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Generation of Langmuir circulations by surface waves— a feedback mechanism

by Christopher Garrett¹

ABSTRACT

Surface gravity waves propagating obliquely through a surface current pattern ($U(y)$, $V(y)$) representative of Langmuir circulations exert a force on the water which tends to produce surface convergences at maxima of $U(y)$. Moreover, the waves achieve their largest amplitude at maxima of $U(y)$ and so are most likely to dissipate there and give up their momentum to the mean flow. A feedback mechanism thus exists, and Langmuir circulations may be regarded as an instability of the surface layer in the presence of waves. A simple model indicates that the largest scale cells grow fastest, but that the transverse circulation is stronger for the smaller scales.

The general theory of the effect of surface waves on mean flows is not restricted in application to Langmuir circulations, but may be important in other oceanographic situations.

1. Introduction

Windrows, surface convergence lines parallel to the wind, were first investigated by Langmuir (1938), who suggested that they were associated with helical circulation patterns in the upper layer of oceans and lakes. Numerous investigations of these "Langmuir circulations" have since been conducted, and many theories advanced. None of these have been entirely satisfactory (Faller, 1971; Leibovich and Ulrich, 1972).

An extensive review of the phenomenon has been given by Faller (1971), who argued, on the basis of laboratory experiments, that breaking waves play an important role in the generation of Langmuir circulations, and drew attention to Myer's (1971) observation on Lake George that wave amplitudes were greater in the convergence lines than away from them.

In this paper we investigate the spatial variation of a steady surface wave field propagating through Langmuir circulation cells, and establish that the back effect of the waves on the mean flow is such as to reinforce the circulations. A positive feedback loop thus exists, so that Langmuir circulations arise as an instability of the surface layer.

1. Department of Oceanography, Dalhousie University, Halifax, N.S., Canada.

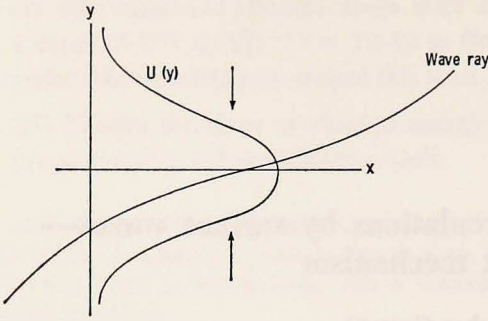


Figure 1. Refraction of a wave ray by a current $U(y)$. The arrows indicate the lateral forces exerted by the waves.

The basic idea is illustrated in Fig. 1. The downwind velocity $U(y)$ refracts a surface wave ray as shown, with two consequences: (i) The waves exert a lateral force on the water, tending to produce an inflow towards a maximum of $U(y)$. (The effect is most obvious for the case where the waves are totally internally reflected by $U(y)$, and so exert a lateral force equal to twice their original momentum flux in that direction.) This effect will produce a coincidence of convergences with maxi-

ma of $U(y)$, as is observed (Langmuir, 1938; Harris and Lott, 1973). (ii) The waves become larger as they are refracted, reaching a maximum at the maximum of $U(y)$ (provided that they are not internally reflected before this), consistent with the observations of Myer (1971). The waves are thus more likely to dissipate at a maximum of $U(y)$, and as they do so they give up part of their momentum to the mean flow, reinforcing $U(y)$. The momentum input in the y -direction cancels if the wave spectrum is symmetric about the downwind direction.

While the above arguments seem plausible, the subject of interactions between waves and mean flows is fraught with difficulties, and a detailed investigation, with careful separation between waves and mean flow, is required.

2. Modification of the waves by the currents

We consider surface gravity waves with wavenumber $\mathbf{k} = (k, l)$ propagating in deep water on currents $\mathbf{U} = (U(y), V(y))$ which are steady, independent of x , and also independent of depth, at least to a depth below which the wave energy is negligible.

The frequency of the waves relative to the water is

$$\omega' = g^{\frac{1}{2}} (k^2 + l^2)^{\frac{1}{4}} \tag{2.1}$$

and the absolute frequency is $\omega = \omega' + \mathbf{U} \cdot \mathbf{k}$. The group velocity relative to the water is

$$\mathbf{c}' = (\partial\omega'/\partial k, \partial\omega'/\partial l) = \frac{1}{2} g^{\frac{1}{2}} (k^2 + l^2)^{-\frac{3}{4}} \mathbf{k} \tag{2.2}$$

and the total group velocity is $\mathbf{U} + \mathbf{c}'$. Assuming that the mean flow varies little over a wavelength, i.e. that $(\omega'^2/g)L \gg 1$, where L is the length scale of variations of \mathbf{U} , the ray equations are (Phillips, 1966, pp. 43, 44)

$$\partial\omega/\partial t + (\mathbf{U} + \mathbf{c}') \cdot \nabla\omega = 0, \partial k/\partial t + (\mathbf{U} + \mathbf{c}') \cdot \nabla k = 0 \tag{2.3}$$

$$\partial l / \partial t + (\mathbf{U} + \mathbf{c}') \cdot \nabla l = -k dU/dy - l dV/dy \quad (2.4)$$

