Interaction of inertia-gravity waves with the wind

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ABSTRACT

An inertia-gravity wave which propagates upwind and upward in the thermocline has a reflection coefficient r which is greater than unity ("overreflection") as a result of the wave current interaction in the mixed layer which overlies the thermocline. The second order amplitude effect of the wave is to produce a mixed layer transport having the same direction as the wind stress $\boldsymbol{\tau}$, whereas the undisturbed (Ekman) transport is perpendicular to $\boldsymbol{\tau}$. The amplification factor $|r|-1$ is proportional to $|\boldsymbol{\tau}|$, and increases as the frequency of the incident wave approaches the Coriolis parameter f. A preferred lateral scale is also given, and it is suggested that the spectral peak near f in the ocean is maintained by successive amplification of long wave packets entering the mixed layer. We also show that the turbulent Ekman flow at the upper boundary of a two-layer density model, such as can be realized in the laboratory, should become unstable with respect to long (hydrostatic) interfacial waves.

1. Introduction

One way in which a wind stress $\rho_0\boldsymbol{\tau}$ can generate inertia oscillations in a mixed layer (Fig. 1) of density ρ_0 is by means of the temporal variations of $\boldsymbol{\tau}$ (Pollard and Millard, 1970; Pollard, Rhines and Thompson, 1972). Thus the sudden application of a horizontally uniform $\boldsymbol{\tau}$ can generate transient currents whose frequency is equal to the Coriolis parameter f. Since the vertical group velocity of these waves vanishes, they cannot account for the prominent spectral peak observed in the stratified thermocline (e.g., Briscoe, 1975; Sanford, 1975), and some kind of horizontal variability must be invoked.

The lateral variation of $\boldsymbol{\tau}$ can produce such an effect. This may be seen by computing the Ekman transport $\mathbf{M}$, or the vertical integral of the ageostrophic shear flow in the mixed layer, from the well-known equation

$$\partial\mathbf{M}/\partial t + f\,\mathbf{k}\times\mathbf{M} = \boldsymbol{\tau}$$

This is valid if $\boldsymbol{\tau}$ (or f) vary "slowly", as is the case on the scale of the wind-driven ocean circulation theory. The solution of the above equation indicates that if curl $\boldsymbol{\tau} \neq 0$, then $\nabla\cdot\mathbf{M}\neq 0$, and thus we see how the Ekman suction velocity $\nabla\cdot\mathbf{M}$ can pump wave energy into the thermocline on scales determined by that of an atmospheric storm. But the validity of this theory becomes questionable when one

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applies it to oceanic inertia oscillations whose horizontal wavelength is believed to be (see observations cited below) much smaller than the gross dimension of a storm. The modifications in the Ekman transport equations which are required when one considers relatively small scales (tens of kilometers) have been presented in previous papers (Stern, 1966, 1975), and in this paper we shall apply the theory to the inertia oscillation problem (See § 4).

The wave generating theory presented here is somewhat similar to an instability mechanism, insofar as it depends on the local $\boldsymbol{\tau}$ (or the local shear) interacting with "pre-existing" wave perturbations. Although our interest is directed mainly toward the prominent inertial peak, we should mention the problem of the semi-diurnal internal oscillation, and refer the reader to Thorpe (1975) for a survey of generating mechanisms.

The following theory is related to previous studies of the instability of the Ekman layer (Kaylor and Faller, 1972; Gammelsrød, 1975), but there are two major differences. The nonhydrostatic force is crucial for the Ekman instability, and also for the turbulent fluctuations which maintain the fully-developed field of mean horizontal velocity $\mathbf{V} = (U(z), V(z))$. In the following theory we examine the interaction of this $\mathbf{V}$ with quasi-hydrostatic waves whose horizontal length scale is much larger than the horizontal scale of the energy containing eddies in the mixed layer. The spectrum of the latter has been measured in the laboratory by Caldwell, Van Atta and Helland, 1972, and most of the turbulent energy appears to be in frequencies well above the Coriolis parameter. Therefore we may ask what happens when this statistically steady Ekman flow is perturbed by an upward propagating inertia-gravity wave (Fig. 1) whose frequency ω is slightly above f, and whose horizontal wavelength $2\pi/l$ is much larger than the wavelength of the turbulent eddies.

Although the several different scales of motion which will enter our problem may be rigorously defined by ensemble averages, it is intuitively clear that the "large" scale wave under discussion is kinematically distinguishable from the turbulent fluctuations, and dynamically distinguishable because of the role of the hydrostatic relation. We shall therefore compute the reflection coefficient r for the thermocline wave (Fig. 1), as a functional of the ageostrophic shear flow in the mixed layer.

In the first of the two theories used for this purpose, we make the bald but conventional assumption that the wave only interacts with the mean shear flow $\mathbf{V}$ (z), and not with the turbulent eddies. These two fields are related in the undisturbed state by $f\,\mathbf{k} \times \mathbf{V} = \partial\Theta_0/\partial z$, where Θ_0 denotes the mean turbulent Reynolds stress. An infinitesimal wave perturbation might change the (statistically averaged) stress to $\Theta(x,y,z,t)$, but our first theory neglects such a contingency. It is true, of course, that the free surface boundary condition $\Theta\ (x,y,0,t) = \boldsymbol{\tau}$ is indeed unaltered by the perturbation, and the same is true near the base of a deep mixed layer where $\Theta = 0$ is assumed as another boundary condition. But it is doubtful that $\Theta = \Theta_0$ at intervening depths, and therefore the entire question is re-examined in § 4 by means of a vertically integrated theory which makes no assumption regarding $\boldsymbol{\Theta}$, and which introduces no pernicious eddy coefficients. The onus is placed on a closure approximation of the integrated theory, and the error is quantifiable. Strong preference is given to this second theory, because it is simpler in the linear regime and extendable into the nonlinear regime. However, the assumption of a deep, homogeneous, and immiscible mixed layer enters as a rather strong assumption which may not be valid in the oceanic case.

The two theories mentioned above are in quantitative agreement in an asymptotic (small $\boldsymbol{\tau}$) limit, despite the different physical assumption in each. They predict that, when the incident wave has an upwind component of phase velocity, the magnitude of the reflection coefficient is greater than unity and $|r|-1$ is proportional to the undisturbed Ekman transport. The scale values of l and $\omega-f$ for "preferred waves" having largest $|r|$ are discussed. Both theories indicate that "spontaneous waves" with $|r| = \infty$ are possible, but this occurs for values of the parameters which are beyond the formal limits of validity of either theory.

A reflection coefficient greater than unity (sometimes called "overreflection") suggests that "packets" (groups) of inertia-gravity waves can be amplified each time they enter the mixed layer with a horizontal phase velocity that is directed upwind (relative to an observer moving with the mean geostrophic velocity of the layer). An amplified packet which travels downward will be reflected subsequently by the ocean bottom or by thermocline inhomogeneities. This new upgoing packet will be re-amplified when it re-enters the mixed layer, and thus we have a potent mechanism for maintaining an equilibrium sea.

Mention should be made of the fact that the phenomenon of overreflection requires very special conditions on a shear flow because of such constraints as are provided by the Eliassen-Palm Theorem (see Lindzen, 1974). These special conditions occur in our problem because of the external force ($\boldsymbol{\Theta}_0$) that maintains the basic flow which is not unidirectional and which does not satisfy the thermal wind equation. (See Andrews and McIntyre, 1976).

