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Streamfunctions for the lateral velocity vector in a compressible ocean

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ABSTRACT

Streamfunctions are known in (i) geopotential surfaces, (ii) isobaric surfaces, (iii) surfaces of constant *in situ* density, ρ, and (iv) surfaces of constant steric anomaly, δ. It is desirable to map a streamfunction in a surface in which most of the mixing and movement of water-masses occurs so that the streamlines obtained in two dimensions will approximate the flow paths of the full three-dimensional flow field. These surfaces are believed to be neutral surfaces, but while a streamfunction exists in a neutral surface, we do not as yet have a closed expression for it in terms of a vertical integral of hydrographic quantities, and quite possibly we never will. An error analysis performed on the use of the Montgomery function (acceleration potential) in a neutral surface shows that the typical error at a depth of 1000 m is about 2 mm/s. To reduce the velocity error below 0.5 mm/s at 1000 m, one would need to map the Montgomery function in a surface that differed in slope from a steric anomaly surface by less than 5×10^{-6}. An error analysis is also performed on the approximate Bernoulli function that is found by integrating $gz\partial\rho_\theta/\partial z$ in the vertical, showing that errors in this Bernoulli function over a depth range of 1000 m are equivalent to a lateral velocity error of 3 mm/s. These examples demonstrate that great care must be taken in calculating a streamfunction in any surface in which an exact expression is unknown. Expressions for the relative slopes of several surfaces (surfaces of constant pressure, steric anomaly surfaces and neutral surfaces) are also derived.

1. Introduction

The geostrophic balance is expressed in terms of the horizontal pressure gradient evaluated at constant z, i.e. evaluated along a geopotential surface. That is

$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}\bigg|_z \qquad \text{and} \qquad -fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_z . \tag{1}$$

A streamfunction, ψ, is a function of x and y in a particular surface and is constant along streamlines of flow in this surface. The streamfunction that we require obeys

$$afu = -\frac{\partial \psi}{\partial y}\bigg|_{surf} \qquad \text{and} \qquad -afv = -\frac{\partial \psi}{\partial x}\bigg|_{surf}, \tag{2}$$

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where the subscript *surf* denotes that the derivative is evaluated by taking changes of ψ in the surface and $a[x, y]$ is an arbitrary function of x and y.

Let the lateral velocity vector, u to the east, v to the north, be written as, $\mathbf{V}_n$, i.e.,

$$\mathbf{V}_n = u\mathbf{i} + v\mathbf{j}. \quad (3)$$

If the height of a general surface is $Z[x, y]$, then the lateral gradient operator in this surface is defined in terms of the corresponding gradient in a geopotential surface by

$$\nabla_{surf} = \left.\frac{\partial}{\partial x}\right|_{surf}\mathbf{i} + \left.\frac{\partial}{\partial y}\right|_{surf}\mathbf{j}, \quad (4)$$

where

$$\left.\frac{\partial}{\partial x}\right|_{surf} = \left.\frac{\partial}{\partial x}\right|_{z} + Z_x\frac{\partial}{\partial z} \quad \text{and} \quad \left.\frac{\partial}{\partial y}\right|_{surf} = \left.\frac{\partial}{\partial y}\right|_{z} + Z_y\frac{\partial}{\partial z}. \quad (5)$$

A streamfunction exists for a particular surface if and only if the lateral divergence of $af\mathbf{V}_n$ vanishes in that surface, that is, if

$$\nabla_{surf} \cdot [af\mathbf{V}_n] = 0. \quad (6)$$

To date, streamfunctions have been found useful only when a is either constant or is equal to ρ. Appendix A contains a condensed proof (from McDougall, 1988) that a streamfunction exists for the geostrophic flow in a neutral surface. Following Haynes and McIntyre (1987), the form of potential vorticity conservation, (6), can be extended to include the relative vorticity, ζ, so that $\nabla_n \cdot [(f + \zeta)\mathbf{V}_n] = 0$.

It can be shown (see McDougall, 1988) that when a is a constant, the divergence, (6), is zero for any surface that contains the line $\nabla p \times \nabla\rho$ (see Fig. 1). That is, the normal to the surface in question must be perpendicular to $\nabla p \times \nabla\rho$. This condition is satisfied by an isobaric surface, a surface of constant *in situ* density, ρ, and by a neutral surface (McDougall, 1988), since the normal to the neutral surface is simply $\rho^{-1}\nabla\rho - \gamma\nabla p$, where γ is the adiabatic and isentropic compressibility of seawater. Also, the normal to a surface of constant steric anomaly, $\delta = \rho^{-1} - (\rho[35, 0, p])^{-1}$, is a linear combination of ∇p and $\nabla\rho$, and so a streamfunction also exists in the steric anomaly surface. That is, we know that

$$\begin{aligned} &\nabla_p \cdot [f\mathbf{V}_n] = 0, \qquad \nabla_\rho \cdot [f\mathbf{V}_n] = 0, \\ &\nabla_n \cdot [f\mathbf{V}_n] = 0 \quad \text{and} \quad \nabla_\delta \cdot [f\mathbf{V}_n] = 0. \end{aligned} \quad (7)$$

However, the same is not true of a potential density surface; rather, the corresponding lateral divergence is

$$\nabla_\sigma \cdot [f\mathbf{V}_n] = \nabla_n p \times \nabla_n\theta \cdot \mathbf{k}[1 - R_\rho^{-1}]\frac{\alpha[\mu - 1]}{f\rho} \neq 0. \quad (8)$$

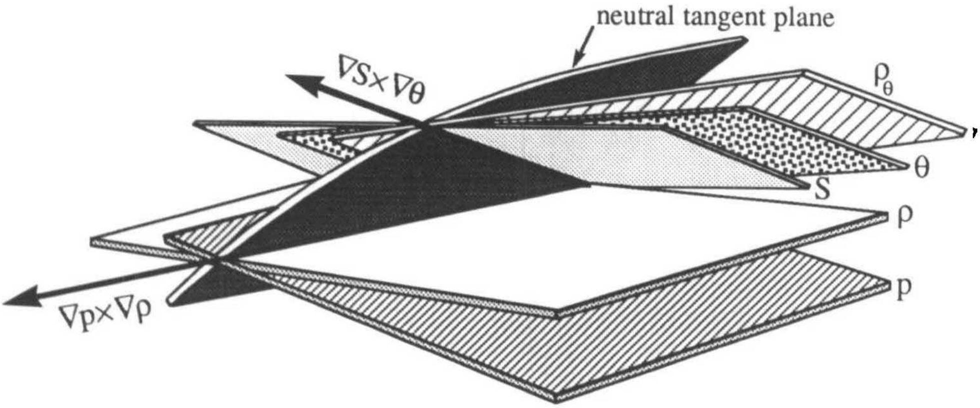

Figure 1. A three-dimensional perspective of the intersection of six surfaces (an isobaric surface, a surface of constant *in situ* density, a surface of constant potential temperature, an isohaline surface, a potential density surface, and a neutral surface). The geopotential surface is not shown but is close to the isobaric surface, although the line $\nabla p \times \nabla \rho$ does not quite lie in a geopotential surface. The surface of constant steric anomaly, δ, is not shown, but it does include the line $\nabla p \times \nabla \rho$, as does the surface of constant p/ρ. Note that $\nabla p \times \nabla \rho$ does not lie in a potential density surface.