The first term on the r.h.s. of (2.4) represents the rotation of wave crests by the shear dU/dy , and the second term a stretching of the length scale in the y -direction. In fact, (2.3) shows that both ω, k are constant, so that variations of l may be obtained from

$$g^{\frac{1}{2}} (k^2 + l^2)^{\frac{1}{2}} + Uk + Vl = \omega \quad (2.5)$$

rather than by integrating (2.4).

The changes in wave energy density E are determined by conservation of wave action (Bretherton and Garrett, 1968)

$$\partial (E/\omega') / \partial t + \nabla \cdot [\mathbf{U} + \mathbf{c}')E/\omega'] = 0 \quad (2.6)$$

which in this case integrates to

$$(V + \partial \omega' / \partial l)E/\omega' = \text{constant}. \quad (2.7)$$

Eqs. (2.6,7) hold even though the basic flow has horizontal divergence dV/dy which must be balanced by upwelling.

We now define $\theta = \tan^{-1}l/k$, $\hat{\mathbf{U}} = \mathbf{U}(k/g)^{\frac{1}{2}}$, so that (2.5,7) become

$$(\sec \theta)^{\frac{1}{2}} + \hat{U} + \hat{V} \tan \theta = (\sec \theta_0)^{\frac{1}{2}} \quad (2.8)$$

$$E = E_0 \sin \theta_0 \cos \theta_0 [2\hat{V}(\cos \theta)^{\frac{1}{2}} + \sin \theta \cos \theta]^{-1} \quad (2.9)$$

The suffix zero refers to values that θ , E would take at a point where $\mathbf{U} = 0$. Eqs. (2.8,9) may now be solved to give $\theta(y)$, $E(y)$ corresponding to any mean flow $\mathbf{U}(y)$. There is no simple solution in closed form, but if $|\hat{\mathbf{U}}|$ is small, i.e. if the current speed is small compared with the phase speed of the waves, we find that correct to first order in \hat{U} , \hat{V} ,

$$\theta = \theta_0 - 2(\cos \theta_0)^{\frac{1}{2}} (\hat{U} \cot \theta_0 + \hat{V}) \quad (2.10)$$

$$E/E_0 = 1 + 4(\cos \theta_0)^{\frac{1}{2}} (\hat{U} \cot \theta_0 \cot 2\theta_0 - \hat{V} \tan \theta_0). \quad (2.11)$$

Thus the wave amplitude is increased by U (for $\theta_0 < \pi/4$) and decreased by V . For $\theta_0 \ll 1$, U is more important than V by a factor $\frac{1}{2}\theta_0^{-3}$, so that for a narrow directional surface wave spectrum about the x -direction, the wave amplitude will be greatest in regions of maximum $U(y)$. Unfortunately the present perturbation solution breaks down as $\theta_0 \rightarrow 0$. Physically, the waves undergo internal reflection for sufficiently small θ_0 , and behave like Airy functions near the turning point (McKee, 1974). This will be discussed further in Section 6, along with other limitations of the theory.

A surprising feature of (2.11) is that the coefficient of U becomes negative for $\theta_0 > \pi/4$.

The extent of the wave energy perturbation by a given current U is weighted with respect to wavenumber by $k^{\frac{1}{2}}$. In other words, the slow-moving, high wavenumber, waves are most affected, but for a typical surface wave spectrum with wavenumber dependence like k^{-4} (Phillips, 1966, p. 115) maximum energy perturbation is still associated with the low wavenumber waves.

3. Effect of waves on currents

We first derive some general results for arbitrary mean flows. In the following the suffixes α, β refer to horizontal components, 3 to the vertical component.

The horizontal momentum equation may be written

$$\frac{\partial}{\partial t} (\rho u_\alpha) + \frac{\partial}{\partial x_\beta} (\rho u_\alpha u_\beta + p \delta_{\alpha\beta}) + \frac{\partial}{\partial x_3} (\rho u_\alpha u_3) = 0. \quad (3.1)$$

