v}_j$ the column element.

	$\mathbf{v}_1[\gamma(y)]$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8[\mu(y)]$
$\mathbf{v}_1[\alpha(y)]$	$\mathbf{v}_1[\gamma]$	$\mathbf{v}_2 + s\mathbf{v}_1[2\alpha + y\alpha']$	$\mathbf{v}_3$	$\mathbf{v}_4 + s\mathbf{v}_1[\alpha]$	$\mathbf{v}_5 + s\mathbf{v}_1[\alpha]$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8[\mu]$
$\mathbf{v}_2$	$\exp\{-s(2 + y\,d/dy)\}\,\mathbf{v}_1[\gamma]$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\exp\{-sy\,d/dy\}\,\mathbf{v}_8[\mu]$
$\mathbf{v}_3$	$\mathbf{v}_1[\gamma]$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4 + s\mathbf{v}_3$	$\mathbf{v}_5$	$\mathbf{v}_6 + s\mathbf{v}_8[1]$	$\mathbf{v}_7 + 2s\mathbf{v}_6 + s^2\mathbf{v}_8[1]$	$\mathbf{v}_8[\mu]$
$\mathbf{v}_4$	$e^{-s}\mathbf{v}_1[\gamma]$	$\mathbf{v}_2$	$e^{-s}\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$e^{s}\mathbf{v}_6$	$e^{2s}\mathbf{v}_7$	$\mathbf{v}_8[\mu]$
$\mathbf{v}_5$	$e^{-s}\mathbf{v}_1[\gamma]$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$e^{-s}\mathbf{v}_6$	$e^{-s}\mathbf{v}_7$	$e^{-s}\mathbf{v}_8[\mu]$
$\mathbf{v}_6$	$\mathbf{v}_1[\gamma]$	$\mathbf{v}_2$	$\mathbf{v}_3 - s\mathbf{v}_8[1]$	$\mathbf{v}_4 - s\mathbf{v}_6$	$\mathbf{v}_5 + s\mathbf{v}_6$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8[\mu]$
$\mathbf{v}_7$	$\mathbf{v}_1[\gamma]$	$\mathbf{v}_2$	$\mathbf{v}_3 - 2s\mathbf{v}_6$	$\mathbf{v}_4 - 2s\mathbf{v}_7$	$\mathbf{v}_5 + s\mathbf{v}_7$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8[\mu]$
$\mathbf{v}_8[\beta(y)]$	$\mathbf{v}_1[\gamma]$	$\mathbf{v}_2 + s\mathbf{v}_8[y\beta']$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5 + s\mathbf{v}_8[\beta]$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8[\mu]$

We next operate with $\mathbf{v}_7$ to obtain

$$\mathbf{w}_2 - 2sc_3\mathbf{v}_6 + s(c_5 - 2c_4)\mathbf{v}_7. \tag{4.14}$$

So, unless $2c_4 = c_5$ we can choose s to cancel the $\mathbf{v}_7$ in $\mathbf{w}_2$. Continuing in this way we find that, provided $c_4 \neq c_5$ and $c_4 \neq 0$ (in addition to the assumptions already made), it suffices to consider the generator

$$\mathbf{v}_4 + a\mathbf{v}_5 + b\mathbf{v}_2 \tag{4.15}$$

instead of (4.6). Here, a and b are arbitrary constants. As explained above, the rules given in Table 1 for transforming solutions into solutions compensate for the components which have been removed from (4.15).

In contrast to (4.6), the differential invariants of (4.15) are easy to calculate. The characteristic equations

$$\frac{dx}{(1 + a + 2b)x} = -\frac{dy}{by} = \frac{dz}{z} = \frac{dM}{aM} \tag{4.16}$$

yield the differential invariants

$$x^{b/(1+a+2b)}y, \qquad x^{1/(1+a+2b)}z^{-1}, \qquad Mx^{-a/(1+a+2b)} \tag{4.17}$$

so that the similarity form is

$$M = x^{a/(1+a+2b)}G\left(zx^{-1/(1+a+2b)}, yx^{b/(1+a+2b)}\right) \tag{4.18}$$

with G to be determined. Of course, (4.17) can be replaced by other combinations, e.g.

$$x^{b/(1+a+2b)}y, \qquad yz^{b}, \qquad Mz^{-a} \tag{4.19}$$

so that (4.18) can be written in many ways. The cases in which the constant factors in the denominators of (4.16) vanish must be separately examined; the cases $a = 0$ and $b = 0$ are correctly obtained from the corresponding limits on (4.18), while $1 + a + 2b = 0$ leads to the similarity form

$$M = y^{2+1/b}G(x, zy^{1/b}). \tag{4.20}$$

The consistency of (4.18) and (4.20) can be verified by direct substitution into (2.3a). When $b = 0$, (4.18) reduces to the similarity form discovered by Young and Ierley (1986), which was itself a generalization of the form given by Robinson and Welander (1963).

We now return to the exceptional cases set aside between (4.6) and (4.15). If, taking the other alternative at (4.10), we have $c_2 = c_5 = 0$ then

$$\mathbf{w} = \mathbf{v}_1[\gamma] + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_6\mathbf{v}_6 + c_7\mathbf{v}_7 + \mathbf{v}_8[\mu]. \tag{4.21}$$

Table 3. The optimal subset of generators for similarity solutions of the thermocline equation (2.3a) and the resulting forms of the similarity solution. Here, a and b are arbitrary constants, $\alpha(y)$ and $\beta(y)$ are arbitrary functions, and G is a function to be determined by substitution in the thermocline equations.

