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## Minimal properties of planetary eddies

**by Melvin E. Stern**[1]

ABSTRACT

An isolated barotropic eddy on the $\beta$-plane can be in equilibrium only if it is composed of a coupled cyclone-anticyclone system, only if it is separated by a vorticity discontinuity (free streamline) from the surrounding fluid, and only if its rms vorticity exceeds $\beta R/\sqrt{2}$, where $R$ is the radius of the free streamline. The eddy having this minimum vorticity is called a "modon", and a close packed array of non-overlapping modons is also an equilibrium solution. The latter is called a "modon-sea" and its rms velocity is $\beta R^2/7.6$. Although the equilibrium modon-sea is probably dynamically unstable, so that non-linear Rossby waves will develop, the total energy is invariant and related to the modon area by the previous relation. The variational principle on which the equilibrium theory is based has also been generalized so that some baroclinic effects can be examined in future work. It is suggested that some of the statistical properties of mid-ocean eddies can be interrelated through the use of the "modon-sea" model.

### 1. Introduction

Any circularly symmetric vortex is an equilibrium solution in the $f$-plane dynamics for an ideal fluid bounded by two infinite walls, each of which is perpendicular to the axis of rotation. But Rossby (1948) pointed out that the situation is entirely different for the $\beta$-plane dynamics (in a spherical annulus) because the average Coriolis force acting on the vortex will not vanish in general. To illustrate this point we assume a steady eddy having its center at some latitude $y = 0$, this center being partially defined by the geometrical condition:

$$\iint y dA = 0 \tag{1}$$

1. Graduate School of Oceanography, University of Rhode Island, Kingston, Rhode Island, 02881, U.S.A.

where the integration is taken over the entire horizontal area. The relative velocity $\mathbf{v}$ on the $\beta$-plane is non-divergent ($\nabla \cdot \mathbf{v} = 0$), so that $\mathbf{v} = \mathbf{k} \times \nabla\psi$ where $\psi(x,y)$ is the stream function and $\mathbf{k}$ the local unit normal. We also suppose that this eddy is "isolated" on the $\beta$-plane, in the sense that $\psi$ and the pressure perturbation vanish "rapidly" with increasing distance from the center of the eddy. This means that the horizontal integral of the pressure gradient force is assumed to be vanishingly small, the integral of the non-linear momentum terms ($\mathbf{v} \cdot \nabla\mathbf{v}$) also vanishes, and the integral of the Coriolis force over the entire plane is

$$\iint f(y)\mathbf{k} \times \mathbf{v} dA = -\iint f(y) \nabla\psi dA = \mathbf{j}\beta \iint \psi dA. \tag{2}$$

where $\mathbf{j}$ is the unit vector in the $y$ (northward) direction. Thus the condition of equilibrium requires the vanishing of (2), or the vanishing of the mean value of the stream function. If $r$ denotes the vector distance of a point from the center of the eddy then the integrated relative angular momentum has the value

$$\iint \mathbf{k} \cdot \mathbf{r} \times \mathbf{v} dA = \iint \mathbf{r} \cdot \nabla\psi dA = -2\iint \psi dA = 0 \tag{3}$$

because (2) vanishes.

The foregoing consideration shows that if an isolated disturbance is to be in equilibrium on the $\beta$-plane, then the eddy must have a dipole character, with equal and opposite amounts of cyclonic and anti-cyclonic angular momentum (3). The fluid parcels must also conserve absolute vorticity ($f + \zeta$), so that the steady state equation of motion is

$$\left.\begin{aligned} &\mathbf{v} \cdot \nabla(\beta y + \zeta) = 0 \\ &\mathbf{v} = \mathbf{k} \times \nabla\psi \\ &\zeta = \mathbf{k} \cdot \nabla \times \mathbf{v} = \nabla^2\psi. \end{aligned}\right\} \tag{4}$$