Here $\nabla_n p$ and $\nabla_n \theta$ are the gradients of pressure and potential temperature in a neutral surface, α is the thermal expansion coefficient of sea-water (and β is the saline contraction coefficient), and R_ρ and μ are defined by

$$R_\rho = \alpha\theta_z/\beta S_z \qquad \text{and} \qquad \mu = \frac{c[R_\rho - 1]}{[R_\rho - c]}, \qquad \text{where } c = \frac{\alpha(p)/\beta(p)}{\alpha(p_r)/\beta(p_r)}. \tag{9}$$

Further details may be found in McDougall (1988). The point here is that since the right-hand side of (8) is nonzero, the streamfunction for the lateral velocity field in a potential density surface may not be simple to interpret. That is, one would need to know both $a[x, y]$ and $\psi[x, y]$ to evaluate u and v. If one defines a potential vorticity variable as f divided by the height between successive potential density surfaces, the right-hand side of (8) contributes a term of magnitude $\beta\Delta v$ (here $\beta = df/dy$) to the potential vorticity equation, where the error in the northward velocity, Δv, can be as large as 2 mm s^{-1} (McDougall, 1988). This is an indication of the nonconservation of this form of potential vorticity following a mean streamline in a potential density surface.

The next two sections present the known streamfunctions in an isobaric surface and in a steric anomaly surface. Section 4 deals with the streamfunction in an *in situ* density surface, while Section 5 is an error analysis of using known streamfunction expressions (appropriate to other surfaces) in a neutral surface. The error resulting from using a Bernoulli function based on the vertical gradient of potential density is

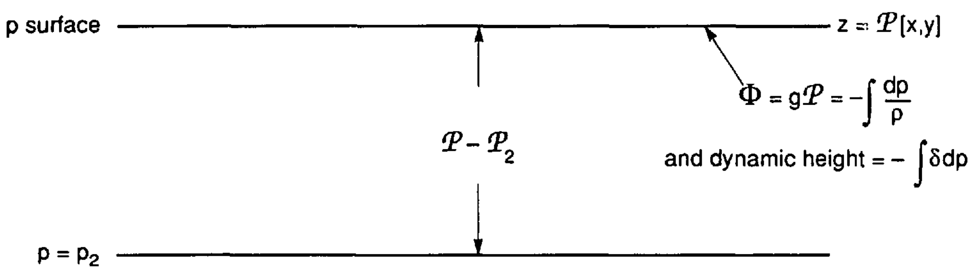

Figure 2. Cross-section through two isobaric surfaces. Streamfunctions are the geopotential, $g\mathcal{P}[x, y]$, and the dynamic height, $-\int \delta\, dp$.

addressed in Section 6, while Section 7 derives simple expressions relating the slopes of the various surfaces.

2. Dynamic height in an isobaric surface

The geopotential, $\Phi = gz = -\int \rho^{-1} dp$, is an exact streamfunction for the geostrophic flow in an isobaric surface (Gill, 1982). If $\mathcal{P}[x, y]$ is the height of an isobaric surface, the geostrophic equations (1) become (using Eq. 5 and the hydrostatic balance, $p_z = -g\rho$),

$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}\bigg|_z = \frac{1}{\rho}\mathcal{P}_y\frac{\partial p}{\partial z} = -g\mathcal{P}_y = -\frac{\partial \Phi}{\partial y}\bigg|_p \tag{10}$$

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_z = \frac{1}{\rho}\mathcal{P}_x\frac{\partial p}{\partial z} = -g\mathcal{P}_x = -\frac{\partial \Phi}{\partial x}\bigg|_p \tag{11}$$

The absolute value of the geopotential on any isobaric surface is not known, but the geopotential at one pressure is expressed relative to its value at another pressure. If p_2 is the deep reference level, the difference in Φ between the two pressure surfaces, p_1 and p_2 is (see Fig. 2)

$$\Phi(p_1) - \Phi(p_2) = -\int_{p_2}^{p_1}\frac{1}{\rho}\,dp = \int_{p_1}^{p_2}\frac{1}{\rho}\,dp. \tag{12}$$

To avoid problems with numerical accuracy in computations, the integrand in (12) is

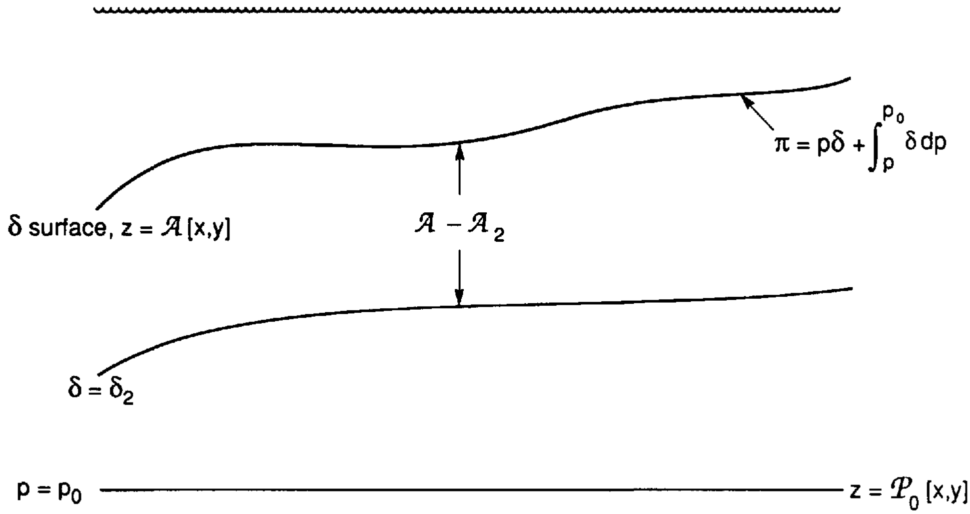

Figure 3. Cross-section through an isobaric reference surface, $p = p_0$, and two surfaces of constant steric anomaly, δ. The streamfunction for the steric anomaly surface is the Montgomery function, or acceleration potential, $\pi = p\delta + \int_p^{p_0} \delta\, dp$.

usually replaced by the specific volume anomaly, or steric anomaly, δ, defined by

$$\delta = \frac{1}{\rho(S, T, p)} - \frac{1}{\rho(35, 0, p)} = \frac{1}{\rho} - \frac{1}{\tilde{\rho}}. \tag{13}$$

Since the extra term

$$-\int_{p_1}^{p_2} \frac{1}{\rho(35, 0, p)}\, dp = -\int_{p_1}^{p_2} \frac{1}{\tilde{\rho}}\, dp \tag{14}$$

is a function only of the constant pressures p_1 and p_2, it is therefore independent of x and y and so the lateral gradient of the integral in (12) is unchanged when δ is used in place of ρ^{-1}. We conclude that the dynamic height, $-\int \delta\, dp$, is an exact streamfunction for the geostrophic flow in an isobaric surface.