We assume that $u_\alpha = U_\alpha + u_\alpha'$, where U_α is the mean flow, assumed independent of depth as far as some depth $-h$ below significant wave activity, and u_α' is associated with the waves. Vertical integration from $-h$ to the free surface ξ , followed by averaging over several wave periods or wavelengths, leads to (Phillips, 1966, p. 46; Hasselmann, 1971)

$$\frac{\partial \bar{M}_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta} \int_{-h}^{\xi} (\rho U_\alpha U_\beta + p^m \delta_{\alpha\beta}) dx_3 = - \frac{\partial}{\partial x_\beta} (U_\alpha M_\beta + U_\beta M_\alpha + S_{\alpha\beta}) \quad (3.2)$$

where

$$\bar{M}_\alpha = \overline{\int_{-h}^{\xi} \rho u_\alpha dx_3} = M_\alpha^m + M_\alpha \quad (3.3)$$

$$M_\alpha^m = \int_{-h}^{\xi} \rho U_\alpha dx_3, \quad M_\alpha = \overline{\int_{\xi}^{\xi} \rho u_\alpha' dx_3}. \quad (3.4)$$

\bar{M}_α is the average momentum of the complete flow, and is made up of the momentum M_α^m of the mean flow plus the wave momentum $M_\alpha = Ek_\alpha/\omega'$. $S_{\alpha\beta}$ is the radiation stress of Longuet-Higgins and Stewart (1964), and for deep water

$$S_{\alpha\beta} = Ek_\alpha c_\beta' / \omega'. \quad (3.5)$$

The part of the average pressure induced by the waves is included in $S_{\alpha\beta}$, so that the pressure p^m of the mean flow satisfies a boundary condition independent of the waves

$$p^m = \bar{p}_\alpha \text{ on } x_3 = \xi \quad (3.6)$$

where \bar{p}_α is the average atmospheric pressure.

We see that the effect of the waves on the complete flow is to exert a force given by the right-hand side of (3.2). However, as emphasized by Hasselmann, the com-

plete flow includes the waves, so that part of the apparent force in (3.2) goes into changing the wave momentum M_a rather than the momentum M_a^m of the mean flow. To obtain an equation for $\partial M_a^m / \partial t$ we need to subtract $\partial M_a / \partial t$ from (3.2). Using $M_a = Ek_a / \omega'$, the wave action conservation equation (2.6) and the equation for the rate of change of k_a (Phillips, 1966, p. 44),

$$\frac{\partial k_a}{\partial t} + (U_\beta + c_\beta') \frac{\partial k_a}{\partial x_\beta} = -k_\beta \frac{\partial U_\beta}{\partial x_a} \quad (3.7)$$

we find

$$\frac{\partial M_a}{\partial t} = -\frac{\partial}{\partial x_\beta} [M_a(U_\beta + c_\beta')] - M_\beta \frac{\partial U_\beta}{\partial x_a} \quad (3.8)$$

and hence the equation for the mean flow is

$$\frac{\partial M_a^m}{\partial t} + \frac{\partial}{\partial x_\beta} \int_{-h}^{\xi} (\rho U_a U_\beta + p^m \delta_{a\beta}) dx_3 = F_a^m \quad (3.9)$$

where

$$F_a^m = \frac{\partial}{\partial x_\beta} (-U_a M_\beta - S_{a\beta} + M_a c_\beta') + M_\beta \frac{\partial U_\beta}{\partial x_a} \quad (3.10)$$

is the force exerted by the waves on the mean flow. For deep water waves (3.5) reduces this to

$$F_a^m = \frac{\partial}{\partial x_\beta} (-U_a M_\beta) + M_\beta \frac{\partial U_\beta}{\partial x_a}, \text{ or } \mathbf{F}^m = -\mathbf{U} \nabla \cdot \mathbf{M} \times (\nabla \times \mathbf{U}) \quad (3.11)$$

Eqs. (3.10,11) are a general result subject to the assumption that the waves are slowly-varying and non-dissipative. (Hasselmann's (1971) Eq. (14), while equivalent to (3.9) here, is not immediately useful as it involves his interaction stress $\overline{u_a' u_\beta'}$, which can only be evaluated by a procedure equivalent to a derivation of (3.8) here. Moreover, Hasselmann's Eq. (17a) lacks the terms $U_a M_\beta + U_\beta M_a$ on the right hand side.)

As mentioned, the pressure p^m satisfies the same equation (3.6) as in the absence of waves, but the kinematic free surface condition for the mean flow is now (Hasselmann, 1971)

$$\frac{\partial \xi}{\partial t} + U_a \frac{\partial \xi}{\partial x_a} - U_3 = -\rho^{-1} \frac{\partial M_a}{\partial x_a} \text{ at } x_3 = \xi \quad (3.12)$$

Physically, an upwelling of the mean flow is required to satisfy the diverging mass flux of the waves.

In summary, the mean flow is driven by a force F_a^m , given by (3.10,11) located near the surface, and a surface source given by $-\partial M_a / \partial x_a$ in (3.12).

