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A simplified linear ocean circulation theory

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ABSTRACT

The linear theory of the wind- and thermally-driven ocean circulation simplifies considerably if the traditional Laplacian viscosity and thermal diffusivity are replaced by a linear-decay friction and heat diffusion. Solutions of the simplified equations display all the physically important features of the standard model.

1. Introduction

The standard equations governing the steady linear wind- and thermally-driven ocean circulation are

$$\begin{aligned} \mathbf{f} \times \mathbf{u} &= -\nabla\phi + A_h\nabla^2\mathbf{u} + A_v\mathbf{u}_{zz} \\ 0 &= -\phi_z - \rho' g/\rho_o + A_h\nabla^2 w + A_v w_{zz} \\ w\bar{\rho}_z &= K_h\nabla^2\rho' + K_v\rho'_{zz} \\ \nabla\cdot\mathbf{u} + w_z &= 0\,. \end{aligned} \tag{1.1}$$

Here, (x, y, z) are Cartesian coordinates in the (east, north, up) direction, $\mathbf{f}$ is the coriolis parameter (times the vertical unit vector), $\mathbf{u} = (u, v)$ is the horizontal velocity, w is the vertical velocity, ϕ is the pressure (divided by a constant representative density ρ_o, in the Boussinesq approximation), $\nabla = (\partial_x, \partial_y)$, A_h and A_v are horizontal and vertical eddy viscosity coefficients, $\bar{\rho}(z)$ is the prescribed mean density, ρ' is the density deviation from $\bar{\rho}(z)$, g is gravity, and K_h and K_v are eddy mixing coefficients. Coordinate subscripts denote differentiation. The boundary conditions that accom-

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pany (1.1) are prescribed density flux at all boundaries, zero velocity vector at all solid boundaries, and no normal velocity, plus prescribed tangential momentum flux (i.e., wind stress) at the ocean surface.

The general solution to (1.1) for asymptotically small Λ and K has been given in two extremely important papers by Pedlosky (1968, 1969). These papers are, in a sense, the culmination of a long search for understanding of the steady, linear ocean circulation that began with the early work of Ekman. However, Pedlosky's two papers are very demanding. The no-slip and nonconduction boundary conditions typically require three nested boundary layers at coastal boundaries.

The purpose of this paper is to draw attention to the fact that, if the standard equations (1.1) are replaced by

$$\begin{aligned} \mathbf{f} \times \mathbf{u} &= -\nabla\phi - \Lambda\mathbf{u} + \tau_z \\ 0 &= -\phi_z - \rho' g/\rho_o - \Lambda w \\ w\bar{\rho}_z &= -K\rho' - Q \\ \nabla \cdot \mathbf{u} + w_z &= 0\,, \end{aligned} \tag{1.2}$$

in which Λ and K are constant eddy decay coefficients, then the arithmetic simplifies tremendously, but the solutions retain the important features discovered by Pedlosky. The simplified equations (1.2) accommodate only the single boundary condition of no normal flow. Therefore, the stress divergence τ_z and diabatic heating Q must be inserted as *prescribed* functions of (x, y, z). The "body force" τ_z represents wind momentum at the point where it enters the steady large-scale circulation, after it has been mixed downward through an "Ekman layer" of prescribed depth by small-scale processes which are neither modelled nor parameterized by the other terms in (1.2). Similar remarks apply to Q. The linear decay terms in (1.2) are certainly artificial, but so are eddy transport terms in (1.1). The latter are frequently justified by an analogy with molecular transports, but this analogy is extremely weak. In fact, the Reynolds flux parameterization in *both* (1.1) and (1.2) is so lacking in fundamental justification that we should probably discount any property of either solution that depends very sensitively on the particular choice of parameterization. From this point of view, it is wise to consider a variety of simple models, but with particular emphasis on the simplest and most physically transparent models. The general strategy of choosing the eddy flux parameterizations to facilitate analysis has been skillfully used by McCreary (1981).

The analysis of (1.2) owes its simplicity to the fact that $\mathbf{u}$, w and ρ' are all expressible as derivatives of ϕ. Then substitutions into (1.2d) yield a second-order elliptic equation for ϕ. The no-normal-flow condition contributes boundary conditions that generally contain both normal and tangential derivatives of ϕ. The resulting ϕ-problem is simple enough that the boundary layer thicknesses and scalings can be easily deduced, almost by inspection.

I certainly agree with most physical oceanographers that models which neglect the advection of momentum and (especially) density cannot adequately explain the general ocean circulation. However, I also share the sentiment of Gill (1985), that the value of a nonlinear theory cannot properly be assessed without a complete understanding of the linear results. The present note is offered in the same pedagogical spirit as Gill's recent paper.

2. Nondimensional equations

Consider a square ocean of side L and depth H, and suppose for simplicity that the average density gradient in (1.2) is a constant. Then if all the variables in (1.2) are re-scaled in the usual way, the resulting *nondimensional* equations are

$$\begin{aligned} -fv &= -\phi_x - \epsilon u - fv_E \\ fu &= -\phi_y - \epsilon v + fu_E \\ 0 &= -\phi_z - \epsilon\delta^2 w + \theta \\ Sw &= -k\theta + Q \\ u_x + v_y &+ w_z = 0 \end{aligned} \tag{2.1}$$

in $0 < x, y, z < 1$. Here $f = 1 + \beta y$ is the nondimensional coriolis parameter in units of (representative value) f_o, θ is the nondimensional temperature deviation (proportional to minus ρ'), $\delta = H/L$, and $S = N^2/N_o^2$ is the ratio of the constant (squared) Väisälä frequency N^2 to a value N_o^2 that is typical of the real ocean. The interesting case is $S = 1$, but we shall also briefly consider the case $S = 0$ of homogeneous fluid. The nondimensional friction and diffusion parameters are

$$\epsilon = A/f_o \text{ and } k = (K/f_o)(f_o^2L^2/N_o^2H^2) .$$

The *prescribed* Ekman velocity $\mathbf{u}_E(x, y, z)$ represents a distributed source of wind momentum that is significant only within the Ekman layer near $z = 1$. The boundary conditions are simply no normal flow.