3. Acceleration potential in a steric anomaly surface

Montgomery (1937) first suggested using the "acceleration potential" function (which is now also called the Montgomery streamfunction), π, defined by (see Fig. 3)

$$\pi = p\delta + \int_p^{p_0} \delta\, dp \equiv p_0\delta_0 + \int_{\delta_0}^{\delta} p\, d\delta, \tag{15}$$

in a surface of constant steric anomaly, δ. The vertical integral is evaluated from an isobaric surface, $p = p_0$, (along which δ_0 varies), to the pressure, p, on the δ surface of concern. The second part of this equation is a geometrical identity, obtained by integrating by parts.

Differentiating π with respect to x along a δ surface of height $\mathcal{A}[x,y]$, we find

$$\begin{aligned}\frac{\partial \pi}{\partial x} &= \left(\frac{1}{\rho} - \frac{1}{\tilde{\rho}}\right)\frac{\partial p}{\partial x}\bigg|_\delta + g\left(\frac{\partial \mathcal{A}}{\partial x} - \frac{\partial \mathcal{P}_0}{\partial x}\right) + \frac{1}{\tilde{\rho}}\frac{\partial p}{\partial x}\bigg|_\delta \\ &= \frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_z - g\frac{\partial \mathcal{P}_0}{\partial x} \\ &= fv - fv_0,\end{aligned} \tag{16}$$

so that π expresses the difference between the lateral velocity field in the δ surface and that in the isobaric surface, $p = p_0$. Similarly,

$$\frac{\partial(\pi_1 - \pi_2)}{\partial x} = fv_1 - fv_2, \tag{17}$$

and so π is the exact streamfunction for the geostrophic flow in the steric anomaly surface. It is interesting to note that if ρ^{-1} were used in (15) in place of δ, an error of magnitude $(g^2/\tilde{c}^2)\ \mathcal{A}\ \mathcal{A}_x$ would occur and that this is equivalent to an error in the northward velocity, v, of 45 mm/s (see the discussion of Eqs. 21–24 below).

4. Streamfunction for an *in situ* density, ρ, surface

Let us take the difference of φ, defined by

$$\varphi = \frac{p}{\rho} - \int \frac{dp}{\rho}, \tag{18}$$

between two surfaces of constant *in situ* density, $\rho = \rho_1$ and $\rho = \rho_2$, finding (see Fig. 4)

$$\varphi_1 - \varphi_2 = \frac{p(\rho_1)}{\rho_1} - \frac{p(\rho_2)}{\rho_2} + g(\mathcal{D}_1 - \mathcal{D}_2). \tag{19}$$

Here $\mathcal{D}[x, y]$ is the height of a surface of constant *in situ* density. Taking the x-derivative of (19) gives

$$\begin{aligned}\frac{\partial(\varphi_1 - \varphi_2)}{\partial x} &= \frac{1}{\rho_1}\frac{\partial p_1}{\partial x}\bigg|_{\rho_1} + g\frac{\partial \mathcal{D}_1}{\partial x} - \frac{1}{\rho_2}\frac{\partial p_2}{\partial x}\bigg|_{\rho_2} - g\frac{\partial \mathcal{D}_2}{\partial x} \\ &= \frac{1}{\rho_1}\frac{\partial p_1}{\partial x}\bigg|_z - \frac{1}{\rho_2}\frac{\partial p_2}{\partial x}\bigg|_z \\ &= fv_1 = fv_2,\end{aligned} \tag{20}$$

as is required of a streamfunction. We conclude that φ, defined by (18), is the exact streamfunction for the geostrophic flow in a surface of constant *in situ* density, ρ. It is

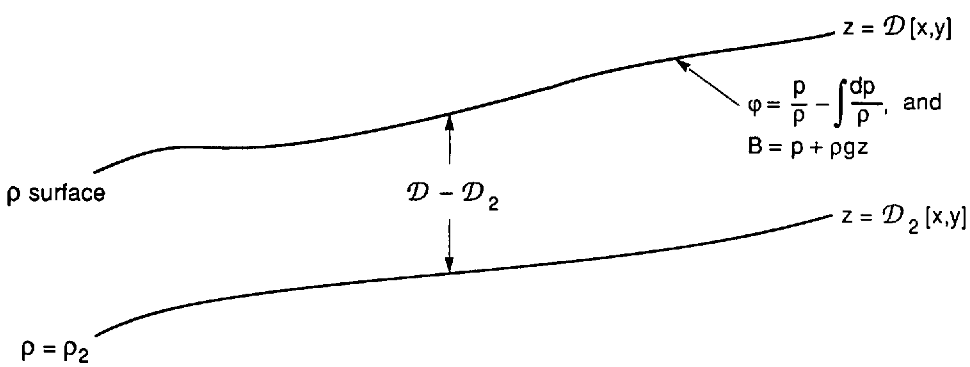

Figure 4. Cross-section through two surfaces of constant *in situ* density, ρ, showing two of the streamfunctions that exist in ρ surfaces.

apparent that the Bernoulli function, $B = p + \rho g z = \rho\varphi$, is also an exact streamfunction for the geostrophic flow in a ρ surface. Instead of using ρ^{-1} in (18), one could use $[\rho^{-1} - 0.001]$ m^3 kg^{-1} to avoid numerical errors involved in integrating ρ^{-1}. This new variable can readily be shown to also be an exact streamfunction for the geostrophic flow in a surface of constant *in situ* density.

5. Errors in using existing streamfunctions in a neutral surface

Neutral surfaces are defined so that a fluid parcel can be moved small distances in this surface without experiencing buoyant restoring forces and so without having to do work against gravity. The lateral gradients of potential temperature and salinity in a neutral surface are related by the thermal expansion and haline contraction coefficients, α and β; that is $\alpha\nabla_n\theta = \beta\nabla_n S$. Since lateral motion and mixing are believed to occur along neutral surfaces, it is natural to seek a streamfunction in these surfaces. However, while it has been proved that such a streamfunction exists, it is not yet known how to write a closed form for it as a vertical integral of some kind, and it is most unlikely that such a closed form exists. One may imagine using either the acceleration potential, π, of (15), or the expression (18) for φ, but neither expression is an exact streamfunction for the geostrophic flow in a neutral surface. The errors resulting from using these two options are investigated here.

