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Finite amplitude edge waves

by R. T. Guza¹ and A. J. Bowen²

ABSTRACT

Large amplitude edge waves are shown to be modified by nonlinear effects in a way very similar to surface waves in deep water (Stokes, 1847); trapped harmonics tend to sharpen the wave crests and the natural frequency increases with wave amplitude, progressive edge waves propagating faster at large amplitude. A standing edge wave exhibits additional properties due to interaction between its two constituent progressive waves. Of particular interest, and the subject of laboratory experiments, is the observation that a standing edge wave, frequency σ , radiates energy at 2σ to the far field. This is a rather special example of the whole class of resonant interactions between edge waves trapped against a coastline and normal, surface waves propagating from, or toward, deep water.

The resonant forcing of edge wave modes has been parameterized by an initial growth rate (Guza and Davis, 1974) which provides an estimate of the modes likely to occur but gives no direct indication of the maximum size the edge waves will attain, a question of obvious practical importance. Edge wave amplitudes are found to be limited by radiation, energy lost to other waves by further nonlinear interaction; by finite amplitude demodulation, the forcing frequency ceasing to lie within the resonant bandwidth as the natural frequency increases with wave amplitude; and by viscosity. For the subharmonic resonance involving the lowest mode edge wave, radiation and demodulation are shown to be of comparable importance in limiting edge wave growth; viscosity is relatively unimportant. In this case, the edge wave amplitude at the shoreline is theoretically three times that of the incoming wave (which is strongly reflected in this case), in good agreement with previous laboratory observations.

1. Introduction

Several mechanisms have recently been suggested for generating waves on beaches, all involving the transfer of energy from the incoming waves by nonlinear interactions. On a shallow, sloping beach, nonlinear interaction occurs at second order in the form of triads, two waves interacting to transfer energy to a third. Gallagher (1971) showed that the interaction between two incoming waves may result in the resonant excitation of edge waves at the beat frequency. Guza and Bowen (1975) demonstrated that a monochromatic wave train, incident on a plane beach and strongly reflected, is unstable to edge wave perturbation. Here,

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the most strongly excited resonance consists of two, mode zero, edge waves (Stokes' edge waves) with different frequencies, travelling in opposite directions along the beach. For the special case of incoming waves normally incident on the beach, the preferred resonance is a standing Stokes' edge wave at the subharmonic, $\sigma/2$, of the incoming wave frequency, σ (Guza and Davis, 1974).

To consider the role that edge waves may play in beach dynamics, particularly the formation of beach features that are rhythmic in the longshore direction, it is necessary to understand not only how edge waves are generated, but how large an amplitude they may reach. Indeed, an insight into the physical processes that limit edge wave growth is desirable not only for estimating the importance of edge waves in the field, but also in assessing the relevance of laboratory experiments to the real world; the relative importance of the various processes that limit edge wave growth might change with the scaling necessary for laboratory studies.

The existing laboratory results show that subharmonic Stokes' edge waves with amplitudes (at the shoreline) substantially larger than the amplitude of the incoming wave may occur when the incoming wave is strongly reflected by the beach (Galvin, 1967; Bowen and Inman, 1971; Guza and Inman, 1975). When the incident wave breaks cleanly this resonance seems to disappear (Galvin, 1965).

Three possible processes which may limit edge wave growth are:

- (i) further nonlinear energy exchange: studies of the excitation of edge waves have neglected nonlinear, edge wave-edge wave interactions as these are initially negligible compared to interactions involving the primary wave. However, after the edge waves have grown to a finite amplitude, significant energy exchange can occur via edge wave-edge wave interactions, resulting in a loss of edge wave energy to other edge wave modes (Kenyon, 1970) or to the far field.
- (ii) detuning: the transfer of energy from incident to edge waves occurs most rapidly when the edge frequency and longshore wave number satisfy the dispersion relation for free edge waves. However, if an edge wave satisfies the dispersion relation initially, when its amplitude is small, it will no longer do so at finite amplitude as the natural frequency at fixed wave-number is amplitude dependent. Continued growth of the edge wave may therefore be inhibited by finite amplitude detuning. In addition, detuning via the boundary conditions is important in a laboratory basin of fixed longshore dimensions where the possible wave-numbers are determined by side wall boundary conditions; initial edge wave growth can occur only in certain frequency bands.
- (iii) frictional dissipation: in a study of the generation of a single edge wave by the interaction of two incident waves, Gallagher (1971) used a linear term (proportional to the surface displacement of the edge wave) to represent the energy dissipation due to bottom friction and scattering off bottom irregularities. For constant forcing by the incident waves, edge growth is then even-

tually limited by dissipation. However, for resonances involving only a single incident wave, a linear dissipative term has the effect of determining only a lower limit on the size of the primary wave which can initiate resonance (Guza and Davis, 1974). Frictional effects apparently do not limit subharmonic edge wave growth with surging incident waves when the assumption of a laminar boundary bottom layer (equivalent to a linear bottom friction) is valid.

To consider the details of the processes which may limit the edge wave amplitude it is useful to derive first some of the properties of edge waves of finite amplitude. In section 2, therefore, the theory for progressive edge waves is extended to include the small, nonlinear corrections due to the finite amplitude; these corrections are generally similar to those known for surface gravity waves in a fluid of constant depth (Stokes, 1847). However, standing edge waves are shown to possess properties which arise due to the nonlinear interaction between the two progressive waves (of equal amplitude, frequency and wave-number but moving in opposite directions) into which the standing wave can be decomposed. This is merely a special case of the general class of possible interactions between edge waves.

One conclusion, that a standing edge wave should radiate energy seawards in the form of an outgoing progressive wave of twice the frequency of the edge wave, was investigated in the laboratory experiments discussed in section 3.

In section 4, the factors that determine the equilibrium amplitude and phase of an actively forced, standing edge wave are considered. The theory is found to be in general agreement with the laboratory observations in suggesting that the edge waves are generally larger than the incoming waves at the shoreline. The effects of finite amplitude detuning and energy radiation due to nonlinear interactions are shown to be formally of the same order and both detuning and radiation seem to be important in determining the equilibrium amplitudes reached by the edge waves. The theoretical results are generally in very good agreement with the equilibrium amplitudes of edge waves observed in laboratory experiments (Guza and Inman, 1975).

2. Edge waves of finite amplitude

The equations governing the edge wave motion are the standard, nonlinear, shallow water equations (Stoker, 1957; Mei and Le Méhauté, 1966) on a plane beach of slope, $\tan\beta$. The velocity potential ϕ , sea surface elevation η , and wave frequency σ are expanded in terms of the (small) parameter ε , where

$$\varepsilon = a\sigma^2/g \tan^2\beta . \quad (1)$$

a is the wave amplitude and g the gravitational acceleration. Then

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots \quad (\text{a})$$

$$\eta = \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \dots \quad (\text{b}) \quad (2)$$

$$\sigma = \sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \dots \quad (\text{c})$$

In addition, the possibility of energy transfer at time scales which are long in comparison with a wave period is anticipated by allowing the wave amplitude to vary slowly in time,

$$\frac{da}{dt} \sim 0(\varepsilon) \quad \text{or} \quad 0(\varepsilon^2) .$$

The lowest order, linear solution for progressive edge waves of mode $n = 0$ (Appendix, Eq. A4) is

$$\phi_0 = \frac{age^{-kx}}{\sigma} \cos(ky - \sigma t) \quad (3)$$

$$\sigma^2 = \sigma_0^2 = gk \tan\beta$$

where x and y are the offshore and longshore coordinates and k is the longshore wave-number. Then the solutions at second order are readily shown (A7) to be,

$$\sigma_1 = 0$$

necessarily, from symmetry arguments, and

$$\phi_1 = 0.$$

Curiously, for the edge wave of mode $n = 0$ only, the velocity potential and sea surface elevation have no harmonic corrections of frequency 2σ . However, there is a second order term in the sea surface elevation, which represents the set-down of mean sea level close to the shore, of magnitude

$$\varepsilon\eta_1 = -\frac{a\varepsilon}{2} e^{-2kx} . \quad (4)$$

Extending the analysis to third order (A11, A12) shows that the wave propagates with constant shape

$$\frac{da}{dt} \sim 0(\varepsilon^3)$$

and

$$\sigma_2 = .25\sigma_0 .$$

The term ϕ_2 contributes a small alteration in the offshore (x) dependence of the basic wave form given by (3). The variation in the wave frequency (at fixed wave-number) is given from (2) and (5) as

$$\sigma^2 = gk(1 + \varepsilon^2/2) \tan\beta . \quad (6)$$

Then, for waves of a given frequency, as the wave height increases the wave number decreases and the phase velocity increases. A similar result for progressive, surface waves in constant depth is well known (Stokes, 1847).