We seek solutions to (2.1) in the asymptotic limit

$$1 \gg \beta \gg \epsilon, \delta \rightarrow 0 .$$

The smallness of β is inessential and merely convenient. However, the friction ϵ *must* be small if the flow outside the Ekman layer is to be in geostrophic balance. It is tempting to assume that the diffusion coefficient k is also asymptotically small. However, we shall see that (if $S = 1$) k must be order one (in ϵ and δ) or the entire flow is unrealistically confined to the Ekman layer. For now we leave the size of k unspecified. If the dependent variables are expanded in appropriate trigonometric series, viz.,

$$\begin{aligned} (\mathbf{u}, \phi, \mathbf{u}_E)(x, y, z) &= \sum_{m=0}^{\infty} (\mathbf{u}, \phi, \mathbf{u}_E)_m(x, y) \cos(m\pi z) \\ (w, \theta, Q)(x, y, z) &= \sum_{m=1}^{\infty} (w, \theta, Q)_m(x, y) \sin(m\pi z) \end{aligned} \tag{2.2}$$

then the boundary condition on w is automatically satisfied and (2.1) are separable into modes. The equations governing the subscript-m variables are

$$\begin{aligned} -fv_m &= -\phi_{m,x} - \epsilon u_m - fv_{Em} \\ fu_m &= -\phi_{m,y} - \epsilon v_m + fu_{Em} \\ 0 &= m\pi\phi_m - \epsilon\delta^2 w_m + \theta_m \\ Sw_m &= -k\theta_m + Q_m \\ u_{m,x} &+ v_{m,y} + m\pi w = 0\,. \end{aligned} \tag{2.3}$$

From (2.3) it follows that

$$\begin{aligned} u_m &= -f^{-1}\phi_{m,y} - \epsilon f^{-2}\phi_{m,x} + u_{Em} \\ v_m &= f^{-1}\phi_{m,x} - \epsilon f^{-2}\phi_{m,y} + v_{Em} \\ w_m &= (S + \epsilon\delta^2 k)^{-1}\,[m\pi k\,\phi_m + Q_m] \\ \theta_m &= (S + \epsilon\delta^2 k)^{-1}\,[-m\pi S\,\phi_m + \epsilon\delta^2 Q_m] \end{aligned} \tag{2.4}$$

for $\epsilon \longrightarrow 0$. Then (2.3e) and (2.4a, b, c) can be combined into a single general equation for the pressure, viz.,

$$(\epsilon\delta^2 k + S)\,[\beta f^{-2}\,\phi_{m,x} + \epsilon\nabla\cdot(f^{-2}\,\nabla\phi_m) - \nabla\cdot\mathbf{u}_{Em}] = m\pi\,[m\pi k\phi_m + Q_m] \tag{2.5}$$

with no-normal-flow boundary condition

$$\partial\phi_m/\partial s + \epsilon f^{-1}\,\partial\phi_m/\partial n = f\mathbf{u}_{Em}\cdot\mathbf{n} \tag{2.6}$$

where $\mathbf{n}$ is the outward unit normal and $\mathbf{s}$ the unit tangent pointing counter-clockwise at the coast.

An inspection of (2.5, 2.6) reveals many properties of its solution. The most important of these are:

(1) The depth-averaged ($m = 0$) motion obeys Stommel's (1948) equation and is independent of δ, k, S, and Q. The Stommel solution is briefly reviewed in Section 3.

(2) The internal ($m \neq 0$) motion depends critically on the size of S. If $S \ll \epsilon\delta^2 k$, then there are coastal boundary layers of thickness $\epsilon\delta$ if either $Q \neq 0$ or the Ekman transport impinges on the coast. These solutions are described in Section 4.

(3) If $S \gg \epsilon\delta^2 k$, then the internal ($m \neq 0$) motion has the same frictional boundary layers as in the case $m = 0$. However, heat diffusion is *everywhere* important. These solutions are described in Section 5.

3. Depth-averaged flow

For $m = 0$ the modal equations (2.3) reduce to Stommel's (1948) equations,

$$\begin{aligned} -fv_o &= -\phi_{o,x} - \epsilon u_o - fv_{Eo} \\ fu_o &= -\phi_{o,y} - \epsilon v_o + fu_{Eo} \\ u_{o,x} &+ v_{o,y} = 0 \end{aligned} \tag{3.1}$$

in which $\mathbf{u}_o$, ϕ_o, $\mathbf{u}_{Eo}$ represent the vertical averages of $\mathbf{u}$, ϕ, $\mathbf{u}_E$. The solution to (3.1) is

$$\mathbf{u}_o = \mathbf{u}_S(x, y) = (-\psi_y, \psi_x) \tag{3.2}$$

where ψ is the unique solution to

$$\beta\psi_x = -\epsilon\nabla^2\psi + \nabla \cdot (f\mathbf{u}_{Eo}) \tag{3.3}$$

and $\psi = 0$ at the boundary. Then ϕ_o can be found from (3.1a, b). The well known solution to (3.3) has an interior part

$$\psi_I = -\beta^{-1} \int_x^1 \nabla \cdot (f\mathbf{u}_{Eo})\, dx',$$

which is Sverdrup flow satisfying the boundary condition at $x = 1$. The other boundaries generally require boundary layers to close this interior flow.