	Generator	Similarity forms
S_1	$\mathbf{v}_4 + a\mathbf{v}_5 + b\mathbf{v}_2$	$M = x^{a/(1+a+2b)}G(zx^{-1/(1+a+2b)}, yx^{b/(1+a+2b)}), M = y^{2+1/b}G(x, zy^{1/b})$
S_2	$\mathbf{v}_5 + a\mathbf{v}_2$	$M = x^{1/(1+2a)}G(yx^{a/(1+2a)}, z), M = y^2G(x, z),$
S_3	$\mathbf{v}_5 + a\mathbf{v}_2 \pm \mathbf{v}_3$	$M = x^{1/(1+2a)}G(yx^{a/(1+2a)}, ye^{\pm az}), M = y^2G(x, ye^{\pm z/2}), M = xG(y, xe^{\pm z})$
S_4	$\mathbf{v}_4 + \mathbf{v}_5 + a\mathbf{v}_2 \pm \mathbf{v}_6$	$M = zG(zx^{-1/(2+2a)}, yx^{a/(2+2a)}) \pm z \ln z, M = zG(x, z/y) \pm z \ln z$
S_5	$\mathbf{v}_4 + 2\mathbf{v}_5 + a\mathbf{v}_2 \pm \mathbf{v}_7$	$M = z^2G(zx^{-1/(3+2a)}, yx^{a/(3+2a)}) \pm z^2 \ln z, M = z^2G(x, z/y^{2/3}) \pm z^2 \ln z$
S_6	$\mathbf{v}_2$	$M = G(xy^2, z)$
S_7	$\mathbf{v}_2 \pm \mathbf{v}_3$	$M = G(xy^2, ye^{\pm z})$
S_8	$\mathbf{v}_2 \pm \mathbf{v}_6$	$M = G(xy^2, z) \pm z \ln y$
S_9	$\mathbf{v}_2 \pm \mathbf{v}_7$	$M = G(xy^2, z) \pm z^2 \ln y$
S_{10}	$\mathbf{v}_2 \pm \mathbf{v}_3 \pm \mathbf{v}_7$	$M = G(xy^2, ye^{\pm z}) \pm \frac{1}{3}z^3$ ($\pm$ taken independently)
S_{11}	$\mathbf{v}_4 + \mathbf{v}_8$	$M = \alpha(y) \ln x + G(x/z, y)$
S_{12}	$\mathbf{v}_1 + \mathbf{v}_8, \mathbf{v}_1 + \mathbf{v}_8 + \mathbf{v}_6$	$M = x\alpha(y) + xz\beta(y) + G(y, z)$
S_{13}	$\mathbf{v}_1 + \mathbf{v}_8 + \mathbf{v}_7$	$M = x\alpha(y) + xz^2\beta(y) + G(y, z)$
S_{14}	$\mathbf{v}_1 + \mathbf{v}_4 - \mathbf{v}_5$	$M = z^{-1}G(x + \alpha(y) \ln z, y)$
S_{15}	$\mathbf{v}_1 + \mathbf{v}_8 + \mathbf{v}_3$	$M = x\alpha(y) + G(x + z\beta(y), y)$
S_{16}	$\mathbf{v}_1 + \mathbf{v}_8 + \mathbf{v}_3 \pm \mathbf{v}_7$	$M = z\alpha(y) + G(x + z\beta(y), y) \pm \frac{1}{3}z^3$

Unless $c_4 = 0$ we can operate with $\mathbf{v}_1$ to remove $\mathbf{v}_1$. Then operations with $\mathbf{v}_3$, $\mathbf{v}_6$, and $\mathbf{v}_7$ remove $\mathbf{v}_3$, $\mathbf{v}_6$, $\mathbf{v}_7$ (respectively) and we are left with

$$\mathbf{v}_4 + \mu(y)\partial_M. \tag{4.22}$$

If on the other hand $c_4 = 0$, we simplify using other operations. Omitting all further details, we eventually find the optimal subset of relatively simple generators given in Table 3. Table 3 also gives the corresponding form of the similarity solution to the thermocline equation. Again, consistency may be checked by direct substitution into (2.3a).

Now, every symmetry generator of the thermocline equation (2.3a) is also a generator of the ideal thermocline equation (2.3b). Therefore all of the similarity forms in Table 3 apply to the special case $\kappa = 0$ of zero temperature diffusion. However, when $\kappa = 0$, we can use the more general set of rules for transforming solutions into solutions, in which (3.24) replaces (3.25). For example, from the second line of Table 3, we see that both (2.3a) and (2.3b) have similarity solutions of the form

$$M = xG(y, z). \tag{4.23}$$

Therefore, by (3.25), (2.3a, b) also have similarity solutions of the more general form

$$M = [x + \alpha(y)]G(y, z) \tag{4.24}$$

where $\alpha(y)$ is an arbitrary function. However, by (3.24), the ideal thermocline equation (2.3b) has similarity solutions of the even more general form

$$M = \alpha(x, y)G(y, z). \tag{4.25}$$

The ideal thermocline equation (2.3b) may have other similarity solutions that cannot be obtained from Tables 1 and 3, because the general generator of similarity solutions to (2.3b), viz.

$$\mathbf{w} = \mathbf{v}_1[\gamma(x, y)] + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 + c_6\mathbf{v}_6 + c_7\mathbf{v}_7 + \mathbf{v}_8[\mu] \tag{4.26}$$

is a generalization of (4.6). In contrast to (4.6), it is generally impossible to remove $\mathbf{v}_1$ from (4.26) by operating with the other generators. Moreover, the characteristic equations corresponding to (4.26), viz.

$$\frac{dx}{\gamma(x, y) + x[2c_2 + c_4 + c_5]} = -\frac{dy}{c_2 y} = \dots \tag{4.27}$$

cannot generally be solved, because the first equation in (4.27) is inseparable for arbitrary $\gamma(x, y)$. Two avenues remain open. First, we can make special choices for $\gamma(x, y)$ for which $\partial\gamma/\partial x \neq 0$ but (4.27) is integrable. Second, we can restrict $c_2 = 0$. Then y itself is a differential invariant and the first differential in (4.27) is exact.

If we set $c_2 = 0$, and simplify (4.26) with the help of Table 2, we obtain the generators in Table 3 with $\mathbf{v}_2$ dropped, and $\mathbf{v}_1[\gamma(x, y)]$ added to every generator. However, tedious calculations show that the similarity forms thus obtained are no more general than those obtained from Tables 1 and 3, when the general transformation rule corresponding to $\mathbf{v}_1[\alpha(x, y)]$ (i.e. the bottom row in Table 1) is included.

We have also examined the point symmetries of the thermocline equations in the form (2.1). These are not *necessarily* the same as for the M-equation (2.3), because the point symmetries of (2.1) are allowed to depend on $x, y, z, u, v, w, \phi, \theta$, (i.e. on $x, y, z, M_{yz}, M_{xz}, M_x, M_z, M_{zz}$), whereas the transformations (3.5) depended only on x, y, z, M. We find, somewhat surprisingly, that the point symmetries of (2.1) are actually a subset of the point symmetries of (2.3);[5] the former lack the gauge symmetry, $\mathbf{v}_8$, that an arbitrary function $\beta(y)$ can be added to M without changing any of the physical variables (2.2). Although the gauge symmetry corresponds to a physically irrelevant change in M, it leads to a number of *physically meaningful* similarity forms in Table 3. In fact, the similarity solutions analyzed in Sections 6–8 depend upon this gauge symmetry in an essential way.

