Journal of Marine Research, 42, 731–735, 1984

On the equivalence of nonlocal and radial-diffusion models for porewater irrigation

by Bernard P. Boudreau[1]

Aller (1977, 1978, 1980) and Aller and Yingst (1978) have demonstrated that the presence of well-irrigated worm-tubes and other animal burrows in sediments can significantly alter the rate of exchange of solutes between porewater and the overlying waters. Specifically, the burrows modify the geometry of the porewater system such that solutes can diffuse toward or away from either the sediment-water interface or the burrow. The burrow constitutes an additional boundary source or sink.

To assess this effect quantitatively in marine sediments, Aller (1978, 1980) has utilized a mathematical model wherein the sediment is idealized as a collection of regularly packed annulli, all of length L. Each annulus has a vertical hollow region corresponding to the burrow or tube and an outer solid region of sediment extending to the half-distance between burrows. Both the radius of the hollow region, r_1, and the outer radius of the solid region, r_2, must be determined from observation. In addition, the model assumes that the only processes affecting a solute are molecular diffusion, advection due to burial and chemical reaction, and that the fluid in the burrow is maintained at the overlying concentration by pumping. The governing conservation equation is of the form:

$$\frac{\partial C}{\partial t} = D_s \frac{\partial^2 C}{\partial x^2} - u \frac{\partial C}{\partial x} + \frac{D_s}{r} \frac{\partial}{\partial r} r \frac{\partial C}{\partial r} + k(C_{eq} - C) + R \tag{1}$$

where t = time

x = vertical distance in the sediment

r = radial distance from the center of a burrow

$C = C(x, r, t)$ = pointwise concentration

D_s = solute molecular diffusion coefficient corrected for tortuosity

k = first order rate constant

C_{eq} = equilibrium concentration

$R = R(x, r, t)$ = an inhomogeneous reaction term

1. Department of Geology and Geophysics, Yale University, P.O. Box 6666, New Haven, Connecticut, 06511, U.S.A.

In writing Eq. (1), we have assumed constant porosity. This is not however a crucial postulate and the results presented here are equally true if porosity is variable. The forms of the reaction terms are not the most general, but certainly allow for a large number of possible phenomena of interest. Furthermore, any diffusive impedance from the burrow linings (Aller, 1983) is ignored. The appropriate boundary conditions are (Aller, 1980)

$$C = C_0 \qquad x = 0 \tag{2a}$$

$$C = C_0 \qquad r = r_1 \tag{2b}$$

$$\frac{\partial C}{\partial r} = 0 \qquad r = r_2 \tag{2c}$$

$$\frac{\partial C}{\partial x} = B \qquad x = L \tag{2d}$$

where B is a given flux. Furthermore, an initial condition is given if steady-state is not assumed.

Eq. (1) is linear and can be solved (in theory) by the standard methods of mathematical physics (i.e., separation of variables, etc.). However, in order to preserve the simplicity of the one space-dimension approach, and to avoid the necessity of making measurements of r_1, r_2 and L, an alternative model for burrow-induced irrigation has been proposed (e.g., Emerson *et al.* this issue). This model suggests that irrigation can be accounted for by a source/sink term that permits the exchange of porewater from any depth with the overlying water. This is not an advective process as envisioned by Hammond and Fuller (1979) or Grundmanis and Murray (1977) but, rather, a nonlocal exchange as defined by Imboden (1981). In this case, the governing conservation equation for a solute is

$$\frac{\partial \overline{C}}{\partial t} = D_s \frac{\partial^2 \overline{C}}{\partial x^2} - u \frac{\partial \overline{C}}{\partial x} - \alpha(\overline{C} - C_0) + k(C_{eq} - \overline{C}) + \overline{R} \tag{3}$$

where $\overline{C} = \overline{C}(x, t)$ = laterally averaged concentration
$\alpha = \alpha(x, t)$ = fraction exchanged per unit time
$\overline{R} = R(x, t)$ = laterally averaged inhomogeneous reaction term

The irrigation effects of the burrows are incorporated in the third term on the right hand side of Eq. (3). The exchange parameter $\alpha(x, t)$ must be determined empirically.

It is the aim of this note to show that the radial-diffusion model (Eq. 1) and the nonlocal model (Eq. 3) are equivalent under a relatively unrestrictive condition. Specifically, we assert that Eq. (3) is simply the radially integrated form of Eq. (1).

To begin the proof, we note that the laterally averaged concentration is simply the radial average over the outer zone of an annulus, i.e.

$$\overline{C} \equiv \frac{2\pi \int_{r_1}^{r_2} rC\,dr}{2\pi \int_{r_1}^{r_2} r\,dr} \equiv \frac{2\pi \int_{r_1}^{r_2} rC\,dr}{2\pi(r_2^2 - r_1^2)} \tag{4}$$

To obtain the conservation equation for $\overline{C}$ from Eq. (1), we begin by multiplying each term of Eq. (1) by $r/r^* = r/(r_2^2 - r_1^2)$. This quantity can enter the derivatives with respect to x and t to produce,

$$\frac{\partial}{\partial t}\frac{rC}{r^*} = D_s \frac{\partial^2}{\partial x^2}\frac{rC}{r^*} - u\frac{\partial}{\partial x}\frac{rC}{r^*} + \frac{D_s}{r^*}\frac{\partial}{\partial r} r \frac{\partial C}{\partial r} + k\left(\frac{rC_{eq}}{r^*} - \frac{rC}{r^*}\right) + \frac{rR}{r^*}. \tag{5}$$