A western boundary layer of thickness ϵ is always present near $x = 0$. Let $\hat{A}$ denote the boundary layer correction to any dependent variable A. That is, let $A = A_I + \hat{A}$. Then near $x = 0$, $\hat{\psi}$ obeys

$$\beta\hat{\psi}_x = -\epsilon\hat{\psi}_{xx} \tag{3.4}$$

with boundary condition

$$\hat{\psi}(0, y) = -\psi_I(0, y) \tag{3.5}$$

and matching condition that $\hat{\psi} \rightarrow 0$ as $x \rightarrow \infty$. The boundary layer solution is simply

$$\hat{\psi} = -\psi_I(0, y) \exp[-\beta x/\epsilon]. \tag{3.6}$$

Unless the wind stress curl $\nabla \cdot (f\mathbf{u}_{Eo})$ is zero at $y = 0$ ($y = 1$) there must also be a southern (northern) boundary layer of thickness $\epsilon^{1/2}$. Near $y = 0$, $\hat{\psi}$ obeys

$$\beta\hat{\psi}_x = -\epsilon\hat{\psi}_{yy} \tag{3.7}$$

and

$$\begin{aligned} \hat{\psi}(x, 0) &= -\psi_I(x, 0) \\ \hat{\psi}(x, \infty) &= 0. \end{aligned} \tag{3.8}$$

Since there is no boundary layer at $x = 1$, we must also require

$$\hat{\psi}(1, y) = -\psi_I(1, y) = 0. \tag{3.9}$$

The solution of (3.7–3.9) is nontrivial, but its general character can be deduced by analogy with a scalar diffusion problem. Let $\kappa = \epsilon/\beta$, $t = 1 - x$, $T(y, t) = \hat{\psi}(1 - t, y)$, $T_o(t) = -\psi_I(1 - t, 0)$, and $T_i(y) = -\psi_I(1, y)$. Then (3.7–3.9) transform to

$$\begin{aligned} T_t &= \kappa T_{yy}, & t, y > 0 \\ T(0, t) &= T_o(t), & t > 0 \\ T(\infty, t) &= 0, & t > 0 \\ T(y, 0) &= T_i(y) & y > 0, \end{aligned} \tag{3.10}$$

which are the equations governing the diffusion of T in a one-dimensional semi-infinite medium with initial scalar concentration $T_i(y)$ and prescribed concentration $T_o(t)$ at $y = 0$. From this analogy it is clear that $\hat{\psi}$ is determined by (3.7–3.9) up to $x = 0$. The boundary layer thickness at $y = 0$ is proportional to $\epsilon^{1/2}(1 - x)^{1/2}$. The western boundary layer extends inside this thicker layer to satisfy the boundary condition at $x = 0$. A similar analysis applies at $y = 1$. The complete solution is summarized in Figure 1.

The vertically-averaged horizontal motion is thus completely determined by the wind curl $\nabla \cdot (f\mathbf{u}_{Eo})$. However, the internal ($m \neq 0$) motion depends on $\mathbf{u}_{Em}$ itself, as well as S, k, δ, and Q. We consider first the case when $S = \theta = Q = 0$.

4. Wind-driven homogeneous flow

If the density is constant, then the modal equations (2.3) and (2.5) reduce to

$$\begin{aligned} -fv_m &= -\phi_{m,x} - \epsilon u_m - fv_{Em} \\ fu_m &= -\phi_{m,y} - \epsilon v_m + fu_{Em} \\ 0 &= m\pi\phi_m - \epsilon\delta^2 w_m \\ u_{m,x} &+ v_{m,y} + m\pi w = 0 \end{aligned} \tag{4.1}$$

and

$$\epsilon\delta^2[\beta f^{-2}\phi_{m,x} + \epsilon\nabla \cdot (f^{-2}\nabla\phi_m) - \nabla \cdot \mathbf{u}_{Em}] = m^2\pi^2\phi_m \tag{4.2}$$

with boundary condition (2.6).

The interior approximation to (4.1) and (4.2) simply drops all terms proportional to ϵ. The interior solution is therefore

$$\mathbf{u}_{mI} = \mathbf{u}_{mE}, \quad \phi_{mI} = 0, \quad w_{mI} = -m^{-1}\pi^{-1}\nabla \cdot \mathbf{u}_{Em}. \tag{4.3}$$

[Note that in this paper the "interior" excludes only *coastal* boundary layers. The Ekman layer, which has a prescribed thickness independent of ϵ and δ, is a part of the interior. There is no bottom boundary layer.] If the Ekman velocity $\mathbf{u}_{Em}$ is nowhere normal to the boundary, then the interior solution (4.3) satisfies the boundary condition (2.6) and no boundary layers are required. However, if the Ekman transport

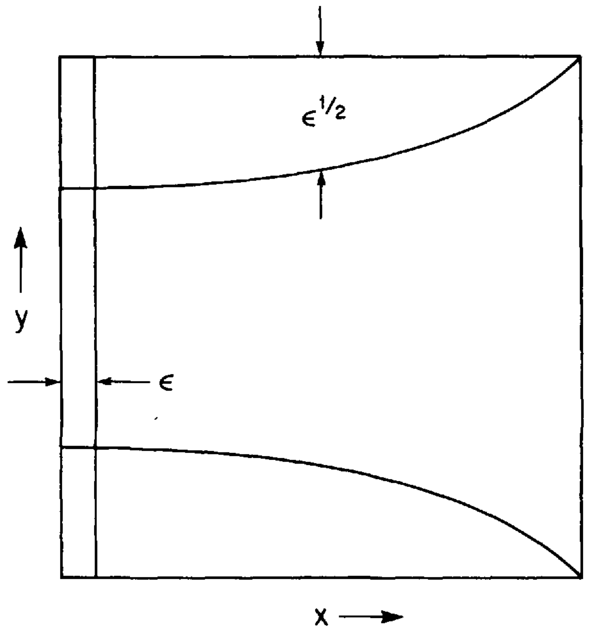


Figure 1. The coastal boundary layers in the Stommel problem.