2. Wave-current interaction theory

The x-axis in Figure 1 is oriented parallel to the wave fronts of the perturbation whose (x, y, z) velocity components are denoted by $(u_1(y,z,t), v_1, w_1)$, and the hydrostatic pressure perturbation in the mixed layer is $p_1\ (y,t)$. If the perturbation in the Reynolds stress is neglected, as previously mentioned, then the linearized perturbation equations for $z > -H$ are

$$\partial u_1/\partial t + V(z)\partial u_1/\partial y + w_1 U_z - f v_1 = 0$$

$$\partial v_1/\partial t + V\partial v_1/\partial y + w_1 V_z + f u_1 = -\rho_0{}^{-1}\partial p_1/\partial y$$

$$\partial p_1/\partial z = 0 \tag{1}$$

$$\partial v_1/\partial y + \partial w_1/\partial z = 0$$

where $U_z \equiv \partial U/\partial z$. By inserting the modes

$$w_1 = Re\, w'(z) \exp i\, l\, y + i\, \omega\, t, \text{ etc.} \tag{2}$$

into (1) we obtain

$$i(\omega + lV)u' - f\, v' + w'\, U_z = 0$$

$$i(\omega + lV)v' + f\, u' + w' V_z = - i\, l\, p'/\rho_0$$

$$p' = \text{constant}$$

$$v' = (i/l)\, dw'/dz$$

The elimination of (u', v') gives the first order inhomogeneous equation

$$\left(\frac{d}{dz} - B(z)\right)\frac{w'(z)}{w'(-H)} = C(z)\,\frac{p'(-H)}{\rho_0\, w'(-H)}$$

in which we use the abbreviations

$$B(z) = \frac{f\, U_z - i(\omega + lV)V_z}{(i/l)\,(f^2 - (\omega + lV)^2)}$$

$$C(z) = \frac{l\,(\omega + lV)}{(i/l)\,(f^2 - (\omega + lV)^2)} \tag{3}$$

This differential equation for $w'(z)$ can be written as

$$\frac{d}{dz}\left[\frac{w'(z)}{w'(-H)} \exp - \int_{-H}^{z} B(\eta) d\eta\right] = \frac{p'(-H)}{\rho_0 w'(-H)}\; C(z) \exp - \int_{-H}^{z} B(\eta) d\eta$$

and since $w'(0) = 0$ is the upper boundary condition, the vertical integral gives

$$-1 = \frac{p'(-H)}{\rho_0\, w'(-H)} \int_{-H}^{0} C(z)\, dz \exp \int_{-H}^{z} - B(\eta) d\eta \tag{4}$$

Because of the continuity of the U, V and density profiles, the vertical velocity $w'(-H)$ and the pressure perturbation are continuous across $z = -H$, and the following wave dynamics for the thermocline will allow us to eliminate these terms in (4).

The static stability s (or the Brunt-Väisälä frequency) is assumed to be constant in the semi-infinite thermocline, and the Boussinesq dynamics supplies us with the dispersion relation. For each (ω,l) this gives us two vertical wavenumbers with opposite sign, one of these waves (w_+', p_+') being associated with an upward energy propagation, and the other one (w_-', p_-') representing a downward radiation of energy. The sum of the two waves gives the total vertical velocity $w = w_+' + w_-'$

and the total pressure $p'(z) = p_+' + p_-'$ at any depth z in the thermocline. If A denotes a velocity amplitude for the upward wave, and rA the corresponding amplitude for the downward wave, then the mathematical expression of the foregoing relations is

$$w_+' = A \exp i\, l(z+H)\, (gs-\omega^2)^{\frac{1}{2}}\, (\omega^2-f^2)^{-\frac{1}{2}}$$

$$\frac{p_+'}{\rho_0} = \frac{(gs-\omega^2)^{\frac{1}{2}}}{l\omega}\, (\omega^2-f^2)^{\frac{1}{2}}\, w_+'$$

$$w_-' = rA \exp - i\, l\, (z+H)\, (gs-\omega^2)^{\frac{1}{2}}\, (\omega^2-f^2)^{-\frac{1}{2}}$$

$$p_-'/\rho_0 = -\frac{(gs-\omega^2)^{\frac{1}{2}}\, (\omega^2-f^2)^{\frac{1}{2}}}{l\omega}\, w_-'$$

$$f < \omega < (gs)^{\frac{1}{2}}, l > 0$$

Since the total vertical velocity field just beneath $z = -H$ is

$$w'(-H) = w_+'(-H) + w_-'(-H) = A\,(1+r)$$

then the total pressure $p'(-H) = p_+' + p_-'$ is

$$\frac{p'(-H)}{\rho_0} = \frac{(gs-\omega^2)^{\frac{1}{2}}\, (\omega^2-f^2)^{\frac{1}{2}}}{\omega l}\left(\frac{1-r}{1+r}\right) w'(-H)$$

Although this relation is valid for nonhydrostatic oscillations, the dynamics of the mixed layer (1) are only valid for hydrostatic conditions. To insure consistency, by making the oscillation in the thermocline hydrostatic, we must restrict the above equation to the range

$$\omega^2 << gs$$
$$(l << 1/H)$$

in which case we get

$$\frac{p'(-H)}{\rho_0\, w'(-H)} = \frac{(gs)^{\frac{1}{2}}\, (\omega^2-f^2)^{\frac{1}{2}}}{\omega l}\left(\frac{1-r}{1+r}\right) \tag{4a}$$

as the hydrostatic boundary condition.

The substitution of the last equation in (4) and the use of (3) gives

$$\frac{1+r}{1-r} = \frac{-i\, l\, (gs)^{\frac{1}{2}}\, (\omega^2-f^2)^{\frac{1}{2}}}{\omega}\int_{-H}^{0} \frac{dz(\omega+lV)}{(\omega+lV)^2-f^2} \exp - i\, l \int_{-H}^{z} \frac{fU_\eta - i(\omega+lV)V_\eta}{(\omega+lV)^2-f^2}\, d\eta$$

$$= -\frac{i\, l(gs)^{\frac{1}{2}}\, (\omega^2-f^2)}{\omega}\int_{-H}^{0} \frac{dz(\omega+lV)}{[(\omega+lV)^2-f^2]^{3/2}} \exp - ilf \int_{-H}^{z} \frac{U_\eta\, d\eta}{(\omega+lV)^2-f^2} \tag{5}$$

$$\equiv - iY - \alpha \tag{5a}$$

where $V(-H) = 0$ has been used in the simplification leading to (5).

Although the form of the turbulent Ekman flow is not known, Eq. (5) can be used to compute the magnitude of the reflection coefficient r for any specified velocity profile. If the right-hand side of (5) is purely imaginary (i.e. $\alpha = 0$), as is the case if $f = 0$, then it follows that $|r| = 1$. The latter conclusion also holds if $U_z \equiv 0$, and thus we see that $|r|$ can be greater then unity in a rotating fluid only if there exists a component of the mean flow which is *parallel* to the wave fronts.