Since $\partial\mathbf{v}/\partial t = 0$ we infer that the *far-field* vorticity must vanish identically, because $\mathbf{v} \cdot \nabla\zeta$ would otherwise be too small to balance the linear $\mathbf{v} \cdot \nabla\beta y$ term. Therefore if we want to investigate the strictly steady solutions then the eddy must be bounded by a free streamline across which the vorticity changes discontinuously. Accordingly, we specify that $\mathbf{v} = 0 = \psi$ for $r > R$, or

$$\psi(R,\theta) = 0 \tag{5}$$

where $(r,\theta)$ are the polar coordinates of the point $(x,y)$ and $R$ is the radius of the free streamline (Fig. 1a). Since the dynamic pressure is uniform for $r \geqslant R$, Bernoulli's equation implies that $\mathbf{v}^2 = (\nabla\psi)^2$ is uniform on $r = R$, and consequently

$$\frac{\partial^2\psi(R,\theta)}{\partial r \partial\theta} = 0 \tag{6}$$

is the second boundary condition for the solution of (4).

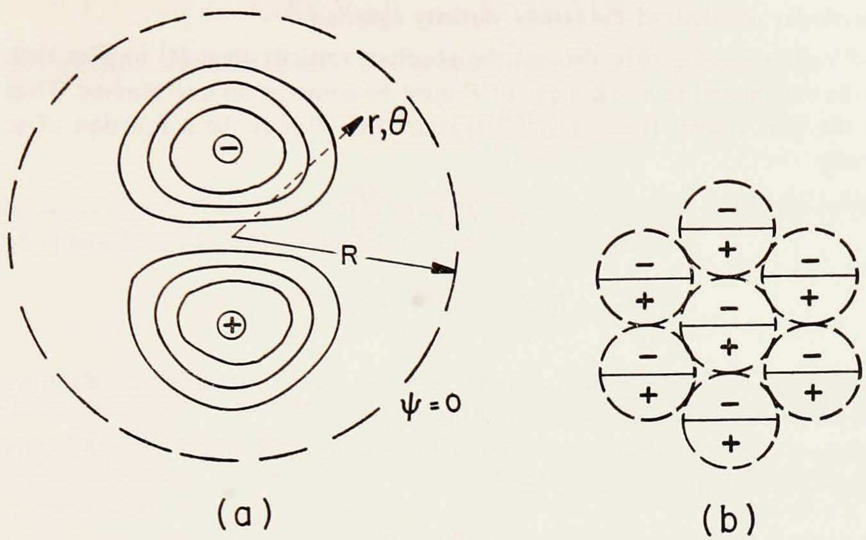


Figure 1. a) Schematic diagram of streamlines in a modon consisting of a cyclone (−), an anticyclone (+), and a free streamline ($\psi = 0$). The center of the modon is at $y = 0$ on the $\beta$-plane, and $\theta$ measures the azimuthal distance of any point from the easterly direction.

b) A close packed array of non-interacting modons with $\mathbf{v} = 0$ in the small "dead spaces" lying between three adjacent circles.

A particular solution of (4)–(6) is discussed in Section 2, and in Section 3 we show that this solution has the smallest rms vorticity (for a given $R$) within the entire class of solutions of (4)–(6). The plausibility of the existence of such a minimal solution is indicated by means of the following analogy with a simple pendulum. If the amplitude of oscillation of the pendulum be small compared to its radius, so that the pendulum *oscillates* back and forth, we say that the system is in a "wave-like" mode. But if the amplitude is large, so that the pendulum rotates monotonically about its pivot, then we say it is in a "current-like" mode. Likewise, a *small* amplitude disturbance on the $\beta$-plane will oscillate like a Rossby wave, whereas a monotonic current can occur if the "amplitude" exceeds a certain critical value. The minimum eddy mentioned above is called a "modon", and a close-packed array is called a "modon-sea". The relevance of this statistical model to the time dependent eddies observed in the mid-ocean is discussed in Section 4. The variational principle upon which the theory is based, is then (Section 5) extended to the case of a fluid having a free upper surface, this being the simplest model in which the (reduced) gravity force enters. Although this extension indicates the feasibility of incorporating the baroclinic effects, the question of the energy source and sink of eddies is beyond the scope of the investigation. Thus the problem at hand may be restated by assuming the mean energy density of the eddies to be given, and by then asking for the related statistical properties, such as the mean eddy radius.