impinges on a coastline, then a boundary layer must exist to bring the normal velocity to zero. The boundary layer thickness is easily inferred from (4.2). The boundary layer correction pressure $\hat{\phi}_m$ clearly obeys

$$\epsilon\delta^2[\beta f^{-2}\hat{\phi}_{m,x} + \epsilon\nabla \cdot (f^{-2}\nabla\hat{\phi}_m)] = m^2\pi^2\hat{\phi}_m \tag{4.4}$$

in which (near $x = 0$, say) the terms are of relative size

$$\epsilon\delta^2[\Phi/\ell \qquad \epsilon\Phi/\ell^2] = \Phi \tag{4.5}$$

where Φ is the scale for $\hat{\phi}_m$ and ℓ is the sought-for boundary layer thickness. Again, only ϵ and δ are asymptotically small. The only possible balance has $\ell = \epsilon\delta$. It then follows from (2.6) and (2.4) that

$$\begin{aligned} \hat{\phi}_m &= 0(\delta) \\ \hat{\mathbf{u}}_m \cdot \mathbf{n} &= 0(1) \\ \hat{\mathbf{u}}_m \cdot \mathbf{s} &= 0(\epsilon^{-1}) \\ \hat{w} &= 0(\epsilon^{-1}\delta^{-1}). \end{aligned} \tag{4.6}$$

The correction pressure therefore obeys

$$\epsilon^2\delta^2\partial^2\hat{\phi}_m/\partial n^2 = f^2m^2\pi^2\hat{\phi}_m \tag{4.7}$$

with boundary condition

$$\epsilon \partial \hat{\phi}_m / \partial n = f^2\, \mathbf{u}_{Em} \cdot \mathbf{n}\,. \tag{4.8}$$

The uniformly valid solution is

$$\phi_m = \delta f / m\pi\, (\mathbf{u}_E \cdot \mathbf{n}) \times \{\exp[-\lambda_m x] + \exp[\lambda_m(x-1)] + \exp[-\lambda_m y] + \exp[\lambda_m(y-1)]\} \tag{4.9}$$

where

$$\lambda_m = m\pi f/\epsilon\delta\,. \tag{4.10}$$

The interior solution (4.3) easily transforms back into physical space. From (3.2), (4.3) and (2.2) we find that

$$\mathbf{u}_I = \mathbf{u}_S(x, y) + \mathbf{u}_E(x, y, z) - \mathbf{u}_{Eo}(x, y) \tag{4.11}$$

and

$$w_I = -\nabla \cdot \left[\int_o^z (\mathbf{u}_E - \mathbf{u}_{Eo})\, dz' \right] \tag{4.12}$$

where $\nabla = (\partial_x, \partial_y)$ and

$$\mathbf{u}_{Eo}(x, y) = \int_0^1 \mathbf{u}_E\, dz'. \tag{4.13}$$

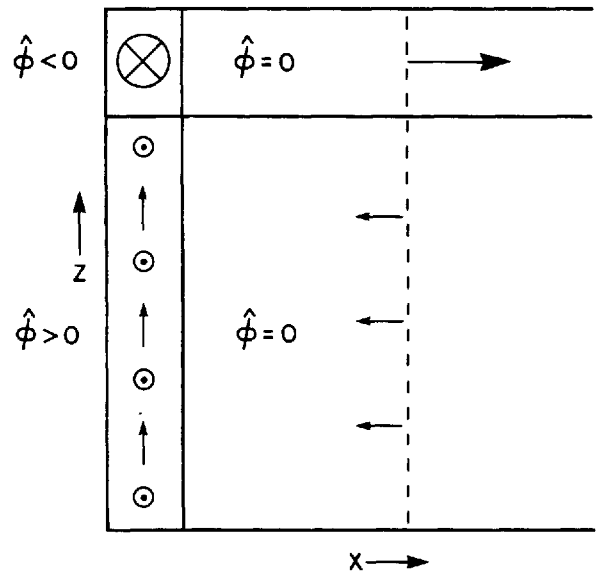
In (4.11) and (4.12) the terms containing $\mathbf{u}_S$ and $\mathbf{u}_{Eo}$ comprise the sub-Ekman layer flow, while the $\mathbf{u}_E$ terms are important only within the Ekman layer. Thus w_I depends linearly on z below the Ekman layer. Note again that the depth average of $\mathbf{u}$ is $\mathbf{u}_S$.

The inverse transformation of (4.9) is not easy, but the boundary layer structure can be deduced from the boundary layer approximations to (2.1). It follows from (4.6) and the known boundary layer thickness $\epsilon\delta$ that, near $x = 0$ (for example), the boundary layer correction variables satisfy

$$\begin{aligned} -f\hat{v} &= -\hat{\phi}_x \\ f\hat{u} &= -\epsilon\hat{v} \\ 0 &= -\hat{\phi}_z - \epsilon\delta^2\hat{w} \\ \hat{u}_x + \hat{w}_z &= 0\,. \end{aligned} \tag{4.14}$$

Thus the correction flow is nondivergent in planes normal to the coastline and the strong longshore boundary current is geostrophic.