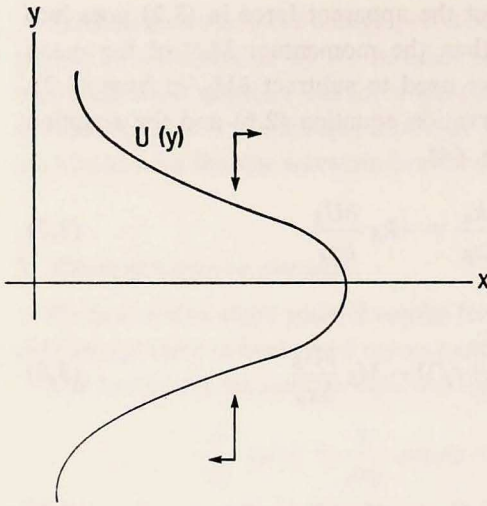


Figure 2a. Arrows indicate the forces exerted on the mean flow by waves propagating in the positive (x,y) directions. For a wave ray at a small angle to the x -axis the force is predominantly towards a maximum of $U(y)$.

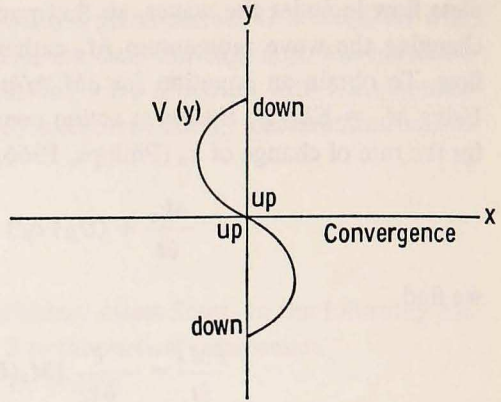


Figure 2b. The source term for the mean flow for a wave ray at a small angle to the x -axis.

4. Application to Langmuir circulations

a. Forces. If the wave momentum and mean flow are independent of x , Eq. (3.11) becomes

$$\mathbf{F}^m = \left[-\frac{d}{dy}(UM_y), -V\frac{dM_y}{dy} + M_x\frac{dU}{dy} \right] \quad (4.1)$$

$$= \left\{ -\frac{d}{dy}[\hat{U}E \tan \theta (\cos \theta)^{\frac{1}{2}}], -\hat{V}\frac{d}{dy}[E \tan \theta (\cos \theta)^{\frac{1}{2}}] + E(\cos \theta)^{\frac{1}{2}}\frac{d\hat{U}}{dy} \right\} \quad (4.2)$$

where we have reverted to the (x,y) notation of Section 2. For weak currents we have

$$\mathbf{F}^m = [-E_0 \tan \theta_0 (\cos \theta_0)^{\frac{1}{2}}, E_0 (\cos \theta_0)^{\frac{1}{2}}] d\hat{U}/dy. \quad (4.3)$$

The direction of this force on either side of the convergence is illustrated in Fig. 2a. F_x^m is much less than F_y^m for small θ_0 , and will disappear if we have equal energy at $\pm\theta_0$. The significant force is F_y^m , which clearly tends to generate a transverse flow towards a maximum of U , and hence create a convergence there. (In general, for weak currents, $\mathbf{F}^m = \mathbf{M}_0 \times (\nabla \times \mathbf{U})$, where \mathbf{M}_0 is the unperturbed wave momentum).

b. *Upwelling.* The driving term in (3.12), which is the upwelling speed if $\zeta = 0$, is $\rho^{-1} dM_y/dy$ where

$$M_y = E(k/g)^{\frac{1}{2}} (\cos \theta)^{\frac{1}{2}} \tan \theta \quad (4.4)$$

$$= E_0(k/g)^{\frac{1}{2}} [(\cos \theta_0)^{\frac{1}{2}} \tan \theta_0 - 3\hat{U} \sin \theta_0 - \hat{V} \sec \theta_0 (2+3 \sin^2 \theta_0)] \quad (4.5)$$

for weak currents. Thus

$$dM_y/dy = -E_0(k/g)^{\frac{1}{2}} [3 \sin \theta_0 d\hat{U}/dy + \sec \theta_0 (2+3 \sin^2 \theta_0) d\hat{V}/dy] \quad (4.6)$$

in which the term in $d\hat{V}/dy$ is dominant for small θ_0 . The direction of this upwelling for Langmuir circulations is illustrated in Fig. 2b. It clearly opposes the circulation.

c. *Typical magnitudes.* The convergence-generating force F_y^m from (4.3) is

$$F_y^m = E_0 (\cos \theta_0)^{\frac{1}{2}} (k/g)^{\frac{1}{2}} dU/dy \approx \frac{1}{2} \rho g a_0^2 (U_0/c_0) L^{-1} \quad (4.7)$$

where a_0 , c_0 are the basic amplitude and phase velocity of the wave (θ_0 is assumed small), and U_0 , L are typical velocity and length scales of $U(y)$. We take $a_0 = 0.25$ m, $c_0 = 5 \text{ ms}^{-1}$ (corresponding to a wave period of 3s and steepness $(\omega'^2/g)a_0 = 0.1$). At a windspeed of 5 ms^{-1} a typical windrow spacing is 20m (Faller and Woodcock, 1964), so that a suitable value for L is 3m. Taking $U_0 = 0.01 \text{ ms}^{-1}$ we find $F_y^m = 0.2 \text{ Nm}^{-2}$. This is certainly significant compared with a wind stress at 5 ms^{-1} of about 0.03 Nm^{-2} .

