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A note on frictional effects in Taylor's problem

by Michele M. Rienecker[1] and Michael D. Teubner[2]

ABSTRACT

Taylor's tidal problem of the reflection of a Kelvin wave in a semi-infinite rotating channel is modified here by considering the effect of the inclusion of friction in the analysis. Results are obtained using Galerkin and Collocation methods to satisfy the end boundary condition, and these are compared with results given by other authors for the nonfriction case.

1. Introduction

The reflection of a Kelvin wave at the closed end of a semi-infinite rectangular channel has been studied by Taylor (1920) and Defant (1925: see Defant 1961, pp. 202-219) who solved the problem without considering any energy dissipation mechanism. Hendershott and Speranza (1971) generalized the model to include localized dissipation at the head of the channel, while Brown (1973) allowed for the presence of propagating Poincaré waves but not for dissipation of energy.

When the effect of bottom friction is included, all Poincaré modes become propagating modes though the amplitudes of all reflected waves decay exponentially away from the end barrier thus limiting the propagation of energy back up-channel. The effect of friction on water elevations and the position of amphidromic points in the channel is studied here. A comparison is also made between Galerkin and Collocation methods used to ensure that the end boundary condition is satisfied.

1. School of Mathematics, University of New South Wales, P.O. Box 1, Kensington, N.S.W. 2033, Australia.
2. Tetra Tech. Inc., 630 North Rosemead Blvd., Pasadena, California 91107, U.S.A.

2. The problem and method of solution

2.1 The equations and the end condition. Consider a rectangular semi-infinite channel of uniform depth h, occupying the region $x \geqslant 0$, $0 \leqslant y \leqslant b$, (x,y) being horizontal coordinates. The response of the water in the channel to an inward travelling wave, of frequency ω or period of oscillation T, is assumed to be governed by the linearized continuity and momentum equations (see, e.g. Heaps, 1969). Manipulation of these equations yields

$$\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} + k^2 Z = 0 , \tag{1}$$

$$(U,V) = \frac{-g}{f^2+\beta^2}\left(\left[\beta\,\frac{\partial Z}{\partial x} + f\,\frac{\partial Z}{\partial y}\right] , \left[-f\,\frac{\partial Z}{\partial x} + \beta\,\frac{\partial Z}{\partial y}\right]\right), \tag{2}$$

where $\mathrm{Re}\{Z(x,y)e^{-i\omega t}\}$ is the elevation of the free surface above mean-sea-level, $\mathrm{Re}\{Ue^{-i\omega t}\}$, $\mathrm{Re}\{Ve^{-i\omega t}\}$ denote the depth-averaged velocities in the x- and y-directions respectively,

f is the Coriolis parameter

g is the acceleration due to gravity $\Big\}$ assumed constant ,

γ is a frictional parameter

$\beta = \gamma - i\omega$

and $k^2 = \dfrac{i\omega}{\beta gh}(f^2 + \beta^2)$.

These equations are solved subject to the conditions

$$V(x,y) = 0 \text{ at } y = 0 \text{ and } y = b, \quad \text{for } x \geqslant 0 , \tag{3}$$

$$U(0,y) = 0 \quad \text{for } 0 \leqslant y \leqslant b \tag{4}$$

and the radiation condition that no other inward travelling waves are excited by the input wave.

The solution to (1) which also satisfies condition (3) and the radiation condition is found to be

$$Z(x,y) = e^{-iK_0 x + \alpha_0 y} + G_0 e^{iK_0 x - \alpha_0 y}$$

$$+ \sum_{n=1}^{\infty} G_n \frac{e^{iK_n x}}{(\beta\alpha_n - ifK_n)}\left\{\beta\alpha_n \cos\frac{n\pi y}{b} - fK_n \sin\frac{n\pi y}{b}\right\} \tag{5}$$

where $\alpha_0 = -ifK_0/\beta$, $K_0 = \{i\omega\beta/gh\}^{\frac{1}{2}}$ and $\alpha_n = in\pi/b$, $K_n = \{\alpha_n^2 + k^2\}^{\frac{1}{2}}$, $\mathrm{Re}\{K_n\} > 0$, $Im\{K_n\} > 0$ for $n \in Z^+$.

The first term on the right-hand side of (5) represents the incident Kelvin wave, the second term the reflected Kelvin wave and the last term the Poincaré waves. The velocities may be found from (2) and so the remaining condition (4) is used to determine the complex coefficients G_n.