**2. Particular solutions of the steady vorticity equation**

If $P = \beta y + \zeta = \beta y + \nabla^2\psi$ denotes the absolute vorticity then (4) implies that $\nabla P$ must be orthogonal to $\mathbf{v} = \mathbf{k} \times \nabla\psi$, or $P$ must be constant on a streamline. Thus we have the well known [Intersoll (1973)] result that $P$ must be a function of $\psi$, or inversely

$$\psi = F'(P) = \frac{dF}{dP} \tag{7}$$

where

$$F(P) = a_2P^2 + \sum_{n \neq 2} a_nP^n \tag{8}$$

is an arbitrarily chosen function, and the constants $a_n$ are the coefficients of the power series expansion in the case where $F(P)$ is analytic.

Fofonoff (1954) obtained a solution for the case in which $F'(P) = -P/\lambda$ is a linear function of $P$, in which case (7) becomes

$$\zeta + \beta y = -\lambda\psi \tag{9}$$

where $-1/\lambda = 2a_2$, and $a_0$, $a_1$, $a_3$, ... all vanish. Fofonoff obtained the free solution for a rigidly bounded ocean basin, so that $\psi = 0$ is the only boundary condition that must be satisfied, whereas in the present problem the isobaric boundary condition (6) must also be satisfied. Since the relative vorticity is $\nabla^2\psi$, and since $y = r\sin\theta$ in polar coordinates, eq (9) can be written as

$$\nabla^2\psi + \beta y + \lambda\psi = 0 \tag{10}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \lambda\right)\psi = -\beta r\sin\theta, \qquad \psi(R,\theta) = 0 = \frac{\partial^2\psi(R,\theta)}{\partial r\partial\theta}. \tag{11}$$

A particular solution of (10) is $\psi = -\beta y/\lambda = -\beta r\sin\theta/\lambda$, and the solution of the homogeneous equation ($\nabla^2\psi + \lambda\psi = 0$) which must be added to satisfy the free boundary conditions (11) is also proportional to $\sin\theta$. By separation of variables we readily find that the relevant homogeneous solution is proportional to $\psi = \sin\theta\, J_1(r\lambda^{1/2})$, where $J_1$ is the Bessel function of the first order and kind. Consequently, the total solution satisfying the first boundary condition in (11) is

$$\psi = -\frac{\beta\sin\theta}{\lambda}\left[r - R\frac{J_1(r\lambda^{1/2})}{J_1(R\lambda^{1/2})}\right]. \tag{12}$$

The final boundary condition in (11) implies $\partial\psi(R,\theta)/\partial r = 0$, and consequently

$$J_1(R\lambda^{1/2}) - R\lambda^{1/2}J_1'(R\lambda^{1/2}) = 0.$$

By using the identity $zJ_1'(z) = J_1(z) - zJ_2(z)$ we then obtain

$$J_2(R\lambda^{1/2}) = 0 \qquad R\lambda^{1/2} = 5.136, \ldots \tag{13}$$

The vorticity is obtained from (9), (12), and for any one of the roots of (13) we find

$$\zeta = -\beta R \sin\theta \frac{J_1(r\lambda^{1/2})}{J_1(R\lambda^{1/2})}. \tag{13a}$$

The mean square vorticity is now evaluated by noting that the indefinite integral of $zJ_1^2(z)$ equals $z^2/2\,[J_1^2(z) - J_0(z)J_2(z)]$. Using (13) we see that

$$\frac{\iint \zeta^2 dA}{\iint dA} = \beta^2 \int_0^R r \left[\frac{J_1(r\lambda^{1/2})}{J_1(R\lambda^{1/2})}\right]^2 dr = \frac{\beta^2 R^2}{2} \tag{14}$$

is independent of $\lambda$, and therefore all of the discrete (and non-superimposable) solutions of (10)–(11) have the same rms vorticity.

What happens if we consider more complicated functions $F(P)$, corresponding to the entire class of possible solutions in a given area $\pi R^2$? This we shall do in the next section, and we shall show that (14) gives the minimum within the entire class. But the argument is rather abstract, and therefore we shall illustrate the point by first considering a slightly more general $F(P)$, but one which is still amenable to explicit calculation.

