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Bottom frictional stresses and longshore currents due to waves with large angles of incidence

by Philip L-F. Liu[1] and Robert A. Dalrymple[2,3]

ABSTRACT

The analytical forms of the time-averaged bottom shear stress are developed in this paper. The effects of the angle between the direction of wave propagation and the mean currents, and a large angle of wave incidence are included in the study. Two different friction models were obtained based on the relative magnitudes of wave orbital velocity and that of mean currents. These two friction models are applied to longshore currents generated by obliquely incident waves. The lateral mixing is ignored and the beach contours are assumed to be straight and parallel. The strong current model, used when the mean currents are greater than the wave's orbital velocity, is compared with laboratory data. Very good agreement is found. The regions of validity of these two theories are discussed in terms of the angle of incident waves, the slope of the beach, and the bottom friction coefficient.

1. Introduction

In recent years, considerable progress has been made in the understanding of nearshore breaking wave-induced currents, largely due to the introduction of the concept of radiation stresses by Longuet-Higgins and Stewart (1964). The investigation of longshore and rip currents has been undertaken by numerous authors (e.g., Birkemeier and Dalrymple, 1975; Bowen, 1969a, b; Jonsson, *et al.*, 1974; Komar, 1975; LeBlond and Tang, 1974; Liu and Mei, 1976; Longuet-Higgins, 1970; Mei and Liu, 1977; Noda, 1973; Noda *et al.*, 1974 and Sonu, 1972) using radiation stresses in the equation of motion for the mean water motion. The equations which are used for the flows (see Phillips, 1966) are averaged over the water depth and also over a wave period. An important term in these equations is the mean bottom friction stress exerted by the oscillating wave motion coupled with a mean current.

In the works mentioned above, the mean bottom friction stress has been derived based on two rather severe assumptions: (1) the magnitude of the mean current is very small in comparison with that of the wave-induced velocity, and (2) the angle

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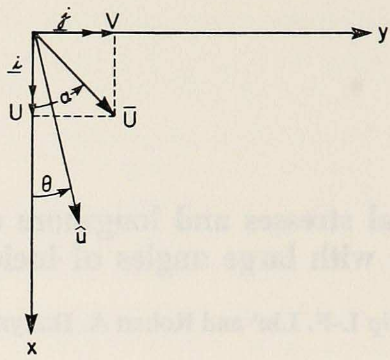


Figure 1. Schematic of coordinate system and wave and current velocity vectors.

of incidence, θ, as defined in Figure 1 is assumed to be very small. Moreover, the effects of large angle between the direction of the waves and the mean current have been ignored. It is clear, however, from the available laboratory data for the measurement of longshore currents, that these hypotheses are rarely satisfied. In Tables 1 and 2 it is shown that in laboratory tests the angle of incidence at the breaker line could reach as high as 57.5° (see Table 2, Putnam, *et al.*, 1949). The maximum orbital velocity at the breaker is roughly estimated by $(H/h)_B\sqrt{gh_B}/2$, while the empirical breaking criterion, $(H/h)_B = \kappa \approx 0.8$, is employed (Bowen, 1969). The ratio between the measured longshore currents and the estimated wave orbital velocity is shown to be consistently an order of one quantity (see the last column of Table 1). Due to many uncertainties of field data collections, it is less conclusive as to whether the previous assumptions are valid. As shown in Table 1, the field data of Inman and Quinn definitely indicate that the mean longshore currents are weak compared with the estimated wave orbital velocity. On the other hand, field data taken by Putnam, *et al.* (1949) are not as conclusive.

With slight modification by including the effects of mean water level set-up, the theoretical formula for longshore currents given by Longuet-Higgins (1970) can be expressed as follows

$$V_{L.H.} = -\frac{5\pi\kappa}{16\,C_f}\left(\frac{\sin\Theta}{C}\right)\frac{dh}{dx}\,g\,(<\eta> + h)/\left(1 + \frac{3\,\kappa^2}{8}\right) \qquad (1)$$

over a uniform sloping beach with slope, $\frac{dh}{dx}$. In the preceding equation $<\eta>$ denotes the mean displacement of the water surface from the still water level and h is the still water depth, $h = -sx$. C represents the wave phase speed and Θ is the angle of wave incidence; they satisfy Snell's Law for refraction,

$$\frac{\sin\Theta}{C} = \text{constant} \qquad (2)$$

Table 1. Summary of Observations: Mean Values.

Investigator (Location)	Type of Beach	$<s>$	$<\Theta_B>$	$V/(u_m)_B$
Putnam *et al.* (1949) (laboratory)	Natural Sand	0.133	14.4	0.52
	Metal or Smooth Concrete	0.171	36.8	1.43
	Gravel	0.123	21.9	0.27
Brebner and Kamphuis (1963) (laboratory)	Roughened Concrete	0.100	16.7	0.91
Galvin and Eagleson (1965) (laboratory)	Smooth Concrete II	0.11	3.7	0.56
	III	0.11	10.3	1.96
	IV	0.11	17.4	2.70
Inman and Quinn (1951) (field)	Torrey Pines and Pacific Beach	0.022	3.58	0.0575
Putnam *et al.* (1949) (field)	Oceanside	0.021	11.1	0.30

In the surf zone and shallow water region, $C \cong \sqrt{g(<\eta> + h)}$ and the maximum orbital breaking wave velocity is

$$u_m = \frac{\kappa}{2} \sqrt{g(<\eta> + h)} \tag{3}$$

Therefore, the longshore current velocity derived by Longuet-Higgins becomes

$$V_{L.H.} = \left[- \frac{5\pi \sin \Theta}{8\, C_f} \frac{dh}{dx} \Big/ \left(1 + \frac{3\,\kappa^2}{8} \right) \right] u_m \tag{4}$$

where C_f is the coefficient of frictional resistance.

Table 2. Laboratory Data by Putnam, Munk and Traylor (1949) and Results.