Exact solutions for these coefficients cannot be obtained because of the infinite summation, but an approximate solution may be found for a finite number of the G_n by using a method of weighted residuals (see Finlayson, 1972) to satisfy (4). An approximation to $U(x,y)$ is given by the truncated series

$$U_T(x,y) = g \; \frac{\alpha_0}{f} \; \{-e^{-iK_0 x + \alpha_0 y} + G_0 e^{iK_0 x - \alpha_0 y}\}$$

$$- ig \sum_{n=1}^{N} G_n \; \frac{K_n \alpha_n e^{iK_n x}}{(\beta \alpha_n - if K_n)} \left\{ \cos \frac{n \pi y}{b} - \phi_n \sin \frac{n \pi y}{b} \right\}, \tag{6}$$

$$\text{where } \phi_n = if\omega/gh\alpha_n K_n, \; n \in Z^+ .$$

Then the $(N+1)$ coefficients in (6) which approximately satisfy (4) are found by solving the $(N+1)$ linear simultaneous equations

$$\int_0^b U_T(0,y) w_m(y) dy = 0, \quad m = 0,1,\ldots,N \tag{7}$$

where the $w_m(y)$ are chosen weighting functions. The solution obtained as $N \to \infty$ will be an exact solution if the $\{w_m(y)\}$ form a complete set.

2.2 The method of Collocation. If the weighting functions are chosen such that $w_m(y) = \delta(y - y_m)$, where $\delta(y)$ is the Dirac delta function and the y_m are chosen points along the boundary $x = 0$, the method is referred to as Collocation. This is the treatment used by Brown (1973) and the system of equations to be solved is simply

$$\frac{\alpha_0}{f} G_0 e^{-\alpha_0 y_m} - \sum_{n=1}^{N} G_n \; \frac{iK_n \alpha_n}{(\beta \alpha_n - if K_n)} \left\{ \cos \frac{n \pi y_m}{b} - \phi_n \sin \frac{n \pi y_m}{b} \right\}$$

$$= \frac{\alpha_0}{f} e^{\alpha_0 y_m}, \quad m = 0,1,\ldots,N. \tag{8}$$

Two sets of Collocation points are used in section 2.4, viz.

$$y_m = (m + \tfrac{1}{2})b/(N+1) \tag{9}$$

$$\text{and } y_m = mb/N. \tag{10}$$

2.3 The Galerkin technique. If the $w_m(y)$ are chosen from the series in equation (6), that is

$$w_0(y) = e^{-\alpha_0 y},$$

$$w_m(y) = \cos \frac{m \pi y}{b} - \phi_m \sin \frac{m \pi y}{b}, \; m = 1,\ldots,N, \tag{11}$$

then the method is called a Galerkin technique. Using these weighting functions in (7) yields the equations

$$G_0 \frac{\alpha_0}{f} [1 - e^{-2\alpha_0 b}] - \sum_{n=1}^{N} G_n \frac{i\alpha_n K_n}{(\beta\alpha_n - ifK_n)} \frac{\left(\alpha_0 - \frac{n\pi}{b} \phi_n\right)}{(\alpha_0^2 - \alpha_n^2)} [1 - (-1)^n e^{-\alpha_0 b}]$$

$$= \frac{\alpha_0 b}{f} \tag{12a}$$

and

$$G_0 \frac{\alpha_0}{f} \frac{\left(\alpha_0 - \frac{m\pi}{b} \phi_m\right)}{(\alpha_0^2 - \alpha_m^2)} [1 - (-1)^m e^{-\alpha_0 b}] - G_m \frac{i\alpha_m K_m b}{2(\beta\alpha_m - ifK_m)} (1 + \phi_m^2)$$

$$- \sum_{\substack{n=1 \\ n \neq m}}^{N} G_n \frac{i\alpha_n K_n}{(\beta\alpha_n - ifK_n)} \frac{\left(\frac{n\pi}{b} \phi_n - \frac{m\pi}{b} \phi_m\right)}{(\alpha_n^2 - \alpha_m^2)} [1 - (-1)^{n+m}]$$

$$= -\frac{\alpha_0}{f} \frac{\left(\alpha_0 + \frac{m\pi}{b} \phi_m\right)}{(\alpha_0^2 - \alpha_m^2)} [1 - (-1)^m e^{\alpha_0 b}], \, m = 1, \ldots, N. \tag{12b}$$