Consider first the function φ defined similarly to (18), but now evaluated on a neutral surface of height $\mathcal{N}[x, y]$. To avoid confusion, the rather clumsy notation, φ^n, will be adopted, so that

$$\varphi^n = \frac{p(\mathcal{N})}{\rho(\mathcal{N})} - \int_{p_0}^{p(\mathcal{N})} \frac{1}{\rho}\, dp. \tag{21}$$

Taking the x-derivative gives

$$\frac{\partial \varphi^n}{\partial x} = \frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_n + g\left(\frac{\partial \mathcal{N}}{\partial x} - \frac{\partial \mathcal{P}_0}{\partial x}\right) - \frac{p}{\rho^2}\frac{\partial \rho}{\partial x}\bigg|_n \tag{22}$$
$$= fv - fv_0 - \frac{p}{\rho^2}\frac{\partial \rho}{\partial x}\bigg|_n,$$

so that the lateral gradient of the difference of φ^n between two neutral surfaces is

$$\frac{\partial(\varphi_1^n - \varphi_2^n)}{\partial x} = [fv_1 - fv_2] - \frac{p_1}{\rho_1^2}\frac{\partial \rho}{\partial x}\bigg|_{n_1} + \frac{p_2}{\rho_2^2}\frac{\partial \rho}{\partial x}\bigg|_{n_2} \tag{23}$$

Along a neutral surface the variations of *in situ* density and pressure are related through the compressibility, γ, so that the extra (error) terms in (23) are

$$-\frac{p_1}{\rho_1^2}\frac{\partial \rho}{\partial x}\bigg|_{n_1} + \frac{p_2}{\rho_2^2}\frac{\partial \rho}{\partial x}\bigg|_{n_2} = -\frac{p_1}{\rho_1}\gamma_1\frac{\partial p}{\partial x}\bigg|_{n_1} + \frac{p_2}{\rho_2}\gamma_2\frac{\partial p}{\partial x}\bigg|_{n_2}$$
$$= -g^2\rho_1\gamma_1\mathcal{N}_1\frac{\partial \mathcal{N}_1}{\partial x} + g^2\rho_2\gamma_2\mathcal{N}_2\frac{\partial \mathcal{N}_2}{\partial x} \tag{24}$$
$$= -\frac{g^2}{c_1^2}\mathcal{N}_1\frac{\partial \mathcal{N}_1}{\partial x} + \frac{g^2}{c_2^2}\mathcal{N}_2\frac{\mathcal{N}_2}{\partial x}.$$

The last line of this equation has used the thermodynamic identity, $\rho\gamma = c^{-2}$, where c is the speed of sound in seawater. This substitution has been made simply because c is more readily found in tables of the properties of seawater than is γ. For a neutral surface slope, $\mathcal{N}_x$, of 10^{-4}, a height, $\mathcal{N}$, of say, -1000 m, and with $c = 1465$ m s^{-1}, $g^2c^{-2}\mathcal{N}\mathcal{N}_x$ is of order 4.5×10^{-6} m s^{-2}. Taking $f = 10^{-4}$ s^{-1}, the error terms in (24) for each of the two surfaces is equivalent to a northward velocity error, Δv, of about 45 mm s^{-1}. Since $\mathcal{N}_x$ may take opposite signs of the two surfaces, this error estimate will be typical of the error in the difference between the two terms in (23). Such an error is, of course, totally unacceptable. This demonstrates that, while (18) may be the exact streamfunction for the geostrophic flow in a ρ surface, it cannot be used as the streamfunction in a neutral surface.

Hogg (1987) performed an inverse study of tracer data in the North Atlantic, using a streamfunction evaluated on two potential density surfaces, each referenced to 1000 db. Since the pressure on each surface in Hogg's study was, on average, within 250 db of the reference pressure, his σ_1 surfaces closely approximated neutral surfaces. In his paper, Hogg describes his streamfunction as in (21) above; that is, a streamfunction based on ρ^{-1}, but evaluated in a σ_1 surface. However, in fact he replaced ρ^{-1} with the steric anomaly (Hogg, 1987; private communication) and so avoided errors of order 45 mm s^{-1}! The next paragraph considers the streamfunction that Hogg actually used.

Next consider the Montgomery function defined similarly to (15), but now evaluated on a neutral surface. Proceeding as above, we find

$$\pi^n = p(\mathcal{N})\,\delta(\mathcal{N}) + \int_{p(\mathcal{N})}^{p_0} \delta\, dp, \tag{25}$$

and

$$\begin{aligned}
\frac{\partial \pi^n}{\partial x} &= \frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_n + g\left(\frac{\partial \mathcal{N}}{\partial x} - \frac{\partial \mathcal{P}_0}{\partial x}\right) - \frac{p}{\rho^2}\frac{\partial \rho}{\partial x}\bigg|_n + \frac{p}{\tilde{\rho}^2}\frac{\partial \tilde{\rho}}{\partial x}\bigg|_n \\
&= fv - fv_0 - \frac{p}{\rho}\gamma\frac{\partial p}{\partial x}\bigg|_n + \frac{p}{\tilde{\rho}^2}\frac{\partial \tilde{\rho}}{\partial p}\frac{\partial p}{\partial x}\bigg|_n \\
&= fv - fv_0 + \frac{p}{g^2\rho^2}\frac{\partial p}{\partial x}\bigg|_n\left[\frac{g^2}{\tilde{c}^2}\frac{\rho^2}{\tilde{\rho}^2} - \frac{g^2}{c^2}\right] \\
&\approx fv - fv_0 + \mathcal{N}\mathcal{N}_x\left[\frac{g^2}{\tilde{c}^2} - \frac{g^2}{c^2}\right].
\end{aligned} \tag{26}$$

The results of Appendix B have been used to replace $\partial\tilde{\rho}/\partial p$ with $(\tilde{c})^{-2}$, and since the difference between ρ and $\tilde{\rho}$ is typically only 3% of the difference between the sound speeds, this variation has been ignored in obtaining the final line of (26). At a depth of 1000 m and a temperature of 8°C, $c = 1498.8$ m/s, while at the same pressure and at 0°C, $\tilde{c} = 1465.5$ m/s. With the slope of the neutral surface to geopotentials of 10^{-4}, the extra term in (26), $\mathcal{N}\mathcal{N}_x[g^2/\tilde{c}^2 - g^2/c^2]$, is 2×10^{-7} m s^{-2}. In midlatitudes where f is about 10^{-4} s^{-1}, this error term corresponds to an error in the northward velocity of 2 mm s^{-1}. The values above have been chosen as typical of the surfaces used by Hogg (1987). This error of 2 mm s^{-1} is surprisingly large — about twice the mean flow obtained by Hogg — which indicates that caution must be used in deciding on appropriate streamfunctions in various surfaces.