In terms of the real $(-\alpha)$ and imaginary parts of (5a) we see that $r = -(1+\alpha+iY)/(1-\alpha-iY)$, and therefore

$$|r|^2 = \frac{(1+\alpha)^2 + Y^2}{(1-\alpha)^2 + Y^2} \tag{5b}$$

is greater than unity, if $\alpha > 0$. If (5) be expanded as power series in the modulus of the basic current $(|U|, |V|)$, then the real and imaginary parts are

$$Y = \frac{l(gs)^{\frac{1}{2}}H}{(\omega^2-f^2)^{\frac{1}{2}}} + \ldots 0\,(|V|)$$
$$\alpha = \frac{l(gs)^{\frac{1}{2}}}{(\omega^2-f^2)^{\frac{1}{2}}}\left(\frac{lf}{\omega^2-f^2}\right)\int_{-H}^{0} dz\, U(z) + \ldots \tag{5c}$$

We therefore obtain the important conclusion that $|r| > 1$ if $\int_{-H}^{0} dz\, U(z)$ is a small positive quantity. The meaning of this becomes clear if the wind stress vector points in the direction of the $+y$ axis, in which case the Ekman transport relation for the mean flow is

$$\int_{-H}^{0} U\, dz = \frac{\tau}{f} \qquad , \int_{-H}^{0} V\, dz = 0$$

The substitution of this in (5c) gives

$$\alpha = \frac{l^2\,(gs)^{\frac{1}{2}}\tau}{(\omega^2-f^2)^{3/2}} \tag{5d}$$

which shows that waves traveling upwind have a reflection coefficient greater than unity. For a further discussion of (5b) see Eq. (30).

Can the reflection coefficient be infinite? If we set $r = \infty$ in (5) we obtain

$$1 = \frac{i\,l\,(gs)^{\frac{1}{2}}\,(\omega^2-f^2)}{\omega}\int_{-H}^{0} \frac{dz(\omega+lV)}{[(\omega+lV)^2-f^2]^{3/2}} \exp - ilf \int_{-H}^{z} \frac{U_\eta\, d\eta}{(\omega+lV)^2-f^2} \tag{6}$$

and the (ω,l) roots of this complex equation give the wavelength and frequency of the "spontaneous" wave. In the following section we will illustrate the pervasive nature of this larger effect by obtaining the roots of (6) for a variety of different profiles, the most realistic of which is in § 3b.

3. Examples of spontaneous waves

a. "Slab" flow in the mixed layer. Suppose that the mean flow in the mixed layer is unidirectional with the axis oriented such that $V = 0$. Then (6) simplifies to

$$1 = \frac{i\,l\,(gs)^{\frac{1}{2}}}{(\omega^2-f^2)^{\frac{1}{2}}} \int_{-H}^{0} dz \exp - \frac{i\,l\,U(z)}{\omega^2-f^2} \tag{7}$$

Suppose further that $U(z)$ increases rapidly from $U(-H) = 0$ to a constant U_0 in a short distance from the interface. This so-called "slab" flow (Pollard, *et al.*, 1973), might be approximately realized for small mixed layer depth, and for such profiles (7) simplifies to

$$1 = \frac{i\,l\,(gs)^{\frac{1}{2}}\,H}{(\omega^2-f^2)^{\frac{1}{2}}} \exp - \frac{i\,l\,f\,U_0}{\omega^2-f^2}$$

The real and imaginary parts of this give

$$\frac{l\,f\,U_0}{\omega^2-f^2} = \frac{\pi}{2}\;,\;\frac{5\pi}{2},\ldots$$

and

$$\frac{l\,(gs)^{\frac{1}{2}}\,H}{(\omega^2-f^2)^{\frac{1}{2}}} = 1$$

It follows that the highest possible frequency of such "spontaneous" waves is given by

$$\frac{(\omega^2-f^2)^{\frac{1}{2}}}{f} = \frac{U_0}{H(gs)^{\frac{1}{2}}}\left(\frac{2}{\pi}\right) \tag{8}$$

and the smallest possible wavenumber is

$$lH = \frac{2}{\pi}\,\frac{f\,U_0}{gs\,H} \tag{9}$$

Since similar nondimensional groups are obtained for the more realistic profiles in the next subsection, we believe (8) and (9) provide significant space-time scales for the oceanic effect, and therefore we shall estimate the order of magnitude by using $\rho_0\,\tau = 2$ dynes/cm^2, $s = 10^{-8}$ cm^{-1}, $f = 10^{-4}$ sec^{-1} and a mixed layer depth of $H = 50$ meters. Note that the equation above (8) implies that the *vertical* wavenumber in the thermocline equals $1/H = 1/50$ meters. If we set U_0H equal to τ/f, then (8) and (9) become

$$\frac{(\omega^2-f^2)^{\frac{1}{2}}}{f} \sim \frac{\tau}{fH^2(gs)^{\frac{1}{2}}}\left(\frac{2}{\pi}\right) \sim \frac{1}{6}$$

$$lH \sim \frac{2}{\pi}\,\frac{\tau}{gsH^2} = \frac{1}{200} \tag{9a}$$

Thus we see that the preferred frequency is slightly greater than f, and the preferred horizontal wavelength is of the order of 60 kilometers. Reference is made to a recent paper by Rossby and Sanford (1976), which supports the idea of a net downward propagation of energy near the inertial frequency, and in which the scaling is not inconsistent with that given above. It must be emphasized, however, that our theory does not imply that waves observed locally receive all their energy locally. We believe that the average inertial oscillation in the ocean is the result of multiple reflections and multiple passages of a wave packet through the mixed layer. Therefore measurements of the average value of l appear to be the most feasible way to test the application of the theory to the ocean.

b. "Spiral" profiles. We shall now show that large values of $|r|$ can also occur when the basic flow is not unidirectional, and for this purpose we may confine attention to waves near the inertial frequency ($\omega \cong f$), with their wavelength being sufficiently long so that the approximation

$$(\omega+lV)^2 - f^2 \cong 2f\,(\omega+lV-f)$$

is valid. Upon introducing this in (6) the equation for the spontaneous waves becomes

$$1 = \frac{i\,l\,(gs)^{\frac{1}{2}}\,(2f)\,(\omega-f)}{(2f)^{3/2}} \int_{-H}^{0} \frac{dz}{(\omega+lV-f)^{3/2}} \exp \int_{-H}^{z} \frac{-i\,l\,U_\eta\,d\eta}{2(\omega+lV-f)} \tag{10}$$

If z be replaced by the new dummy variable

$$\phi = \frac{l}{2} \int_{-H}^{z} \frac{U_\eta\,d\eta}{\omega+lV-f} \tag{11}$$

so that

$$\frac{d\phi}{dz} = \frac{l}{2}\,\frac{U_z}{(\omega+lV-f)}$$

then (10) transforms to

$$1 = \frac{i\,l\,(gs)^{\frac{1}{2}}\,(\omega-f)}{(2f)^{\frac{1}{2}}} \int_{0}^{\phi(0)} d\phi\,\frac{2e^{-i\phi}}{l\,U_z\,(\omega+lV-f)^{\frac{1}{2}}} \tag{12}$$

where the upper limit of integration is obtained by setting $z = 0$ in (11).