Consider the situation shown in Figure 2, in which the wind blows into the page at a constant speed. The interior Ekman transport is offshore and there is a weak compensating flow at depth. In order to cancel the interior normal flow, the boundary layer correction $\hat{u}$ must be strong and negative near the surface, and weak and positive below. Then by (4.14b) the longshore boundary current must be strong and into the



Figue 2. Internal coastal boundary layers in a homogeneous ocean. The offshore Ekman transport creates an upwelling boundary layer with a surface-intensified longshore current.

page in the Ekman layer, and weak and outward below. For these signs of $\hat{v}$, (4.14a) requires that the pressure $\hat{\phi}$ increase with depth. The upward pressure gradient force balances the frictional drag on the vertical velocity in (4.14c).

5. Stratified flow

Now consider the case $S = 0(1)$ of realistic stratification, but suppose for now that $Q = 0$. We begin by showing that the thermal diffusivity k must be order one, or the circulation is unrealistically confined to the surface Ekman layer. This conclusion is unsurprising because $Q = k = 0$ imply $w = 0$ by (2.1d). However, the case $k \rightarrow 0$ has a pedagogical value worth pursuing.

Let k be asymptotically small and expand all dependent variables A in power series in k:

$$A(x, y, z; k, \epsilon, \delta) = A^0(x, y, z; \epsilon, \delta) + k\, A^1(x, y, z; \epsilon, \delta) + \dots \tag{5.1}$$

At the leading order in k, (2.1) reduce to

$$\begin{aligned} -fv^0 &= -\phi_x^0 - \epsilon u^0 - fv_E \\ fu^0 &= -\phi_y^0 - \epsilon v^0 + fu_E \\ u_x^0 + v_y^0 &= 0 \\ w^0 &= 0 \\ \theta^0 &= \phi_z^0 . \end{aligned} \tag{5.2}$$

The first three of (5.2) are, at every level z, identical to (3.1). Thus

$$\mathbf{u}^0(x, y, z) = (-\psi_y, \psi_x) \tag{5.3}$$

where $\psi(x, y, z)$ is the solution to "the Stommel problem at every z," viz.

$$\beta\psi_x = -\epsilon\nabla^2\psi + \nabla \cdot (f\mathbf{u}_E) \tag{5.4}$$

and $\psi = 0$ at coastal boundaries. Again there is a western boundary layer of thickness ϵ and (generally) latitudinal boundary layers of thickness $\epsilon^{1/2}$. However, the motion is *everywhere* confined to the Ekman layer. At greater depths the powerful constraint $w^0 = 0$ prevents vertical vortex stretching and keeps the deep fluid at rest. Since such flow is very unrealistic, we conclude that k must be order one.

Once ψ is known, the pressure ϕ^0 is determined by (5.2a, b) up to a function only of z. We easily find that

$$\phi^0 = f\psi + \int_0^y [fu_E(1, y', z) - \epsilon\psi_x(1, y', z)]dy' + \int_x^1 [fv_E(x', y, z) - \epsilon\psi_y(x', y, z)]dx' + G(z) \tag{5.5}$$

where

$$G(z) = \phi^0(1, 0, z) \tag{5.6}$$

is the undetermined function of z. To determine $G(z)$ (and the leading contribution to w) we must proceed to the next order in k. However, a complete analysis is not required. The important equation is

$$Sw^1 = -\theta^0 = -\phi_z^0 . \tag{5.7}$$

Since

$$\iint dx\, dy\, w^1 = 0 \tag{5.8}$$

at every z, we must have

$$\iint dx\, dy\, \phi_z^0 = 0 . \tag{5.9}$$

Equation (5.9) determines $G(z)$ up to an irrelevant constant.

The *interior* approximations to ψ and ϕ^0 are simply

$$\psi_I = -\beta^{-1}\int_x^1 \nabla \cdot (f\mathbf{u}_E)\, dx' \tag{5.10}$$

and

$$\phi_I^0 = -\beta^{-1}f^2\int_x^1 \nabla \cdot \mathbf{u}_E\, dx' + \int_0^y fu_E(1, y', z)\, dy' + G(z) . \tag{5.11}$$

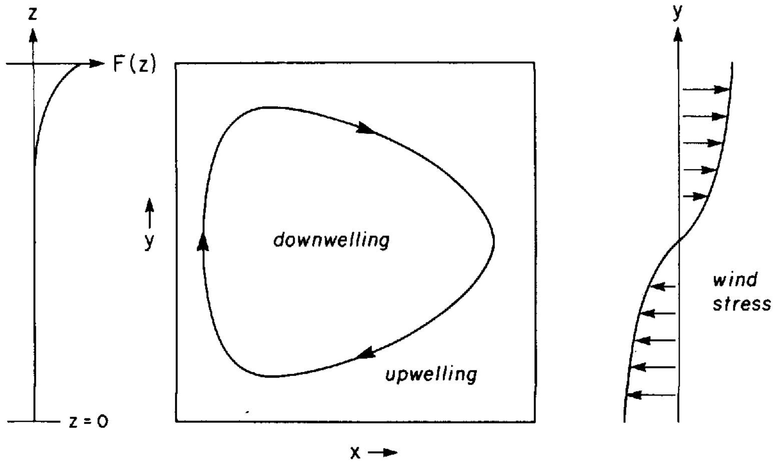


Figure 3. Circulation of a stratified ocean with $k \to 0$, driven by the classical subtropical wind gyre. Only the isobar corresponding to $w = 0$ is drawn.

The latter satisfies the "interior boundary condition"

$$\partial\phi^0/\partial y = fu_E \qquad \text{at } x = 1. \tag{5.12}$$

It is easy to see that the interior region makes the dominant contribution to (5.9). Therefore $G(z)$ is determined by

$$dG/dz = \iint dx\, dy \left\{\beta^{-1}f^2 \int_x^1 \nabla \cdot \mathbf{u}_{E,z}\, dx' - \int_0^y fu_{E,z}(1, y', z)\, dy'\right\}. \tag{5.13}$$

This completes the solution for $k \to 0$.