6. Error analysis of the Bernoulli function, $g \int z\partial\rho_\theta/\partial z\, dz$

An interesting new method for inverting hydrographic data has been developed by Killworth (1986). This method uses the ideal fluid result that the Bernoulli function, $B = p + \rho g z$, is constant along mean streamlines in an incompressible fluid, as is the density, ρ, and the potential vorticity, q. By drawing ρ-q diagrams for several casts, Killworth notes that at the many points where these curves intersect, the Bernoulli function must be the same on both casts. This requirement is used in a least-squares scheme to determine the lateral map of the Bernoulli function and hence the lateral flow field. If the ocean were incompressible (like the data of Cox and Bryan, 1984, that were used in part of Killworth's study) this method would be exact (subject to the restrictions of linear dynamics, as with most inverse models). In an effort to accommodate the compressible nature of seawater, Killworth used isopycnal potential vorticity, $-f\partial\rho_\theta/\partial z$, evaluated on potential density surfaces, as the potential vorticity variable, and an approximate Bernoulli function defined by the vertical integral,

$$B = \bar{B} + g \int_{Z_0}^{\mathcal{R}} z \frac{\partial \rho_\theta}{\partial z} dz, \tag{27}$$

which was evaluated on a potential density surface of height $\mathcal{R}[x, y]$ where Z_0 is a constant reference height. In Section 1 of this paper, it was pointed out that isopycnal potential vorticity is not conserved in a potential density surface along the direction of the mean lateral velocity vector. In this section a completely different matter is addressed; namely, the amount by which B, as defined by (27), is not conserved following the mean flow in a potential density surface.

Adding and subtracting $z\rho_z$ to the integrand in (27) leads to

$$B - \bar{B} = \int_{Z_0}^{\mathcal{R}} [p + gz\rho]_z \, dz + g \int_{Z_0}^{\mathcal{R}} z \left(\frac{\partial \rho_\theta}{\partial z} - \frac{\partial \rho}{\partial z} \right) dz. \tag{28}$$

The second integral is now integrated by parts, giving

$$B - \bar{B} = [p + gz\rho_\theta] Z_0^{\mathcal{R}} - g \int_{Z_0}^{\mathcal{R}} (\rho_\theta - \rho) dz. \tag{29}$$

Taking the lateral derivative in the x-direction, the integral term becomes

$$\frac{\partial \left(\int_{Z_0}^{\mathcal{R}} (\rho_\theta - \rho) dz \right)}{\partial x} = (\rho_\theta - \rho) \mathcal{R}_x + \int_{Z_0}^{\mathcal{R}} \frac{\partial (\rho_\theta - \rho)}{\partial x} dz, \tag{30}$$

and so the lateral gradient of B is

$$\left. \frac{\partial B}{\partial x} \right|_{\rho_\theta} - \frac{\partial \bar{B}}{\partial x} = \left. \frac{\partial p}{\partial x} \right|_{\rho_\theta} + g\rho \mathcal{R}_x + g \int_{Z_0}^{\mathcal{R}} \frac{\partial (\rho - \rho_\theta)}{\partial x} dz - \left. \frac{\partial p}{\partial x} \right|_{Z_0} - gZ_0 \left. \frac{\partial \rho_\theta}{\partial x} \right|_{Z_0}. \tag{31}$$

The last two terms here are evaluated at $z = Z_0$ and are functions of x and y, but not of z. These terms do not affect Killworth's method since they can be absorbed into the depth-independent term, $\bar{B}_x$. The first two terms on the right-hand side of (31) are equal to $\rho f v$, and the integral in (31) represents the inherent error in using (27) as a streamfunction. Note that if one wanted B to be approximately $\rho f v - \rho_0 f v_0$, then the origin of z must be at Z_0, so that the last term in (31) is zero.

To evaluate the offending integral in (31), an expression is needed for $(\rho - \rho_\theta)$. Consider the isentropic and adiabatic compression of a water parcel at salinity, S, and potential temperature, θ, from the reference pressure of the potential density variable, p_r, to the *in situ* pressure, p. During this compression process, the variation of ρ is related to the pressure change by $d\rho = \rho\gamma dp$, so that

$$\rho - \rho_\theta = \int_{p_r}^{p} \rho\gamma \, dp' = \int_{p_r}^{p} \frac{1}{[c(S, \theta, p')]^2} dp', \tag{32}$$

where p' is the dummy variable of integration. The integration variable may be changed from dp' to $(-g\rho)dz'$, using the hydrostatic equation, and the x-derivative of (32) becomes

$$\left.\frac{\partial(\rho - \rho_\theta)}{\partial x}\right|_z = \int_{z(p_r)}^{z} \frac{2g\rho}{c^3} \left.\frac{\partial c}{\partial \theta}\right|_{S,p} \left.\frac{\partial \theta}{\partial x}\right|_z dz', \tag{33}$$

where the following three excellent approximations have been made: (i) the lateral gradient of $z(p_r)$ has been ignored, (ii) the relative variations of ρ have been neglected in comparison to those of c, and (iii) the variations of c with S and p have been ignored in comparison with the dependence of c on potential temperature. Taking $\partial\theta/\partial x|_z$ to be 3×10^{-6} K m^{-1} as typical of the upper ocean (e.g., a slope of potential isotherms to the horizontal of 3×10^{-4} and $\theta_z = 10^{-2}$ K m^{-1}), and with $\partial c/\partial\theta$ equal to 3.3 m s^{-1} K^{-1}, the integrand in (33) is 6×10^{-11} kg m^{-5}. The error term in the lateral gradient of B in (31) is then

$$g\int_{Z_0}^{\mathcal{R}} \frac{\partial(\rho - \rho_\theta)}{\partial x} dz = gx6x10^{-11}[0.5\mathcal{R}^2 - 0.5Z_0^2 - (\mathcal{R} - Z_0)z(p_r)]. \tag{34}$$

At a depth, $\mathcal{R}$ of -1000 m, and with Z_0 and $z(p_r)$ both zero, (31) is 3×10^{-4} kg m^{-2} s^{-2}, which is equal to $\rho f \Delta v$, with a northward velocity error, Δv, of about 3 mm s^{-1}.

This error analysis shows that the Bernoulli function as used by Killworth (1986) is prone to errors because of the compressible nature of seawater. While he has called the method the "Bernoulli method," it would work with any streamfunction being used in place of the Bernoulli function. For example, if surfaces of constant steric anomaly were used instead of potential density surfaces, there would be two immediate advantages. First, the potential vorticity variable, $f\partial\delta/\partial z$, or f/h^δ (where h^δ is the height between successive steric anomaly surfaces), is conserved following the mean flow, assuming that the total velocity vector lies in the steric anomaly surface. Second, an exact streamfunction is known for the geostrophic flow in these surfaces; namely the Montgomery function.