Let us consider a class of profiles for which $U_z > 0$ and U_z decreases monotonically upward, but U is otherwise unrestricted. We shall now prove that if the profile normal to the wavefronts is given by

$$V(z) = \beta\left[\frac{1}{U_z^2} - \frac{1}{U_z^2(-H)}\right] \tag{13}$$

where $U_z(-H)$ is the value of U_z at $z = -H$, and where

$$\beta = \frac{1}{2\pi} \int_{-H}^{0} U_z^3\,dz \tag{14}$$

then

$$\left(\frac{\omega-f}{f}\right)^{\frac{1}{2}} = \frac{U_z(-H)}{2^{3/2}(gs)^{\frac{1}{2}}}$$
$$l = \frac{(\omega-f)\, U_z^2(-H)}{\beta} \tag{15}$$

are roots of (12). When the last equation is used we see that (13) can be written as

$$l\, V(z) + \omega - f = \frac{l\beta}{U_z^2} = \frac{(\omega-f)\, U_z^2(-H)}{U_z^2} \tag{15a}$$

and therefore (12) simplifies to

$$1 = \frac{i\,(gs)^{\frac{1}{2}}\,(\omega-f)^{\frac{1}{2}}}{(2f)^{\frac{1}{2}}\, U_z(-H)} \int_0^{\phi(0)} 2e^{-i\phi}\, d\phi$$

When the first equation in (15) is used this becomes

$$1 = \frac{1-e^{-i\phi(0)}}{2}$$

and the first root is $\phi(0) = \pi$. Therefore when (11) is evaluated at $z = 0$, and when (15a) is used, we obtain

$$\pi = (l/2) \int_{-H}^{0} \frac{U_z\, dz}{\omega+lV-f} = \frac{1}{2\beta} \int_{-H}^{0} U_z^3\, dz$$

which is identical to (14). This completes the proof that (15) gives the frequency and wavenumber of a spontaneous wave for the class of profiles considered.

Since U_z decreases monotonically for this class, and $V(z)$ increases monotonically from $V(-H) = 0$ to its maximum value at $z = 0$, we see that the vector formed from (U,V) will lie in the same quadrant and will rotate with height. The hodograph is a plausible representation of a turbulent boundary layer, and we have shown that such a profile allows large reflection coefficients. Our subsequent work [cf. Eq. (30)] will indicate, however, that the assumption (on Θ) of this theory cannot be relied upon when $|r| = \infty$, and thus (15) merely suggests the scales having maximum amplification.

If one forms an energy equation for stationary waves from (1) then it follows that the net downward radiation (pressure work) at the base of the mixed layer (or the top of the thermocline) must equal the average of w_1, u_1, $U_z + w_1$, v_1, V_z. The latter quantity, or the rate at which the mean field does work on the perturbation, must be positive when $|r|>1$. But this does not imply that the mean flow kinetic energy decreases with time (to second order) because the body force (or $\boldsymbol{\tau}$) can resupply the mean flow, as discussed in § 5.

4. The vertically integrated theory

The weak point of the foregoing theory, as mentioned in § 1, is the assumption that the interaction of the waves with the turbulence (which maintains the basic flow) can be neglected. One of the advantages of the following integrated theory of the mixed layer is that it removes this assumption, and replaces it with another one which occurs in the closure of the theory, and which is quantifiable. In this way we are able to formulate a consistent asymptotic theory for the interaction of the internal waves with $\boldsymbol{\tau}$. Moreover the final result of the integrated theory is not only simpler than the previous theory, but it also may be extended to finite amplitude waves. After reviewing the basis for this theory (Stern, 1966; 1975) it will be applied to the problem of the generation of inertia-gravity waves by the wind.

We will assume that the local thickness h of our mixed layer is greater than $0.39\ \boldsymbol{\tau}^{\frac{1}{2}} f^{-1}$, the latter being the depth at which Caldwell *et al.* (1972) find that the ageostrophic shear flow decreases by 99%. Thus the vertical shear and the turbulent stress $\boldsymbol{\Theta}$ are negligible near and below $z = -h$. In the presence of an interfacial perturbation $h(x,y,t)$, the statistical (ensemble) average of the horizontal velocity in the mixed layer may be expressed as $\mathbf{V}_b(x,y,z,t) + \mathbf{V}_0(x,y,t)$, where $\mathbf{V}_0$ represents the realized flow at $Z = -h(x,y,t)$ and $\mathbf{V}_b(x,y,z,t)$ is the departure of the realized flow from $\mathbf{V}_0$ at any larger[2] Z. The generalized Ekman transport $\mathbf{M}$ is defined by the integral of the shear component, or

$$\mathbf{M} = \int_{-h(x,y,t)}^{0} \mathbf{V}_b\, dz \tag{16}$$

The total vertical velocity $w_0(x, y, z, t) + w_b(x, y, z, t)$ is also resolved into two components, the first of which (w_0) is computed from $\partial w_0/\partial z = -\nabla_2 \cdot \mathbf{V}_0$ and the interfacial boundary condition $w_0(z{=}-h) = -dh/dt$. Therefore this w_0 gives the realized vertical velocity near $z = -h$. From the linearity of the continuity equation it follows that w_b must satisfy $\partial w_b/\partial z = -\nabla_2 \cdot \mathbf{V}_b$, and $w_b(x,y,-h,t) = 0$ provides the boundary condition. Since $w_0(x,y,0,t) + w_b(x,y,0,t) = 0$ at the free surface, we must have

$$w_0(x,y,0,t) = \int_{-h(x,y,t)}^{0} \nabla_2 \cdot \mathbf{V}_b\, dz = \nabla_2 \cdot \mathbf{M} \tag{16a}$$

where the interchange of the integration with ∇_2 is permissible because $\mathbf{V}_b\,(x,y,-h,t) = 0$. Since w_0 is a linear function of z, the continuity equation for w_0 can be written as

$$\nabla_2 \cdot \mathbf{V}_0 = -\frac{w_0(x,y,0,t) - (-dh/dt)}{h}$$

and by using (16a) this becomes

2. The continuation of $\mathbf{V}_0$ in the thermocline will, of course, vary with z, but explicit consideration of the thermocline dynamics is unnecessary here.

$$\frac{\partial h}{\partial t} + \nabla_2 \cdot (\mathbf{V}_0 h) = - \nabla \cdot \mathbf{M} \tag{17}$$

where the last term is merely the generalized Ekman suction velocity. Let us now turn to the hydrostatic momentum equations which arise after the smaller scales of motion are averaged over.

Since the turbulent stress at $z = - h$, as well as $\mathbf{V}_b$, is assumed to be negligible, the horizontal momentum equation at the base of the mixed layer must reduce to the familiar form

$$\frac{\partial \mathbf{V}_0}{\partial t} + \mathbf{V}_0 \cdot \nabla_2 \mathbf{V}_0 + f \mathbf{k} \times \mathbf{V}_0 = - \rho_0^{-1} \nabla_2 p(x,y,t) \tag{18}$$

but this equation (together with the conservative dynamics of the underlying thermocline) and (17) are obviously incomplete since the Ekman suction velocity in (17) must be determined by another equation.