For a standing Stokes' edge wave (A4), additional terms occur in the second order equations which can be regarded as the cross interactions of the two progressive waves forming the standing wave. The second order solutions for the $n = 0$ (Stokes) mode of a standing wave are, from (A9),

$$\phi_1 = \frac{ag}{\sigma} [p(\chi) \cos 2\sigma t + q(\chi) \sin 2\sigma t]$$

$$\chi^2 = \frac{4(2\sigma)^2 x}{g \tan \beta}$$

where $p(\chi)$, $q(\chi)$, complex functions of the offshore distance, are given by (A8). However, at offshore distance greater than a few wavelengths,

$$\lim_{\chi \rightarrow \infty} \phi_1 = -\frac{age_1(\infty)}{8\sigma} \cdot \pi (J_0(\chi) \cos 2\sigma t + Y_0(\chi) \sin 2\sigma t) \quad (7)$$

where $e_1(\infty) = .541$ and J_0 and Y_0 are zero order Bessel functions. ϕ_1 is therefore an outgoing, progressive wave of frequency 2σ . A standing edge wave therefore leaks energy to the far field at second order. This energy exchange can be regarded as being the interaction between two progressive edge waves of longshore wave-number $+k$ and $-k$, and frequency σ , resulting in a free wave propagating directly seawards of zero longshore wave-number and frequency 2σ . These three waves, satisfying the interaction condition that the sums of the longshore wave-numbers and frequencies are both zero, provide a special example of the general class of interactive wave triads that may occur on a beach.

The second order solution for the surface elevation is

$$\eta_1 = -\frac{1}{g} \frac{\partial \phi_1}{\partial t} - \frac{a}{4} e^{-2kx} (1 + 2 \cos 2\sigma t) .$$

The second term gives a set-down, steady in time (equivalent to the sum of the set-downs associated with two progressive waves of amplitude $a/2$, and a forced oscillation of frequency 2σ . The first term describes the oscillations of frequency 2σ associated with the radiated waves.

The extension of the calculations to third order shows that for a standing, Stokes edge wave (A15)

$$\sigma_2 = .055\sigma_0$$

where σ_1 is again zero. The dispersion relation then becomes

$$\sigma^2 = gk \tan \beta (1 + .11\epsilon^2) \quad (8)$$

and again, at fixed frequency, the wave-number decreases with increasing wave

amplitude. At third order, the edge wave amplitude is a slowly varying function of time where (A15)

$$\frac{da}{dt} = -a\sigma\epsilon^2 e_1(\infty)2\pi\alpha; \quad \alpha = 0.0169 \quad (9)$$

consistent with the intuitive expectation that if a wave at the harmonic frequency is radiated, energy is continuously lost to the far field. In the absence of an energy source, the edge wave amplitude slowly decays. The offshore energy flux associated with the radiated wave (7) is readily shown to be

$$-\frac{\rho g a^2 \pi \epsilon^2 e_1^2(\infty) \sigma}{64k} . \quad (10)$$

The edge wave energy, per unit longshore length is given by

$$E = \frac{\rho g a^2}{8k} \quad (11)$$

so that the rate of change of E , from (9), is given by

$$\frac{\rho g a}{4k} \frac{da}{dt} = -\frac{\rho g a^2}{2k} \pi \alpha e_1(\infty) \epsilon^2 \sigma . \quad (12)$$

As the rate of loss of energy from the edge wave (12) must be exactly balanced by the energy flux in the radiated waves (10), α and $e_1(\infty)$ satisfy the condition

$$e_1(\infty) = 32.0 \alpha$$

and the independently calculated values do, indeed, satisfy this equation (providing a useful check on the numerical methods).

An expression for the instantaneous rate of decay of the edge wave energy is then

$$\frac{1}{E} \frac{dE}{dt} = -\frac{\pi \epsilon^2}{8} e_1^2(\infty) \sigma = -0.115 \epsilon^2 \sigma .$$

However, as $\epsilon^2 \sim a^2 \sim E$, the rate of decay decreases as the edge waves become smaller; the nonlinear processes being most important at large amplitudes. In the absence of other dissipative effects, the full solution for the energy decay due to radiation is then

$$1/E - 1/E_0 = 0.92 \frac{k^3 \sigma (t-t_0)}{\rho g}$$

where E_0 is the edge wave energy at an initial time t_0 .

3. Laboratory experiments

The concept of offshore radiation at the harmonic of a standing edge wave was investigated in a simple laboratory experiment. In a wave tank the solution for the

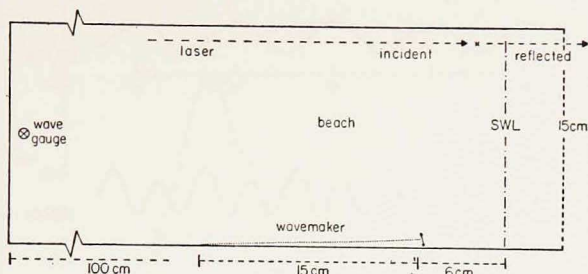


Figure 1. Experimental configuration: edge waves were generated by a wavemaker on the beach face and the resulting sea surface slope tilt was measured by changes in the reflection plane of a laser beam. Far field sea surface displacements were measured with a capacitance gauge.

radiated wave is modified by the boundary conditions imposed offshore by the end of the tank at $x = x_0$. The theory is complicated by the possibility of longitudinal resonances in the tank, between the beach and the offshore wall. However, for tank lengths which are several wavelengths (of the 2σ wave) long, the bandwidth of this resonance is sufficiently wide that there are, in practice, no observable resonances.

A small, rectangular, plexiglass wave basin (15 cm wide, 200 cm long, and 30 cm deep, and of very similar dimensions to that used by Ursell (1952) in his edge wave experiments) was fitted with a plane beach of variable slope. A paddle (15 cm long) was hinged to the tank wall 21 cm from the shoreline and made to oscillate periodically with a maximum displacement of 0.75 cm (Fig. 1). The tank was filled with tap water treated with Photoflow 200 to reduce unwanted surface tension effects at the shoreline and sidewalls. The maximum edge wave response would be expected (and was observed) when the paddle frequency forced a mode whose wavelength divided into twice the width b of the tank, is close to an integral number m . Using Ursell's formula, which is more exact on the steep beaches used in these experiments,

$$\sigma_0^2 = gk \sin(2n + 1)\beta \quad (13)$$

where

$$k = m\pi/b .$$

The edge wave response was determined by measuring the on-offshore slope of the sea surface very near the shoreline with a laser, the angular displacement of the reflected laser beam being proportional to the slope. The wave motion in the far field was measured 1 cm from the offshore wall by a sensitive capacitance wave gauge. Some typical results for the measurement of the far field are shown in Fig. 2.