Run	H_B cm	T sec	h_B cm	$\kappa = H_B/h_B$	Θ_B deg.	s	s/f	V[1] measured m/sec	V[1] cal. m/sec	V[1] L.H. m/sec
15	7.32	0.99	9.75	0.75	28.0	0.100	1.94	0.51	0.67	2.89
16	6.71	1.32	8.23	0.81	22.8	0.100	2.34	0.44	0.62	2.78
17	4.88	1.63	7.01	0.70	18.8	0.100	2.36	0.29	0.52	1.92
18	4.88	1.98	6.71	0.73	18.4	0.100	2.67	0.23	0.55	2.18
19	8.53	0.83	13.11	0.65	56.6	0.139	2.41	0.75	0.92	6.25
20	7.01	0.91	10.06	0.70	45.3	0.139	2.46	0.70	0.85	5.30
21	6.71	1.00	8.84	0.76	38.8	0.139	2.65	0.68	0.82	4.96
22	6.10	1.12	7.32	0.83	33.2	0.139	2.85	0.59	0.75	4.82
23	6.10	1.35	7.62	0.80	31.1	0.139	3.19	0.46	0.71	4.81
24	10.36	0.80	18.90	0.55	57.5	0.260	4.34	1.15	1.49	12.47
25	8.84	0.90	13.11	0.67	52.5	0.260	4.88	1.02	1.36	12.93
26	8.53	0.98	12.50	0.68	47.2	0.260	5.11	0.91	1.38	12.31
27	6.10	1.23	7.92	0.77	32.5	0.260	5.54	0.58	1.08	3.03
28	6.71	1.27	7.01	0.96	31.9	0.260	6.28	0.54	1.07	11.06

[1] Mean longshore currents over surf zone.

Using the analogy between longshore currents and uniform flow over a rough horizontal plate, Longuet-Higgins argued that the friction coefficient should be of the order of 0.005 (see Huntley, 1976). The values inside the bracket on the right-hand side of the above equation indicate the ratio of the longshore current velocity to the local maximum orbital velocity. In order to satisfy the assumption (1) as stated before, it establishes the range of the validity of Longuet-Higgins' theory as

$$\frac{V_{L.H.}}{u_m} = -\frac{5\pi \sin \Theta}{8\, C_f} \frac{dh}{dx} \Big/ \left(1 + \frac{3\,\kappa^2}{8}\right) << 1$$

or

$$\sin \Theta << \frac{8\, C_f \left(1 + \frac{3\,\kappa^2}{8}\right)}{5\pi\, s} = 0.63 \left(\frac{C_f}{s}\right) \tag{5}$$

where $\kappa = 0.8$ has been used. For a typical laboratory test (see Table 1); $s = 0.1$, $C_f = 0.005$ and $C_f/s = 0.05$, equation (5) suggests that Longuet-Higgins' theory is applicable when $\Theta << 1.80°$ (Huntley, 1976). On the other hand, for the field measurement $s = 0.01$, $C_f = 0.005$, and $C_f/s = 0.5$, then $\Theta << 18.36°$. Since the limitation of the theory developed by Longuet-Higgins is created by the assumptions imposed on the mean bottom friction, it appears that a more general and accurate model for the mean shear stress is necessary.

In this paper, analytical forms for the time-averaged bottom shear stress and associated longshore currents are developed. The effects of large angles between the direction of wave propagation and the mean currents are included in the study. Moreover, the angle of incidence is assumed to be order of one, 0(1), throughout the analysis. In the first part of this paper, efforts are made to extend the Longuet-Higgins' friction model so as to include the large angle of incidence, while the mean currents are assumed to be small in comparison with the orbital wave velocity. The application of this model to the prediction of longshore currents and its limitations are presented and discussed. The second part of this paper proposes mean bottom shear stress and longshore current models for the case when the mean currents are in the same order of magnitude as the wave orbital velocity.

The regions of validity of the models presented herein are discussed. The strong current theory is compared with some available data. Reasonable agreements are observed.

2. Theoretical Development for Mean Stress

The total shear stress exerted on the ocean bottom is due to the instantaneous wave orbital velocity and the mean currents. In Figure 1, the coordinate axis and velocity vectors are shown. The mean current is designated as $\bar{U}$, while the oscillatory velocity is $\hat{u}$. The angle of incidence of wave propagation is measured counterclockwise from the x-axis and is denoted by Θ, while the mean current makes an angle α with the x-axis as indicated in Figure 1. The bottom shear stress then is due to the total velocity, $\underline{U}_t$, where

$$\underline{U}_t = (\bar{U}\cos\alpha + \hat{u}\cos\Theta)\,\underline{i} + (\bar{U}\sin\alpha + \hat{u}\sin\Theta)\,\underline{j} \tag{6}$$

and $(\underline{i}, \underline{j})$ are the unit vectors in the x and y directions, respectively.

The bottom stress can be expressed as

$$\underline{\tau}_B = \rho\,\frac{f}{8}\,\underline{U}_t|\underline{U}_t| \tag{7}$$

where ρ is the fluid density and f is a Darcy-Weisbach type friction factor. The absolute value sign is necessary to insure the stress reverses with velocity reversals. Values of f that have been used in literature are those taken from steady state flow experiments (cf. Longuet-Higgins, 1970); it is clear that f can be related to the friction coefficient C_f used by Longuet-Higgins,

$$f = 8\,C_f \tag{8}$$

or from values developed by Jonsson (1966) and Kamphuis (1975). Kajiura (1968) proposed a different form of f,

$$f = 1.41\left(\frac{Tu_m}{2\pi\,k_e}\right)^{-2/3} \tag{9}$$

based on an oscillating turbulent boundary layer argument. In this formula, T is the incident wave period, k_e is the equivalent roughness of the bottom material. It should be noted here that the friction coefficient, strictly speaking, depends on the local wave amplitude and the local beach profile. Nevertheless, it is assumed to be a constant over a wave cycle and throughout the entire surf zone.