To obtain the required dynamical equation for M one proceeds as follows: The realized horizontal acceleration (i.e. the Lagrangian derivative of the total field $\mathbf{V}_0 + \mathbf{V}_b$) equals the sum of the corresponding Coriolis force, the pressure gradient force, and $\partial\Theta/\partial z$. By subtracting (18) from the latter relation, by integrating the result from $z = - h(x,y,t)$ to $z = 0$, and by simplifying one obtains (Stern, 1975)

$$\frac{\partial \mathbf{M}}{\partial t} + f \mathbf{k} \times \mathbf{M} + (\mathbf{V}_0 \cdot \nabla_2) \mathbf{M} + (\mathbf{M} \cdot \nabla_2) \mathbf{V}_0 + \mathbf{M} (\nabla_2 \cdot \mathbf{V}_0) + \mathbf{Q} = \boldsymbol{\tau} \tag{19}$$

$$\mathbf{Q} = \nabla_2 \cdot \int_{-h}^{0} (\mathbf{V}_b \, \mathbf{V}_b) \, dz \tag{20}$$

where $\mathbf{V}_b \, \mathbf{V}_b$ is a dyad. Equation (19) reduces to the classical Ekman transport relation when the fields are horizontally homogeneous ($\nabla_2 = 0$) or slowly varying. A new dynamical system arises, however, when the horizontal fluctuations in $\mathbf{V}_b$ (or $\mathbf{M}$) are small compared to the $\mathbf{V}_0 H$ fluctuations. In this case the quadratic term (20) is small compared to the bilinear terms in (19), and we have the "small stress" approximation

$$\frac{\partial \mathbf{M}}{\partial t} + f \mathbf{k} \times \mathbf{M} + (\mathbf{V}_0 \cdot \nabla_2) \mathbf{M} + (\mathbf{M} \cdot \nabla_2) \mathbf{V}_0 + \mathbf{M} (\nabla_2 \cdot \mathbf{V}_0) = \boldsymbol{\tau} \tag{21}$$

We shall subsequently compare the magnitude of the neglected term $\mathbf{Q} \sim (\boldsymbol{\tau}/fH) |\nabla_2 \mathbf{M}|$ with the terms involving $(\mathbf{V}_0, \mathbf{M})$ which have been retained in (21). This latter equation together with (17), (18), and the nonturbulent dynamical equations for $z \leqslant - h$ constitute a complete set for the investigation of a much wider class of motions than is encompassed by the classical wind-driven theory. Notice that no strong dynamical assumption regarding the turbulent stress $\Theta(x,y,z,t)$ has been made in this formalism, and the onus has been transferred to $\mathbf{Q}$.

Let us apply the "exact" equation (19) to the inertia-gravity wave problem (Fig. 1) considered previously. Suppose that the basic state consists of a wind stress $\boldsymbol{\tau}$ acting in the $+y$ direction, with $\bar{M}_x = \boldsymbol{\tau}/f$, $\bar{M}_y = 0$ denoting the (x,y) components of the undisturbed Ekman transport. Suppose that the interface at the base of the mixed layer is then given an infinitesimal perturbation $h'' = h - H$ which is independent of x, and let $\mathbf{V}_0 = (u'', v'')$, $\mathbf{M}'' = (M_x'', M_y'')$ denote the corresponding velocity and transport perturbations. The linearized value of the bilinear term $\mathbf{M}\ (\nabla_2 \cdot \mathbf{V}_0)$ in (19) will then have an x component equal to $(\tau/f)\ \partial v''/\partial y$ and a vanishing y component. The $\mathbf{Q}$ term, on the other hand, has an order of magnitude $\boldsymbol{\tau}/fH\ \partial M''/\partial y$, it being assumed that the undisturbed $\mathbf{V}_b$ is of order $\boldsymbol{\tau}/fH$. Therefore the ratio of $\mathbf{Q}$ to the bilinear term is

$$\epsilon \equiv \frac{|\mathbf{Q}|}{|\mathbf{M}\ \nabla_2 \cdot V_0|} \sim \frac{|\mathbf{M}''|}{H\ v''} \tag{22}$$

For a given amplitude of the incident wave in the thermocline it is obvious (and verified below) that as τ approaches zero, M'' approaches zero, but v'' does not. Thus the nondimensional stress parameter (22) approaches zero, thereby implying that $\mathbf{Q}$ is indeed negligible compared to $M(\nabla_2 \cdot \mathbf{V}_0)$. Furthermore $\mathbf{V}_0 \cdot \nabla\mathbf{M}$ vanishes to first order in the perturbation amplitude, and $\mathbf{M} \cdot \nabla_2\ \mathbf{V}_0 = \tau/f\partial\mathbf{V}_0/\partial x = 0$ because the perturbation is independent of x. Therefore the perturbed version of (19) can be written as

$$\frac{\partial M_y''}{\partial t} + f\,M_x'' = 0$$

$$\frac{\partial M_x''}{\partial t} - f\,M_y'' + \frac{\tau}{f}\ \frac{\partial v''}{\partial y}\ (1+0(\epsilon)) = 0$$

where the $0(\epsilon)$ term corresponds to $\mathbf{Q}$ in (15) and (22). The elimination of M_x'' then gives

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right) M_y'' = \tau\,\frac{\partial v''}{\partial y}\ (1+0(\epsilon)) \tag{23}$$

This clearly shows that $|M_y''| \sim (\tau l/\omega^2 - f^2)\ |v''|$, and thus the value of (22), is

$$\epsilon = \frac{\tau l}{(\omega^2 - f^2)H} \to 0 \tag{24}$$

as $\tau \to 0$, for any fixed $\omega > f, l$.

When (23) is combined with the linearization of (17), or

$$\frac{\partial h''}{\partial t} + H\ \frac{\partial v''}{\partial y} = -\ \frac{\partial M_y''}{\partial y} \tag{25}$$

we get

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\left(\frac{\partial h''}{\partial t} + H\frac{\partial v''}{\partial y}\right) = -\tau\frac{\partial^2 v''}{\partial y^2}(1 + 0(\epsilon)) \tag{26}$$

When modes of the form

$$v'' = Re\, v^* \exp i\,\omega t + i\,l\,y$$

$$h'' = Re\, h^* \exp i\,\omega t + i\,l\,y$$

are substituted in (26), the result may be written as

$$\omega h^* + l\,H_i\,v^* = 0 \tag{27}$$

$$H_i = H - \frac{i\,\tau\,l}{\omega^2 - f^2}(1 + 0\,(\epsilon))$$

The form of (27) is the same as the ordinary continuity equation for a homogeneous layer when $\tau = 0$, but in the present problem the "equivalent depth" H_i has an imaginary part proportional to τ. The analogy shows that the reflection coefficient for Figure 1, which we shall compute below, will be complex.

First linearize the momentum equations (18), and then insert the mode forms to get

$$i\,\omega\,u^* - f\,v^* = 0$$

$$-\frac{i\,l\,p^*}{\rho_0} = i\,\omega\,v^* + f\,u^* = (i\,\omega + f^2/i\omega)\,v^* \tag{28}$$

where p^* is the amplitude of the hydrostatic pressure perturbation in the mixed layer. The latter quantity is also given by the radiation boundary condition (4a) which was obtained from the dynamics of the thermocline. With $w'(-H) = -i\,\omega\,h^*$ (4a) can be written as

$$\frac{p^*}{\rho_0} = \frac{(gs)^{\frac{1}{2}}\,(\omega^2-f^2)^{\frac{1}{2}}}{\omega l}\left(\frac{1-r}{1+r}\right)(-i\,\omega h^*)$$

Therefore (28) becomes

$$(gs)^{\frac{1}{2}}\,(\omega^2-f^2)^{\frac{1}{2}}\left(\frac{1-r}{1+r}\right)h^* = \frac{\omega^2-f^2}{i\,\omega}\,v^*$$

By combining the latter equation with (27) we obtain the following result for the reflection coefficient

$$\frac{1+r}{1-r} = -\frac{i\,(gs)^{\frac{1}{2}}\,l}{(\omega^2-f^2)^{\frac{1}{2}}}\left[H - \frac{i\,\tau l}{\omega^2-f^2}\,(1+0(\epsilon))\right]$$

This is identical to the result obtained from (5a), (5c), (5d), thereby implying equivalence of the two theories when τ is small, and establishing confidence in the conclusion that an inertia-gravity wave which propagates upwind can obtain energy from $\boldsymbol{\tau}$. Note that the contribution of the $\mathbf{Q}$ term to the above equation is proportional to $(\tau\epsilon) \propto \tau^2$, and is therefore negligible compared to the contribution of the $\mathbf{M} \nabla \cdot \mathbf{V}_0$ term.