Suppose, for example, that

$$f\mathbf{u}_E = (0, F(z) \cos \pi y) \tag{5.14}$$

with $F(z) > 0$, $dF/dz > 0$, and $F(z)$ nonzero only near $z = 1$. Refer to Figure 3. This is the classical subtropical wind gyre. Note that $\nabla \cdot (f\mathbf{u}_E) = 0$ at $y = 0, 1$. The uniformly valid solution,

$$\phi^0 = \beta^{-1}f[\pi \sin \pi y\,(1 - x - e^{-\beta x/\epsilon}) - 1]\, F(z) \tag{5.15}$$

(assuming $\beta \ll 1$) has the familiar isobar pattern shown in Figure 3. The vertical velocity is proportional to $-\partial\phi^0/\partial z$ and has zero horizontal average at every z. Thus the isobars are also contours of vertical velocity. There is downwelling in the center of the gyre and upwelling everywhere on the periphery. The upwelling in the western boundary layer has the same magnitude as in the interior, but it makes a negligible contribution to the net vertical mass flux because the boundary layer is so thin. There is no "Ekman suction," but none is needed, because the Ekman layer carries the full Sverdrup transport.

Now suppose $k = 0(1)$ and allow $Q \neq 0$. Since $S = 0(1)$, a uniformly valid approximation to (2.5) is

$$\beta f^{-2} \phi_{m,x} + \epsilon \nabla \cdot (f^{-2} \nabla \phi_m) - m^2 \pi^2 k S^{-1} \phi_m = \nabla \cdot \mathbf{u}_{Em} + m\pi Q_m / S \quad (5.16)$$

with boundary condition (2.6). We assume for convenience that the right-hand side of (5.16) vanishes for m greater than some *finite* m_{max}. This assumption does not excessively restrict $\mathbf{u}_E$, but it does require that $Q = 0$ at top and bottom boundaries. We return below to the case of general $Q(x, y, z)$. For finite m_{max}, there are ϵ or $\epsilon^{1/2}$ Stommel boundary layers everywhere except $x = 1$. The interior pressure $\phi_I(x, y, z)$ can be written explicitly in the form of a vertical convolution integral that reduces to (5.11) in the limit $k \rightarrow 0$. However, it is more illuminating to consider the interior ($\epsilon = 0$) limit of (2.1), viz.

$$\begin{aligned} u_I &= -f^{-1}\phi_{I,y} + u_E \\ v_I &= f^{-1}\phi_{I,x} + v_E \\ S w_I &= -k\,\phi_{I,z} + Q \end{aligned} \quad (5.17)$$

which on substitution into (2.1e) yield

$$\phi_{I,x} + k f^2 \beta^{-1} S^{-1} \phi_{I,zz} = \beta^{-1} f^2 (\nabla \cdot \mathbf{u}_E + Q_z / S) \quad (5.18)$$

with boundary conditions

$$k\phi_{I,z} = Q \quad \text{at } z = 0, \ 1. \quad (5.19)$$

Since there is no boundary layer at $x=1$, ϕ_I must also satisfy

$$\partial \phi_I / \partial y = 0 \quad \text{at } x = 1 \quad (5.20)$$

where we have assumed (for convenience) that $u_E(1, y, z) = 0$.

The general character of ϕ_I can be deduced by analogy to a scalar diffusion problem very similar to that of Section 3. Let $t = 1 - x$, $T(z, t; y) = \phi_I(1 - t, y, z)$, $\kappa(y) = k f^2 / \beta S$ and

$$q = -\beta^{-1} f^2 (\nabla \cdot \mathbf{u}_E + Q_z / S). \quad (5.21)$$

Then (5.18 – 5.20) transform to the diffusion equation,

$$T_t = \kappa T_{zz} + q, \quad t > 0, \quad 1 > z > 0 \quad (5.22)$$

with boundary conditions

$$T_z = Q/k \quad at\ z = 0, 1 \quad (5.23)$$

and initial condition

$$T(z, 0; y) = T_i(z) \tag{5.24}$$

where the initial scalar concentration $T_i(z)$ remains to be given. The latitude y enters (5.22–5.24) only as a parameter. Obviously $T(t, z; y)$ is determined once $T_i(z)$ is given.

To determine $T_i(z)$, we note (from (5.17c), which applies uniformly in x, y, z) that the vertical velocity has the same magnitude throughout the flow. The coastal boundary layers thus make a negligible contribution to the requirement

$$\iint dx\, dy\, w = 0 \tag{5.25}$$

which therefore reduces to

$$\iint dx\, dy\, \phi_{1,z} = \iint dx\, dy\, Q/k, \tag{5.26}$$

that is,

$$d/dz \iint dt\, dy\, T = \iint dx\, dy\, Q/k \tag{5.27}$$

at every z. Equation (5.27) determines $T_i(z)$ and closes the problem.

To see how, suppose (for example) that $Q = 0$ and $\nabla \cdot \mathbf{u}_E$ is negative and independent of $\mathbf{x}$ corresponding to a simple subtropical gyre. Then $q > 0$ in (5.22). Suppose further that the y-dependence of $\kappa(y)$ is negligible (i.e., that $\beta \ll 1$). Then T and ϕ_I are independent of y and (5.27) reduces to

$$\int_0^1 dt\, T = 0. \tag{5.28}$$

The source q is independent of time and acts only near $z = 1$. By time t, this source has penetrated a vertical depth $(\kappa t)^{1/2}$. To satisfy (5.28) the initial scalar concentration $T_i(z)$ must be large near $z = 0$ and small near $z = 1$, so that T has opposite signs near $t = 0$ and $t = 1$. The isolines of T must be as in Figure 4a, which shows the solution to (5.22, 5.23, 5.28) with $Q = 0$, $q = .1 \times \exp[(z - 1)/.15]$ and $\kappa = 1/16$ (corresponding to representative oceanic values). Note that the lowest T is found at $z = 1$ up until some time t_1 at which the source q overcomes the initial T gradient. At time t_1, $\partial T/\partial z$ changes sign.