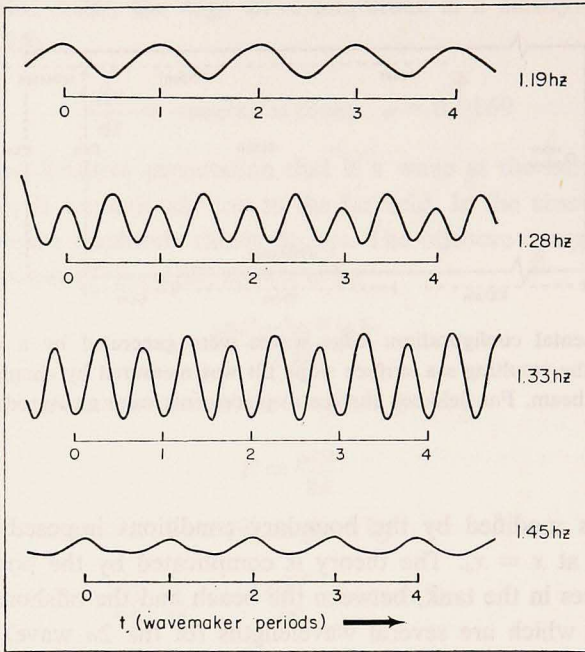


Figure 2. Far field, sea surface displacements: when large edge waves are present on the beach face (1.28, 1.33 Hz.), the far field consists primarily of motions at the 2nd harmonic. The time axes are in units of edge wave (and wavemaker) periods.

Periods of oscillation of the wavemaker are shown on the time axis; the elevation is in arbitrary units (maximum displacements are approximately 3 mm). For this particular experiment ($\tan\beta = .38$) the maximum response of the edge wave was observed at 1.33 Hz, close to the theoretical value of 1.35 Hz given by (13) when $n = 0$, $m = 1$. Fig. 2 shows that, away from the observed edge wave resonance (1.19 or 1.45 Hz), the far field consists primarily of oscillations at the paddle frequency. Close to resonance (1.33, 1.28 Hz) a large component of the motion occurs at twice the paddle (and edge wave) frequency.

Experiments using several different beach slopes and tank lengths showed that, regardless of the configuration of the wave tank, the 2σ component of the far field was strongly correlated with the size of the edge wave observed on the beach. Fig. 3 shows the relation between the normalized edge wave slope (S_N) near the shoreline and the normalized amplitude of the 2σ oscillation at the deep end of the tank ($\tan\beta = 0.30$). The results are very similar for two tank lengths, one theoretically resonant, suggesting that the tank was long enough to eliminate variations in the 2σ wave due to longitudinal resonances. The positions of the theoretical edge wave resonances (13) for the modes ($m = 1, n = 0$), $m = 2, n = 0$) are

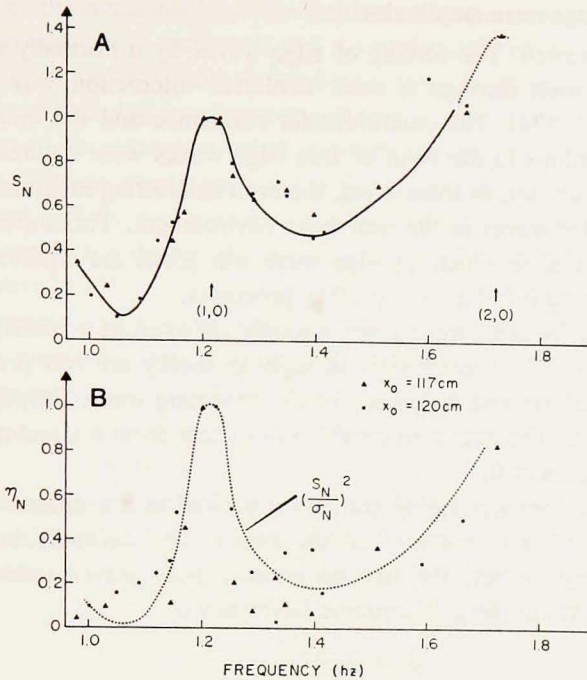


Figure 3. Normalized edge wave slopes S_N and the sea surface displacement of the harmonic η_N for $\tan\beta = 0.3$. x_0 is the distance from the beach to the offshore wall. The dashed line in B) is the theoretical estimate of the far field from eq. 14, appropriate for the smoothed observed curve for the edge waves shown by the solid line in A). The theoretical resonant frequencies for edge waves are indicated by the arrows.

indicated by the arrows. The far field elevation η_∞ at frequency 2σ , theoretically from (1) and (7), varies as

$$\eta_\infty \sim a\epsilon = g(ak)^2/\sigma^2 = g\left(\frac{S}{\sigma}\right)^2, \quad S = ak \quad (14)$$

where S is the maximum slope of the surface at $x = 0$, due to the edge wave motion. This slope, measured directly, is shown in Fig. 3a, normalized by the response at the ($m = 1, n = 0$) resonance. In Fig. 3b the theoretical value of η_∞ (calculated from the observed values of the slope), again normalized by the value at (1, 0), is shown in conjunction with the measured values. The agreement is generally good, although there is some scatter at small values of η_∞ . The actual magnitude of η_∞ at the (1, 0) resonance was again of the order of 3 mm and the small values therefore represent very small disturbances. There is no doubt that the general trend of the results shows the elevation at 2σ in the far field varying as the square of the edge wave slope, in accord with the theoretical predictions.

4. Equilibrium edge wave amplitudes

a. Resonant growth. The forcing of edge waves by a normally incident, monochromatic wave train through a weak nonlinear interaction was investigated by Guza and Davis (1974). The conditions for resonance and the initial growth ratio of small perturbations in the form of free edge waves were determined. However, the growth rates are not, in themselves, the most interesting measure of the possible importance of edge waves in the nearshore environment. The important parameter is the maximum size to which an edge wave will grow, the equilibrium amplitude at which the forcing is balanced by other processes.

For a normally incident wave train, strongly reflected at a beach, the edge wave perturbations which are forced most strongly in theory are two progressive Stokes waves, at the subharmonic frequency of the incoming wave, propagating in opposite directions along the beach; in combination, these form a standing, subharmonic edge wave of mode $n = 0$.

The basic equations describing the initial excitation are obtained by expanding the lowest order velocity potential as the sum of the normally incident, primary wave of frequency 2σ and the two progressive edge waves (which appear as a standing edge wave) at the subharmonic frequency σ .

$$\phi_0 = \phi_i + \phi_e$$

where

$$\phi_i = \frac{a_i g}{2\sigma} J_0(\chi) \sin 2\sigma t \quad (15)$$

$$\phi_e = \frac{a_e g}{\sigma} e^{-kx} \cos ky \cos(\sigma t + \theta) .$$

The expansion of the potential to second order gives the terms involving the self-interaction of the edge wave which have been discussed in section 2, terms involving the self-interaction of the incoming wave (which give set-down and harmonics at 4σ), and the terms arising from the cross-interaction between the incoming waves and edge waves which describe the resonant excitation of the edge wave. The growth of the edge wave is (A20)

$$\begin{aligned} \frac{da_e}{dt} = a_e [2\alpha\sigma\varepsilon_i \{1 - \left(\frac{\Delta\sigma}{2\alpha\varepsilon_i\sigma_f}\right)^2\}^{1/2} - \frac{C_v v^{1/2} \sigma^{5/2}}{2^{1/2} g \tan^2 \beta} \\ + 0(\varepsilon_i^2, \varepsilon_e^2, \varepsilon_i \varepsilon_e)] \quad (16) \end{aligned}$$

where

$$\varepsilon_i = \frac{a_i (2\sigma)^2}{g \tan^2 \beta}$$

and

$$\Delta\sigma = \sigma_f - \sigma .$$

ε_i is a measure of the nonlinearity of the incident waves, σ_f is the natural frequency of free edge waves of wave-number k and $\Delta\sigma$ is the difference between the actual edgewave frequency and the natural frequency. Initially, when $\varepsilon_e^2 \cong 0$, $\sigma_f \cong \sigma_0$. Viscous effects are included by computing the edge wave damping due to laminar boundary layers, C_ν is a constant which depends on the type of boundary layer at the free surface. With an uncontaminated surface, damping in the bottom boundary layer is dominant and $C_\nu = 1$; as in Guza and Davis (1974). With contaminated surfaces the damping is substantially increased and C_ν may reach values of up to $C_\nu = 3$ (McGoldrick, 1970).