7. Geometry of the various surfaces

The reason for seeking a streamfunction in a particular surface is to capture the spreading and mixing of water masses in this surface. By choosing the correct "density" surface one hopes that the streamlines will not only represent the direction of the lateral components of the the three-dimensional velocity field, but that the total velocity vector will lie close to one's surface, thus making the two-dimensional streamlines in this surface an approximation to the flow directions of the full three-dimensional flow. Apart from the very small dianeutral advection velocities caused by vertical mixing processes, the surfaces in which the lateral movement and mixing of water parcels occur are neutral surfaces (McDougall, 1987a). Potential density surfaces and surfaces of constant steric anomaly approximate neutral surfaces to varying degrees.

In this section the slopes of the surfaces considered above are investigated to discover the degree to which these surfaces approximate neutral surfaces. Beginning with the surface of constant *in situ* density, and regarding the *in situ* density to be a function of salinity, potential temperature and pressure (rather than of S, T and p), the two-dimensional spatial gradient of ρ in a ρ surface can be expressed as

$$\frac{1}{\rho}\nabla_\rho \rho = 0 = \beta\nabla_\rho S - \alpha\nabla_\rho\theta + \gamma\nabla_\rho p. \tag{35}$$

The lateral gradient operator in the neutral surface can be related to that in the ρ surface by (from (5))

$$\nabla_n = \nabla_\rho + [\nabla_n\mathcal{N} - \nabla_\rho\mathcal{D}]x\frac{\partial}{\partial z}\bigg|_{xy}. \tag{36}$$

Re-expressing the lateral gradients in (35) in terms of the epineutral gradients, and using the fact that $\alpha\nabla_n\theta = \beta\nabla_n S$, we find that

$$[\nabla_n\mathcal{N} - \nabla_\rho\mathcal{D}]x\left[\frac{N^2}{g} - \gamma p_z\right] = -\gamma\nabla_n p. \tag{37}$$

The epineutral gradient of pressure, $\nabla_n p$ is equal to $-g\rho[\nabla_n\mathcal{N} - \nabla_p\mathcal{P}]$, so that

$$[\nabla_n\mathcal{N} - \nabla_p\mathcal{P}] = [\nabla_\rho\mathcal{D} - \nabla_p\mathcal{P}]\,x\left[1 + \frac{g^2/c^2}{N^2}\right]. \tag{38}$$

This relationship between the tangent of the angles between (i) a neutral surface and an isobaric surface, and (ii) an *in situ* density surface and an isobaric surface, is illustrated in Figure 5. In the upper ocean where N^2 is larger than $g^2/c^2 \approx (4.3 \times 10^{-5}\ \mathrm{s}^{-2})$, the slope of the neutral surface is similar to that of the ρ surface, but in the deep ocean where N^2 is only 1% of g^2/c^2, the slope of the neutral surface will be 100 times that of an *in situ* density surface. At a depth of 1000 m, where N^2 is about $10^{-5}\ \mathrm{s}^{-2}$, an *in situ* density surface has only one fifth of the slope of the neutral surface. It is apparent that a ρ surface is a very poor approximation to a neutral surface. It is interesting to note that a totally incompressible float (this is an approximation to a SOFAR float) moves around the ocean on a surface of constant *in situ* density.

Next, the slope of a surface of constant steric anomaly is examined by taking the lateral derivative of steric anomaly in a δ surface,

$$\nabla_\delta\delta = 0 = -\frac{1}{\rho^2}\nabla_\delta\rho + \frac{1}{\tilde{\rho}^2}\nabla_\delta\tilde{\rho}, \tag{39}$$

and the first lateral gradient is expressed in terms of the epineutral gradient of ρ, while