Next, we integrate each term of Eq. (5) from r_1 to r_2, inverting the order of differentiation and integration for those terms involving derivatives of x and t. Finally, we apply the definition given by Eq. (4) to arrive at

$$\frac{\partial \overline{C}}{\partial t} = D_s \frac{\partial^2 \overline{C}}{\partial x^2} - u\frac{\partial \overline{C}}{\partial x} + \frac{D_s}{r^*}\left[r\frac{\partial C}{\partial r}\right]_{r_1}^{r_2} + k(C_{eq} - \overline{C}) + \overline{R}. \tag{6}$$

The elimination of the remaining derivative in r is a two-step procedure. First, using boundary condition (2c), we obtain

$$\left[r\frac{\partial C}{\partial r}\right]_{r_1}^{r_2} = -r_1 \left.\frac{\partial C}{\partial r}\right|_{r=r_1}. \tag{7}$$

The value of the radial gradient at the burrow wall, i.e. $r = r_1$, must now be evaluated in terms of either known parameters and/or $\overline{C}(x, t)$. To accomplish this task, we make use of the Mean Value Theorem for integrals which states (Pearson, 1974):

> if $C(x, r, t)$ is continuous on the closed interval r_1 to r_2, then the mean value, i.e., $\overline{C}(x, t)$, occurs at a point $\bar{r}$ within this interval, i.e. $r_1 < \bar{r} < r_2$.

Given this fact, it is reasonable to believe that the linear gradient between the burrow concentration, C_0, and the average value $\overline{C}(x, t)$ may adequately approximate the actual gradient at $r = r_1$. A formal derivation of this approximation begins by expressing $\overline{C}(x, t)$ as a Taylor Series expansion of $C(x, r, t)$ about the point $r = r_1$,

$$\overline{C}(x, t) = C_0 + \left.\frac{\partial C}{\partial r}\right|_{r=r_1}(\bar{r} - r_1) + \left.\frac{\partial^2 C}{\partial r^2}\right|_{r=r_1}\frac{(\bar{r} - r_1)^2}{2!} + \cdots \tag{8}$$

If and only if the second and higher order terms in the series given by Eq. (8) are neglected, can the desired approximation be obtained

$$\left.\frac{\partial C}{\partial r}\right|_{r=r_1} \simeq \frac{\overline{C} - C_0}{(\overline{r} - r_1)}. \tag{9}$$

Substitution of Eq. (7) and Eq. (9) into Eq. (6) produces:

$$\frac{\partial \overline{C}}{\partial t} = D_s \frac{\partial^2 \overline{C}}{\partial x^2} - u \frac{\partial \overline{C}}{\partial x} - \frac{D_s r_1}{r^* (\overline{r} - r_1)} (\overline{C} - C_0) + k(C_{eq} - \overline{C}) + \overline{R}. \tag{10}$$

Comparing Eqs. (3) and (10), we see that these equations (and models) are identical if we simply write the equality

$$\alpha(x, t) = \frac{D_s r_1}{(r_2^2 - r_1^2)(\overline{r} - r_1)}. \tag{11}$$

The radial-diffusion model for porewater irrigation can be reduced to a nonlocal transport model under the restriction imposed by Eq. (9). This conversion is not simply academic but affords a dramatic mathematical simplification when steady-state is operative (e.g., Emerson *et al.,* this issue). The validity of the approximation given by Eq. (9) will prove difficult to establish conclusively without an extensive and demanding sampling program; however, it is the opinion of this author that Eq. (9) will prove to be surprisingly robust.

Finally, we note that although the radial-diffusion model can be reduced to a nonlocal transport model, not all nonlocal models are equivalent to radial-diffusion models. A large class of possible transport phenomena can be represented by nonlocal models.

Acknowledgments. The author wishes to acknowledge Dr. Steven Emerson whose work catalysed this research. Research supported by NSF Grant OCE 8219580.

REFERENCES

Aller, R. C. 1977. The influence of macrobenthos on chemical diagenesis of marine sediments. Ph.D. dissertation, Yale University. New Haven, CT, 600 pp.

——1978. Experimental studies of changes produced by deposit feeders on pore water, sediment, and overlying water chemistry. Am. J. Sci., *278,* 1185–1234.

——1980. Quantifying solute distributions in the bioturbated zones of marine sediments by defining an average microenvironment. Geochim. Cosmochim. Acta. *44,* 1955–1966.

——1983. The importance of the diffusive permeability of animal burrow linings in determining marine sediment chemistry. J. Mar. Res., *41,* 299–322.

Aller, R. C. and J. Y. Yingst. 1978. Biogeochemistry of tube-dwellings: A study of the sedentary polychaete *Amphitrite Ornata* (Leidy). J. Mar. Res., *36,* 201–254.

Grundmanis, V. and J. W. Murray. 1977. Nitrification and denitrification in marine sediments from Puget Sound. Limnol. Oceanogr., *22,* 804–813.

Hammond, D. E. and C. Fuller. 1979. Use of Radon-222 to estimate benthic exchange and

atmospheric exchanges in San Francisco Bay, *in* San Francisco Bay, The Urbanized Estuary, T. J. Conomos, ed., Amer. Assoc. Adv. Sci., San Francisco, CA. 213–230.
Imboden, D. M. 1981. Tracers and mixing in the aquatic environment. Habilitation Thesis, Swiss Federal Institute of Technology, Dubendorf, Switzerland, 137 pp.
Pearson, H. L. 1974. Elements of analysis, *in* Handbook of Applied Mathemetics, C. E. Pearson, ed., Van Nostrand Reinhold Co., NY, 82–126

Received: *17 April, 1984.*