5. This conclusion is unchanged if (2.1) are augmented by the defining equation $q = y\theta_z$ so that the transformations also depend on the potential vorticity q.

In view of all the foregoing results, and particularly those summarized in the preceding three paragraphs, we conclude that Tables 1 and 3 contain a relatively broad class of similarity solutions to the thermocline equations.

5. Summary of the results in Tables 1 and 3

Tables 1 and 3 contain a broad class of similarity solutions to the thermocline equations (2.3). In Table 3, $G(\, , \,)$ is a function to be determined by substitution into the thermocline equations, $\alpha(y)$ and $\beta(y)$ are arbitrary functions, and a and b are arbitrary constants. The consistency of all the forms in Table 3 is guaranteed. Once G is determined, then the resulting solution can be further generalized by applying any combination of the transformations given in Table 1, excluding the bottom row. If $\kappa = 0$ (the ideal thermocline equation), we may also use the bottom row of Table 1.

For example, setting $a = 0$ in the similarity form S_2, we seek solutions of (2.3a) in the form (4.23) used by Salmon (1990, Section 5). The resulting equation,

$$y[-G_{zy}G_{zz} + G_z G_{zzy}] + GG_{zzz} = y^2 \kappa G_{zzzz} \tag{5.1}$$

has one fewer independent variable than (2.3a).

Two of the similarity forms in Table 3 (S_{12} and S_{13}) lead to *linear* equations for G and can therefore be thoroughly analyzed. All the other forms lead to nonlinear equations, like (5.1). For the nonlinear equations, there are two ways to proceed. We can solve the equation numerically as did Salmon (1990), or we can seek special analytical solutions. In this paper we are interested in analytical solutions.

Special solutions may be sought by a variety of techniques, including a further similarity reduction to an ordinary differential equation. A special solution of (4.23, 5.1) is

$$M = x[\gamma(y) \exp(Cz/y) + \kappa C y] \tag{5.2}$$

where $\gamma(y)$ is an arbitrary function and C is an arbitrary constant. We obtain other solutions by applying the transformations in Table 1. For example, applying the transformation corresponding to $\mathbf{v}_1[\alpha(y)]$ (and remembering that M is arbitrary up to a function only of y), we generalize (5.2) to

$$M = x[\gamma(y) \exp(Cz/y) + \kappa C y] + \alpha(y) \exp(Cz/y) \tag{5.3}$$

where $\alpha(y)$ is another arbitrary function. If $\kappa = 0$ we can use the more general transformation corresponding to $\mathbf{v}_1[\alpha(x, y)]$ to obtain

$$M = \alpha(x, y) \exp(Cz/y) \qquad (\kappa = 0). \tag{5.4}$$

The solutions (5.2–5) are the simplest members of a general family of solutions found by Welander, Needler, Blandford and others, in which the temperature varies exponentially with depth. For a review of this earlier work, see Veronis (1969, 1973) or Pedlosky (1987). These similarity solutions have led to other, closely related solutions such as

$$M = \alpha(x,y)\exp(Cz/y) + \kappa Cxy \tag{5.5}$$

which generalizes (5.2-4). All of these solutions are, like (5.5), characterized by a balance between temperature diffusion and the advection by a depth-independent component of the vertical velocity. In the following sections, we show that there are other analytical solutions of the thermocline equations with strikingly different properties.

6. The similarity form S_{12}

We begin by analyzing the similarity form S_{12} from Table 3, the first of two similarity forms that lead to linear equations for the undetermined function G. Setting

$$\alpha(y) = y^2 w_0(y), \qquad \beta(y) = y^2[w_1(y) - w_0(y)] \tag{6.1}$$

we have

$$M = xy^2 w_0(y) + xzy^2[w_1(y) - w_0(y)] + G(y,z) \tag{6.2}$$

where $w_0(y)$ and $w_1(y)$ are arbitrary functions and G must be determined. The northward and vertical velocities are

$$v = y^{-1}M_{xz} = y[w_1(y) - w_0(y)] \tag{6.3}$$

and

$$w = y^{-2}M_x = w_0(y) + z[w_1(y) - w_0(y)]. \tag{6.4}$$

Thus $w_0(y)$ and $w_1(y)$ are the vertical velocities at $z = 0$ and $z = 1$, respectively. If we take the ocean surface to be at $z = 1$, then we can regard $w_1(y)$ as a prescribed Ekman upwelling velocity and $w_0(y)$ as the abyssal (or bottom) velocity. The eastward velocity u is nonzero, but it has no effect upon the temperature,

$$\theta = M_{zz} = G_{zz} \tag{6.5}$$

because $\theta_x = 0$. Thus the thermocline equation (2.3a) reduces to the linear equation

$$v\theta_y + w\theta_z = \kappa\theta_{zz} \tag{6.6}$$

for θ with v and w given by (6.3) and (6.4). The temperature enters (6.6) as would a passive scalar, but the thermal wind equations are satisfied because θ_x and v_z are zero; this is the special significance of the similarity form (6.2).