To translate these results to the circulation problem, it is only necessary to recall that $\phi = T$, $\theta = T_z$, and $w = -kT_z/S$. The *thermocline* depth at longitude x is

$$D(x) = (\kappa t)^{1/2} = \{kf^2(1 - x)/\beta S\}^{1/2}. \tag{5.29}$$

The geostrophic velocity is concentrated above the thermocline and is predominantly southward. The geostrophic velocity increases with increasing x, as the constant

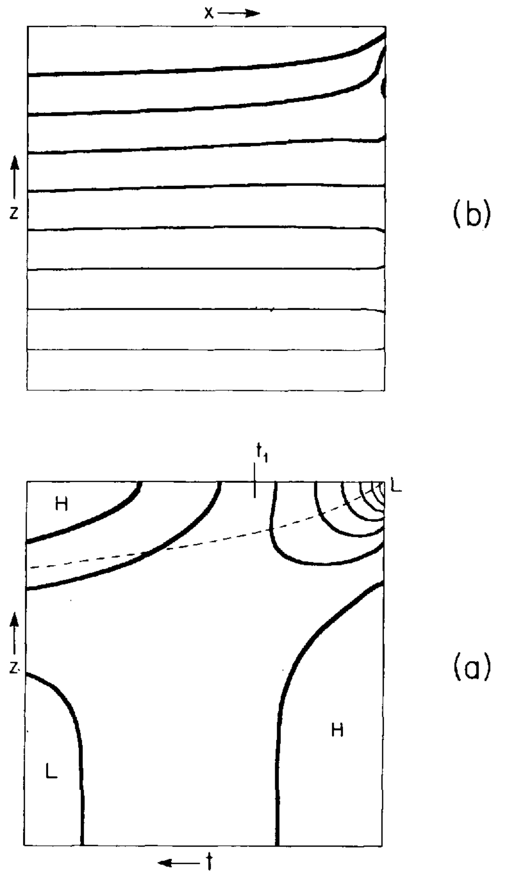


Figure 4. (a) The solution to the scalar diffusion analog (5.22, 5.23, 5.28) with $Q = 0$, $q = .1 \times \exp[(z - 1)/.15]$ and $\kappa = 1/16$. Darker isolines correspond to larger T. The scalar T is analogous to ϕ. The dashed line is at depth $D(x)$. (b) The corresponding ocean temperature θ.

Sverdrup transport is confined to a layer of decreasing thickness. The vertical velocity is negative west of longitude $x_1 = 1 - t_1$, and there is relatively strong upwelling to the east. The corresponding ocean temperature θ is shown in Figure 4b. This temperature includes a mean $\overline{\theta}(z)$ whose vertical gradient is near the minimum required to maintain static stability at $x = 1$.

The temperature equation (5.22) has a similarity solution analogous to that found by Gill (1985) for the standard model. However, the similarity solution applies only to the region west of x_1. The region east of x_1 is misrepresented by an artificial singularity. The singularity has nothing to do with the neglect of boundary layers (there is none at $x = 1$) and is a defect in the similarity solution itself. Since the region east of x_1 is an important part of the interior flow, the similarity solution has a very limited value.

Pedlosky (1969) showed that the *complete* interior solution of the standard model (1.1) contains a function analogous to $T_i(z)$ above that is also determined by the vertical mass flux condition (5.25). In Pedlosky's solution, the interior vertical velocity makes the dominant contribution to (5.25) *except* very near $z = 1$, where contributions from the boundary layers at $y = 0, 1$ must be included. This is logical because (5.25) would otherwise contradict the *standard model* result that the "Ekman suction velocity" depends *solely* on the wind stress curl. In the present example, the interior vertical velocity actually overwhelms the Ekman downwelling east of x_1. This is possible because, in the simplified model, the asymptotic limit $\epsilon \to 0$ does not affect the Ekman layer thickness. The Ekman layer can therefore carry a significant share of the Sverdrup transport. In Pedlosky's solution, this transport is carried *below* the Ekman layer, but, on the other hand, the Ekman layer is infinitesimally thick. There is no real contradiction between the two solutions.

As a final example, suppose that $\mathbf{u}_E = 0$ and

$$Q = 1/2 - y \tag{5.30}$$

corresponding to a depth-independent heating (cooling) in the southern (northern) half of the ocean. Now, because $Q \neq 0$ at $z = 0, 1$, (5.19) contradicts (5.20) at the corners $(x, z) = (1, 0)$ and $(1, 1)$. In this case, tiny corner regions of horizontal thickness ϵ and vertical depth $\epsilon^{1/2}$ must be present. These corner regions, which play no role except to remove the contradiction between (5.19) and (5.20) at the corner, are absent if only the sine expansion (2.2b) of Q truncates at some arbitrarily large but finite $m_{\max}$. In that case the "flux" (5.19) is replaced by an equivalent "source" (cf. Eq. 5.18) within an arbitrarily small distance m_{max}^{-1} of the top and bottom boundaries, and the solution ϕ is negligibly affected.