Let $F'(P) = -(P+\lambda_1)/\lambda$, and let $\hat{\zeta} = \nabla^2\hat{\psi}$ denote the vorticity in this problem, where $\lambda_1$ is another arbitrary constant. Equation (7) then becomes

$$\beta y + \hat{\zeta} + \lambda\hat{\psi} + \lambda_1 = 0. \tag{15}$$

We seek a solution having the form

$$\hat{\psi} = \psi(r,\theta) + \psi^*(r) \tag{16}$$

where the azimuthally varying part ($\psi$) satisfies (10), (12), and thus the permitted values of $\lambda$ in (15) are still determined by (13). By subtracting (10) from (15) we find that the new component in (16) must satisfy

$$\nabla^2\psi^* + \lambda\psi^* + \lambda_1 = 0$$

or

$$\frac{1}{r}\frac{d}{dr} r \frac{d\psi^*}{dr} + \lambda\psi^* = -\lambda_1 \qquad \psi^*(R) = 0$$

and this $\psi^*$ automatically satisfies the second boundary condition (6). (If an azimuthal dependence in $\psi^*$ were included then it would not have been possible to satisfy both boundary conditions). The solution of the above equation is

$$\psi^* = -\frac{\lambda_1}{\lambda}\left[1 - \frac{J_0(r\lambda^{1/2})}{J_0(R\lambda^{1/2})}\right]$$

and the associated vorticity is $\nabla^2\psi^* = -\lambda_1 - \lambda\psi^*$, or

$$\zeta^* = \nabla^2\psi^* = -\lambda_1 \frac{J_0(r\lambda^{1/2})}{J_0(R\lambda^{1/2})}. \tag{17}$$

Since (17) is orthogonal to (13a), the mean square of $\hat{\zeta} = \zeta + \zeta^*$ equals the sum of (14) with the mean square of (17), and thus we see that the mean square vorticity increases with $\lambda_1^2$. The minimum mean square vorticity corresponds to $\lambda_1 = 0$, this being a particular example of the general proposition discussed below.

### 3. Variational principle

Let $\psi$ denote a solution of the non-linear partial differential equation (7) for any assigned $F$ (or any assigned $a_0, a_1, a_2 \ldots$), let $\delta\zeta$ denote any variation of the vorticity from the equilibrium solution, and let $\delta\psi$ be computed from $\nabla^2\delta\psi = \delta\zeta$ with $\delta\psi = 0$ on the unvaried boundary. Since $0 = \delta(\beta y) = \delta a_1 = \delta a_2 \ldots$, and since $\psi = 0$ on the boundary, we have the relation

$$\left.\begin{aligned} \delta \iint dA\,[\tfrac{1}{2}(\nabla\psi)^2 + F(P)] &= \iint dA\,[\nabla\psi\cdot\delta\nabla\psi + F'(P)\delta P] \\ &= \iint dA[\nabla\cdot(\psi\delta\nabla\psi) - \psi\delta\nabla^2\psi + F'(P)\delta\zeta] \\ &= \iint dA[-\psi\delta\zeta + \psi\delta\zeta] = 0 \end{aligned}\right\} \tag{18}$$

when (7) is used. Thus we see that a solution $\psi$ extremizes the first integral in (18). Conversely, if that integral is stationary for an arbitrary $\delta\zeta = \nabla^2\delta\psi$ with $\psi = 0 = \delta\psi$ on the boundary, then $\psi = F'(P)$ is a steady solution of the vorticity equation. Notc that the free streamline condition (6) has not yet been used. This variational principle is a particular case of a result obtained by Blumen (1971) for a barotropic equilibrium in a uniformly stratified fluid.