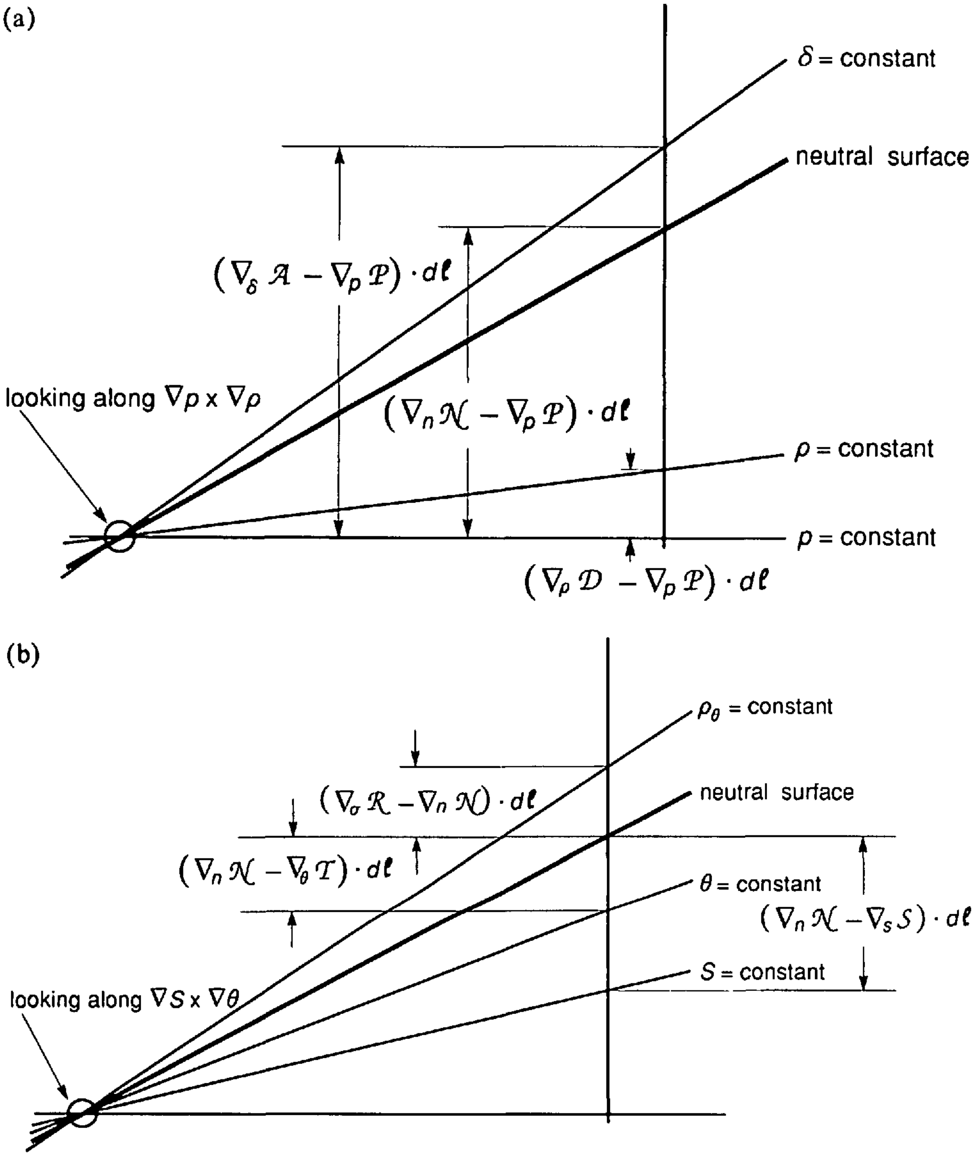

Figure 5. (a) Cross-section showing the relative slopes of four surfaces that all intersect along the line $\nabla p \times \nabla \rho$. This vector points directly into the page. The two-dimensional vector, dl, is the horizontal line element, $dx\mathbf{i} + dy\mathbf{j} + 0\mathbf{k}$. The relative slopes are drawn appropriate to a pressure of about 1000 db; this is, the slopes (with respect to the isobaric surface) of the ρ and δ surfaces are 25% and 125%, respectively, of the slope of the neutral surface with respect to the isobaric surface. (b) Cross-section showing the relative slopes of a potential density surface, a neutral surface, a potential isotherm, and an isohaline surface.

the second gradient is expressed in terms of the isobaric gradient of $\tilde{\rho}$ using (from 5)

$$\nabla_n = \nabla_\delta + [\nabla_n \mathcal{N} - \nabla_\delta \mathcal{A}] \, x \frac{\partial}{\partial z}\bigg|_{x,y}$$

and (40)

$$\nabla_p = \nabla_\delta + [\nabla_p \mathcal{P} - \nabla_\delta \mathcal{A}] \, x \frac{\partial}{\partial z}\bigg|_{x,y}$$

This procedure leads to

$$[\nabla_n \mathcal{N} - \nabla_p \mathcal{P}] = [\nabla_\delta \mathcal{A} - \nabla_p \mathcal{P}] x \left[1 + \frac{g^2/c^2}{N^2} - \frac{g^2/\tilde{c}^2}{N^2}\right]. \tag{41}$$

Taking N^2 to be 10^{-5} s^{-2}, the sound speed, c, of 1498.8 m/s appropriate to a temperature of 8°C and a pressure of 1000 db, while at 0°C, $\tilde{c}$ is 1465.5 m/s, the square bracket in (41) is 0.8, implying that the steric anomaly surface has a steeper slope with respect to isobars than does the neutral surface by 25%. Deeper in the water column, the situation becomes worse. (Note that the large square bracket in (41) is equal to $\rho g N^{-2} \partial\delta/\partial z$.)

The relative slopes of these surfaces are sketched in Figure 5(a). All meet along the line in three-dimensional space, $\nabla p \times \nabla \rho$. Not shown in this figure are the geopotential surface, z = constant, and a potential density surface, because neither of these surfaces includes the line $\nabla p \times \nabla \rho$. For many purposes, the geopotential and isobaric surfaces may be considered coincident, while the potential density surface intersects the neutral surface along the line $\nabla S \times \nabla \theta$ (see Fig. 1). A potential density surface that is referenced to a pressure more than 1000 db different to the *in situ* pressure, will often slope quite differently to that of the neutral surface (McDougall, 1987b).

To compare these slopes with those of other relevant surfaces, Figure 5(b) shows a section in three-dimensional space normal to $\nabla S \times \nabla \theta$ along which potential density surfaces, neutral surfaces, isohaline surfaces, and surfaces of constant potential temperature intersect. The slopes of these surfaces are related by (see McDougall, 1988 for the derivation of these equations),

$$\begin{aligned} [\nabla_\sigma \mathcal{R} - \nabla_n \mathcal{N}] &= [\mu - 1] x \, [\nabla_n \mathcal{N} - \nabla_\theta \mathcal{T}] = [\mu - 1] x \frac{\nabla_n \theta}{\theta_z} \\ [\nabla_n \mathcal{N} - \nabla_S \mathcal{S}] &= R_\rho x \, [\nabla_n \mathcal{N} - \nabla_\theta \mathcal{T}] = R_\rho x \frac{\nabla_n \theta}{\theta_z}, \end{aligned} \tag{42}$$

where $\mathcal{T}[x, y]$ and $\mathcal{S}[x, y]$ are the heights of the surfaces of constant potential temperature and salinity, respectively. $R\rho$ and μ are defined in Eq. (9). Taking a value for μ of 1.5 at a pressure, $(p\text{-}p_r)$, of 1000 db, and a slope between the neutral surface and the potential isotherm of 10^{-4}, gives a slope between the potential density surface and the neutral surface of 0.5×10^{-4}.

Table 1. Streamfunctions in various surfaces.

Surface	Lateral divergence	Known streamfunctions
Geopotential surface, z = constant	$\nabla_z \cdot (\rho f V_n) = 0$, but $\nabla_z \cdot (f V_n) \neq 0$.	Pressure p is the streamfunction for $\rho f V_n$.
Isobaric surface, p = constant	$\nabla_p \cdot (f V_n) = 0$	(i) Dynamic height, $-\int \delta\, dp$, (ii) Geopotential, $gP[x, y] = -\int dp/\rho$.
Steric anomaly surface, $\delta = (1/\rho) - (1/\tilde{\rho})$ = constant	$\nabla_\delta \cdot (f V_n) = 0$	Montgomery function = acceleration potential = π $= p\delta - \int \delta\, dp$.
In situ density surface, ρ = constant.	$\nabla_\rho \cdot (f V_n) = \nabla_\rho \cdot (\rho f V_n) = 0$	(i) Streamfunction for $f V_n$ is $\varphi = p/\rho - \int dp/\rho \equiv -\int p/\rho^2\, d\rho$. (ii) Streamfunction for $\rho f V_n$ is $B = p + \rho g z = p - \rho \int dp/\rho$.
Surface of constant p/ρ	$\nabla_{p/\rho} \cdot (f V_n) = 0$	Streamfunction exists, but as yet, an expression for it is not known.
Neutral surface, $1/\rho\, \nabla_n \rho = \gamma \nabla_n p$	$\nabla_n \cdot (f V_n) = 0$	Streamfunction exists, but as yet, an expression for it is not known.
Potential density surface, ρ_θ (or σ_1) = constant	$\nabla_\sigma \cdot (f V_n) \neq 0$	No useful streamfunction exists.

8. Discussion

The principal message of this paper is that one must be very careful when selecting a streamfunction for the lateral velocity in a compressible fluid. Table 1 lists all the known streamfunctions in several different surfaces. Note that in two cases there are more than one known streamfunction. Although a streamfunction is known to exist in a neutral surface, we do not have a closed expression for it. By contrast, a potential density surface does not possess a useful streamfunction (i.e. one from which the lateral velocity components can be readily calculated).