If $\kappa = 0$, the solution of (6.6) has θ constant along characteristic lines given by

$$\frac{dz}{dy} = \frac{w}{v} = \frac{z}{y} - \frac{w_0(y)}{yW(y)} \tag{6.7}$$

where

$$W(y) \equiv -w_1(y) + w_0(y). \tag{6.8}$$

That is,

$$\theta = C(s) \qquad (\kappa = 0) \tag{6.9}$$

where C is an arbitrary function of the characteristic coordinate

$$s = \frac{z}{y} + \int^y \frac{w_0(y')}{(y')^2 W(y')} dy'. \tag{6.10}$$

If κ is nonzero but small, we anticipate that the solution may contain boundary layers or fronts. In the simplest case w_0 and w_1 are constants, θ is independent of y, and the solution to (6.6) is simply

$$\frac{d\theta}{dz} = C_1 \exp\left\{-\frac{W(z - z_0)^2}{2\kappa}\right\} \tag{6.11}$$

where C_1 is an arbitrary constant (positive for static stability), and

$$z_0 \equiv \frac{w_0}{W} \tag{6.12}$$

is the depth at which the vertical velocity is zero. Figure 1 shows the character of this solution as $\kappa \rightarrow 0$, with C_1 and the integration constant of (6.11) chosen to satisfy boundary conditions of prescribed temperature at the top and bottom of the ocean. If $W > 0$ (converging vertical velocity) and $z = z_0$ lies within the ocean,[6] then (6.11) corresponds to two layers of constant temperature separated by a sharp front of thickness $\kappa^{1/2}$ at $z = z_0$. If $W < 0$ the temperature gradient increases in both directions from z_0. Then if the ocean lies above z_0 the vertical velocity is positive throughout the ocean and (6.11) corresponds to a boundary layer of thickness κ at the ocean surface (i.e. at the bottom of an Ekman layer with prescribed temperature.) If z_0 lies within the ocean, there are both top and bottom boundary layers. These solutions are very similar to those discussed by Salmon (1990).

For general $w_0(y)$ and $w_1(y)$, we regard θ as a function of y and the characteristic coordinate s. Then (6.6) transforms to

$$-y^3 W(y)\theta_y = \kappa\theta_{ss}. \tag{6.13}$$

6. We could regard the location of the ocean as fixed and translate the solutions vertically by an arbitrary amount, but here it is easier to keep the solutions fixed and regard the location of the ocean as arbitrary.

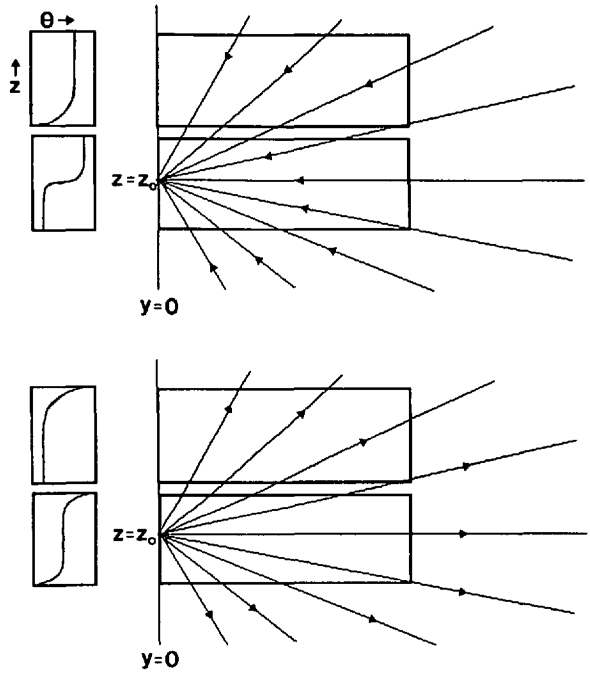

Figure 1. Schematic drawing of the solution (6.11) to (6.3, 6.4, 6.6) in the case of y-independent vertical velocity. The rays emanating from the stagnation point $(y, z) = (0, z_0)$ correspond to the projections of particle trajectories onto the yz-plane, and the arrows indicate the direction of flow. (The three-dimensional flow is nondivergent.) Superimposed rectangles represent hypothetical ocean boundaries, and the corresponding temperature profiles are shown on the left. In the case of convergence (upper diagram), a front occurs if the level z_0 lies within the ocean (lower rectangle). Otherwise (upper rectangle) there is a boundary layer where the fluid exits the domain. In the case of divergence (lower diagram), only boundary layers are present.

A further transformation to the coordinate

$$\xi \equiv -\int^{y} \frac{1}{(y')^3 W(y')} dy' \tag{6.14}$$

yields the "heat equation"

$$\theta_\xi = \kappa\theta_{ss} \tag{6.15}$$

for $\theta(\xi, s)$. The coordinate ξ increases in the direction of the flow along characteristics. Thus, according to (6.15), the temperature on each characteristic is determined by the "initial" temperature at the end where flow enters from the Ekman layer or abyss, and by the diffusion across characteristics.

In the previously considered case of constant w_0 and W, the characteristics are straight lines emanating from the point at $z = z_0$ on the equator $y = 0$. Refer again to Figure 1. The solution (6.9) is multi-valued at this point, but (6.11) shows how even small diffusion removes this singularity. The front at z_0 in (6.11) occurs because all the characteristics originating above z_0 have a uniform "initial" temperature that is higher than the temperature of those characteristics originating below z_0. In fact, the solution (6.11) corresponds to the fundamental solution,

$$\theta_s = \frac{C_1}{\sqrt{2W\xi}} \exp\left\{-\frac{s^2}{4\kappa\xi}\right\} \tag{6.16}$$

of (6.15), with

$$s = \frac{(z - z_0)}{y}, \qquad \xi = \frac{1}{2Wy^2}. \tag{6.17}$$

At the opposite extreme lies the solution

$$\theta = C_1 s = C_1 \frac{z - z_0}{y} \tag{6.18}$$

in which the boundary temperature distribution is so smooth that the diffusion is everywhere unimportant. Other solutions of (6.6) could be obtained by writing down solutions of (6.15) and then inverting the transformations (6.10) and (6.14), but the general nature of the solutions is already clear from (6.6) or (6.15).

7. The similarity form S_{13}

We next investigate the similarity form S_{13} in Table 3. We find that the problem again reduces to a linear advection-diffusion problem in two dimensions. In this case, however, the "passive" scalar quantity turns out to be the Bernoulli functional or potential vorticity.