The horizontal wave orbital velocity, $\hat{u}$, can be expressed at any point as an oscillatory function of time, t, for small amplitude waves,

$$\hat{u} = u_m\cos\left(\frac{2\pi t}{T}\right) \tag{10}$$

The amplitude, u_m, is the maximum value of the wave orbital velocity. From equation (7), the mean bottom shear stress can then be expressed as

$$<\underline{\tau}_B> = \rho\,\frac{f}{8} <\underline{U}_t|\underline{U}_t|> \tag{11}$$

where

$$<\ > = \frac{1}{T}\int_t^{t+T}(\ \ )\,dt \tag{12}$$

is the operator for time average over one wave period.

Approximate mean shear stress forms are derived for two cases, (a) weak currents, $\frac{\bar{U}}{u_m} << 1$ and (b) strong currents, $\frac{\bar{U}}{u_m} \geqslant 1$. The applications of these two models on the predictions of longshore currents are presented herein.

3. Weak currents

In the case of very weak mean currents, Longuet-Higgins (1970) derived the mean shear stress formula specifically for the longshore current problem with requirement that the incident wave angle Θ is small. Modifications are made here to include the effects of large wave incidence angle and of the angles between the directions of wave propagation and mean flows. Referring to equation (6), one can approximate the absolute value of the total velocity

$$|U_t| \cong \left[|\hat{u}| + \bar{U}\frac{|\hat{u}|}{\hat{u}} \cos(\Theta - \alpha) \right] \tag{13}$$

under the assumption that

$$\frac{\bar{U}}{u_m} << 1 \tag{14}$$

Substitution of equations (6) and (13) into equation (11) and the use of the following relations

$$<|\hat{u}|> = 2\, u_m/\pi, \; <|\hat{u}|\hat{u}> = 0 \tag{15}$$

yield the mean bottom shear stress

$$<\tau_B> = \frac{\rho f u_m}{4\pi}\{[U(1 + \cos^2\Theta) + V \sin\Theta\cos\Theta]i + [V(1 + \sin^2\Theta) + U \sin\Theta\cos\Theta]j\} \tag{16}$$

with

$$U = \bar{U} \cos\alpha, \; V = \bar{U} \sin\alpha \tag{17}$$

where (U, V) are the mean flows in the x and y directions, respectively. For longshore currents, $U = 0$ and the proceeding equation reduces to

$$<\tau_B> = \frac{\rho f u_m}{8\pi}\{[V \sin2\Theta]i + [2V(1 + \sin^2\Theta)]j\} \tag{18}$$

For fixed V, $<\tau_B>$ increases as Θ goes from 0 to $\pi/2$. In the situation when the angle of wave incidence, Θ, is very small, Longuet-Higgins' early result is recovered, i.e.

$$<\tau_B> = \frac{\rho f u_m}{4\pi} V \underline{j} \tag{19}$$

To illustrate the effects of large angle of wave incidence and the large mean currents, the longshore current theory due to Longuet-Higgins (1970) is reexamined.

Consider a long, straight coastline with parallel offshore contours. Neglecting the horizontal turbulent mixing and wave-current interaction, it can be shown that the mean flows must satisfy the following equations

$$\frac{\partial}{\partial x}\{[<\eta> + h]U\} + \frac{\partial}{\partial y}\{[<\eta> + h]V\} = 0$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -g\frac{\partial<\eta>}{\partial x} - \frac{1}{\rho\,[<\eta> + h]}\left\{\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} + <\tau_{B_x}>\right\}$$

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = -g\frac{\partial<\eta>}{\partial y} - \frac{1}{\rho\,[<\eta> + h]}\left\{\frac{\partial S_{yx}}{\partial x} + \frac{\partial S_{yy}}{\partial_y} + <\tau_{B_y}>\right\} \tag{20}$$

where $S_{i,j}$ $(i, j = x, y)$ represents the radiation stress component which is the excess flux of i-directed wave momentum in the j-direction. For progressive small amplitude waves,

$$S_{xx} = E[(2n - \tfrac{1}{2}) - n\sin^2\Theta]$$

$$S_{yy} = E[(n - \tfrac{1}{2}) + n\sin^2\Theta]$$

$$S_{xy} = S_{yx} = En\sin\Theta\cos\Theta \tag{21}$$

with $E(= \rho g H^2/8)$, the local wave energy intensity and $n(= C_g/C)$, the ratio of the wave group velocity to the phase speed.

Since the incident wave system is uniform in the y-direction (alongshore), with a uniform beach profile the mean currents are also expected to have the same property. Thus, $U = 0$, and $\frac{\partial(\ )}{\partial y} = 0$ and the governing equations can be simplified

$$0 = -g\frac{d<\eta>}{dx} - \frac{1}{\rho\,[<\eta> + h]}\left\{\frac{dS_{xx}}{dx} + <\tau_{B_x}>\right\} \tag{22}$$

$$0 = -\frac{1}{\rho\,[<\eta> + h]}\left\{\frac{dS_{yx}}{dx} + <\tau_{B_y}>\right\} \tag{23}$$

In the shallow water region, $n \approx 1$, $C \cong \sqrt{g[<\eta> + h]}$, and therefore the radiation stresses are reduced to the following forms

$$S_{xx} = E\left(\frac{3}{2} - \sin^2\Theta\right)$$

$$S_{yy} = E\left(\frac{1}{2} + \sin^2\Theta\right)$$

$$S_{xy} = S_{yx} = E\sin\Theta\cos\Theta \tag{24}$$

Furthermore, as is customarily assumed (Bowen, 1969a,b, Longuet-Higgins, 1970), the following spilling breaker condition will be valid throughout the surf zone; the wave height, $H = \kappa\,(<\eta> + \mathrm{h})$ where κ is the breaking parameter.