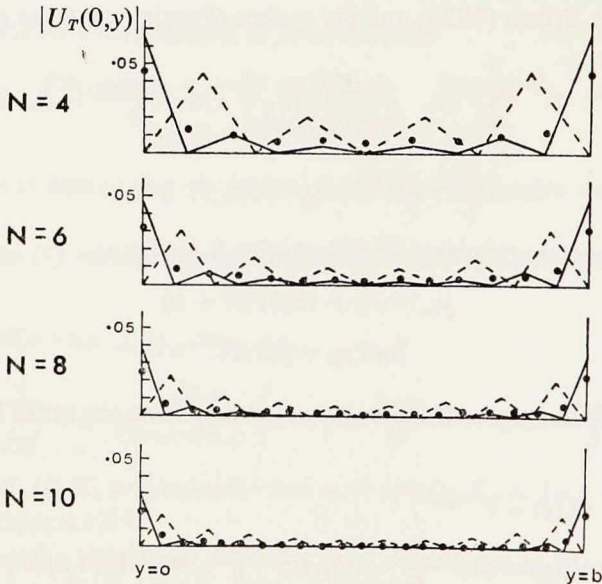These equations have been solved numerically.

Figure 1. The values (in m/sec) of $|U_T(0,y)|$ for $T = 12.4$ hrs and $\gamma = 0.0$. • indicates the results from the Galerkin technique, ———— indicates results from Collocation using equation (9) and – – – – – results from Collocation using equation (10).
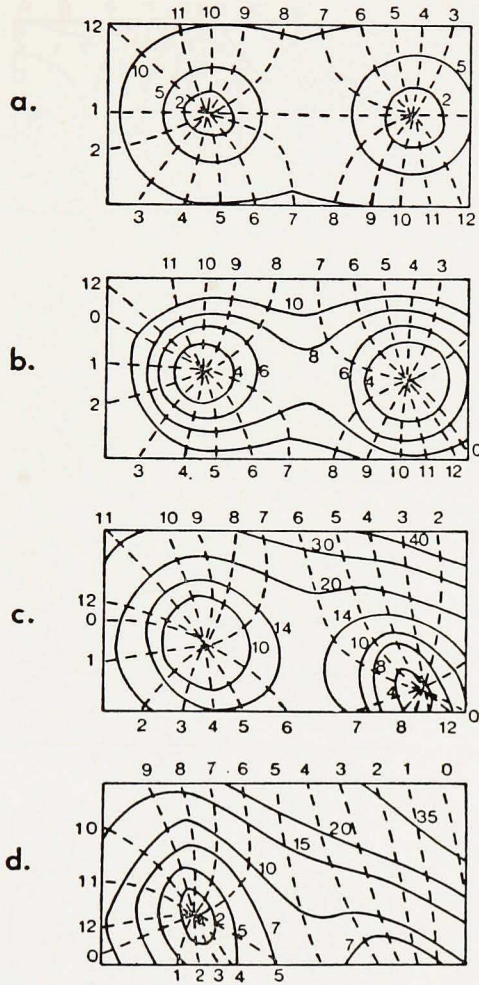
Figure 2. Co-amplitude (————) and co-tidal (– – – –) lines for $T = 12.4$ hrs. The phases are shown in hours and the amplitudes in any units (with $Z(2b,0)$ shown as 10 units). (a) $\gamma = 0.0$, (b) $\gamma = 0.00001$, (c) $\gamma = 0.00005$, (d) $\gamma = 0.0001$.