When edge waves are forced in a laboratory basin of fixed longshore dimensions b , the possible longshore wave-numbers are restricted by the condition that the longshore orbital velocities must vanish at the boundaries, hence

$$k = m\pi/b; m = 1, 2, 3, \dots$$

For a given longshore wave-number, edge waves can be excited over a range of frequencies determined by (16). The frequency band within which the edge waves will grow is centered on $\sigma_f = \sigma_0$, where the initial growth rate is maximum and has band-width.

$$1 - \delta \leq \frac{\sigma}{\sigma_f} \leq 1 + \delta \quad (17)$$

where

$$\delta = 2\alpha\varepsilon_i \left[1 - \frac{C_\nu^2 \nu \sigma^3}{8g^2 \alpha^2 \varepsilon_i^2 \tan^4 \beta} \right]^{\frac{1}{2}} .$$

The reduced forcing away from $\sigma = \sigma_0$ leads to an increase in the minimum incident wave amplitude, a_c , which is needed to overcome the viscous damping. However, once $a_i > a_c$ resonant growth occurs, provided the frequency is within the band-width given by (17). Growth is not limited by viscous effects as the ratio between the forcing and viscous terms in (16) remains constant. This is very different from Gallagher's (1971) suggestion for the forcing of edge waves by the interaction of two incoming waves, amplitudes $a_i^{(1)}$, $a_i^{(2)}$ where the resonance equation takes the form (very schematically)

$$a_{e_t} = G a_i^{(1)} a_i^{(2)} - C a_e$$

where G is a coupling coefficient and C a damping parameter. In this case, the forcing may remain approximately constant and the linear damping limits the edge wave amplitude to $G a_i^{(1)} a_i^{(2)} / C$.

One of the interesting parameters in (15) is the phase between the incoming wave and the edge wave. For resonant forcing of the edge waves (neglecting viscous effects),

$$\tan \theta = (-2\alpha\epsilon_i\sigma + \Delta\sigma)((2\alpha\epsilon_i\sigma)^2 - (\Delta\sigma)^2)^{-\frac{1}{2}} + O(\epsilon_i^2) . \quad (18)$$

At the lowest frequency at which resonance is possible, $\sigma = \sigma_f(1-\delta)$, $\Delta\sigma = \delta\sigma = 2\alpha\epsilon_i\sigma$ so $\theta = 0$, at the highest frequency $\theta = -\pi/2$, and at the central frequency of the band ($\Delta\sigma = 0$), $\theta = -\pi/4$. The inclusion of a small viscous term alters the phase by approximately $a_c/2a_i$.

b. Edge wave-edge wave interactions. Initially, when σ is sufficiently close to the natural frequency, the edge wave grows exponentially. The natural frequency of the edge wave, however, depends on the wave amplitude via the nonlinear term ϵ_e^2 in (8), as does the rate of energy loss via radiation (12). This is crucial, for it means that instead of growing indefinitely, the edge wave will approach a steady state at some finite amplitude if σ is within the frequency band for resonance.

Consider an initially small edge wave which satisfies the condition for resonant growth having frequency σ (which satisfies (17) at small ϵ_e), that is

$$\frac{\sigma}{\sigma_0} = 1 + c\delta + O(\epsilon_e^2) \quad (19)$$

where

$$-1 < c < 1 .$$

In the absence of radiative losses, the edge wave amplitude, and natural frequency would increase until the resonance condition (17) is no longer satisfied. Growth ceases when the edge wave frequency is at the lower end of the resonant band

$$\frac{\sigma}{\sigma_f} = \frac{\sigma}{\sigma_0(1+.055\epsilon_e^2)} = 1 - \delta .$$

Therefore, at equilibrium, with radiative losses ignored,

$$.055\epsilon_e^2 = (1 + c)\delta \quad (a)$$

and if viscosity is negligible ($a_i \gg a_c$)

$$\epsilon_e^2 = .615(1 + c)\epsilon_i, \quad \theta = 0. \quad (b)$$

At equilibrium, $0(\epsilon_e) \sim 0(\epsilon_i^{1/2})$ and since $\epsilon_i \leq 0(1)$ it follows that the edge wave amplitude is larger than the incident wave. The edge wave equilibrium amplitude is maximum when $c = +1$.

Radiation limits growth, if natural frequency changes are temporarily ignored, when the edge wave amplitude decay due to radiation (12) exactly balances the forcing by the incident wave (which remains constant since the natural frequency is now assumed constant)

$$\epsilon_e^2 = .588\epsilon_i \left[1 - \left(\frac{c\delta}{2\alpha\epsilon_i} \right)^2 \right]^{\frac{1}{2}} - \frac{C_\nu v^{1/2} \sigma^{3/2}}{2^{3/2} \pi g a \epsilon_1(\infty) \tan^2 \beta} . \quad (21)$$

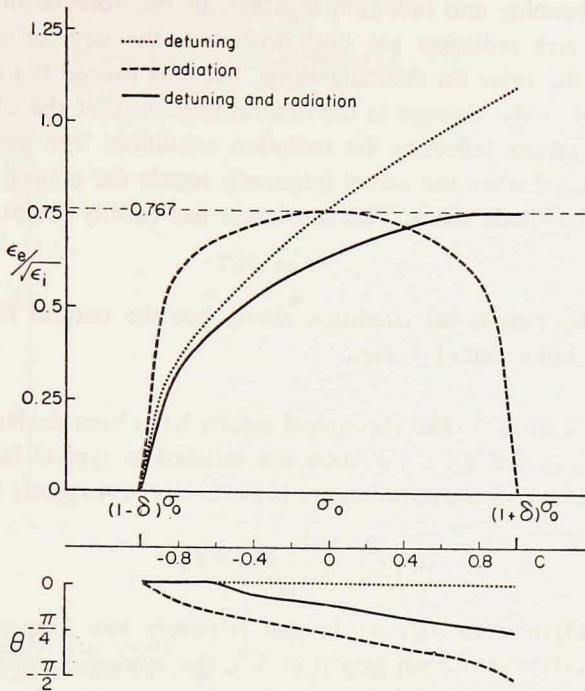


Figure 4. Equilibrium amplitude for a subharmonic, Stokes edge wave and its phase in relation to the incoming wave as a function of frequency, when the resonance is limited by detuning or radiation or detuning and radiation together.

If the viscous effects are small, then at equilibrium

$$\epsilon_e^2 = .588\epsilon_i (1 - c^2)^{\frac{1}{2}} \tag{22}$$

the phases being given by (18). The maximum edge wave amplitudes occur when $c = 0$ ($\sigma = \sigma_0$). Again, $0(\epsilon_e) \sim 0(\epsilon_i^{\frac{1}{2}})$, the edge wave grows to be larger than the primary wave.

It is clear that detuning and radiation, taken separately, both predict equilibrium amplitudes of very much the same size and are therefore of roughly equal importance in limiting edge wave growth. It is therefore necessary to consider the combined effects of detuning and radiation. Equilibrium amplitudes then occur when

$$2\alpha\sigma\epsilon_i \left[1 - \left(\frac{\Delta\sigma}{2\alpha\epsilon_i\sigma} \right)^2 \right]^{\frac{1}{2}} = \epsilon_e^2\pi\alpha e_1(\infty)\sigma + \frac{C_{\nu}v^{1/2}\sigma^{5/2}}{2^{1/2}g \tan^2\beta} \tag{23}$$

where

$$\sigma_f = (1 + .055\epsilon_e^2)\sigma_0 .$$

Fig. 4 shows the theoretical response curves for the amplitude and phase of the edge wave at equilibrium as functions of frequency for detuning alone, radiation

alone, and for detuning and radiation together, in the absence of viscous effects. When detuning and radiation are both included, the equilibrium amplitude is always less than the value for detuning alone, but may exceed the radiation limited value for $c \cong 0.4$ as the changes in the natural frequency of the edge wave due to finite amplitude effects influence the radiation condition. The maximum possible amplitude is reached when the actual frequency equals the natural frequency, now including finite amplitude terms. This maximum has exactly the same value

$$\varepsilon_e \varepsilon_i^{-\frac{1}{2}} = .767$$

as is predicted by radiational condition alone, but the natural frequency at this amplitude is now not σ_0 but $(1 + \delta)\sigma_0$.