Adopting the mean shear stress model given by equation (18), one obtains the following governing equations

$$0 = -g\frac{d<\eta>}{dx} - \frac{g\kappa^2}{8\,(<\eta> + h)}\frac{d}{dx}\left[(<\eta> + h)^2\left(\frac{3}{2} - \sin^2\Theta\right)\right]$$

$$- \frac{f\kappa\sqrt{g(<\eta> + h)}}{8\pi\,(<\eta> + h)}\,V\sin\Theta\cos\Theta \tag{25}$$

$$0 = -\frac{g\kappa^2}{8\,(<\eta> + h)}\frac{d}{dx}[(<\eta> + h)^2\sin\Theta\cos\Theta] - \frac{f\kappa\sqrt{g\,(<\eta> + h)}}{8\pi\,(<\eta> + h)}V\,(1 + \sin^2\Theta) \tag{26}$$

Note that $u_m = \dfrac{\kappa}{2}\sqrt{g(<\eta> + h)}$ has been employed. Elimination of V from equations (25) and (26) and after a considerable manipulation, a nonlinear ordinary differential equation is obtained for the mean free surface displacement

$$\left[\left(1 + \frac{3\kappa^2}{8}\right) + \left(1 - \frac{5\kappa^2}{16}\right)\left(\frac{\sin\Theta}{C}\right)^2 g\,(<\eta> + h)\right.$$

$$\left. - \frac{3\kappa^2}{4}\left(\frac{\sin\Theta}{C}\right)^4 g^2\,(<\eta> + h)^2\right]\frac{d(<\eta> + h)}{dx} - \left(\frac{\sin\Theta}{C}\right)^2\frac{dh}{dx}\,\cdot$$

$$g(<\eta> + h) = \frac{dh}{dx} \tag{27}$$

The boundary conditions are

$$(<\eta> + h) = 0, \text{ on } X = x - x_s = 0 \tag{28}$$

$$(<\eta> + h) = (<\eta> + h)_B, \text{ on } X = x_B - x_s \tag{29}$$

where $x = x_s$ is the position of the mean shoreline and is part of the solution. The total mean water depth at the breaker, $(<\eta> + h)_B$, as well as the position of breaker line, x_B, may be computed from the first order wave theory and the breaking criterion.

Let the solution to equations (27-29) be composed of two parts

$$(<\eta> + h) = (<\eta> + h)_H + (<\eta> + h)_P \tag{30}$$

with

$$(<\eta> + h)_P = -\left[\;g\,\left(\frac{\sin\Theta}{C}\right)^2\;\right]^{-1} \tag{31}$$

which is a constant. It may be shown that homogeneous solution satisfies the following equation.

$$-\frac{3\kappa^2}{8}\left(\frac{\sin\Theta}{C}\right)^4 g^2\,(<\eta> + h)^2{}_H + \left(1 + \frac{19\kappa^2}{16}\right)\left(\frac{\sin\Theta}{C}\right)^2 g\,\cdot$$

$$(<\eta> + h)_H - \frac{\kappa^2}{16}\ ln\,(<\eta> + h)_H = g\left(\frac{\sin\Theta}{C}\right)^2 \ .$$

$$\frac{dh}{dx} X + 1 + \frac{13\kappa^2}{16} + \frac{\kappa^2}{16}\ ln\left[\ g\ \left(\frac{\sin\Theta}{C}\right)^2\ \right] \tag{32}$$

Where the boundary condition, equation (28), has been employed. To determine the location of the mean shoreline the second boundary condition, equation (29), is used

$$x_s = -\left\{ -\frac{3\kappa^2}{8}\left(\frac{\sin\Theta}{C}\right)^4\ g^2\,[(<\eta> + h)_B - (<\eta> + h)_P]^2 + \left(1 + \frac{19\kappa^2}{16}\right) .\right.$$
$$\left(\frac{\sin\Theta}{C}\right)^2\ g\,[(<\eta> + h)_B - (<\eta> + h)_P] - \frac{\kappa^2}{16}\ .$$
$$\left. ln\left[\left(\frac{\sin\Theta}{C}\right)^2\ g\,((<\eta> + h)_B - (<\eta> + h)_P)\right] - \left(1 + \frac{13\kappa^2}{16}\right)\right\}$$
$$\Big/\left[\ g\ \left(\frac{\sin\Theta}{C}\right)^2\ \frac{dh}{dx}\right] + x_B \tag{33}$$

The longshore current velocity can be obtained directly from equation (26) with the use of Snell's Law and equation (27),

$$V = -\frac{g\kappa\pi}{2f}\left(\frac{\sin\Theta}{C}\right)\frac{dh}{dx}\ (<\eta> + h)\left[\ 5 - 6\left(\frac{\sin\Theta}{C}\right)^2\ g\,(<\eta> + h)\right] .$$
$$\left[\ 1 - \left(\frac{\sin\Theta}{C}\right)^2\ g\,(<\eta> + h)\right]^{-1/2}\left\{\left(1 + \frac{3\kappa^2}{8}\right) + \left(1 - \frac{5\kappa^2}{16}\right) .\right.$$
$$\left.\left(\frac{\sin\Theta}{C}\right)^2\ g\,(<\eta> + h) - \frac{3\kappa^2}{4}\left(\frac{\sin\Theta}{C}\right)^4\ g^2\,(<\eta> + h)^2\right\}^{-1} \tag{34}$$

For very small angle of incidence, this equation may be reduced to

$$V = -\frac{5g\kappa\pi}{2f}\left(\frac{\sin\Theta}{C}\right)\frac{dh}{dx}\ (<\eta> + h)\Big/\left(1 + \frac{3\kappa^2}{8}\right) \tag{35}$$

which is the same as that derived by Longuet-Higgins (1970); also see equation (1) with $f = 8\ C_f$.