2.4 Comparison between the two methods. Figure 1 shows, for the case $T = 12.4$ and $\gamma = 0$, a comparison of the values $|U_T(0,y)|$ calculated using the Galerkin and the Collocation techniques for increasing values of N. Both methods show satisfactory convergence as N increases, the Galerkin method giving better results toward the side walls than the Collocation method which uses y_m as defined by (9). If the y_m are chosen according to (10), the solution is exact for the side walls, but the modulus of the error is much larger away from the corner positions. Hence, the Galerkin technique gives a better over-all result and is the method used to obtain
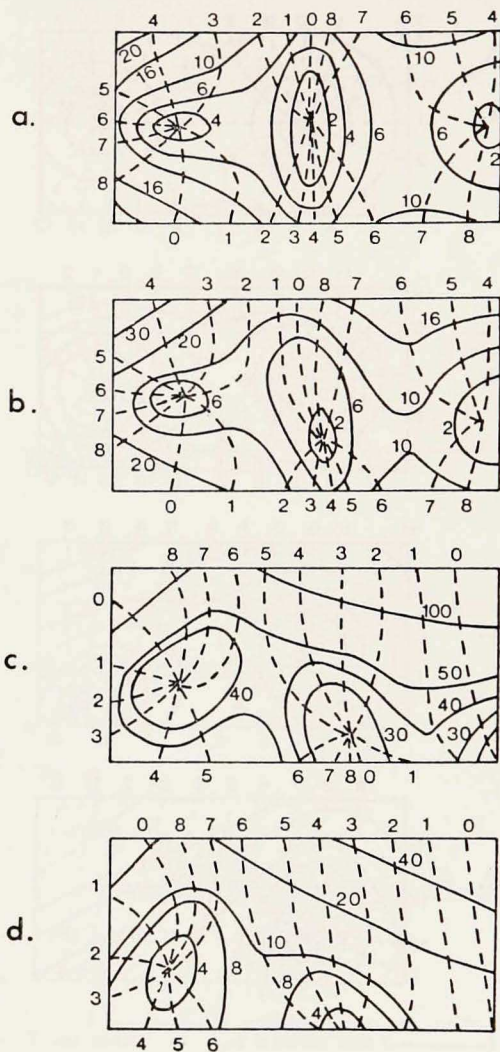
Figure 3. Co-amplitude (———) and co-tidal (– – – –) lines for $T = 9.0$ hrs. (a) $\gamma = 0.0$, (b) $\gamma = 0.00001$, (c) $\gamma = 0.00005$, (d) $\gamma = 0.0001$.

the results presented in the next section. For the case $\gamma = 0$, the resulting co-amplitude and co-phase patterns coincide with those obtained by Brown.

3. Results

The solution determined in the previous section has been applied to a rectangular channel of width $b = 500.5$ km and depth $h = 74$ m; the value of f was taken to be .000119 sec^{-1}, corresponding to the Coriolis parameter at latitude 54.46°N. These values coincide with those used by Taylor (1920) and Brown (1973). The results

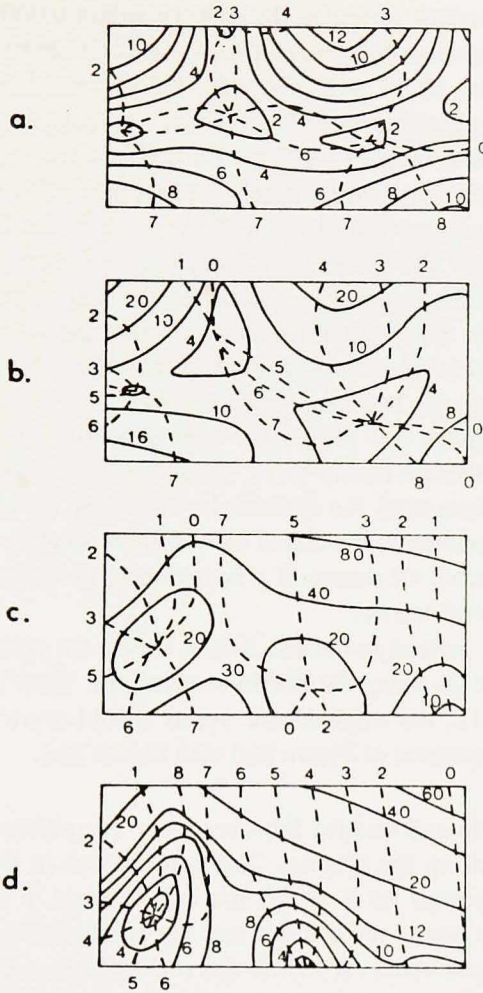Figure 4. Co-amplitude (———) and co-tidal (– – – –) lines for $T = 8.3$ hrs. (a) $\gamma = 0.0$, (b) $\gamma = 0.00001$, (c) $\gamma = 0.00005$, (d) $\gamma = 0.0001$.

were scaled so that $Z(2b,0) = 1.0$; this does not affect the relative amplitude and phase throughout the channel, nor does it affect the position of amphidromic points.