In the scalar diffusion analog, (5.30) corresponds to $q = 0$. If the y-dependence of κ can again be neglected, then the initial condition $T(z, 0; y) =$ constant satisfies (5.24) and (5.27) by symmetry. The scalar diffusion analog is therefore driven solely by the surface and bottom fluxes of T given by (5.23). These correspond to surface influx/bottom efflux of T on $y < 1/2$ and surface efflux/bottom influx of T on $y > 1/2$. The solution must resemble the sketches in Figure 5. Remember that T is analogous to the pressure ϕ. The boundary fluxes of T cause top and bottom thermocline layers of vertical thickness $D(x)$ (cf. Eq. 5.29). The horizontal fluid motion is confined to these layers. South of $y = 1/2$, the horizontal velocity is southward in the surface thermocline layer and northward at the bottom. The vertical velocity is positive, as is most easily deduced from the vorticity equation,

$$w_{I,z} = \beta v_I / f. \tag{5.31}$$

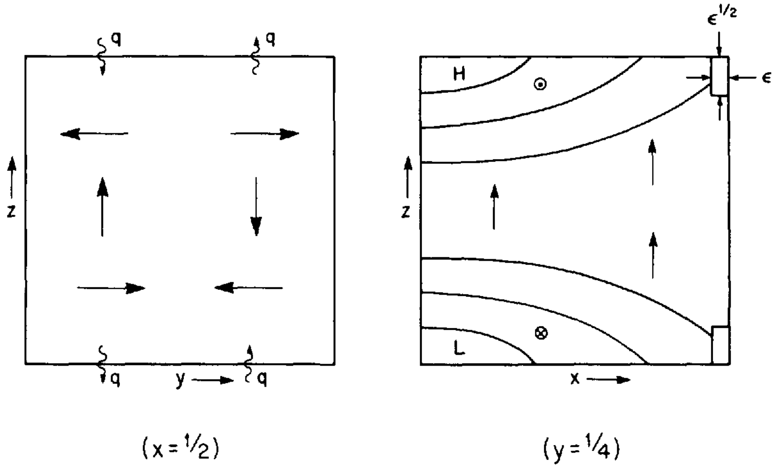


Figure 5. The circulation resulting from a depth-independent heating (cooling) in the southern (northern) half of the ocean. The curves in the lower panel are analogous to isobars. The symbol q represents the surface flux in the scalar diffusion analog.

All velocities change sign north of $y = 1/2$. Stommel boundary layers of thickness ϵ and $\epsilon^{1/2}$ close the horizontal flow at every z. Again, these boundary layers make a negligible contribution to the vertical mass flux.

Outside the thermocline layers, T and hence ϕ is uniform, θ is zero, and the dominant balance in the heat equation (5.17c) is between the diabatic heating Q and the vertical advection of the mean temperature gradient. The vertical velocity has the z-independent value $w = Q/S$. Within the thermocline layers, w adjusts to meet the boundary conditions $w = 0$ at $z = 0, 1$. The resultant vortex stretching causes horizontal motion by the Sverdrup relation (5.31). Since w changes by Q/S across both thermocline layers, the Sverdrup transport of both layers is independent of x. Within the thermocline layers (and particularly near $z = 0, 1$ where $w = 0$) the balance in the heat equation is between the diabatic heating and the temperature diffusion $k\theta$.

The stratified solutions of this section are all strongly controlled by the temperature diffusion parameter k. Unlike ϵ, k must be order one, and temperature diffusion is therefore important both near and far from boundaries. However, the temperature equation (2.1d) reveals that the important dynamical role of temperature diffusion is an artifact of linear theory. If $Q = 0$ there can be no vertical motion, and hence no vertical vortex stretching, without a compensating temperature diffusion. But this tight connection between diffusion and vertical velocity would disappear if the horizontal temperature advection terms had been retained in (2.1d). Thus it is the linearization of the heat equation about a state with *flat* isopycnals that is really responsible for the unrealistically diffusive character of the present solutions.

A sequel paper will report numerical solutions of the "nonlinear thermocline equations" analogous to (2.1), in which the linear heat equation (2.1d) is replaced by the nonlinear equation

$$D\theta/Dt = Q + K_h\nabla^2\theta + K_v\theta_{zz}. \tag{5.32}$$

Eq. (5.32) steps θ forward in time, and the velocity at the new time is determined from θ and the remaining equations (2.1a, b, c, e). The latter combine to yield an equation analogous to (2.5), viz.,

$$\epsilon\delta^2[\beta f^{-2}\phi_{m,x} + \epsilon\nabla \cdot (f^{-2}\nabla\phi_m) - \nabla \cdot \mathbf{u}_{Em}] = m\pi[m\pi\phi_m + \theta_m] \tag{5.33}$$

with boundary condition (2.6). In (5.33) θ_m is a forcing term analogous to Q_m in (2.5). The vertically-averaged ($m = 0$) velocity is the same as in Section 3. For $m \neq 0$, coastal boundary layers occur wherever the interior solution

$$\phi_{mI} = -\theta_m/m\pi \tag{5.34}$$

fails to satisfy (2.6). Eq. (5.34) is a statement of hydrostatic balance. The coastal boundary layers have the same thickness $\epsilon\delta$ and the same physical balances as the nonhydrostatic upwelling layers described in Section 4 for the *unstratified* linear model. The dominant balance in the boundary condition (2.6) is

$$\begin{aligned}\epsilon f^{-1}\partial\hat{\phi}_m/\partial n &= f\mathbf{u}_{Em} \cdot \mathbf{n} - \partial\phi_{mI}/\partial s \\ &= f\mathbf{u}_{Em} \cdot \mathbf{n} + m^{-1}\pi^{-1}\partial\theta_m/\partial s\end{aligned} \tag{5.35}$$

where $\hat{\phi}$ is the boundary layer correction pressure. Thus an upwelling boundary layer exists wherever either the Ekman flow or the geostrophic thermal wind impinges on a coastline. Preliminary numerical experiments suggest that these nonhydrostatic upwelling layers are important components of the large-scale circulation even in cases where the horizontally-averaged Ekman suction is zero and there is no Ekman flow into any coast.

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