In Figure 2, the longshore current velocity distributions inside the surf zone are presented for different values of the angle of incidence at the breaker line, Θ_B. The velocity has been nondimensionalized by the longshore current velocity at the breaker line predicted by Longuet-Higgins. The value, $X = 0$, represents the mean shoreline and $X = -1$, the breaker line. It is seen that the velocity profiles for moderate angles of incidence deviate from the linear profile (which is the result obtained by Longut-Higgins) significantly. The effects of the large angle of incidence on the longshore currents are further demonstrated in Figure 3, where the velocity

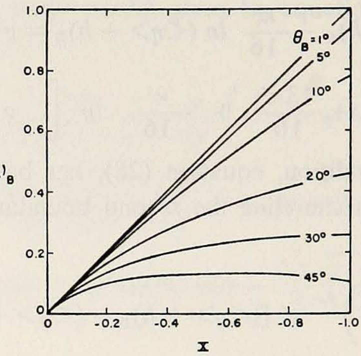


Figure 2. Non-dimensional longshore current profiles as given by equation (34) for a sequence of values of angle of incidence at the breaker line; $(V_{L.H.})_B$ = longshore current velocity at breaker line given by Longuet-Higgins (1970), and $X = (x - x_s)/(x_B - x_s)$.

profile (equation (34)) is normalized by the Longuet-Higgins' solution (equation (35)) locally. It is noticed that approximately a 20% difference is observed even for $\Theta_B = 10°$. On the other hand, the effects of currents and bottom friction on $<\eta>$ are relatively insignificant, yielding only about maximum 2% difference from the value used by Longuet-Higgins for a large angle of incidence. In order to find out the regions of validity of the present theory as well as that of Longuet-Higgins' results, the longshore current velocity is divided by the local orbital wave velocity, u_m, and is plotted in Figure 4. The values of the velocity ratio have been scaled according to different values of s/f. As shown in the figure, for milder beach slope and/or larger bottom friction, the smaller values of the velocity ratio are predicted. It is also interesting to point out the fact that the maximum velocity ratio for the moderate angle of incidence does not necessarily occur at the breaker line as suggested by Longuet-Higgins' theory. Moreover, comparison of Figures 3 and 4 re-

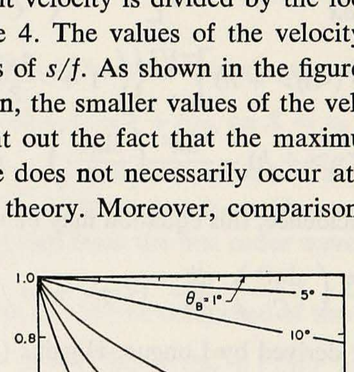


Figure 3. Comparisons between the local longshore current velocity given by equation (34) and that given by Longuet-Higgins (1970).

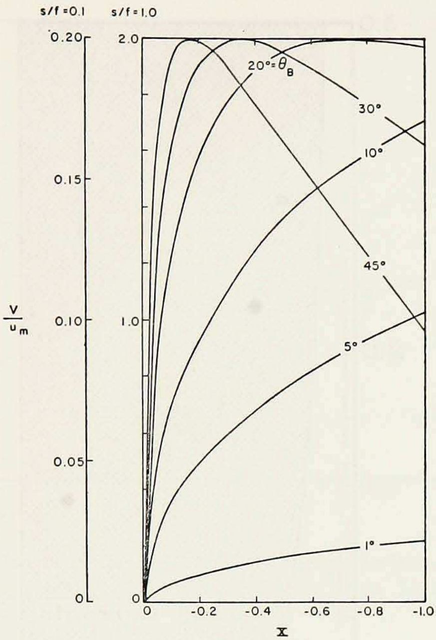


Figure 4. Longshore current velocity in the surf zone (given by equation (34)) nondimensionalized by the local wave orbital velocity, u_m.

veals that the velocity ratio calculated by Longuet-Higgins' result can be several times greater than that shown in Figure 4.

Since the weak current theory is developed under the assumption that the velocity ratio, V/u_m, is much smaller than one, it is important to show the regions of validity of the theory. It is clear that two factors, namely, s/f and Θ_B, play very important roles in the determination. In Figure 5, the upper dashed line represents the contour line for the velocity ratio $V_{\max}/u_m = 0.5$ as predicted by the present theory. On the other hand, the lower dashed line represents the ratio calculated by Longuet-Higgins' result. If one accepts that $V_{\max}/u_m = 0.5$ is the upper limit of the weak current theory (rigorously, this value should be much smaller), then the area underneath these curves indicates the regions of validity of the weak current theories. Although the present theory does increase the region of validity for larger angles of incidence, it is still desirable to develop a parallel theory which allows the mean current to be in the same order of magnitude or greater than that of the wave orbital velocity.

4. Strong currents

For this case the wave orbital velocity is assumed to be smaller than the mean current,

$$\frac{\bar{U}}{u_m} \geqslant 1 \tag{36}$$

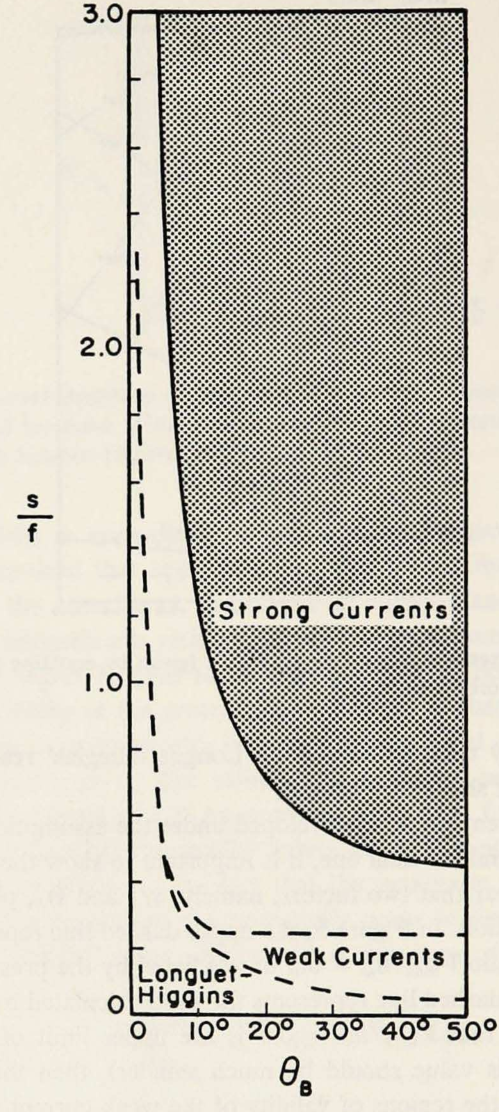


Figure 5. Validity diagram for longshore current theories as a function of slope to friction factor ratio and breaker line value of wave angle of incidence. Upper limit of $\frac{V}{(u_m)_B}$ for the weak current theory is 0.5 and the lower limit of this velocity ratio was taken as 1.25 for the strong current cases.