The variational principle can be more readily interpreted by introducing the expansion (8) in (18), and thus we have

$$\iint dA\,[\tfrac{1}{2}(\nabla\psi)^2 + F(P)] = \tfrac{1}{2}\iint dA\,(\nabla\psi)^2 + \textstyle\sum a_n P_n$$

where

$$P_n = \iint \cdot P^n dA \qquad (n = 0, 1, 2, \ldots)$$

are the moments of the absolute vorticity. If we keep all these $P_n$ constant in the variation then

$$\delta \iint \tfrac{1}{2}(\nabla\psi)^2 dA = 0 \tag{19}$$

and thus we see that the extremals of the kinetic energy determine the steady solutions for the case of a fixed ($\psi = 0$) boundary. The physical significance of the $P_n$ constraints may be indicated by the following (time dependent) adjustment problem. Suppose we have a flow field which is in equilibrium within a cavity whose lateral boundaries are *rigid* but deformable. Let us then change the shape of the boundary adiabatically, so as to produce a new equilibrium state. The conservation of potential vorticity implies that all the $P_n$ have the same value in the final as in the initial state, but the energy is not an adibatic invariant since work is done in changing the shape of the boundaries. Thus we may solve this adjustment problem by using the given initial $P_n$ as the constraints in extremizing the kinetic energy of the final equilibrium state.

Alternatively, we can fix the energy as well as $P_1, P_3, P_4 \ldots$ and extremize $P_2$. With a circular boundary of radius $R$ this can be expressed as

$$\delta \iint (\beta y + \zeta)^2 dA = 0 \tag{20}$$

$$\tfrac{1}{2}\iint (\nabla\psi)^2 dA = \text{constant} \qquad \psi(R,\theta) = 0 \qquad \text{I}$$

$$\iint (\beta y + \zeta)\, dA = \text{constant} \qquad \iint (\beta y + \zeta)^3 dA = \text{constant} \ldots . \qquad \text{II}$$

Now we turn to the case of a *free streamline*, in which the foregoing "constants" must, themselves, be constrained so as to satisfy the dynamic boundary condition (11), or

$$\frac{\partial^2 \psi(R,\theta)}{\partial r \partial \theta} = 0 \qquad \text{III}$$

The two boundary conditions imply that

$$\iint y\zeta dA = \iint dA \left[\nabla \cdot (y \nabla \psi) - \frac{\partial \psi}{\partial y}\right] = \frac{R \partial \psi(R,\theta)}{\partial r} \int_0^{2\pi} y(R,\theta)\, d\theta = 0 \qquad \text{IIIa}$$

because the mean valve of $y(R,\theta)$ vanishes on a circle.

The class of free eddy solutions may be obtained by varying (20) with respect to functions satisfying the constraints I–II, and by then restricting the values of the Lagrange multipliers so that III is satisfied by the extremals. Let

$$P_2{}^* = \min \iint (\beta y + \zeta)^2 dA \tag{21}$$

denote the smallest value of the mean square vorticity in this dynamical class (note that the trivial solution $\psi \equiv 0$ is excluded by the constraint on $(\nabla\psi)^2$ in I). The solution that minimizes (21) will also be the one that minimizes $\iint \zeta^2 dA$, because IIIa implies that $\iint y\zeta dA = 0$ for all solutions.

If we *relax* any of the constraints, or any of the boundary conditions, then the minimum value of $\iint (\beta y + \zeta)^2 dA$ so computed must clearly be less than or equal to $P_2{}^*$. Accordingly, we consider a class of functions $\zeta'$ (and $\nabla^2 \psi' = \zeta'$) with

$$\begin{aligned} \psi'(R,\theta) &= 0 \\ \tfrac{1}{2}\iint (\nabla\psi')^2 dA &= \text{constant} \end{aligned} \tag{22}$$

where the permitted values of the constant are such as to satisfy the weak boundary condition IIIa, or

$$\iint y\zeta' dA = 0. \tag{22a}$$

If we now minimize the mean $(\beta y + \zeta')^2$ in this class, the result will not exceed $P_2{}^*$, or

$$\min \iint (\beta y + \zeta')^2 dA \leqslant P_2{}^*. \tag{23}$$

The equality sign in (23) will apply only if the extremal $\zeta'$ happens to be a dynamical solution (satisfying III).