The value of (5b) becomes large when $\alpha = 1$ and when Y is small. When $\alpha = 1$ (5d) implies

$$\frac{1}{l} = \left[\frac{(gs)^{\frac{1}{2}} \tau}{(\omega^2 - f^2)^{3/2}} \right]^{\frac{1}{2}} \tag{29}$$

and for any Y Eq. (5b) then becomes

$$|r|^2 = 1 + 4/Y^2$$

When the asymptotic ($\tau \to 0$) value of Y given in (5c) is inserted here, and when the optimum wavelength (29) is used we get

$$|r|^2 = 1 + \frac{4(\omega^2 - f^2)}{gsl^2H^2} = 1 + \frac{4\tau}{(gs)^{\frac{1}{2}} H^2 (\omega^2 - f^2)^{\frac{1}{2}}} \tag{30}$$

with the error (cf 24) being of the order of the square of the last term in (30). Although consistency requires the latter to be less than unity[3], we may conclude from (30) that $|r|^2$ increases as the inertial frequency is approached, and these waves are therefore "preferred" at small τ.

It is interesting to examine the implication of our theory for a two-layer density model, such as may be realized in the laboratory. The upper layer of mean depth H has a turbulent Ekman layer, as previously, but the region below the interface now contains a layer having uniform $\rho_0 + \Delta\rho$ and thickness H_2. We shall examine the growth rate $- Im\omega$ of the normal modes. The essential result (27) has already been obtained, and we previously pointed out that this equation for the upper layer is formally identical to the continuity equation when there is no stress. A little reflection will reveal that the dispersion relation for the frequency ω when $\tau > 0$ can be obtained from the well-known dispersion relation when $\tau = 0$, by merely replacing the upper layer thickness in the latter relation with H_i as given by (27). Accordingly, we find that the normal modes in our present problem satisfy the dispersion relation

$$(\omega^2 - f^2) \left[\frac{1}{H_i} + \frac{1}{H_2} \right] = \frac{l^2 \, g\Delta\rho}{\rho_0}$$

When (27) is substituted in this, and when terms of order τ^2 are neglected (for consistency) we get

$$\omega^2 - f^2 = \frac{l^2 \, g\Delta\rho/\rho_0}{\dfrac{1}{H}\left[1 - \dfrac{i\,\tau l}{H(\omega^2 - f^2)} \right]^{-1} + \dfrac{1}{H_2}}$$

3. Note that the first equation in (9a) does not satisfy this consistency relation, and therefore the prediction of the "spontaneous waves" cannot be relied upon. Nevertheless (9a) suggests optimum values of (l, ω).

$$= \frac{l^2 \, g\Delta\rho/\rho_0}{\dfrac{1}{H} + \dfrac{1}{H_2}} - \frac{i\,\tau l}{H(1+H/H_2)} + \ldots 0\,(\epsilon^2)$$

Thus we see that the asymptotic growth rate

$$-\,Im\,\omega = \frac{\dfrac{\tau l}{H(1+H/H_2)}}{2\left[\, f^2 + \dfrac{g\Delta\rho}{\rho_0} \cdot l^2 \left(\dfrac{1}{H} + \dfrac{1}{H_2} \right)^{-1} \right]^{\frac{1}{2}}}$$

is positive when τ, l are positive, thereby implying *exponential* amplification of interfacial waves, which propagate upwind. This relation might be tested experimentally.

5. Remarks on energetics

We have used two different theories to show that internal waves near the inertial frequency are substantially amplified when they enter the mixed layer with a horizontal component of the phase velocity which is directed upwind. (In the case where there is a uniform geostrophic velocity superimposed on the mixed layer the waves must propagate opposite to $\boldsymbol{\tau}$ relative to an observer moving with this geostrophic velocity). When the amplified wave leaves the mixed layer and propagates downward in the thermocline, it will eventually be reflected upward again because of the ocean bottom or inhomogeneities in the thermocline. Thus the waves enter the mixed layer a second time and abstract more energy from the wind. The amplification process will therefore continue until some equilibrium is reached, in which the rate of transfer of energy from the wind is balanced by the transfer of energy from the near inertia frequency to higher frequency waves. Equation (30) suggests a maximum spectral peak near the Coriolis frequency, and (9a) is a more detailed estimate of the relevant space-time scales. Reference has been made to the measurements of Rossby and Sanford which indicate a net downward flux of inertial energy; their estimates of horizontal wavelength are not inconsistent with our scaling.

At the end of § 3 we alluded to the peculiar energetics of the "overreflected" wave, and the following discussion of the nonlinear momentum and energy integrals obtained from Eqs. (17)-(21) will clarify the mechanism.

Let us assume a finite amplitude perturbation field which is cyclic in the unbounded horizontal direction; a bar will denote an average over (x,y). Thus $\overline{\mathbf{Q}} = 0$ in (19), $\overline{\mathbf{V}_0 \cdot \nabla_2 \mathbf{M}} = -\overline{\mathbf{M}\, \nabla_2 \cdot V_0}$, and it follows that the average of (19) is

$$\frac{\partial \overline{\mathbf{M}}}{\partial t} + f\,\mathbf{k} \times \overline{\mathbf{M}} - \overline{\mathbf{V}_0 (\nabla_2 \cdot \mathbf{M})} = \boldsymbol{\tau} \tag{31}$$

This equation and (34) show how the wave Reynolds stress modifies the horizontal average of $\mathbf{M}$ and $\mathbf{V}_0$. If $-z = +H_0 < \min h(x,y,t)$ be any level surface in the mixed layer then the average of (18) can be written as

$$\frac{\partial}{\partial t} \overline{H_0 \mathbf{V}_0} + f \mathbf{k} \times \overline{\mathbf{V}_0 H_0} - \overline{H_0 \mathbf{V}_0 (\nabla_2 \cdot \mathbf{V}_0)} = 0 \tag{32}$$

Since $\nabla_2 \cdot \mathbf{V}_0$ is independent of z and since (16a) states $w_0(x,y,0,t) = \nabla_2 \cdot \mathbf{M}$, we can write

$$\nabla_2 \cdot \mathbf{V}_0 = -\frac{\partial w_0}{\partial z} = -\frac{w_0(x,y,0,t) - w_0(x,y - H_0,t)}{H_0}$$

and thus

$$\nabla_2 \cdot \mathbf{M} + H_0 \nabla_2 \cdot \mathbf{V}_0 = w_0 (x,y{-}H_0,t) \tag{33}$$

Therefore the sum of (31) and (32) is

$$\left(\frac{\partial}{\partial t} + f \mathbf{k}\, x\right) (\overline{\mathbf{M}} + \overline{\mathbf{V}_0} H_0) = \boldsymbol{\tau} + \overline{\mathbf{V}_0\, w_0(x,y,-H_0,t)} \tag{34}$$

The interpretation of this mean momentum equation is simplified if we assume that the $z = -H_0$ datum may be chosen such that $\mathbf{V}_b(x,y,-H_0,t)$ is negligible, in which case $\mathbf{M} + \mathbf{V}_0 H_0$ is the total horizontal momentum above the $z = -H_0$ datum level. Equation (34) then states that the rate of change of this momentum plus the Coriolis force equals the difference between $\boldsymbol{\tau}$ and the downward flux of momentum at $z = -H_0$. Since this agrees with first principles, we see that our (approximate) theory (21) conserves momentum exactly.