Examining the laboratory data cited in Table 1, most of the measurements belong to this case. The absolute value of the total velocity can be expressed as

$$|U_t| = \{(\hat{u})^2 + 2\hat{u}\bar{U}\cos(\Theta - \alpha) + (\bar{U})^2\}^{1/2}$$

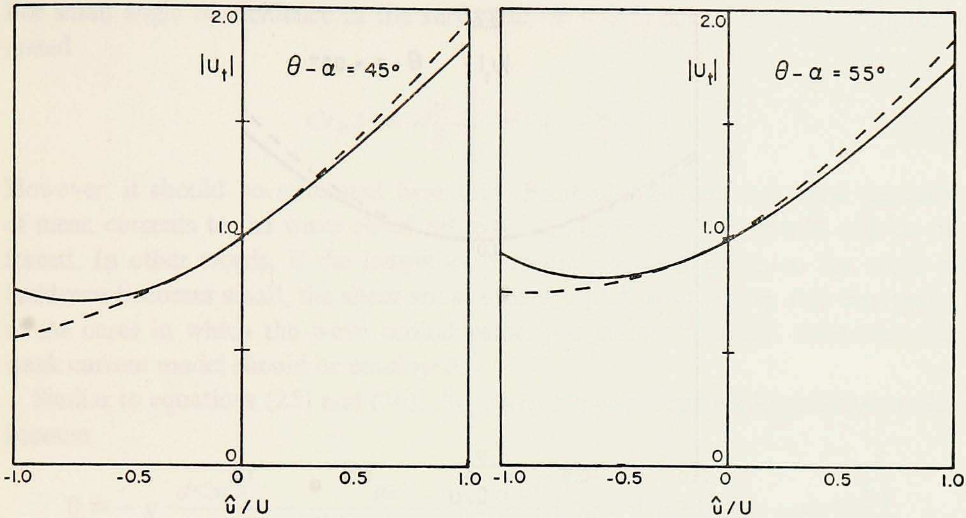
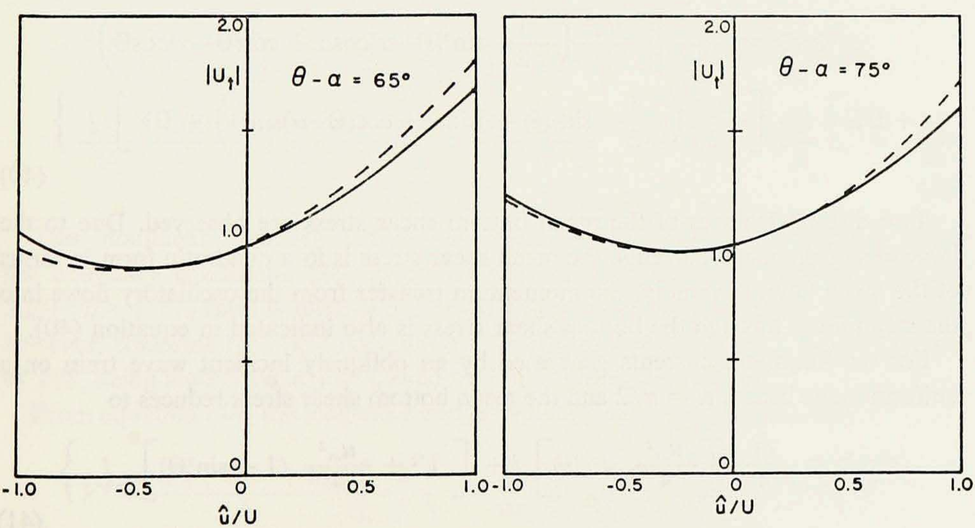


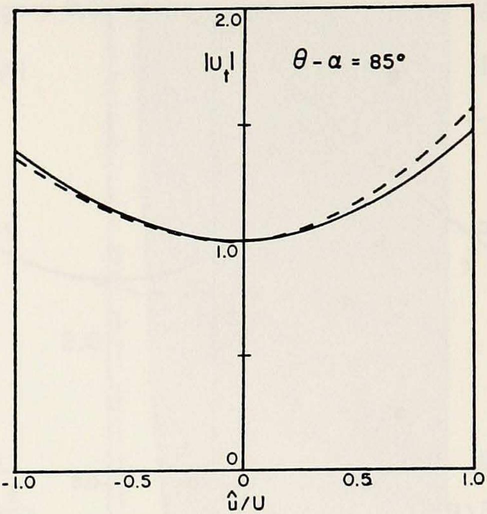
Figure 6. Comparisons between the exact solution of the absolute values of the total velocity field (solid lines), $|U_t|$, and the approximate solution given by equation (38) (dashed lines) for a sequence of values of $(\Theta - \alpha)$.

$$= \bar{U}\left\{\left(\frac{\hat{u}}{\bar{U}}\right)^2 + 2\left(\frac{\hat{u}}{\bar{U}}\right)\cos(\Theta-\alpha) + 1\right\}^{1/2} \tag{37}$$

Using equation (36) a very good approximation is obtained for $|U_t|$, by truncating the binomial expansion for $|U_t|$.

$$|U_t| \cong \bar{U}\left\{1 + \left(\frac{\hat{u}}{\bar{U}}\right)\cos(\Theta-\alpha) + \frac{1}{2}\left(\frac{\hat{u}}{\bar{U}}\right)^2 \sin^2(\Theta-\alpha)\right\} \tag{38}$$





The accuracy of this approximation is shown in Figure 6, for $45° < |\Theta - \alpha| < 90°$. It is noticed that excellent agreements between equations (37) and (38) are achieved for $|\hat{u}/\bar{U}| < 0.5$, while less than 10% error is committed as $|\hat{u}/\bar{U}| > 0.5$. It should also be pointed out that since the errors change signs as $\hat{u} \to \pm\bar{U}$, after time averaging, the total error committed in the mean quantities is actually less.