Let $\lambda'$ denote the Lagrange multiplier in the variational problem (22)–(23), so that for arbitrary $\delta\zeta'$ we have the Euler equation

$$\left.\begin{aligned} \beta y + \zeta' + \lambda'\psi' = 0, \qquad \zeta' = \nabla^2\psi' \\ \psi'(R,\theta) = 0 \end{aligned}\right\} \tag{23a}$$

and the partial integration of (22a) gives

$$\int_0^{2\pi} \sin\theta \frac{\partial\psi'}{\partial r}(R,\theta)\, d\theta = 0. \tag{23b}$$

A solution of (23a) which satisfies (23b) is given by (12) provided $\lambda' = \lambda$. Moreover there is no additional analytic solution of the homogeneous equation $(\nabla^2\psi' + \lambda\psi' = 0)$ having this eigenvalue. Therefore the minimum value of the mean $(\zeta')^2$ is equal to (14), and the extremal solutions $(\psi')$ are, in fact, *dynamical* solutions satisfying the strong boundary condition III. Thus we have shown that the equality sign in (23) applies, or

$$(\overline{\zeta^2})^{1/2} = \frac{\beta R}{\sqrt{2}} \tag{24}$$

is the smallest value of the rms vorticity within a (circular) boundary of given area. These minimal solutions are called *modons*, and the bar indicates an average taken over the modon.

## 4. Statistical aspects and possible oceanic applications

The modon in Fig. 1a is a sketch of the stream function (12) for the smallest root of (13), and the higher roots will correspond to a larger number of vortices within the free streamline boundary. The modon is composed of a low pressure vortex lying to the north of an anti-cyclonic eddy, and the east-west width of either vortex is roughly twice as large as the north-south width. The dipole structure is the simplest manifestation of the fact that each oceanic vortex must interact with the surrounding fluid. Since two or more non-overlapping modons have no dynamical interaction, they may be superimposed to form the equilibrium "modon-sea" shown by the close packed array in Fig. 1b.

The average kinetic energy $^1/_2\overline{(\nabla\psi)^2}$ in each modon bears a simple relation to the mean square vorticity, as can be seen by multiplying (9) with $\zeta = \nabla^2\psi$ and averaging the result over the area of the modon. By using IIIa, (24), and (13) we then obtain

$$\overline{(\nabla\psi)^2} = \frac{\overline{\zeta^2}}{\lambda} = \frac{\beta^2 R^4}{2(5.136)^2}. \tag{25}$$

Also note that $\beta\overline{y^2} = -\lambda\overline{\psi y}$ follows from (9), and therefore $\overline{\psi y} < 0$, as shown in Fig. 1a.

We can compute the rms velocity $\langle \mathbf{v}^2\rangle^{1/2}$ over the entire $x$–$y$ plane by multiplying (25) with the fractional area covered by the modons, this area being distinguished from the "dead-space" ($\mathbf{v} = 0$) lying between three closepacked circles. Accordingly, we connect the centers of three adjacent circles, thereby constructing an equi-lateral triangle whose side equals $2R$ and whose area equals $R^2\sqrt{3}$. This triangle contains 1/6 of the area of three individual modons, each being of area $\pi R^2/6$. Therefore $\pi R^2/2$ is the area of the modons within the equi-lateral triangle, and $(\pi/2\sqrt{3})$ is the fractional area occupied by modons in the entire $x,y$ plane. Thus the rms velocity in the entire $(x,y)$ plane is

$$\langle \mathbf{v}^2\rangle^{1/2} = \frac{\beta R^2}{\sqrt{2}(5.136)}\left(\frac{\pi/2}{\sqrt{3}}\right)^{1/2} = \frac{\beta R^2}{7.6}. \tag{26}$$

The observations of quasi-geostrophic eddies in mid-ocean suggest an rms velocity of $\langle \mathbf{v}^2\rangle^{1/2} \simeq 10$ cm/sec, and reference is made to Koshlyakov and Grachev (1973) who analyzed five months of velocity measurements in a closely spaced network and obtained a picture of a single anticyclone drifting slowly thru the network. A "smaller dimension" of 90 Km was cited for the elliptical vortex, and if this ocean eddy can be associated with half of a modon then I estimate the corresponding modon radius to be $R = 3(90) = 270$ Km. The latter figure will now be compared to a theoretical estimate of $R$ obtained from (26) by using the observed $\langle \mathbf{v}^2\rangle^{1/2} \simeq 10$ cm/sec and $\beta = 2.2(10^{-13})$ cm$^{-1}$ sec$^{-1}$. Thus we compute $R = 190$ Km, in acceptable agreement with the observed dimension. In contrast with our model, however, the oceanic anti-cyclone was time dependent, baroclinic, and its axis was obliquely inclined to the meridian. Therefore we must now consider the rationale for applying our model to the real ocean.