c. Experimental results. The theoretical results have been derived under the assumption that $\varepsilon_i, \varepsilon_e \ll 1.$, a condition not satisfied in typical laboratory experiments. Indeed, if an edge wave resonance is to occur (16) suggests that, for $\sigma = \sigma_0$,

$$a_i \left(\frac{2\sigma}{\nu} \right)^{\frac{1}{2}} \cong 7.4 C_\nu$$

and even with clean water ($C_\nu = 1$) and relatively low frequency edge waves (0.2 Hz) on a fairly steep beach (say $\beta = 5^\circ$), the minimum value of ε_i at which resonance may occur is $\varepsilon_i = 0.4$. Viscous damping cannot be overcome unless the primary waves are moderately nonlinear. Garrett (1970) derived finite amplitude detuning results, conceptually similar to (20), for the excitation of subharmonic cross waves. These compared well with the amplitude-frequency measurements for small primary waves. For large primary waves, however, the response curves were of different shape as higher order terms became significant. For edge wave resonances, the problem of strong primary wave nonlinearities is more complex; not only is the convergence of the formal expansion in doubt, there is the additional complication that as the incoming waves increase in height they eventually begin to break, forming dissipative bores. The nonlinear theory of Carrier and Greenspan (1958) predicts that the transition to breaking occurs when $\varepsilon_i = 1.0$, compared with an early, linear theory of Miche (1944) which suggested that the reflectivity of the beach decreases, from a value of 1.0 when $\varepsilon_i \leq 0$, with increasing ε_i . As the reflectivity decreases when $\varepsilon_i > 1$, it is clear that formal analysis to $O(\varepsilon_i^2)$ is pointless at large values of ε_i , even if the solution converges, unless the assumption of total reflection is relaxed. However, any modelling of the breaking processes must involve rather gross approximations even in the lowest order solutions.

However, most of the laboratory data deals with the situation with $\varepsilon_i \cong 1.0$, a surging wave beginning to break at the shoreline, and this seems also to be a condition in which features such as beach cusps are frequently observed to form in

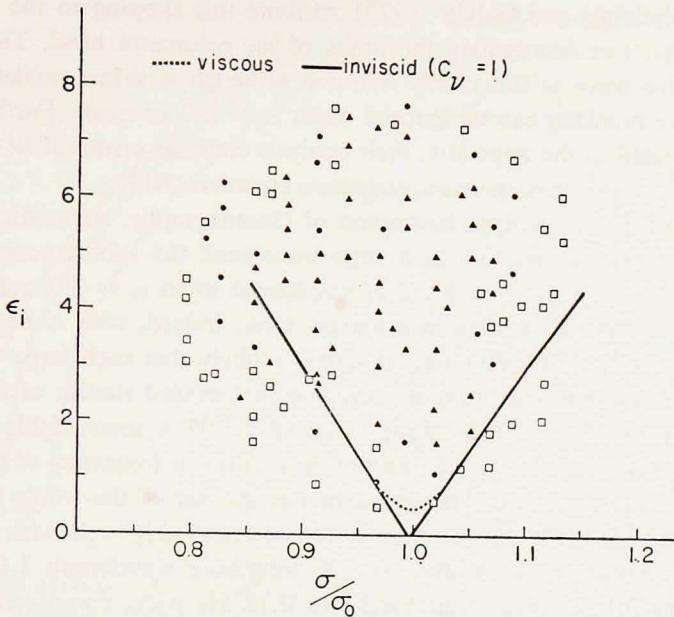


Figure 5. Subharmonic edge wave resonant band-widths: data from Birchfield and Galvin (1975), \blacktriangle spontaneous resonance, \bullet induced resonance, \square no resonance; theory from (17).

the field. Although the theory is reaching the limits of validity, it is clear that its predictions are the first step in understanding the processes that limit edge wave amplitudes in moderately nonlinear cases. If the wave breaking introduces entirely new physics, as in the generation of longshore currents, the present theory will not provide appropriate predictions.

The present theory provides three predictions which can be compared with the experimental data.

(i) the band width of the resonance (Fig. 5)

(ii) the edge wave amplitudes at equilibrium (Fig. 4)

(iii) the phase between the incoming wave and edge wave at equilibrium (Fig. 4).

Fig. 5 shows the data on the resonant band-width given by Birchfield and Galvin (1975) for the case $n = 0$, $\tan \beta = 0.132$, $\sigma_0 = 3.15$ compared to the theory (17) for $C_v = 0$ (inviscid) and $C_v = 1$ (viscous, clean surface). The triangles indicate spontaneous resonance and fall generally within the theoretical band-width; closed circles indicate resonances which did not occur spontaneously but, if initially forced, would subsequently maintain themselves; these tend to occur at the fringes of the band. At large values of ϵ_i , the resonance is skewed toward lower frequencies. This may be associated with the set-up associated with the breaking waves, the decrease in the effective beach slope decreasing the natural frequency of the

edge wave. Birchfield and Galvin (1975) attribute this skewing to the importance of terms of $O(\varepsilon_i^2)$ in determining the limits of the resonance band. They have to assume that the wave is still totally reflected although it seems unlikely that the effects of wave breaking can be ignored when $\varepsilon_i \cong 2.0$ or more. Furthermore, as discussed in detail in the appendix, their analysis contains errors of $O(\varepsilon_i)$ resulting from incorrect averaging of the basic resonance equation (A17).

Data collected at the Scripps Institution of Oceanography, summarized in Guza and Inman (1975), suggest that in a large wave tank the subharmonic resonance generally ceases (or is much reduced in amplitude) when $\varepsilon_i \cong 2.0$, rather than at $\varepsilon_i \cong 7.0$ as observed by Galvin in a narrow tank. Indeed, with ε_i determined by the wave amplitude at the shoreline, it seems unlikely that such large values of ε_i are possible. The Scripps data are not suitable for detailed studies of the resonant band as the variations in beach slope (even of $\pm .05^\circ$), unavoidable on a large beach, lead to uncertainties in the location of the central frequency of the resonant band σ_0 which are typically of the order of one quarter of the whole band-width. However, the total band-widths observed agree reasonably well with the predictions of (17). For example, with $\varepsilon_i \cong 1.6$, longshore wavelength 1.62 m, on a beach of slope 5.1° , the resonant band was 0.12 Hz wide, compared to a theoretical width of 0.13 Hz.

Fig. 6 shows the comparison between the laboratory observations (Guza and Inman, 1975) of the equilibrium, horizontal edge wave displacement at the shoreline R_0 where

$$R_0 = 2a_e/\tan\beta$$

and the theoretical values obtained from (23), independent of beach slope, for various values of C_v . The value shown is at the center of the band σ_0 , and could be as much as 15% greater at $\sigma = (1 + \delta)\sigma_0$. The actual location of the observations relative to the band-center is uncertain due to the uncertainties in estimating the beach slope. The agreement in Fig. 6 is good considering the crudeness of the data and the rather large values of ε_i and ε_e involved. However, Galvin's (1967) data seems to show similar trends for $\varepsilon_i < 2.2$.