To determine the mean bottom stress, the shear stress must be averaged over a wave period as defined in equations (11) and (12). The following relations hold.

$$<\hat{u}> = 0, <(\hat{u})^2> = u_m{}^2/2, <(\hat{u})^3> = 0 \tag{39}$$

Hence, the mean bottom shear stress, $<\tau_B>$, can be expressed

$$<\tau_B> = \frac{\rho f}{8}\left\{\left[\frac{u_m{}^2}{2}\left(\frac{1}{2}\sin^2(\Theta-\alpha)\cos\alpha + \cos(\Theta-\alpha)\cos\Theta\right)\right.\right.$$
$$\left.\left. + \bar{U}U\right]\underline{i} + \left[\frac{u_m{}^2}{2}\left(\frac{1}{2}\sin^2(\Theta-\alpha)\sin\alpha + \cos(\Theta-\alpha)\sin\Theta\right) + \bar{U}V\right]\underline{j}\right\} \tag{40}$$

Two distinct features of the mean bottom shear stress are observed. Due to the largeness of the mean currents, the mean shear stress is in a quadratic form in terms of the mean flows. Secondly, the momentum transfer from the oscillatory flows into the mean flows through the bottom shear stress is also indicated in equation (40).

For the longshore currents generated by an obliquely incident wave train on a uniform plane beach, $\alpha = \pi/2$ and the mean bottom shear stress reduces to

$$<\underline{\tau_B}> = \frac{\rho f}{8}\left\{\left[\frac{u_m{}^2}{4}\sin 2\Theta\right]\underline{i} + \left[V^2 + \frac{u_m{}^2}{4}(1 + \sin^2\Theta)\right]\underline{j}\right\} \tag{41}$$

For small angle of incidence in the surf zone, $\Theta << 1$, it can be further approximated

$$<\tau_B> = \frac{\rho f}{8}\left[V^2 + \frac{u_m^2}{4}\right]\underline{j} \tag{42}$$

However, it should be reiterated here that the restriction imposed upon the ratio of mean currents to the wave obital velocity, i.e., equation (36), should still be enforced. In other words, if the longshore currents indeed diminish as the angle of incidence becomes small, the shear stress model, equation (42), can only be applied to the cases in which the wave orbital velocity is also very small. Otherwise, the weak current model should be employed.

Similar to equations (25) and (26), the governing equations for longshore currents become

$$0 = -g\frac{d<\eta>}{dx} - \frac{g\kappa^2}{8(<\eta>+h)}\frac{d}{dx}\left[(<\eta>+h)^2\left(\frac{3}{2} - \sin^2\Theta\right)\right] - \frac{f\kappa^2 g}{128}\sin 2\Theta \tag{43}$$

$$0 = -\frac{g\kappa^2}{8(<\eta>+h)}\frac{d}{dx}\left[(<\eta>+h)^2\sin\Theta\cos\Theta\right] - \frac{f}{8(<\eta>+h)}\left\{ V^2 + \frac{\kappa^2 g(<\eta>+h)}{16}(1+\sin^2\Theta)\right\} \tag{44}$$

Again, the use of Snell's Law in the x-momentum equation, equation (43), leads to a nonlinear, first-order ordinary differential equation for the total mean water depth inside the surf zone,

$$\left[\left(1+\frac{3\kappa^2}{8}\right) - \frac{3\kappa^2}{8}\left(\frac{\sin\Theta}{C}\right)^2 g(<\eta>+h)\right]\frac{d(<\eta>+h)}{dx} + \frac{\kappa^2 f}{64}\left(\frac{\sin\Theta}{C}\right)\sqrt{g(<\eta>+h)}\left[1-\left(\frac{\sin\Theta}{C}\right)^2 g(<\eta>+h)\right]^{1/2} = \frac{dh}{dx} \tag{45}$$

This nonlinear ordinary differential equation was solved numerically using the boundary conditions (28, 29). The solution was obtained iteratively as the $X_B/(<\eta> + h)_B$ ratio is unknown a priori and was adjusted to satisfy (29). Again, however, solutions for $<\eta>$ only deviate from that developed by Longuet-Higgins by 2%, being always slightly less, even for large angles of incidence.

From equation (44) the longshore current velocity can be represented by

$$V^2 = \frac{\kappa^2 g(<\eta>+h)}{4}\left\{\left[\frac{2}{f}\left(-\frac{dh}{dx}\right)\left(\frac{\sin\Theta}{C}\right)\sqrt{g(<\eta>+h)} + \right.\right.$$

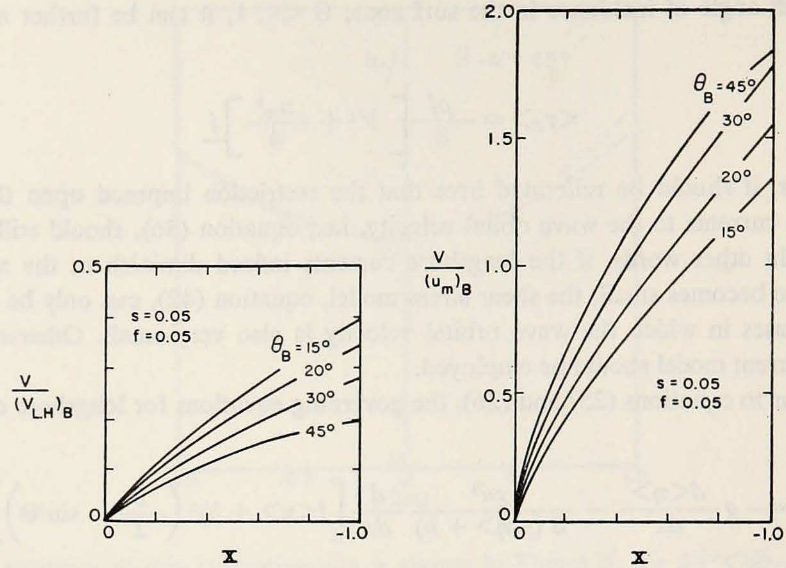


Figure 7a. Nondimensional longshore current velocity profiles as given by equation (46) for a sequence of values of angle of incidence at the breaker line; $(u_m)_B$ represents the wave orbital velocity at the breaker line.