As oceanographers, we are grappling with the first-order dynamical balances of the ocean circulation, and tend to think that the subtle nonlinearities of the equation of state are not of prime importance. However, for the evaluation of streamfunctions, these rather esoteric nonlinear terms are surprisingly important. For example, in Section 5 it was shown that using the Montgomery function in a neutral surface often causes a velocity error of 2 mm s^{-1}, despite the fact that the slope of this neutral surface is within 25% of that of the surface of constant steric anomaly in which the

Montgomery function is an exact streamfunction for the geostrophic flow. To use the Montgomery function while keeping the velocity error below 0.5 mm/s, one would need to use a surface that differed in slope from a steric anomaly surface by less than 5×10^{-6}. Similarly, the approximate expression for a Bernoulli function based on a vertical integral of z times the gradient of potential density was shown to lead to errors of about 3 mm s^{-1} in the horizontal velocity. These velocity errors are as large as the mean flow in quiet regions of the ocean at depths of about 1000 m. Inverse methods can sometimes determine the mean flow with rms uncertainities of about this magnitude, although much work remains to understand the reasons for the much larger errors that often occur in energetic flow regimes. For example, Killworth and Bigg's inversions of Cox's eddy-resolved general circulation model data gave mean velocities with an rms uncertainty of 2 mm s^{-1} in the relatively quiet Eastern Atlantic, but with uncertainties of about 13 mm s^{-1} in the more active regions such as the Gulf Stream extension and the homogenized mid-gyre regions (see their Figures 3, 8 and 13).

Of the surfaces discussed in this paper, the steric anomaly surface has much to recommend it. Linear potential vorticity (based on the height between steric anomaly surfaces, i.e. based on $f\partial\delta/\partial z$, not on fN^2) is conserved in this surface, the Montgomery function is an exact streamfunction for the geostrophic flow, and this surface is as close to a neutral surface as any other in common usage. This closeness can easily be further improved by forming a steric anomaly variable based on constant values of salinity and temperature that are the area-averaged values, $\overline{S}$ and $\overline{T}$, for the surface under consideration. The new "local" steric anomaly would be defined by

$$\delta = \frac{1}{\rho(S,T,p)} - \frac{1}{\rho(\overline{S},\overline{T},p)}. \tag{43}$$

The Montgomery streamfunction, (15), is exact in the δ surface, no matter what constant values are taken for $\overline{S}$ and $\overline{T}$ in the definition of the steric anomaly variable. So long as these values are fixed during each individual vertical integration from the reference pressure, p_0, different values could be used for the different surfaces in which streamfunctions are evaluated. If the temperature changed by, say 1°C in the region of interest on such a surface, the estimate for the difference in slopes between this surface and a neutral surface is an eighth of that given following Eq. (41); that is, at a depth of 1000 m, one would expect these surfaces to differ in slope by only 3%. This suggests that a careful and flexible definition of steric anomaly may prove to be a workable solution to some of the annoying problems caused by the nonlinear nature of the equation of state.

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APPENDIX A

A streamfunction exists in a neutral surface

In a steady state, the vertical velocity past geopotentials, w, and the dianeutral velocity, e, are related by

$$w = e + \mathrm{V}_n \cdot \nabla_n \mathcal{N}, \tag{A1}$$

where V_n is the two-dimensional lateral velocity vector (see (3)) and $\mathcal{N}[x, y]$ is the height of a neutral surface. Vertically differentiating (A1) and using the thermal wind equation, we find that

$$w_z = e_z + \mathrm{V}_n \cdot \nabla_n[\ln(h)], \tag{A2}$$

where h is the height between closely-spaced neutral surfaces. The continuity equation is

$$\mathrm{V}_n \cdot \nabla_n[\ln(h)] + \nabla_n \cdot \mathrm{V}_n + e_z = 0. \tag{A3}$$

Combining the linear vorticity equation, $\beta/f\, v = w_z$, with (A2) and (A3) gives

$$\nabla_n \cdot [f\mathrm{V}_n] = 0, \tag{A4}$$

so that a streamfunction exists in a neutral surface.

APPENDIX B

Evaluation of $\partial\tilde{\rho}/\partial p$

The adiabatic and isentropic compressibility of seawater, γ, is defined by a pressure derivative at constant salinity and *potential* temperature, θ, and is related to the *in situ* density and the sound speed by

$$\gamma = \frac{1}{\rho}\frac{\partial \rho}{\partial p}\bigg|_{S,\theta} = \frac{1}{\rho c^2}. \tag{B1}$$

From the calculus of variations we find that γ is related to the derivative at constant salinity and *in situ* temperature, T, by (see McDougall, 1987a, Eq. 11)

$$\gamma = \frac{1}{\rho}\frac{\partial \rho}{\partial p}\bigg|_{S,\theta} = \frac{1}{\rho}\frac{\partial \rho}{\partial p}\bigg|_{S,T} + \alpha\frac{\partial \theta}{\partial p}\bigg|_{S,T} . \tag{B2}$$

Now $\tilde{\rho} = \rho\ (35,\ T = 0,\ p)$ is a function of pressure alone and its derivative is equal to $\partial\rho/\partial p|_{S,T}$ at $S = 35$ psu and $T = 0°C$, so that

$$\begin{aligned} \frac{1}{\tilde{\rho}}\frac{\partial \tilde{\rho}}{\partial p} = \frac{1}{\rho}\frac{\partial \rho}{\partial p}\bigg|_{S=35,T=0} &= \frac{1}{\rho}\frac{\partial \rho}{\partial p}\bigg|_{S,\theta} - \alpha(35,\theta,p)\frac{\partial\theta}{\partial p}\bigg|_{S=35,T=0} \\ &= \gamma(35,\theta,p) - \alpha(35,\theta,p)\frac{\partial\theta}{\partial p}\bigg|_{S=35,T=0} \end{aligned} \tag{B3}$$

Here θ is the potential temperature (with a reference pressure of 0 db) of the fluid parcel at $S = 35$ psu, $T = 0°C$, and at the *in situ* pressure, p. At this temperature, $\alpha \approx 0.8 \times 10^{-4}\ K^{-1}$ and $\partial\theta/\partial p \cong -5 \times 10^{-5}\ K\ (db)^{-1} = 5 \times 10^{-9}\ K\ (Pa)^{-1}$ (Gill, 1982 Appendix 3), so that $1/\tilde{\rho}\ \partial\tilde{\rho}/\partial p$ differs from the compressibility of the $\tilde{\rho}$ fluid parcel by just $4 \times 10^{-13}\ (Pa)^{-1}$. This is about 0.1% of $\gamma(35, \theta, p)$ and so can be ignored for our purposes. (By way of comparison, a 1°C change causes the compressibility of seawater to change by about 0.6%.) We conclude that $\partial\tilde{\rho}/\partial p$ is very well approximated by $(\tilde{c})^{-2}$, where $\tilde{c}$ is the sound speed at $S = 35$ psu, $T = 0°C$, and at the *in situ* pressure, p.

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