Setting

$$M = xy^2 w_0(y) - xz^2y^2W(y) + G(y,z), \tag{7.1}$$

we have

$$\begin{aligned} u &= 4xzW(y) + 2xzyW'(y) - y^{-1}G_{yz} \\ v &= -2zyW(y) \\ w &= w_0(y) - z^2W(y) \end{aligned} \tag{7.2}$$

and

$$\theta = -2xy^2W(y) + G_{zz}. \tag{7.3}$$

The thermocline equation reduces to

$$2yW(y)G_{yz} - 2zyW(y)G_{yzz} + [w_0(y) - z^2W(y)]G_{zzz} = \kappa G_{zzzz}. \tag{7.4}$$

All three components of velocity enter (7.4), because the temperature (7.3) has gradients in all three directions. Hence (7.4) does not fit the form of an advection-diffusion equation in two dimensions. However, the equivalent equation for the Bernoulli functional does fit the advection-diffusion form because, by (7.1),

$$B = M_z - zM_{zz} = G_z - zG_{zz} \tag{7.5}$$

is independent of x. Thus (2.4) reduces to

$$vB_y + wB_z = \kappa z\left[\frac{B_z}{z}\right]_z \tag{7.6}$$

with v and w given by (7.2b, c). Eqs. (7.4) and (7.6) are equivalent. Then, since

$$B = F - zF_z = -z^2\frac{\partial}{\partial z}\left(\frac{F}{z}\right) \tag{7.7}$$

where

$$F \equiv G_z \tag{7.8}$$

is the contribution of G to the pressure, the solution to (7.6) determines the whole flow.

From the general relation

$$q = -\frac{y}{z}B_z \tag{7.9}$$

between the potential vorticity q and the Bernoulli functional, we see that $q_x = 0$ if $B_x = 0$. Thus, (2.5) reduces to

$$vq_x + wq_z = \kappa q_{zz} \tag{7.10}$$

and the potential vorticity also obeys a two-dimensional advection-diffusion equation with prescribed velocity. If q is known from (7.10), then by (7.9) we know B_z. Then B_z and (7.6) determine B_y so that B is known everywhere, and the velocity and temperature are determined as before.

It is clear that the reasoning of Section 6 again applies, with the Bernoulli functional or the potential vorticity now replacing temperature as the "passive scalar." If $\kappa = 0$, then B is constant along characteristics of constant

$$s = \frac{z^2}{y} + \int^y \frac{w_0(y')}{(y')^2W(y')}dy' \tag{7.11}$$

and the general solution of (7.6) is

$$B(y, s) = -C_1(s) \tag{7.12}$$

where $C_1(s)$ is an arbitrary function. Then, using (7.7), the general solution for $\kappa = 0$ is given by (7.2), (7.3) and

$$F = G_z = z \int_0^z \frac{1}{(z')^2} C_1 \left[\frac{(z')^2}{y} + \int^y \frac{w_0(y')}{(y')^2 W(y')} dy' \right] dz' + z C_2(y) \tag{7.13}$$

where C_1 and C_2 are arbitrary functions of their respective arguments. Moreover, since $\kappa = 0$, the factor x in (7.2–3) can be replaced by an arbitrary function $\alpha(x, y)$. The arbitrary function C_1 is determined by values of the Bernoulli functional at the "inflow" ends of characteristics. The arbitrary function C_2 corresponds to an arbitrary component of the temperature that depends only on y, and a corresponding component of eastward thermal wind. These two features produce cancelling effects in the advective derivative, $(\mathbf{u} \cdot \nabla\theta)$. It is straightforward to verify that (7.2), (7.3) and (7.13) are a solution to the ideal thermocline equations.

By (7.9), the potential vorticity is

$$q = yG_{zz} = 2C_1'(s). \tag{7.14}$$

Thus, as anticipated above, surfaces of constant potential vorticity coincide with surfaces of constant Bernoulli functional when $\kappa = 0$. By (7.12) and (7.14) the potential vorticity and Bernoulli functional are *proportional* if

$$2C_1'(s) = -aC_1(s) \tag{7.15}$$

for some constant a. That is, if

$$C_1(s) = e^{-as/2} \tag{7.16}$$

then

$$q = aB. \tag{7.17}$$

Now, within this solution, we can replace z by $z - c/a$, where c is another constant. Then since

$$B = G_z - zG_{zz} \rightarrow G_z - \left(z - \frac{c}{a}\right) G_{zz} = B + \frac{c}{a}\theta \tag{7.18}$$

(7.17) becomes

$$q = aB + c\theta \tag{7.19}$$

which is the linear relationship between potential vorticity, Bernoulli functional, and temperature postulated by Welander (1971). However, the general solution (7.13) is far less restricted.

As a simple example, suppose

$$w_0(y) \equiv 0, \; W(y) \equiv 1. \tag{7.20}$$

This example is indeed special, because, by (7.2c), the vertical velocity is everywhere of one sign, but with (7.20) the rather complicated expression (7.13) can be rewritten

$$G_z = A(z^2/y) + zC(y) \tag{7.21}$$

where now A and C are arbitrary functions. From (7.21) we obtain the solution

$$\begin{aligned} u &= 4xz + z^2y^{-3}A'(z^2/y) - zy^{-1}C'(y) \\ v &= -2zy \\ w &= -z^2 \\ \theta &= -2xy^2 + 2zy^{-1}A'(z^2/y) + C(y). \end{aligned} \tag{7.22}$$

It is easy to verify that (7.22) solve the ideal thermocline equations for arbitrary functions A and C. Again, the factor x can be replaced by an arbitrary function $\alpha(x, y)$.

We now turn to the more realistic case $\kappa \neq 0$. Thus far, our viewpoint has been that the equations determining G, such as (7.4), should be solved before we apply the transformations in Table 1 to further generalize the solutions. However, this strategy is sensible only if we intend to write down the general solution of (7.4). When $\kappa \neq 0$, this procedure is inconvenient.

If special solutions are sought, in which we wish to exercise our prejudices about boundary conditions, then it is far more efficient to generalize the similarity forms at the outset. Now it can be easily shown that, with only two exceptions, the transformations in column 3 of Table 1 add nothing to the similarity forms S_{12} and S_{13} that is not already contained in the arbitrary functions $\alpha(y)$, $\beta(y)$ and the undetermined function $G(y, z)$. The two exceptions are the shift transformation $\mathbf{v}_3$ in the case of S_{13} only, and the transformation $\mathbf{v}_1[\gamma(y)]$ in the case of both S_{12} and S_{13}. Applying the shift transformation to S_{13} we obtain

$$M = x\alpha(y) + x(z + c)^2 \beta(y) + G(y, z + c) \tag{7.23}$$

which, after suitable redefinitions of α, β, and G is

$$M = x\alpha(y) + x(z^2 - z_0z) \beta(y) + G(y, z) \tag{7.24}$$

where z_0 is a constant proportional to c. Now having incorporated into (7.24) an arbitrary shift and scaling of z, we can assume without loss of generality that our model ocean lies on

$$0 < x, y, z < 1. \tag{7.25}$$

The vertical velocity at the ocean bottom is proportional to $\alpha(y)$, which must therefore be zero. The vertical velocity at $z = 1$ is

$$w_E(y) \equiv (1 - z_0)\, y^{-2}\, \beta(y). \tag{7.26}$$

We regard (7.26) as a prescribed Ekman upwelling velocity, and seek solutions to the thermocline equation in the form (8.1).