The extent to which our system conserves energy may be seen by forming the following integrals. The scalar product of $H_0\, \mathbf{V}_0$ with (18) yields

$$\frac{1}{2} \frac{\partial}{\partial t} \overline{H_0 \mathbf{V}_0^2} - \frac{1}{2} \overline{H_0 \mathbf{V}_0^2 \nabla_2 \cdot \mathbf{V}_0} = \frac{H_0}{\rho_0} \overline{p \nabla_2 \cdot \mathbf{V}_0} \tag{35}$$

and the scalar product of $\mathbf{M}$ with (18) yields

$$\overline{\mathbf{M} \cdot \frac{\partial \mathbf{V}_0}{\partial t}} + \overline{\mathbf{M} \cdot (\mathbf{V}_0 \cdot \nabla_2)\, \mathbf{V}_0} - \overline{\mathbf{V}_0 \cdot f \mathbf{k} \times \mathbf{M}} = -\overline{\mathbf{M} \cdot \nabla_2 p}\, \rho_0^{-1}$$

On the other hand, the scalar product of $\mathbf{V}_0$ with (21) yields

$$\overline{\mathbf{V}_0 \cdot \frac{\partial \mathbf{M}}{\partial t}} + f\, \overline{\mathbf{V}_0 \cdot \mathbf{k} \times \mathbf{M}} + \overline{\mathbf{V}_0 \cdot (\mathbf{V}_0 \cdot \nabla_2)\, \mathbf{M}} + \overline{(\mathbf{V}_0 \cdot \mathbf{M})\, \nabla_2 \cdot \mathbf{V}_0}$$
$$+ \overline{\mathbf{V}_0 \cdot (\mathbf{M} \cdot \nabla_2)\, \mathbf{V}_0} = \overline{\mathbf{V}_0 \cdot \boldsymbol{\tau}}$$

The sum and simplification of the last two equations gives

$$\frac{\partial}{\partial t} \overline{(\mathbf{V}_0 \cdot \mathbf{M})} + \overline{\mathbf{V}_0 \cdot \nabla_2 (\mathbf{M} \cdot \mathbf{V}_0)} + \overline{(\mathbf{V}_0 \cdot \mathbf{M})\, \nabla_2 \cdot \mathbf{V}_0} + \tfrac{1}{2}\, \overline{\mathbf{M} \cdot \nabla_2 \mathbf{V}_0^2} =$$
$$\overline{\mathbf{V} \cdot \boldsymbol{\tau}} + \overline{p \nabla_2 \cdot \mathbf{M}}\, \rho_0^{-1}$$

or

$$\frac{\partial\, \overline{\mathbf{V}_0 \cdot \mathbf{M}}}{\partial t} - \overline{(\mathbf{V}_0^2/2 + p/\rho_0)\, \nabla_2 \cdot \mathbf{M}} = \overline{\mathbf{V}_0 \cdot \boldsymbol{\tau}}$$

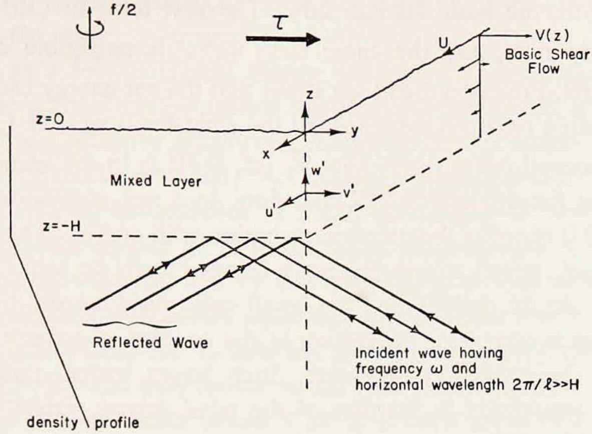

Figure 1. Schematic diagram of a wind-driven mixed layer overlying a semi-infinite thermocline of uniform density gradient. The undisturbed state consists of an arbitrary shear flow $U(z)$ in the x direction, and $V(z)$ in the y direction, with $U = V = 0$ for $z \leqslant -H$ (the base of the mixed layer). This shear flow is maintained by the small scale turbulence produced by the wind stress $\boldsymbol{\tau}$. The perturbation consists of an oblique plane wave propagating energy upward in the thermocline, and the wavelength $2\pi/l$ is very much greater than H. We compute the amplitude of the reflected wave, or the reflection coefficient r, as the result of the wave-current interaction in $z > -H$. The Coriolis parameter is f and (w', u', v') are the infinitesimal amplitude perturbations associated with the wave.

The sum of this and (35) is

$$\frac{\partial}{\partial t}\left(\frac{H_0}{2}\overline{\mathbf{V}_0{}^2} + \overline{\mathbf{V}_0 \cdot \mathbf{M}}\right) - \overline{(\mathbf{V}_0{}^2/2 + p/\rho_0)\ (\nabla_2 \cdot \mathbf{M} + H_0\, \nabla_2 \cdot \mathbf{V}_0)} = \overline{\mathbf{V}_0 \cdot \boldsymbol{\tau}}$$

or

$$\frac{\partial}{\partial t}\left(\frac{H_0}{2}\overline{\mathbf{V}_0{}^2} + \overline{\mathbf{V}_0 \cdot \mathbf{M}}\right) - \overline{w_0(x,y,-H_0,t)\ (\mathbf{V}_0{}^2/2 + p/\rho_0)} = \overline{\mathbf{V}_0 \cdot \boldsymbol{\tau}} \qquad (36)$$

where (33) has been used.

Since the total kinetic energy above $z = -H_0$ is

$$\frac{1}{2}\int_{-H_0}^{0} \overline{(\mathbf{V}_0 + \mathbf{V}_b)^2}\, dz = \frac{H_0}{2}\overline{\mathbf{V}_0{}^2} + \overline{\mathbf{V}_0 \cdot \mathbf{M}} + \frac{1}{2}\int_{-H_0}^{0} \overline{\mathbf{V}_b{}^2}\, dz$$

we may interpret the energy Equation (36) as follows:

The kinetic energy associated with the barotropic component (i.e. $H_0/2\ \overline{\mathbf{V}_0{}^2}$) plus the "interaction kinetic energy" $\overline{\mathbf{V}_0 \cdot \mathbf{M}}$ increases at a rate equal to the rate at which the wind does work on $\mathbf{V}_0$, minus the energy transported downward at $z = -H_0$. It is in this sense that the small stress approximation (21) conserves energy.