Figure 7b. Longshore current profiles as given by equation (46); $(V_{L.H.})_B$ is the Longuet-Higgins' result at the breaker line.

$$\left(\frac{\sin\Theta}{C}\right)^2 g\,(<\eta> + h)\ .$$

$$\left(1-\left(\frac{\sin\Theta}{C}\right)^2 g\,(<\eta> + h)\right)^{1/2}/32\Bigg]\left[5-6\left(\frac{\sin\Theta}{C}\right)^2 g\,(<\eta> + h)\right].$$

$$\left[1-\left(\frac{\sin\Theta}{C}\right)^2 g\,(<\eta> + h)\right]^{-1/2}\left[\left(1+\frac{3\kappa^2}{8}\right)\right.$$

$$\left.-\frac{3\kappa^2}{8}\left(\frac{\sin\Theta}{C}\right)^2 g\,(<\eta> + h)\right]^{-1} - \frac{1}{4}\left[1+\left(\frac{\sin\Theta}{C}\right)^2 g\,(<\eta> + h)\right. \tag{46}$$

For very small angle of incidence, the longshore current velocity can be approximated

$$V=\left\{\frac{\kappa^2\, g\,(<\eta> + h)}{2}\left[\frac{5}{f}\left(-\frac{dh}{dx}\right)\left(\frac{\sin\Theta}{C}\right)\sqrt{g\,(<\eta> + h)}\right.\right.$$

$$\left.\left./\left(1+\frac{3\kappa^2}{8}\right)-\frac{1}{8}\right]\right\}^{1/2} \tag{47}$$

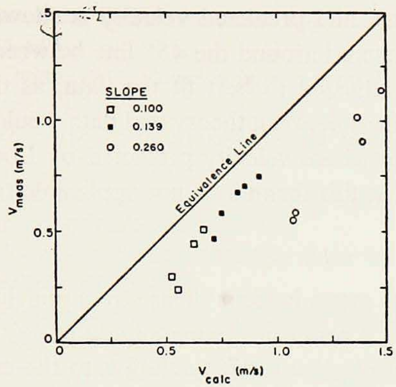


Figure 8. Comparisons between the measured longshore currents (Putnam, *et al.*, 1949, also see Table 2), $\overline{V}_{meas.}$ and the predicted mean currents as given by equation (46), $V_{calc.}$

Eagleson (1965) derived a formula for the mean longshore current velocity also using the concept of mean momentum flux. His result bears a strong resemblance to equation (47), except that the contribution from the orbital velocity, u_m, is ignored. Moreover, in Eagleson's analysis, the mean water level set-up in the surf zone was not considered and the total velocity field was assumed to vary linearly with time.

Examples of the longshore current calculated for various angles of incidence at the breaker line are plotted in Figure 7 for a pair of (s, f) values. These values have been nondimensionalized, again, by the Longuet-Higgins' solution (35), and the maximum orbital velocity, both evaluated at the breaker line. Clearly, the calculated result (46) is much smaller than that predicted by Longuet-Higgins.

The validity of the strong currents theory is presented in Figure 5. The boundary line of the shaded area is the contour line for $V/(u_m)_B = 1.25$ where V is computed from equation (46). The strong currents theory is valid for all the combinations of s/f and Θ_B which are located in the shaded area.

5. Comparisons with laboratory data

To validate the strong current theory, results calculated by equation (47) are compared with a set of data collected by Putnam, *et al.* (1949). These data are shown in Table 2, and represent the data set with the largest known breaking angles. The beach was covered with sheet metal or smooth cement and in computing the friction coefficient f, the equivalent roughness is chosen to be 0.0013 ft. As shown in the table, the calculated f based on equation (9) is surprisingly close to the value recommended by Longuet-Higgins (1970), which is 0.04. The averaged longshore current velocity is computed by integrating the flow rate, $V(<\eta> + h)$ over the cross-sectional area of the entire surf zone and dividing by the cross-sectional area. It is seen that the computed values are consistently greater than the data. In Fig. 8, agreement between observed and predicted velocity is shown qualitatively by how closely the plotted data clustered around the 45° line between the axes. Note that if the friction factor, f, was adjusted to best fit the data, as done by Putnam *et al.* (1949), an excellent agreement between theory and data would occur.

In Table 2, the mean longshore velocity predicted by Longuet-Higgins' formula is also shown. It is clear that this formula is not applicable to the present situation.

6. Conclusions

Two different models for mean bottom shear stress and longshore currents have been developed, differing as a result of the magnitude of (V/u_m). The weak current model extends the Longuet-Higgins (1970) solution to the case for large angles of incidence and furthermore, it shows that the influence of large incident wave angles affects the velocity profile variation across the surf zone. The strong current model, which is primarily valid for such steep beaches and large angles of wave incidence as would occur in laboratory situations, agrees quite well with the laboratory data of Putnam, Munk and Traylor (1949).

Regions of validity of each model are defined, and it appears that the weak current model is generally good for the field application. Balsillie (1975) has demonstrated this fact for one California site. Komar (1976) based on field data obtained at the Scripps Institution of Oceanography, has estimated that the s/f ratio is a constant and equal to 0.172. This case again would correspond to the weak current case.

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