8. Examples

We seek solutions to the thermocline equation (2.3a) in the form

$$M = xy^2 \frac{(z^2 - z_0 z)}{(1 - z_0)} w_E(y) + G(y, z) \tag{8.1}$$

where z_0 is a constant, $w_E(y)$ is an arbitrary function, and $G(y, z)$ remains to be determined. The form (8.1) satisfies the boundary conditions

$$w(x, y, 0) = 0, \ \ w(x, y, 1) = w_E(y) \tag{8.2}$$

at the ocean bottom and the base of the surface Ekman layer. The corresponding velocity and temperature fields are

$$\begin{aligned} u &= -\frac{x\,(y^2 w_E)'}{y h_0}(2z - z_0) - \frac{1}{y} G_{yz} \\ v &= \frac{y w_E(y)}{h_0}(2z - z_0) \\ w &= \frac{w_E(y)}{h_0}(z^2 - z_0 z) \\ \theta &= \frac{2xy^2 w_E(y)}{h_0} + G_{zz} \end{aligned} \tag{8.3}$$

where

$$h_0 \equiv 1 - z_0. \tag{8.4}$$

It is easily verified that (8.3) satisfy the Sverdrup and thermal wind relations; the temperature equation is satisfied if

$$\frac{w_E(y)}{h_0}[-2yG_{yz} + y(2z - z_0)G_{zzy} + (z^2 - z_0 z)G_{zzz}] = \kappa G_{zzzz} \tag{8.5}$$

which is a third-order equation for G_z. The simplest solution of (8.5) has w_E constant and $G_y = 0$. We easily find that

$$\frac{d^3G}{dz^3} = \frac{\partial\theta}{\partial z} = C_1 \exp\left\{\left(\frac{1}{3}z^3 - \frac{1}{2}z_0z^2\right)\frac{w_E}{\kappa h_0}\right\} \tag{8.6}$$

with C_1 a positive constant. If $w_E < 0$ and $0 < z_0 < 1$, then θ_z has a local maximum at $z = z_0$, within the ocean. Then, for $\kappa \rightarrow 0$, (8.6) corresponds to a front of thickness $\kappa^{1/2}$ separating regions with vertically uniform temperature. The jump in temperature across the front is proportional to the arbitrary constant C_1.

The two further integrations required to produce G_z from (8.6) contribute only a physically irrelevant constant to the temperature. Thus, for $\kappa \rightarrow 0$, we can abbreviate the solution corresponding to (8.6) as

$$G_z = (z - z_0)CH(z - z_0) \tag{8.7}$$

where C is a constant related to C_1, and H is the Heaviside function (zero for negative argument and unity for positive argument).

Eqs. (8.3) with (8.7) (and w_E a negative constant) are one solution to the thermocline equations. However, we can obtain another solution by applying the transformation corresponding to $\mathbf{v}_1$ in Table 1. (This is the only such transformation not already incorporated in the form (8.1).) That is, we can generalize (8.7) to

$$G_z = (z - z_0)CH(z - z_0) + (2z - z_0)\gamma(y) \tag{8.8}$$

where $\gamma(y)$ is an arbitrary function. This arbitrary function can be used to satisfy another of our prejudices about boundary conditions: that the depth-averaged eastward flow be zero at the eastern boundary $x = 1$. With $\gamma(y)$ so chosen, the complete solution is

$$\begin{aligned} u &= \frac{2w_E}{h_0}(2z - z_0)(1 - x) \\ v &= \frac{yw_E}{h_0}(2z - z_0) \\ w &= \frac{w_E}{h_0}(z^2 - z_0z) \\ \theta &= -\frac{2y^2w_E}{h_0}(1 - x) + CH(z - z_0) \end{aligned} \tag{8.9}$$

where, again, w_E is a negative constant.

It can easily be checked that (8.9) satisfy the thermocline equations. This solution, shown schematically in Figure 2, has interesting features that correspond to the southern part of a subtropical gyre. The vertical velocity is zero at the bottom but

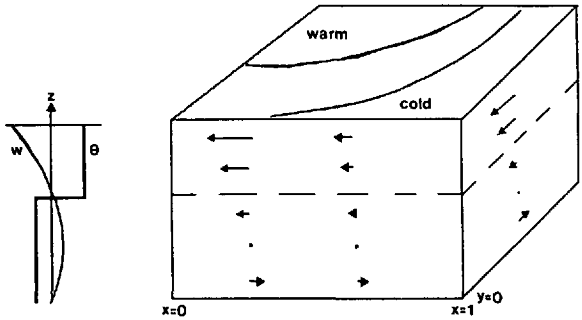

Figure 2. Schematic drawing corresponding to the solution (8.9) of the thermocline equations. A front (dashed) separates regions of vertically uniform temperature. The vertical velocity (profile at left) is negative throughout the upper layer, positive throughout the lower layer, and zero at the bottom. The horizontal velocities change sign midway through the lower layer.

positive throughout the lower layer below z_0. This deep upwelling drives a poleward flow in the bottom half of the lower layer. The southward Sverdrup flow is confined to the upper layer and to the upper half of the lower layer. The eastward velocity is negative in the upper region of southward flow, and positive below. The temperature increases to the northwest, and the horizontal velocity is along the isotherms on level surfaces, which are the same at all levels. Of course (8.9) also has unrealistic features, such as the constant frontal depth, which cannot be overcome within the framework of similarity form S_{13}. If w_E is a positive constant, then the solution corresponding to (8.6) has a minimum stratification at z_0 and a boundary layer at $z = 1$, where flow departs into the Ekman layer. This could correspond to a subpolar gyre.