Let us apply (36) to the problem (Fig. 1), in which an infinitesimal inertia-gravity wave interacts with the basic Ekman flow. The first term in (36) vanishes in the statistically steady state, and the cubic term $\overline{w_0 \mathbf{V}_0{}^2}$ is negligible compared to the quadratic $\overline{w_0 p}$ term. Equation (36) then states that the net energy radiated downward into the thermocline must be balanced by the right-hand side of (36), thereby implying that the second order mean flow $\overline{\mathbf{V}_0}$ [cf. (34)] is in the same direction as $\boldsymbol{\tau}$. Thus we see that the undisturbed Ekman flow does not continuously lose energy, even though (§2) it supplies the wave perturbation with energy. The basic shear flow acts like a catalyst, which allows the body forces ($\boldsymbol{\tau}$) to do more work than they would ordinarily do (in maintaining the small scale turbulence). The second order effect of the wave is extremely interesting in the context of generating larger scales of motion than the inertia-gravity wave. Such larger scales, ranging up to that which has been considered in theories of the wind driven general circulation, are also encompassed by (21).

6. Summary

The two models which have been used to demonstrate the systematic interaction between long inertia-gravity waves and the wind induced shear flow have some common assumptions and some notable differences, these being summarized as follows.

There are similar scaling and kinematic assumptions in each model, such that the turbulent stress and vertical shear vanish near and below the homogeneous upper layer. There is a distinct scale separation between the turbulence and the hydrostatic disturbance, such that the pressure field associated with the latter is independent of depth in the mixed layer. Thus the vertical shear of the horizontal motion in this layer would vanish when $\boldsymbol{\tau} = 0 = \mathbf{M}$, in which case (17) and (18) reduce to the classical nonlinear shallow water equations. When $\boldsymbol{\tau} \neq 0$ the vertical shear on the scale of the disturbance will not vanish, and we therefore inquire as to whether such an interaction can systematically transfer energy from the wind to the near inertial oscillations in the oceanic thermocline.

In the first theory the wave perturbation of the wind-driven stress field is completely neglected. [We have rejected an alternate approach, in which one uses an eddy coefficient parameterization of Θ and then assumes this coefficient to be constant. The latter approach is, of course, completely appropriate in the related problem of the instability of laminar Ekman layers having constant molecular viscosity]. It is worth emphasizing that our first model is equivalent to a "thought experiment" in which the basic shear flow is produced by applying fixed lateral body forces to the rotating fluid. Eq. (1) is certainly "dynamically consistent" in such a system, in the sense that no conservation laws are violated. In this sense the conclusion regarding overreflected waves, and also $|r| = \infty$, is unobjectionable. But the asymptotic, or qualitative, validity of our conclusions is not assured in the context of the wind-driven model, wherein the body force $\partial\Theta/\partial z$ is not fixed. We obviously need some measure of the error.

Therefore we turn to the second theory wherein (17)-(20) appear as rigorous deductions, in the context of the scaling assumptions mentioned above. Eqs. (17)-(20) arise from the "primitive equations" (Stern, 1975) by means of a convenient formal separation of the total velocity $\mathbf{V}_0\ (x,y,t) + \mathbf{V}_b\ (x,y,z,t)$ into two components, the first of which ($\mathbf{V}_0$) is independent of z and the second of which ($\mathbf{V}_b$) vanishes at $z = -\ h(x,y,t)$. The choice of this separation was dictated by the hydrostatic character of the total velocity, and by the desirability of maintaining "correspondence" with the classical hydrostatic equations for $\mathbf{V}_0$ when $\boldsymbol{\tau} = 0 = \mathbf{V}_b$. Thus the equations which define (sic!) the $\mathbf{V}_0$ field are given by (18), (17), these being chosen such that when $\boldsymbol{\tau} = 0 = \mathbf{M}$ they reduce to the well-known shallow water dynamics. Thus we see that the separation (when $\boldsymbol{\tau} \neq 0$) is such as to shift the "dynamical burden" to the equation for $\partial\mathbf{V}_b/\partial t$ (cf Stern, 1975), which equation arises by subtracting (18) from the dynamical equation for $\partial\ (\mathbf{V}_0 + \mathbf{V}_b)/\partial t$. The vertical integral of the $\partial\mathbf{V}_b/\partial t$ equation then gives the equation (19) for $\partial\mathbf{M}/\partial t$. It is the $\mathbf{Q}$ term in the latter equation which prevents closure of the system for the fields ($\mathbf{V}_0$, $\mathbf{M}$). But we find that this $\mathbf{Q}$ term may be neglected compared to the new bilinear terms in two different asymptotic regimes, one of which is dealt with in this paper. In this regime we found that infinitesimal perturbations to the undisturbed Ekman transport ($|\mathbf{M}| = \boldsymbol{\tau}/f$) give rise to perturbations in $\mathbf{Q}$ which are small compared to those in $\mathbf{M}\ \nabla \cdot \mathbf{V}_0$ [by order ϵ (24)], in consequence of which the leading term in the expression for the amplification coefficient ($|r| - 1$) can be computed by discarding the term $\mathbf{Q} \propto \boldsymbol{\tau}^2$ while retaining the term $\mathbf{M}\ \nabla \cdot \mathbf{V} \propto \boldsymbol{\tau}$. Although the amplification coefficient obtained from (30) is asymptotic, thereby justifying the main result obtained from the first theory, we have no assurance that the detailed vertical structure of the wave in the first theory is asymptotic. Likewise, the suggestive information about the "preferred" lateral scale (9a) of the inertia oscillation cannot be relied upon because of these confidence limits. The author would also make the same criticism of any model which utilized an eddy coefficient whose magnitude was quantitatively significant.

Both theories imply that a turbulent Ekman layer in a vertically bounded system is unstable with respect to infinitesimal amplitude long waves, and the energy near the inertial frequency will therefore increase with time. We may be able to address the question of the statistically steady state which eventually arises in a real fluid by means of the following alternative view of (21). Suppose that $\boldsymbol{\tau} = 0$ and $\mathbf{V}_0$ is large at some time $t \leqslant 0$. For $t < 0$ the conservative nonlinear dynamics of the system are then governed by (18), (17) with $\mathbf{M} = 0$, and by the well-known dynamics of the thermocline. The total energy in such a system ($t < 0$) cannot increase in time, and must be arbitrarily specified *ab initio*. At time $t > 0$ we apply a "small" uniform wind stress $\boldsymbol{\tau}$. Eq. (19) applies to the resulting weak interaction problem because **Q** is small compared to the bilinear terms in (19) by a factor which is proportional to τ. "Weak interaction" means that the nonlinear inertial waves will be approximately conservative over short periods of time (f^{-1}) but will gain a significant amount of energy from the wind over long periods of time ($>> f^{-1}$). In the stationary state the energy gained by the inertial waves must equal the energy transferred to other scales of motion in the thermocline, and if this "dissipative" aspect can be formulated then we can address the fundamental problem of relating the r.m.s. amplitude of the intertia wave to the driving force of the wind and to the other scales of motion to which it is coupled. This is an ambitious problem, of course, but it seems worthwhile to consider it in the context of a homogeneous model [e.g. Fig. 1] before turning to the real ocean case in which $\boldsymbol{\tau}$ varies in space and time.

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