The primary importance of the oceanic eddies undoubtably lies in their average transport and dissipation properties. Therefore our original intent was to formulate a first approximation to the statistically steady state, such as would apply to independent "snapshots" of the entire ocean taken at large time intervals, and such as would describe the relationship between average eddy amplitude and dimension. The ensemble which we allude to needs to be distinguished from the synoptic realization and its detailed evolution over a relatively short time interval like "five months". On the other hand, we only have a *small* number of synoptic observations, and therefore some speculation on the connection of our model with observables is in order.

Reference will also be made to Rhines' (1973, 1974) numerical integration of the barotropic vorticity equation, wherein the arbitrarily specified initial state is composed of relatively small horizontal wavelengths. Consequently the initial vorticity balance is between the $\mathbf{v}\cdot\nabla\zeta$ and $\partial\zeta/\partial t$ terms. But Rhines observes the well known "infra-red cascade" as time increases, whereby energy is systematically transferred to longer wavelengths. Thus the $\beta$ term eventually becomes important in the vorticity equation, and the associated planetary restoring force inhibits the energy cascade. A kind of equilibrium spectrum then appears, with $(v\beta, \mathbf{v}\cdot\nabla\zeta, \partial\zeta/\partial t)$ all having (roughly) equal magnitudes, and with the spectral amplitude being consistent with (26). Our starting point, on the other hand, involves a detailed balance between the $\mathbf{v}\cdot\nabla\zeta$ and $\beta v$ terms. But such a realization (Fig. 1b) is undoubtedly dynamically unstable because $F''(P) = -1/\lambda < 0$ (cf. eq. (9)), and the Rayleigh-Fjortoft condition for stability is *not* satisfied (Blumen (1968)). Thus an infinitesimal perturbation introduced into Fig. 1b will probably lead to finite values of $\partial\zeta/\partial t$, and the subsequent evolution of the modons will probably resemble Rhines' results. Note that the modon area and the total energy are temporal invariants of our model. The mean square vorticity will also be an invariant because of the statistical homogeneity (the statistical average of any Reynolds stress must be non-divergent). These invariants will provide the parameters upon which the temporal statistics of the modon sea depends.

## 5. Variational principle for a baroclinic eddy

The purpose of this section is to show how the model can be extended so as to incorporate some baroclinic effects. The simplest case is a two-layer fluid, with the deep lower layer being at rest, and with $g$ denoting the reduced value of gravity (based on the density difference between the two fluids). This system is dynamically equivalent to that of a single fluid having a free surface at a height $h(x,y)$ above a level rigid bottom surface. Accordingly, we now examine the steady solutions in the latter system, for which the continuity equation becomes $\nabla\cdot(\mathbf{v}h) = 0$. If $\psi\ (x,y)$ is now used to denote the mass transport function, and $\zeta$ the vertical component of relative vorticity, then we have

$$\left.\begin{aligned} \mathbf{v}h &= \mathbf{k}\times\nabla\psi \\ \zeta &= \mathbf{k}\cdot\nabla\times\mathbf{v} = \nabla\cdot h^{-1}\nabla\psi \end{aligned}\right\} \quad (27)$$