For constant, w_E, another special solution of (8.5) is

$$G_z = \frac{C_1}{y}\left\{1 + \frac{2w_E}{\kappa h_0}\int_0^z dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} d\xi \exp\left[\left(\frac{1}{3}\xi^3 - \frac{1}{2}z_0\xi^2\right)\frac{w_E}{\kappa h_0}\right]\right\} + C_2\frac{z^2}{y} + \gamma(y)(2z - z_0) \quad (8.10)$$

where C_1, C_2 are arbitrary constants and $\gamma(y)$ is an arbitrary function. The solution (8.10) corresponds to a jump in potential vorticity $y\,G_{zzz}$ at z_0. Again the last term in (8.10) corresponds to the transformation generated by $\mathbf{v}_1$. For negative w_E and small κ, we can abbreviate (8.10) as

$$G_z = \frac{C}{2y}(z - z_0)^2 H(z - z_0) + C_2\frac{z^2}{y} + \gamma(y)(2z - z_0) \quad (8.11)$$

where H is the Heaviside function, and C is a constant related to C_1. According to (8.11), the stratification θ_z—but not the temperature θ—is discontinuous at z_0. We choose $C_2 = 0$ to make the stratification in the lower layer vanish, and we choose $\gamma(y)$

to make the vertical average of the eastward velocity be zero at $x = 1$. With these choices, the complete solution is

$$
\begin{aligned}
u &= \frac{2w_E}{h_0}(2z - z_0)(1 - x) + \frac{C}{2y^3}\left[(z - z_0)^2 H(z - z_0) - \frac{1}{3}h_0^2(2z - z_0)\right] \\
v &= \frac{yw_E}{h_0}(2z - z_0) \\
w &= \frac{w_E}{h_0}(z^2 - z_0 z) \\
\theta &= -\frac{2y^2 w_E}{h_0}(1 - x) + \frac{C}{y}\left[(z - z_0)H(z - z_0) - \frac{1}{3}h_0^2\right].
\end{aligned}
\tag{8.12}
$$

The arbitrary constant C is proportional to the constant potential vorticity $y\theta_z$ of the upper layer, and must be positive for static stability. Above the front at z_0, diffusion is unimportant and the temperature varies in all three directions. Thus (8.12) offers a counterexample to Salmon's (1990) conjecture that fronts always separate regions of vertically uniform temperature.

It is logical to think of the w_E-terms in (8.9) or (8.12) as the wind-driven circulation, and the C-terms as the thermohaline circulation. The thermohaline circulation adds nothing to the vertically averaged transport, but in (8.12) it induces a component of the eastward thermal wind. This thermal wind component is eastward near the surface and bottom, and westward at mid-depth. Since (8.5) is linear, the thermohaline circulations can be superposed. For example, by combining (8.9) and (8.12) we obtain a solution with discontinuities in both temperature and stratification.

We next want to relax the requirement that w_E and θ_z be independent of y. Applying the operator $y\partial_z$ to (8.5) we obtain the potential vorticity equation

$$v q_y + w q_z = \kappa q_{zz} \tag{8.13}$$

where

$$q = yG_{zzz} \tag{8.14}$$

and v and w are given by (8.3b, c). Once again, (8.13) is a linear advection-diffusion equation in two dimensions for the "passive" scalar q. Now, however, the shapes of the characteristic lines defined by

$$\frac{dz}{dy} = \frac{w}{v} = \frac{(z^2 - z_0 z)}{y(2z - z_0)} \tag{8.15}$$

are independent of $w_E(y)$. These characteristic lines are the lines of constant

$$s = \frac{(z^2 - z_0 z)}{y} \tag{8.16}$$

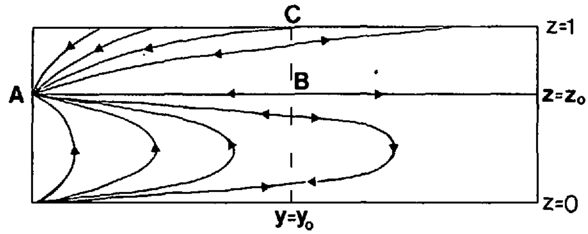

Figure 3. A north-south section showing the characteristic lines corresponding to (8.3b, c) and (8.13). These characteristic lines are streamlines for the velocity field projected onto the yz-plane. The arrows indicate the direction of the flow along characteristics when the Ekman velocity $w_E(y)$ is negative (positive) to the south (north) of y_0.

shown in Figure 3. All of the characteristics pass through a stagnation point at depth z_0 on the equator. All of the characteristics below z_0 pass through a second stagnation point at the bottom. All of the characteristics above z_0 pass into the Ekman layer.

The sign of $w_E(y)$ determines the direction of the flow along the characteristics. If w_E is negative, as assumed above, then the flow throughout the ocean converges on the stagnation point at z_0 on the equator. If $\kappa = 0$, the potential vorticity is generally multivalued at this point, but as the preceding solutions demonstrate, even a small nonzero κ removes this singularity. The solution (8.9) corresponds to a potential vorticity that is zero except for a "delta-function" at $z = z_0$. The delta-function solution represents a balance between advection by the flow converging on the equatorial stagnation point and very small diffusion across characteristics. The solution (8.12) with layers of uniform potential vorticity has a similar interpretation.

We now assume that $w_E(y)$ is negative in a subtropical gyre south of y_0 and positive to the north. We continue to assume that $0 < z_0 < 1$, to keep the possibility of a front. Then the flow along characteristics is as shown by the arrows on Figure 3. There is no flow either along or across the vertical stagnation line at $y = y_0$. Let the boundary conditions on (8.13) be prescribed θ_z at the ocean surface and bottom. It is logical to take $\theta_z = 0$ at the bottom ($z = 0$) and also at the surface ($z = 1$) in the subpolar gyre ($y > y_0$). Then, for $\kappa \to 0$, the stratification θ_z is zero throughout the region below z_0, and throughout the subpolar gyre. Above the characteristic line AC, θ_z is determined by its prescribed value at the surface: the potential vorticity $y\theta_z$ is constant along characteristics. Within the characteristic triangle ABC, $y\theta_z$ is an arbitrary function of s, the characteristic identifier, because there is no flow through, or diffusion across, the line $y = y_0$. However, this arbitrary function must be zero, or θ_{yz} (and hence u_{zz}) is infinite at $y = y_0$. Thus θ_z is zero everywhere below the characteristic line AC, and the solution does not resemble the ocean.