As indicated in Fig. 4, the predicted maximum edge wave amplitudes are very similar to those which would occur if radiation was the only limiting factor. However, phase measurements provide further insight into the actual mechanism. The sea surface displacement at the shoreline is (15),

$$\eta_i = -a_i \cos 2\sigma t$$

$$\eta_e = a_e \sin(\sigma t + \theta) \cos ky$$

and the displacement of the water line on the beach depends on θ . The time history at an antinode ($\cos ky = 1$) for $\theta = 0, -\pi/4, -\pi/2$ is shown in Fig. 7 for the

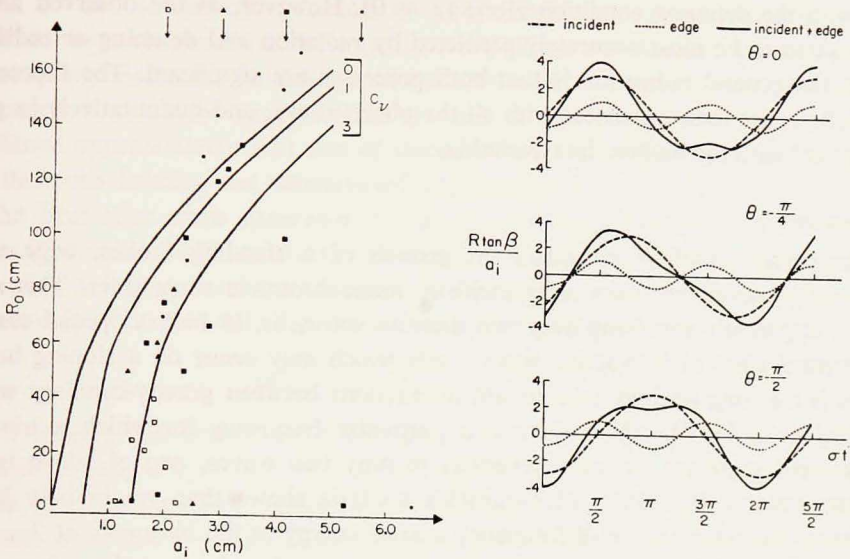


Figure 6. (Left) Maximum horizontal displacement at the shoreline of Stokes edge wave at equilibrium amplitude; data from Guza and Inman (1975), with $T_e = 5.54$ sec., and beach slopes; $\square = 4.17^\circ$, $\blacktriangle = 5.08^\circ$, $\blacksquare = 6.0^\circ$, $\bullet = 6.83^\circ$. Vertical lines indicate the onset of wave plunging, $\varepsilon_i = 2.0$; theory from (23) when $\sigma = \sigma_0$, for each slope.

Figure 7. (Right) Horizontal displacement at the shore R for various phase differences, θ , between the edge and incident waves (24), for $a_e/a_i = 3.0$.

case $\varepsilon_e \varepsilon_i^{-1/2} = .75$ ($a_e = 3a_i$). The horizontal displacement at the shoreline, which is the most easily seen feature in laboratory experiments, is then

$$R = \frac{\eta_i + \eta_e}{\tan\beta} = a_i \tan\beta^{-1}(-\cos 2\sigma t + 3 \sin(\sigma t + \theta)) .$$

When $\theta = 0$, the edge wave is maximum (or minimum) when the incident wave has its maximum, positive value (i.e., is furthest up the beach) and zero displacement at maximum run-down of the incident wave. For $\theta = -\pi/2$, the pattern is reversed with zero displacement of the edge wave at the time of maximum run-up. When $\theta = -\pi/4$, edge wave maxima occur when the incident wave is at its mean position, the total displacement being skewed so that the uprush is noticeably more sudden than the backwash.

Detailed phase measurements are not available, but it has been generally noted that the displacement of large edge waves are very noticeable at the maximum run-up of the incident wave (Harris, 1967; Bowen and Inman, 1969; Guza and Inman, 1975). A run-up pattern of the form given by $\theta = -\pi/2$ has not been reported. The laboratory observations are therefore consistent with the theoretical suggestion that $0 > \theta > -\pi/4$ for detuning and radiation, but would also be in

line with the detuning condition alone ($\theta = 0$). However, as the observed amplitudes seem to be most accurately predicted by radiation and detuning or radiation alone, the general indication is that both processes are significant. The theoretical predictions are then consistent with all the observations, and quantitatively in good agreement with the limited data available.

5. Discussion

The present analysis considers the growth of a standing Stokes, edge wave resonantly excited by a normally incident, monochromatic wave train. This resonance, apparently involving only two discrete waves, is, in fact, a special case of the general class of interactive wave triads which may occur on a sloping beach. There is a strong analogy to resonant interactions between gravity-capillary waves (McGoldrick, 1970), where there is a particular frequency for which a triad of progressive, capillary waves degenerates to only two waves, one of which is the subharmonic of the other. McGoldrick's analysis shows that, an initially large, progressive capillary wave of frequency σ loses energy to the harmonic at 2σ , even if the harmonic initially has zero amplitude. This transfer of energy from frequency σ to 2σ is also characteristic of the standing, Stokes edge wave, in the absence of forcing. Furthermore, inspection of McGoldrick's equations reveals that a primary wave of frequency 2σ is unstable to subharmonic perturbations, again in very close analogy to the case of waves normally incident on a beach.

The detailed calculation of the equilibrium values of the amplitude and phase of the edge wave have been shown for the particular case of normally incident waves and the lowest mode ($n = 0$), subharmonic edge wave. However, the same physical processes of radiation and finite amplitude detuning will limit the growth of any edge waves which are members of resonant triads. If a normally incident wave excites two edge waves of different frequency and mode number then formally, at equilibrium,

$$0(\varepsilon_i) = 0(\varepsilon_e^{(1)} \varepsilon_e^{(2)})$$

where $\varepsilon_e^{(n)}$ is the nonlinearity parameter of a particular mode. However, the usefulness of this type of general statement is limited because the relative importance of the forcing, radiation and detuning depend on the numerical values of the appropriate coupling coefficients. The subharmonic, Stokes mode, resonance is known to be the most strongly forced (Guza and Davis, 1974), other resonances cannot reach comparable amplitudes unless the numerical values associated with radiation and detuning are comparatively small. In practice, none of these other, possible resonances has ever been observed in the laboratory.

A non-normally incident wave, ϕ^i , may form a resonant triad with two progressive edge waves travelling in opposite directions (ϕ^+ , ϕ^-), generally with different frequencies, and possibly different mode numbers. In the most easily excited reso-

nance, ϕ^+ and ϕ^- are of mode number zero, and have frequencies slightly different from the subharmonic of the incident wave (Guza and Bowen, 1975). Because the coupling coefficient for this resonance is so similar to the subharmonic Stokes resonance, it would be expected that ϕ^+ and ϕ^- will each have an equilibrium amplitude approximately half that of resonant standing subharmonic Stokes wave, and that both detuning and radiation will be important.

The beat edge wave generated by the non-linear interaction of two incident waves (Gallagher, 1971) interacts with each of these waves to radiate energy at the frequency and longshore wave-number of the other. In this case, radiation limits growth only when $\varepsilon_e = 0(1)$, but detuning is limiting when

$$0(\varepsilon_e^3) = 0(\varepsilon_i^{(1)} \varepsilon_i^{(2)}); \varepsilon_i^{(1,2)} < 1$$

and it appears that detuning will limit growth at a lower order than the radiation. At equilibrium, if $\varepsilon_i^{(1)} \cong \varepsilon_i^{(2)}$, $0(\varepsilon_e) \sim 0(\varepsilon_i^3)$ and the edge wave may become larger than the primary waves. However, the edge waves forced by a single incident wave are formally larger, $0(\varepsilon_e) \sim 0(\varepsilon_i^{\frac{1}{2}})$. The precise relations and the importance of viscous effects again depend strongly on the numerical values of the coupling coefficients involved.

All these triad interactions involve energy exchange between the far field and the modes trapped in shallow water. The total energy in the nearshore region is not necessarily conserved, energy from the far field may be fed into edge wave modes both by the direct interaction of the edge waves with the primary waves and by further interactions among the edge waves themselves (Kenyon, 1970). At a given frequency, the energy in an edge wave spectrum is determined by a very complex balance between nonlinear processes and viscous effects. The analogy to surface gravity waves suggests that the important interactions will involve a continuous shifting of edge wave energy from the source frequency to lower frequencies. Of course, all interactions arising from the instability of a single wave, either an incoming wave (Guza and Bowen, 1975) or a progressive edge wave (Kenyon, 1970), necessarily involve a shift to low frequency; the growing perturbations must have lower frequencies than the initial wave to satisfy Haselmann's (1967) rule.