The magnitude of the upwelling, for small θ_0 , is given from (4.6) by

$$\rho^{-1} dM_y/dy \approx -2\rho^{-1} E_0(k/g) dV/dy \approx g a_0^2 (V_0/c_0^2) L^{-1} \quad (4.8)$$

where V_0 is a typical transverse velocity. For $V_0 = 0.01 \text{ ms}^{-1}$ and the other parameters as before, this upwelling speed is $8 \times 10^{-5} \text{ ms}^{-1}$, which is totally negligible compared with observed downwelling speeds of about 0.05 ms^{-1} (Sutcliffe et al., 1963).

Thus, in evaluating the effect of the waves on the Langmuir circulation, we may ignore the kinematically induced upwelling.

d. *Effect of wave dissipation.* So far we have seen how waves refracting through a current field independent of x will exert forces on the mean flow tending to generate convergence towards a maximum of the current $U(y)$ in the x -direction. In other words, the existence of U leads to the generation of V . But the generation of Langmuir circulations through a positive feedback process requires a means of reinforcing U . Any form of dissipation will provide this, for, as the waves lose their energy, they must give up their lost momentum to the mean flow, at a rate which is greatest where the wave energy is greatest, i.e. at the maxima of $U(y)$.

Let us suppose that the dissipative rate of loss of energy at any point is γE , with γ a constant, although any process involving wave breaking is likely to be a much

stronger function of E . A term $-\gamma M_x$ must then be added to the right hand side of (3.8), and γM_x to F_x^m in (3.10). For small θ_0 the dominant force is γM_x in the x -direction. Approximate symmetry between $\pm\theta_0$ will tend to cancel out γM_y anyway, though, if wave dissipation occurs preferentially in the convergences, waves emerging from a convergence may be slightly smaller than those entering, resulting in a very small net inwards force towards the convergence.

The momentum input to the waves by the wind will enter both Eqs. (3.2) and (3.8), so that the effect on Eq. (3.9) for the mean flow is zero.

Thus for a generating, dissipating, wave field, the significant force is $\gamma \mathbf{M}$ where γ is a dissipation rate. For the present situation with weak currents

$$M_x = E_0(k/g)^{\frac{1}{2}} (\cos \theta_0)^{\frac{1}{2}} [1 + 2\hat{U} \cot^2 \theta_0 (\cos \theta_0)^{\frac{1}{2}} (1 - \frac{1}{2} \tan^2 \theta_0) - 3\hat{V} \sin \theta_0 (\cos \theta_0)^{-\frac{1}{2}}] \quad (4.9)$$

which for small θ_0 is dominated by the term in \hat{U} . The force γM_x thus fluctuates about its mean value $\gamma E_0(k/g)^{\frac{1}{2}} (\cos \theta_0)^{\frac{1}{2}}$ by a fraction of approximately $2(k/g)^{\frac{1}{2}} \hat{U} \theta_0^{-2}$. The theory breaks down as $\theta_0 \rightarrow 0$, but we see that even for $\theta_0 = 0.1$, and U_0, c_0 as before, the fraction is 0.4. It is clear that significant lateral variations of the force associated with wave dissipation can occur.

In summary, for a fairly narrow directional spectrum symmetric about Ox , the dominant forces exerted by the waves on the mean flow consist of (i) a force in the y -direction proportional to dU/dy , and (ii) a force in the x -direction, proportional to the dissipation rate, consisting of a y -independent part plus a part proportional to U . If the waves are short compared with the scale of variation of U , as we assume (but see Section 6), these forces effectively act as surface stresses on the mean flow.

5. Instability

The simplest model is now a viscous fluid of infinite depth, with a velocity field $\mathbf{U}(y, z)$, acted upon by a surface stress

$$\tau(y) = (aU, b dU/dy) \text{ at } z = 0. \quad (5.1)$$

(The average stress τ_x generates a y -independent flow which is independent of the flow generated by (5.1).) We look for solutions

$$\mathbf{U}(y, z, t) = \text{Re } \mathbf{u}(z) e^{i\beta y} e^{at}, \quad P(y, z, t) = \text{Re } p(z) e^{i\beta y} e^{at} \quad (5.2)$$

of the equations

$$\partial \mathbf{U} / \partial t + (1/\rho) \nabla P = \nu \nabla^2 \mathbf{U}, \quad \nabla \cdot \mathbf{U} = 0 \quad (5.3)$$

with boundary conditions

$$\mu \partial U / \partial z = aU, \quad \mu \partial V / \partial z = b \partial U / \partial y, \quad W = 0 \text{ at } z = 0 \quad (5.4)$$

and $\mathbf{U} \rightarrow 0$ as $z \rightarrow -\infty$.