Our failure to obtain a two-gyre similarity solution of the form (8.1) may reflect only the inadequacy of the particular form (8.1). We speculate, however, that it represents a general failing of the thermocline equations. At the boundary between gyres, isotherms slope steeply, and friction or inertia probably become important as the flow assumes some of the character of a separated western boundary layer. Horizontal temperature diffusion may also become important. Its neglect is responsible for the arbitrariness of the flow inside the triangle ABC, but the addition of horizontal diffusion cannot, by itself, cure the deficiencies noted above. Nevertheless, solutions of the form (8.1), exemplified by (8.9) and (8.12), offer textbook examples of simple flows within individual gyres that share interesting properties with the real ocean.

9. Discussion

We have applied symmetry group methods to the oceanic thermocline equations in the form (2.3) to obtain the wide class of similarity forms summarized in Tables 3 and 1. In Table 3, G is a function of two independent variables that depend upon the Cartesian coordinates. By substituting these similarity forms back into the thermocline equation, we obtain the equation for G. The consistency of this equation (i.e. the property that only the similarity variables appear in it) is automatically guaranteed by the theory. Once G is determined, the similarity solution can be further generalized by means of any of the transformations summarized in column 3 of Table 1. Of these, the transformation corresponding to generator $\mathbf{v}_1$ is especially important. In the diffusive ($\kappa \neq 0$) case, it permits the solution to be shifted in the x-direction by an amount that depends arbitrarily on y. In the nondiffusive ($\kappa = 0$) case the allowed generalization is even greater; the x-dependence can be replaced by a dependence on an arbitrary function of x and y.

In 14 of the 16 cases in Table 3, the equation for G turns out to be nonlinear (but of a lower dimensionality than the original equation for M). These 14 cases provide the raw material for future numerical studies. In two cases, however, the G-equation is linear, and the whole dynamics reduces to a linear, two-dimensional advection-diffusion equation for the temperature or potential vorticity (respectively). In these equations, the temperature or potential vorticity behave like passive scalars, because they affect only the x-component of velocity, and there is no change in temperature or potential vorticity in the x-direction. Although it is difficult to write down the general solutions exactly, the general character of the solutions is obvious, and the special cases given in Sections 6–8 are prototypical. From a more mathematical viewpoint, these solutions are interesting because they arise from the gauge symmetry of the M-equation, and thus would not have been obtained from a symmetry group analysis of the thermocline equations in the form (2.1), in which only physical variables appear. That is, a transformation that produces no change in any physical quantity leads to similarity solutions of physical importance.

Why look for similarity solutions in the first place? Evidently Nature likes "special solutions," and the current intense interest in "coherent structures" and "patterns" in fluid mechanics is really a belated recognition of this fact. Complicated general solutions are often mosaics in which special solutions form the individual pieces. In any case, this is our view of the way in which solutions like those presented in Sections 6–8 will likely contribute to our understanding of the general ocean circulation. On this view, one cannot hope to successfully fit individual similarity solutions to the circulation of the whole ocean (or even, perhaps, of a single gyre) by adjusting the parameters available to a single solution.

Two courses of further study suggest themselves. First, one could look at similarity solutions of the time-dependent, non-diffusive form of (2.1), obtained by setting $\kappa = 0$ and adding $\partial\theta/\partial t$ to (2.1d). In these equations, fronts would correspond to developing singularities in the derivatives of temperature, and questions about stability and attracting states (which are outside the scope of the steady-state theory) could be addressed in a natural way. Second, one could look for similarity solutions of the steady, viscous form of (2.1), obtained by adding friction terms to (2.1a–c). As we speculate in Section 8, friction is likely important at the poleward edge of the subtropical gyres, and it would be interesting to seek similarity solutions corresponding to the separated western boundary current. We speculate that such solutions would have a multiple, frictional-diffusive boundary-layer structure.

After this paper was first submitted, Roger Samelson kindly drew our attention to the work of Filippov (1968), who applied symmetry group methods to the thermocline equations including horizontal diffusion. For these equations, Filippov lists five generators (not including the gauge symmetry) and gives seven specific similarity solutions. In the first pair of these, the advection terms and the diffusive terms separately cancel. The second pair are very similar to the first pair, but there is some cross-cancellation. Two of the remaining solutions satisfy our Eq. (2.1). However, none of these solutions exhibit fronts, or resemble any of the solutions we have discussed.

Acknowledgments. This work was supported by the National Science Foundation under grants OCE-8601399 and OCE-8901720. During a part of this time, RH was supported by the Kenneth W. Nelson Fellowship of Arete Associates. RS began this project while a participant in the 1988 summer program in Geophysical Fluid Dynamics at the Woods Hole Oceanographic Institution.

REFERENCES

Bluman, G. W. and J. D. Cole. 1974. Similarity Methods for Differential Equations, Springer-Verlag, 332 pp.

Bluman, G. W. and S. Kumei. 1989. Symmetries and Differential Equations, Springer-Verlag, 412 pp.

Filippov, U. G. 1968. Application of invariant-group method to solution of the problem of non-homogeneous ocean current determination. Meteorologiia i Gidrologiia, *9,* 53–62 (in Russian).

Olver, P. J. 1986. Applications of Lie Groups to Differential Equations, Springer-Verlag, 497 pp.

Pedlosky, J. 1987. Thermocline theories, *in* General Circulation of the Ocean., H. D. I. Abarbanel and W. R. Young, eds., Springer-Verlag, 291 pp.

Robinson, A. R. and P. Welander. 1963. Thermal circulation on a rotating sphere; with application to the oceanic thermocline. J. Mar. Res., *21,* 25–38.
Salmon, R. 1990. The thermocline as an "internal boundary layer." J. Mar. Res., *48,* 437–469.
Schwarz, F. 1988. Symmetries of differential equations: from Sophus Lie to computer algebra. SIAM Review, *30,* 450–481.
Veronis, G. 1969. On theoretical models of the thermohaline circulation. Deep Sea Res., *16* (Suppl.), 301–323.
—— 1973. Large scale ocean circulation. Adv. Appl. Mech., *13,* 1–92.
Welander, P. 1971. Some exact solutions to the equations describing an ideal-fluid thermocline. J. Mar. Res., *29,* 60–68.
Young, W. R. and G. R. Ierley. 1986. Eastern boundary conditions and weak solutions of the ideal thermocline equations. J. Phys. Oceanogr., *16,* 1884–1900.

Received: *8 November, 1990;* revised: *7 March, 1991.*