For steady motion we know that the Bernoulli function

$$B = gh + \mathbf{v}^2/2 \quad (28)$$

is conserved by each fluid column, and also the steady horizontal momentum equation can be written as

$$(f(y)+\zeta)\mathbf{k}\times\mathbf{v} = -\nabla B. \tag{29}$$

The use of (27) then gives

$$\nabla\psi = \frac{\nabla B}{P} \tag{30}$$

where

$$P = \frac{f+\zeta}{h} \tag{31}$$

now denotes the *potential* vorticity. For steady motion the conservation of potential vorticity gives

$$h\mathbf{v}\cdot\nabla P = 0 \tag{32}$$

and consequently $\psi$ must equal some explicit function of $P$, or

$$\psi = F'(P). \tag{33}$$

We readily verify that the solutions of (33) satisfy (32) because $h\mathbf{v} = \mathbf{k}\times\nabla\psi = \psi''(P)\mathbf{k}\times\nabla P$ is orthogonal to $\nabla P$. Since (28) is also conserved, the Bernoulli function must also equal some explicit function $B(P)$ of the potential vorticity. The substitution of these two functional relations in (30) then gives

$$PF''(P) = B'(P)$$

or

$$PF'(P) - F(P) - B(P) = \text{constant} \tag{34}$$

Consider a steady flow with $\psi = 0$ on the boundary, let $\delta\zeta$ be an arbitrary variation in vorticity, let $\delta h$ be an arbitrary variation in the layer thickness, and let $\delta\psi$ be computed from (27), or

$$\nabla\cdot h^{-1}\nabla\delta\psi = \delta\zeta + \nabla\cdot\left(\frac{\delta h}{h^2}\right)\nabla\psi$$

with $\delta\psi = 0$ on the fixed boundary. The three integrals listed below then have the following variations:

$$\left.\begin{aligned}
\delta\iint hF(P)dA &= \iint dA\delta h[F - PF'] + \iint dA(\delta\zeta)F' \\
\delta\iint \tfrac{1}{2}h\mathbf{v}^2 dA &= \iint dA[h^{-1}\nabla\psi\cdot\delta\nabla\psi - \tfrac{1}{2}h^{-2}(\nabla\psi)^2\delta h] \\
&= \iint dA[\nabla\psi\cdot\delta(h^{-1}\nabla\psi) + \tfrac{1}{2}h^{-2}(\nabla\psi)^2\delta h] \\
&= -\iint dA\psi\delta\zeta + \tfrac{1}{2}\iint dA\mathbf{v}^2\delta h \\
\delta\iint \tfrac{1}{2}gh^2 dA &= \iint dA gh\delta h
\end{aligned}\right\} \tag{35}$$

where the boundary condition $\psi = 0$ has been used in obtaining the last line in (35). The sum of these three integral relations is

$$\delta\iint dA[hF(P)+\tfrac{1}{2}h\mathbf{v}^2+\tfrac{1}{2}gh^2] = \iint dA(F'-\psi)\delta\zeta + \iint dA[F-PF'+B]\delta h. \quad (36)$$

Equation (34) implies that $F-PF'+B$ is independent of $(x,y)$ and therefore if the volume of the eddy be held constant in the variation, or

$$\iint hdA = \text{constant} \quad (37)$$

then the last term in (36) vanishes, while the preceding term also vanishes according to (33). Therefore any steady solution yields the extremum

$$\delta\iint dA[hF(P)+\tfrac{1}{2}h\mathbf{v}^2+\tfrac{1}{2}gh^2] = 0 \quad (38)$$

and the converse is readily established.

By expanding $F(P)$ in a power series, and by introducing the constraints

$$\iint\left(\frac{f+\zeta}{h}\right)^n hdA = P_n = \text{constants} \quad (39)$$

we then conclude that

$$\delta\iint(\tfrac{1}{2}h\mathbf{v}^2+\tfrac{1}{2}gh^2)dA = 0 \quad (40)$$

or the sum of the kinetic and potential energy is an extremum. The significance of the constraints (39), as mentioned in Section 3, is that they are adiabatic invariants for a time dependent adjustment problem.

## 6. Conclusion

We have obtained variational formulations for the steady motion of a fluid on the $\beta$-plane, in the belief that the minimal states will be useful for the formulation of a statistical model of mid-ocean eddies. In the simplest case (Sec 2) we showed that no steady state solution is possible unless the rms vorticity exceeds a lower bound. The mean energy density of the modon sea is related to the mean radius by a relation which is in reasonable agreement with observations, and a possible connection with Rhines (1973) numerical calculation was suggested. The generalized variational formulation given in Section 5 suggests the feasibility of incorporating baroclinic effects.

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