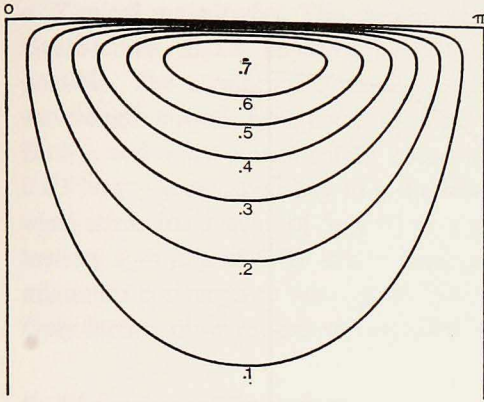


Figure 3a. Contours of $\sin \gamma \eta (e^{\gamma \zeta} - e^{\zeta})$ for $\gamma = 0.1$. The horizontal axis is $\gamma \eta$, the vertical axis is $\gamma \zeta$.

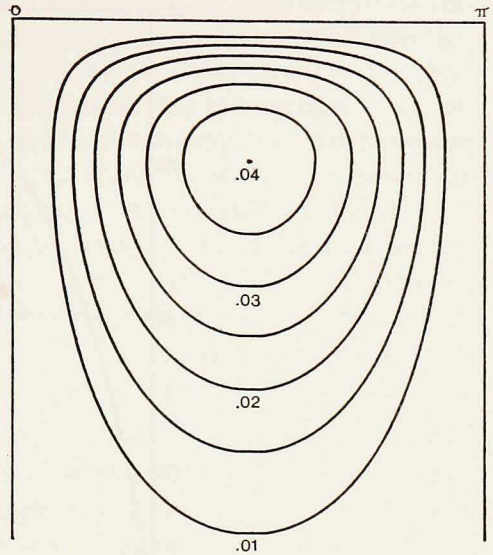


Figure 3b. As in 3a, but with $\gamma = 0.9$.

The equations governing U are independent of those for the transverse circulation involving V , W . The problem for U becomes

$$qu = \nu(d^2u/dz^2 - \beta^2u) \quad (5.5)$$

with $\mu du/dz = au$ at $z = 0$. The solution is

$$u = u_0 \exp[(a/\mu)z], \quad q = \nu[(a/\mu)^2 - \beta^2] \quad (5.6)$$

so that the current has scale depth (μ/a) and fastest growth rate for large transverse scales, with a high wavenumber cutoff at $\beta = a/\mu$. If a is itself proportional to μE_0 , then the high wavenumber cutoff increases with increasing wave energy.

The transverse circulation has a streamfunction $\Psi(y, z, t) = \text{Re } \psi(z)e^{i\beta y}e^{qt}$ such that $v = d\psi/dz$, $w = -i\beta\psi$ and

$$\left[\frac{d^2}{dz^2} - (\beta^2 + g/\nu) \right] \left[\frac{d^2}{dz^2} - \beta^2 \right] \psi = 0 \quad (5.7)$$

with boundary conditions

$$\psi = 0, \quad \mu d^2\psi/dz^2 = i\beta b u_0 \text{ at } z = 0. \quad (5.8)$$

The solution is

$$\psi(z) = i\beta b (q\rho)^{-1} u_0 \{ \exp[(\beta^2 + q/\nu)^{1/2} z] - \exp(\beta z) \} \quad (5.9)$$

and we note that $(\beta^2 + q/\nu)^{1/2} = a/\mu$, which is also the coefficient of z in the depth dependence of u in (5.6).

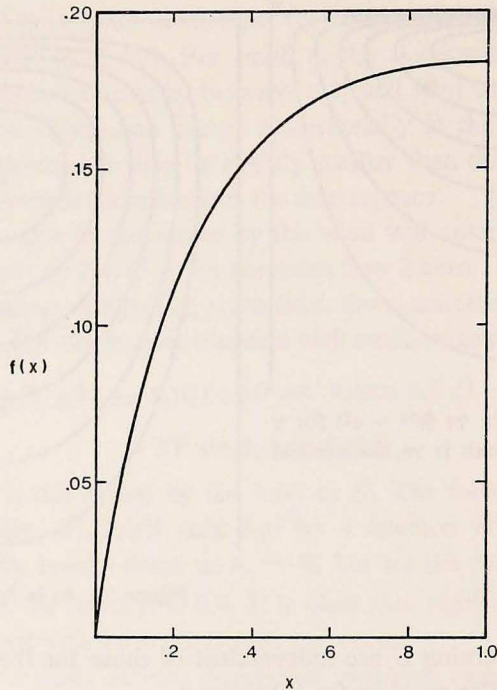


Figure 4. Plot of $f(x) = x(1-x^2)^{-1} [x^{x/(1-x)} - x^{1/(1-x)}]$.