6. Conclusions

The principal result of the present analysis is the identification of the processes that limit the growth of resonantly forced edge waves. Equilibrium is reached for the subharmonic, Stokes wave generated by a normally incident wave when, approximately (Figs. 4, 6),

$$\varepsilon_e \cong 0.77 \varepsilon_i^{1/2}; \varepsilon_i \leq 2.0$$

Table 1. Total horizontal displacement at the shoreline of an edge wave generated by surging incident waves ($\varepsilon_i = 1$.)

Frequency Hz	Period T_e (sec)	Slope $\tan\beta$	Displacement R_0 (m)
1	1	.01 (.1)	.0038 (.038)
0.2	5	.01 (.1)	.095 (.95)
0.1	10	.01 (.1)	.382 (3.82)
0.05	20	.01 (.1)	1.52 (15.20)

a value in good agreement with the existing laboratory measurements. The total horizontal displacement at the shoreline R_0 associated with this wave, when the incoming wave is just beginning to break and surge, $\varepsilon_i \cong 1.0$, is then given by

$$R_0 = 2a_e/\tan\beta = 1.54 g \tan\beta / \sigma_e^2 .$$

As shown in Table 1, long period swell on a steep beach may generate large edge waves. Typical laboratory waves ($T_e = 5$ secs, $\tan\beta = .1$) should have amplitudes of $0(1 m)$ as observed. Even though there are many additional complications which may become important in various field situations, the present results suggest that low frequency edge waves are likely to be a significant factor in the generation of sedimentary features in the nearshore region. For the case of a wave gently surging on a beach, a situation reported to be favorable for the generation of beach cusps, the theoretical results predict that at the shoreline of a plane beach the edge wave should reach an amplitude a factor of three larger than that of the incoming wave, a prediction in accord with the laboratory observations.

APPENDIX

The basic equations are the standard, nonlinear, shallow water equations which are valid close to the shore on a sloping beach. The approximations made in deriving these equations are discussed in detail by Mei and Le Méhauté (1966). Mass conservation gives

$$\eta_t + [u(\eta+h)]_x + [v(\eta+h)]_y = 0 \quad (A1)$$

where η is the displacement of the free surface, h the depth and x, y (u, v) the offshore and long-shore coordinates (velocities). The motion is assumed irrotational so

$$\mathbf{u} = \nabla \cdot \phi$$

where ϕ is the velocity potential. The momentum equations may be integrated to obtain a Bernoulli equation for shallow water (Stoker, 1957).

$$\eta = -\frac{1}{g} [\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2)] . \quad (A2)$$

Then, eliminating η from (A1) and (A2)

$$L(\phi) = Q(\phi, \phi) + C(\phi, \phi, \phi) \quad (A3)$$

where

$$\begin{aligned}
 L(\phi) &= -\phi_{tt} + (gh\phi_x)_x + (gh\phi_y)_y \\
 Q(\phi, \phi) &= 2\phi_x\phi_{xt} + 2\phi_y\phi_{yt} + \phi_t(\phi_{xx} + \phi_{yy}) \\
 C(\phi, \phi, \phi) &= (\phi_{xx} + \phi_{yy}) \left(\frac{\phi_x^2 + \phi_y^2}{2} \right) + \phi_x^2\phi_{xx} + \phi_y^2\phi_{yy} + 2\phi_x\phi_y\phi_{xy} .
 \end{aligned}$$

The velocity potential, sea surface elevation, and natural frequency are expanded in the non-linear ordering parameters, ϵ (1,2). The amplitude, a , is allowed to have slow time variations; $a_t = 0(\epsilon)$ or $0(\epsilon^2)$. Lowest order (linear) edge wave solutions ($L(\phi_0) = 0(\epsilon)$) are given by

$$\phi_0 = \frac{ag}{\sigma} L_n(2kx) e^{-kx} \left\{ \begin{array}{l} \cos ky \cos \sigma t \\ \cos(ky - \sigma t) \end{array} \right\} \tag{A4}$$

where (A4) is the solution for either a standing or progressive edge wave, L_n is the LaGuerre polynomial of order n (Eckart, 1951), and

$$\sigma_0^2 = gk(2n + 1)\tan \beta . \tag{A5}$$

Linear incoming waves, approaching normal to the beach and therefore having no longshore dependence are

$$\phi_0 = \frac{a_1g}{\sigma_1} (J_0(\chi)\cos \sigma_1 t - Y_0(\chi)\sin \sigma_1 t) \tag{A6}$$

where

$$\chi^2 = 4\sigma_1^2 x / g \tan \beta$$

and J_0 and Y_0 are Bessel functions of zero order. An out-going progressive wave would have a plus sign before Y_0 . A wave totally reflected from the beach has a solution in J_0 only, being a standing wave of the form

$$\phi_0 = \frac{a_1g}{\sigma_1} J_0(\chi)\cos \sigma_1 t .$$

Second order edge wave solutions are given by substitution of (A4) and (A5) into (A3).

$$\begin{aligned}
 \epsilon L(\phi_1) &= \epsilon Q(\phi_0, \phi_0) - 2\epsilon ag\sigma_1 e^{-kx} L_n(2kx) \left\{ \begin{array}{l} \cos ky \cos \sigma t \\ \cos(ky - \sigma t) \end{array} \right\} \\
 &\quad - 2a_1g L_n(2kx) e^{-kx} \left\{ \begin{array}{l} \cos ky \sin \sigma t \\ \sin(ky - \sigma t) \end{array} \right\}
 \end{aligned} \tag{A7}$$

where

$$Q(\phi_0, \phi_0) = \left\{ \begin{array}{l} f(x) + g(x)\cos 2ky \\ F(x)\sin 2(ky - \sigma t) \end{array} \right\} \left\{ \begin{array}{l} \sin 2\sigma t \\ -e^{-2kx} a g \sigma \end{array} \right\}$$

and the upper and lower terms in curly brackets refer to standing and progressive edge waves respectively. It is obvious by inspection that $a_t = 0.$, to $O(\epsilon)$, and $\sigma_1 = 0$. For mode $n = 0$, the most algebraically simple case, $L_n(2kx) = 1.$; $F(x) = g(x) = 0.$, $f(x) = 1$. Only the Stokes mode, $n = 0$, will be explicitly considered at high order. For a progressive Stokes edge wave, $\phi_1 = 0$, and the elevation correction, η_1 , obtained from the expansion of (A2), is given by (4).

The second order correction to the standing edge wave is more interesting as, although the longshore variation in (A7) has again dropped out, a time and x dependence remains. The general solution, using the standard method of variation of parameters, is

$$\begin{aligned}
 \phi_1 &= -\frac{ag\pi}{8\sigma} [\{(e_1(\chi)+r_2)Y_0(\chi) - (e_2(\chi)-r_1)J_0(\chi)\} \sin 2\sigma t \\
 &\quad + \{r_3J_0(\chi) + r_4Y_0(\chi)\} \cos 2\sigma t]
 \end{aligned} \tag{A8}$$

where

$$\chi^2 = 4(2\sigma)^2 x/g \tan \beta$$

$$\{e_1(\chi), e_2(\chi)\} = \int_0^\chi Z\{J_0(Z), Y_0(Z)\}e^{-x^2/8}dZ$$

and the r_n are constants associated with the homogeneous solutions to be determined by the boundary conditions. The boundary condition of finiteness everywhere requires $r_2 = r_4 = 0$. The offshore boundary condition, when the beach slopes into deep water, is that there is no incoming wave energy at large values of χ . Then $r_1 = e_2(\infty) = .854$, $r_3 = e_1(\infty) = .541$, numerically, and

$$\begin{aligned} \phi_1 = -\frac{ag\pi}{8\sigma} [\{e_1(\chi)Y_0(\chi) + (e_2(\infty)-e_2(\chi))J_0(\chi)\}\sin 2\sigma t \\ + e_1(\infty)J_0(\chi)\cos 2\sigma t] \end{aligned} \tag{A9}$$

so that offshore, as $\chi \rightarrow \infty$

$$\lim_{\chi \rightarrow \infty} \phi_1 = -\frac{ag\pi e_1(\infty)}{8\sigma} (J_0(\chi)\cos 2\sigma t + Y_0(\chi)\sin 2\sigma t) \tag{A10}$$

which is an outgoing progressive wave of frequency 2σ .