For the nonfrictional case, for each rectangular channel there is a critical period T_{cr} such that if $T > T_{cr}$ the incident wave is perfectly reflected, whereas if $T < T_{cr}$ Poincaré waves propagate back up the channel. For the channel described above, Brown found that $T_{cr} \approx 8.46$ hrs.

(a) $T > T_{cr}$

Solutions obtained for the case $T > T_{cr}$, using $T = 12.4$ hours are shown in Figure 2. When $\gamma = 0$ (Fig. 2(a)), the co-amplitude lines are symmetric about the

central channel line; as γ is increased through the values 0.00001, 0.00005, 0.0001 (Figs. 2(b)-(d) respectively), this symmetry is lost as the amplitude of the reflected waves becomes smaller. For $T = 12.4$ hrs, the Poincaré modes decay too quickly to be considered as propagating any reflected energy away from the end barrier; their presence is more obvious for the case $T = 9.0$ hrs (Fig. 3).

As γ is increased the long-channel spacing of amphidromic points is little changed but the amphidromes move toward the wall along which the reflected Kelvin wave travels. This result was also found by Hendershott and Speranza (1971); however with an energy-absorbing barrier the distance of the amphidromes from the wall is constant with x, while with bottom friction, as x increases the amphidrome moves closer to the wall (this effect is suggested in an Appendix of Hendershott and Speranza) until it actually becomes virtual (beyond the channel boundaries) and eventually lost completely (see Fig. 2(d)). This is because the presence of bottom friction causes the reflected Kelvin wave, as well as the Poincaré waves, to decay up-channel. As γ is increased, the distance in which any of the reflected waves decays to e^{-1} of its amplitude at the closed end becomes smaller until there is little or no reflected energy at all. Of course, if γ is too large, the input wave itself dies out before it reaches the end barrier.

For the case of a perfectly-reflected Kelvin wave, the spacing between amphidromic points is $\lambda_0/2$ (λ_0 being the Kelvin wavelength). Since $\lambda_0 \propto \omega^{-1}$ or T, as T is decreased toward T_{cr}, the amphidromic points should move closer together. This is borne out by a comparison of Figure 2(a) with Figure 3(a).

(b) $T < T_{cr}$

The co-amplitude and co-tidal lines are more complicated when the Poincaré waves propagate back up the channel. This can be seen in Figure 4 which shows the amplitudes and phases for $T = 8.3$ hrs and different γ values. When $\gamma = 0$, there is no symmetry across the channel and the amphidromic points no longer lie along the central channel line. As well as the real amphidromes (situated inside the channel boundaries) there are virtual ones which were not present in the non-frictional results for $T > T_{cr}$. However, the effect of increasing γ is similar to the case $T > T_{cr}$ since, for $\gamma > 0$ the reflected modes decay up-channel. As γ increases the interference pattern tends to be governed more by γ than by T.

4. Conclusion

This paper has shown the effect of including friction in Taylor's tidal problem for a rotating semi-infinite channel. The results presented coincide with those of Brown (1973) for the situation $\gamma = 0$ and confirm a suggestion of Hendershott and Speranza (1971) regarding the effect of bottom friction on the position of amphidromic points. As friction increases, the amphidromes tend to move toward the wall along which the reflected Kelvin wave travels, until they are lost completely. For large values of

γ, the co-amplitude and co-phase patterns tend to be independent of T, until γ becomes too large and the input wave dies out before reaching the end barrier.

REFERENCES

Brown, P. J. 1973. Kelvin wave reflection in a semi-infinite canal, J. Mar. Res., *31*, 1–10.

Defant, A. 1961. Physical Oceanography, Vol. 2, Pergamon Press, Oxford, 598 pp.

Finlayson, B. A. 1972. The Method of Weighted Residuals and Variation Principles, Academic Press, New York, 412 pp.

Heaps, N. S. 1969. Some notes on tidal theory and its possible relevance to a program of deep-sea tidal measurement, Dtsche Hydrograph. Zeit., *22*, 11–25.

Hendershott, M. C. and A. Speranza. 1971. Co-oscillating tides in long narrow bays; the Taylor problem revisited, Deep Sea Res., *18*, 959–980.

Taylor, G. I. 1920. Tidal oscillations in gulfs and rectangular basins, Proc. Lond. Math. Soc., (2), *20*, 148–181.