Defining $(\eta, \zeta) = (a/\mu)(y, z)$, $\gamma = (u/a)\beta$, the streamfunction is

$$\Psi(y, z, t) = u_0 b a^{-1} \sin(\gamma \eta) \gamma (1-\gamma^2)^{-1} (e^{\gamma \zeta} - e^{\zeta}) \exp[vt(a/\mu)^2(1-\gamma^2)] \quad (5.10)$$

Streamlines, corresponding to contours of $\sin(\gamma \eta)[e^{\gamma \zeta} - e^{\zeta}]$, are shown in Fig. 3 for $\gamma = 0.1$ (low viscosity) and $\gamma = 0.9$ (high viscosity). Growing solutions only occur for $\gamma < 1$. We see that in both cases the aspect ratio of the cells is of order unity.

The maximum value of $e^{\gamma \zeta} - e^{\zeta}$ is $B(\gamma) = \gamma^{\gamma/(1-\gamma)} - \gamma^{1/(1-\gamma)}$ at a depth $\zeta = (1-\gamma)^{-1} \ln \gamma$, so that the strength of the transverse circulation is given by $\gamma(1-\gamma^2)^{-1} B(\gamma)$. This is plotted in Fig. 4. It increases with γ as far as the maximum at $\gamma = 1$. Thus although the growth rate is greatest for the largest cells, small ones have a larger coefficient, and so will appear first for a white initial perturbation spectrum of U , though they give way later to the larger scales. A reduction of the growth rate of large scale transverse circulations is presumably introduced by the presence of a thermocline.

Advective terms in the equations of motion will tend to make the downwelling zone stronger than the upwelling, and presumably the cells will also stabilize at some finite amplitude in practice. These problems will not be pursued further here.

a. Typical magnitudes. The present stability model, with its many unknowns and idealisations, makes anything more than the crudest estimates of magnitudes impossible. The significant quantities are the penetration depth of u and the short wavelength cut-off, both equal to μ/a , and the growth time of large cells $\nu^{-1}(\mu/a)^2$. Both μ and a are almost totally unknown, and here we merely note that a stress of 0.03 Nm^{-2} (corresponding to a doubling, at the maximum of $U(y)$, of the average wind stress for a wind of 5 ms^{-1}) at a maximum U of 0.01 ms^{-1} and an eddy diffusivity $\nu = \mu/\rho$ of $0.01 \text{ m}^2\text{s}^{-1}$ correspond to a length scale of 3m (and hence a minimum convergence spacing of 20m) and a growth time of $10^3 \text{ s} = 17 \text{ minutes}$. Only further observations will establish whether values of this order are appropriate.

6. Discussion and limitations

The results for a slowly varying wave train in Section 2 assume that total internal reflection does not occur. However, this will happen if the solution θ of (2.8) goes to zero. Neglecting \mathcal{V} , we ask for what angle θ_0 does a given U produce internal reflection? For small θ_0 we have $\theta_0 = 2U^{1/2} = 0.09 \text{ radians} = 5^\circ$ for the parameters of section 4. Thus a surprisingly wide angular spread of wave rays originating at a place where $U = 0$, can be internally reflected by a weak current. However, this does not alter the principle of the feedback process we have proposed. Internally reflected waves will exert a lateral force, towards the line of maximum U , equal to twice their original momentum flux. Thus convergence will still be generated. On the other hand the maximum wave amplitude will now be near the line of reflection, rather than at the maximum of U which is never reached. The forces reinforcing U will then be greatest at either side of the maximum U , spreading the region of strong U against the tendency of V to confine it.

Quite apart from the problem of internal reflection, the assumption of slow variation requires that $(\omega'^2/g)L$ should be large (e.g. McKee, 1974) though in the present problem it is only 1.2 for the parameters of Section 4. In practice with problems of this kind the assumption can be considerably relaxed (Kulsrud, 1957), and Evans (1975) has shown that a discontinuity of $U(y)$ gives a surprisingly small reflection coefficient with the transmitted energy not substantially different from that calculated on the assumption of slow variation of $U(y)$.

The results of Section 2 assumed that the mean flow was steady, though we subsequently investigated the development of the mean flow with time. Although this is inconsistent, the assumption seems reasonable as the mean flow develops slowly compared with the time for a wave packet to traverse a circulation cell.