Third order equations are

$$\begin{aligned} \varepsilon^2 L(\phi_2) = \varepsilon^2 C(\phi_0, \phi_0, \phi_0) + \varepsilon^2 (Q(\phi_1, \phi_0) + Q(\phi_0, \phi_1)) \\ - 2ag\varepsilon^2 \sigma_2 e^{-kx} \left\{ \begin{matrix} \cos \sigma t \cos ky \\ \cos(ky - \sigma t) \end{matrix} \right\} - 2a_t g e^{-kx} \left\{ \begin{matrix} \sin \sigma t \cos ky \\ \sin(ky - \sigma t) \end{matrix} \right\} . \end{aligned} \tag{A11}$$

For the progressive Stokes edge wave, $\phi_1 = 0$, and

$$C(\phi_0, \phi_0, \phi_0) = ag\sigma e^{-3kx} \cos(ky - \sigma t) .$$

It is clear that (A11) is satisfied with $a_t = 0 + 0(\varepsilon^3)$, and that

$$\phi_2 = \frac{ag}{\sigma} f(x)\cos(ky - \sigma t) .$$

It can be shown, using a normal mode expansion for $f(x)$, that ϕ_2 (finite everywhere) exists only if

$$2\sigma_2 \int_0^\infty e^{-kx} \cdot e^{-kx} dx = \sigma \int_0^\infty e^{-3kx} \cdot e^{-kx} dx \tag{A12}$$

so

$$\sigma_2 = \sigma_0/4 .$$

The third order equation for the standing Stokes edge wave is algebraically complex because ϕ_1 (A9) enters into Q . The result is, schematically,

$$\begin{aligned} \varepsilon^2 L(\phi_2) = g \cos ky \sin \sigma t \{-2a_t e^{-kx} + a\sigma \varepsilon^2 d_1\} \\ + g \cos ky \cos \sigma t \{-2a\varepsilon^2 \sigma_2 + a\sigma \varepsilon^2 d_2\} \\ + ag\sigma \varepsilon^2 \cos ky \{d_3 \sin 3\omega t + d_4 \cos 3\omega t\} \end{aligned} \tag{A13}$$

where the d_n are complicated functions of x . The 3σ terms are generally small harmonic corrections to the velocity potential. However, it is possible for 3σ and k to satisfy the linear dispersion relation for free edge waves on very shallow beaches where

$$(3\sigma)^2 = gk(2n + 1)\tan \beta; n = 4$$

and on any beach for which there happens to be a value of n for which the exact linear dispersion relation (Ursell 1952) is satisfied

$$(3\sigma)^2 = gk \sin(2n + 1)\beta .$$

In these cases energy may be transferred to higher frequency and mode number. This interaction will not be considered further because it is weaker ($0(\epsilon^3)$) than the second order interactions ($0(\epsilon)$) that transfer energy between edge wave modes (Kenyon, 1970) or between edge waves and incoming or outgoing waves. The terms of frequency σ and longshore wave-number k satisfy the dispersion relation for the first order solution (A5), and ϕ_2 finite everywhere, and not growing with time, can exist only if

$$2a_1\epsilon^2 \int_0^\infty e^{-kx} \cdot e^{-kx} dx = a\sigma \int_0^\infty d_1 \cdot e^{-kx} dx \quad (A14)$$

$$2\sigma_2 \int_0^\infty e^{-kx} \cdot e^{-kx} dx = \sigma \int_0^\infty d_2 \cdot e^{-kx} dx .$$

That is, the parts of the "forcing term" which are not orthogonal to the free solution, e^{-kx} , result in amplitude and natural frequency variations. Numerical solution of (A14) gives

$$\sigma_2 = .055 \sigma_0 \quad (A15)$$

$$a_1 = -a\sigma\epsilon^2 e_1(\infty)2\pi a; a = .0169$$

To model the interaction of a normally incident wave and a standing edge wave let

$$\phi_0 = \phi^t + \phi^o \quad (a)$$

where

$$\phi^t = \frac{a_1 g}{2\sigma} J_0(\chi) \sin 2\sigma t \quad (b)$$

$$\phi^o = \frac{ag}{\sigma} e^{-kx} \cos ky \quad (c)$$

(a) contains edge wave amplitude, phase, and frequency information. Temporarily neglecting cubic terms and the interactions of each wave with itself, (A3) becomes,

$$L(\phi^o) = Q(\phi^t, \phi^o) + Q(\phi^o, \phi^t)$$

and substitution of (A16) gives

$$e^{-kx}(a_{1t} + agk \tan \beta) = -\epsilon_1(2\sigma a_1 f_1(\chi) \sin 2\sigma t \quad (A17)$$

$$+ agk \tan \beta f_1(\chi) \cos 2\sigma t)$$

where

$$f_1(\chi) = (4\chi^{-2} J_2(\chi) + 4gk(2\sigma)^{-2} \chi^{-1} J_1(\chi)) e^{-kx}$$

$$f_2(\chi) = (4\chi^{-1} J_1(\chi)) e^{-kx} .$$

In the absence of incident waves ($\epsilon_1 = 0$), (A17) describes free linear oscillations with lowest order natural frequency

$$\sigma_0^2 = gk \tan \beta .$$

The x dependence of (A17) is removed, leaving an equation for $a(t)$, by the condition that only the portion of the r.h.s. which is not orthogonal to the free solution, e^{-kx} , can contribute to resonant forcing. This statement is formally justified in Guza and Davis (1974), and follows from the fact that the linear operator L (with fixed σ and k_y) is of the self-adjoint, Sturm-Liouville type possessing a complete set of eigenfunctions and eigenvalues. If the non-linear interaction terms, i.e., the right hand side of (A17), are orthogonal to the free edge wave solution, it can be proved that no resonance occurs, only small, forced corrections expressible in terms of normal mode solutions. The nonorthogonal portion of the interaction terms, the part which can drive the resonance, is obtained by multiplying (A17) by e^{-kx} and integrating from $x = 0$ to $x = \infty$. The subharmonic resonance theory of Birchfield and Galvin (1975) is incorrect, at lowest order, because they directly spatially average an equation similar to (A17), implicitly assigning a weighting function of 1, rather than e^{-kx} . Applying the orthogonality condition results in

$$a_{,t} + \sigma_0^2 a + 2\varepsilon_1 \sigma_0^2 (2\sigma)^{-2} \{2\sigma a_l f_l^I \sin 2\sigma t + a_{\sigma_0} f_{\sigma_0}^I \cos 2\sigma t\} = 0 \quad (\text{A18})$$

$$\text{where } f_l^I = \int_0^{\infty} e^{-kx} f_l d(kx); l = 1, 2 .$$

With the transformation (Garrett, 1970)

$$a = b \exp[\varepsilon_1 \sigma_0^2 (2\sigma)^{-2} f_l^I \cos 2\sigma t]$$

(A18) becomes

$$b_{,t} + b \sigma_0^2 [1 + 2\varepsilon_1 (2\sigma)^{-2} \cos 2\sigma t (f_{\sigma_0}^I \sigma_0^2 + 2\sigma^2 f_l^I)] + 0(\varepsilon_1^2) = 0 , \quad (\text{A19})$$

which is the Mathieu equation. Following Bogoliubov and Mitropolsky (1961, ch. 17), and Garrett (1970), resonantly growing solutions for b and hence for a , exist when

$$\sigma/\sigma_f = 1 \pm 2\varepsilon_1$$

where σ_f is the natural edge wave frequency. The amplitude, a , is proportional to $e^{\theta t}$ where

$$S = ((2\varepsilon_1 a \sigma)^2 - (\sigma_f - \sigma)^2)^{\frac{1}{2}} \quad (\text{A20})$$

so the growth rate is maximum when $\sigma = \sigma_f$. The phase θ is given by (18).

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