Another possibly weak assumption of the theory is that the surface current is constant over the depth of the waves. The e -folding depth for energy is $(2k)^{-1} = 1.25 \text{ m}$ for the 3s period wave assumed earlier. However, even if the current is not uniform over this depth or more, and the interaction stresses exerted by the waves

on the mean flow really generate some vertical shear, the basic mechanism proposed here must still operate to some extent.

It has been implicitly assumed in much of the preceding work that the cells that appear through the proposed instability will be oriented with axes downwind. However, the theory applies to any orientation and we account for the downwind orientation of observed rolls by pointing out that the dissipative mechanism that reinforces U is strongest for small θ_0 , so that the fastest growing rolls will be those oriented with the maximum of the directional surface wave spectrum, i.e. downwind.

The analysis of this paper is readily extended to capillary waves, with very similar results, so that under suitable conditions (such as in a small puddle) circulations driven by capillary waves can be expected.

7. Vorticity

Faller (1971) has emphasized that the vorticity of Langmuir circulations cannot be generated by an inviscid process from purely irrotational flow. The primary source of vorticity must be the wind stress at the sea surface. It is thought (Stewart, 1967) that nearly all the momentum flux from the wind goes into the irrotational surface waves, which only generate vorticity in the mean flow when they dissipate. If wave dissipation is independent of y , only a y -component of vorticity, $\partial U/\partial z$, is generated. Leibovich and Ulrich (1972) showed that Craik's (1970) mechanism for the generation of Langmuir circulations amounts to distortion of the vortex lines by a Stokes drift that varies in the y -direction, and that this produces surface divergence at maxima of the downwind mass flux, in conflict with observations.

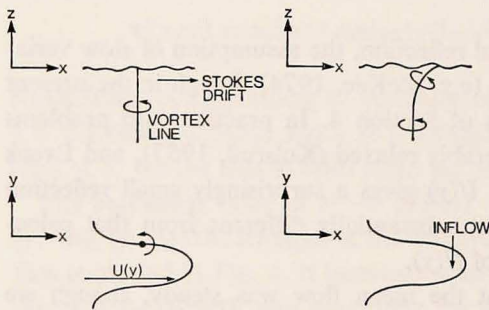


Figure 5. A vertical vortex line associated with $U(y)$, as shown in (x, z) and (x, y) planes on the left, is bent in the x -direction by the x -component of the Stokes drift of the surface waves, generating an x -component of vorticity which causes surface convergence towards a maximum of $U(y)$, as shown in (x, z) and (x, y) planes on the right.

The point of the present paper is that any initial perturbation of downwind velocity U (due, perhaps, to a whitecap or a downwind sequence of whitecaps) will be amplified by subsequent wave dissipation occurring preferentially at the same y . Thus wave dissipation can be a source of z -vorticity, $-\partial U/\partial y$, as well as y -vorticity $\partial U/\partial z$.

As vortex lines in an inviscid fluid move with the fluid, the tendency of the downwind Stokes drift is to tilt the vertical vortex lines and create x -vorticity of the right sign to produce a surface convergence at maxima of $U(y)$ (see Fig. 5).

In summary, the x -vorticity of the rolls comes from 1) creation of z -vorticity since wave dissipation is a function of y , and 2) the tilting of vertical vortex lines by the downwind Stokes drift of the waves.

8. Experimental tests and other applications

The theory developed in this paper accounts for many of the features of Langmuir circulation, but further observational tests are required. Perhaps the most important test of the theory will come from an investigation of whether the greatest wave dissipation occurs at maxima of the downwind velocity U , i.e. in the convergences of the Langmuir circulations. Myer's (1971) observation of greater wave amplitudes inside the convergences than outside makes this appear very likely, but further tests through, for example, aerial photography of the location of whitecapping in relation to convergences, or measurement of turbulence levels cross-wind, are desirable. It would also be useful to check in the laboratory or field that waves refracting through a surface current exert the calculated forces (3.10,11) on the mean flow even without dissipation.

The interactions of surface waves and currents discussed in this paper are relevant to more than Langmuir circulations. For example, swell propagating through any current jet will tend to create a convergence or divergence there, depending on the relative direction of waves and current, and may enhance the jet if breaking occurs.

9. Conclusions

The interaction between waves and currents appears capable of generating Langmuir circulations through a feedback process. Large scale cells are the fastest growing, but smaller scales tend to have a stronger transverse circulation relative to the downwind flow, and so may be the first to appear. The mechanism is closely tied to the overall dynamics of the mixed layer, in that the details of the cells depend on properties of the turbulence in the water. The cells in turn, as first stressed by Langmuir (1938), may play an important role in the deepening of the mixed layer.

We stress not only that mixed layer theories must take account of Langmuir circulation, but also that observational programs should not assume horizontal homogeneity. Any observational platform that is swept into a convergence will measure atypical statistics of waves, turbulence and other properties.

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