A two-layer model for the separation of inertial boundary currents[1]

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1. Introduction

The problem of the steady circulation in an inviscid, two-layer ocean is solved in the limit of small mid-oceanic Rossby number. The model is not new. Certain aspects of the problem have been treated by Fofonoff (1954 and 1962), Deacon *et al* (1955), Charney (1955), Morgan (1956), Stommel (1958), Robinson (1963), Blandford (1965), Jacobs (1968), and Schmitz (1969). Their efforts to explain the relevance of the model to the real ocean are considered more than adequate, and no such effort will be made here. At various stages of the analysis, the connection with some of these previous works will be pointed out. The main thrust of this paper is the mathematical treatment of the separation of the western boundary current from the coast and its subsequent steady meandering across the ocean basin. The solution to the circulation problem is obtained by the method of matched asymptotic expansions. In this model, the flow is confined to the upper layer, in which the potential vorticity is everywhere positive.

2. Equations of the model

The equations of motion and continuity in non-dimensional form are

$$\varepsilon(UU_x+VU_y)-fV+D_x=0, \tag{2.1}$$

$$\varepsilon(UV_x+VV_y)+fU+D_y=0, \tag{2.2}$$

and

$$(UD)_x+(VD)_y=0, \tag{2.3}$$

(cf. equations 1-3 of Charney, 1955).

The dimensionless velocity component in the eastward (x) direction is U and in the northward (y) direction is V, and the dimensionless depth of the

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upper layer is D. The dimensionless Coriolis parameter is given by $f = 1 + y$. Horizontal distances have been non-dimensionalized with respect to $R \tan \theta_0$, where R is the earth's radius and θ_0 is the latitude where $y = 0$. The depth of the upper layer is non-dimensionalized with respect to H_0, the nominal depth of the warm layer. The velocity scale is $\dfrac{\Delta\varrho g H_0}{\varrho_0 f_0 R \tan \theta_0}$, where ϱ_0 is the density of the lower (motionless) layer, $\Delta\varrho$ is the density difference between the two layers, g is the acceleration due to gravity, and f_0 is the value of the Coriolis parameter at $y = 0$. The Rossby number is given by $\varepsilon = \dfrac{\Delta\varrho g H_0}{\varrho_0 f_0^2 R^2 \tan^2 \theta_0}$, which is also the square of the ratio of the internal Rossby radius of deformation to the length scale of the basin.

The boundary conditions are that $U = 0$ on $x = 0$ and $x = x_0$ and $V = 0$ on $y = 0$ and $y = y_0$, provided that $D > 0$ on these boundaries. Furthermore, there is a natural boundary condition on any line along which $D = 0$, namely that there is no transport across such a line. The above equations and boundary conditions do not constitute a well-set problem without some specification of the vorticity distribution in the fluid. Note that all non-trivial solutions to the above equations with the specified boundary conditions will consist of a closed gyre or gyres.

It is convenient to introduce the transport stream function ψ defined by

$$UD = -\psi_y, \tag{2.4}$$

and

$$VD = \psi_x, \tag{2.5}$$

(cf. equation 6 of Charney, 1955).

There are two exact first integrals of equations (2.1) – (2.3) which may be obtained very simply in the following manner. Equations (2.1) and (2.2) may be rewritten in the form

$$\frac{\partial}{\partial x}\left\{\frac{\varepsilon}{2}(U^2 + V^2) + D\right\} - VD\left\{\frac{\varepsilon(V_x - U_y) + f}{D}\right\} = 0, \tag{2.6}$$

and

$$\frac{\partial}{\partial y}\left\{\frac{\varepsilon}{2}(U^2 + V^2) + D\right\} + UD\left\{\frac{\varepsilon(V_x - U_y) + f}{D}\right\} = 0. \tag{2.7}$$

It is then convenient to define the Bernoulli energy

$$B = \frac{\varepsilon}{2}(U^2 + V^2) + D,$$

and the potential vorticity

$$P = \frac{\varepsilon(V_x - U_y) + f}{D}.$$

Using these definitions and the definition of the transport stream function allows equations (2.6) and (2.7) to be written as

$$B_x - \psi_x P = 0, \tag{2.8}$$

and

$$B_y - \psi_y P = 0. \tag{2.9}$$

Multiplying equation (2.8) by $-\psi_y$ and equation (2.9) by ψ_x and adding gives

$$\psi_x B_y - \psi_y B_x = J(\psi, B) = 0, \tag{2.10}$$

which implies that $B = B(\psi)$. It then follows from either equation (2.8) or (2.9) that $P = P(\psi) = \frac{dB}{d\psi}$ (cf. Fofonoff (1962), equations 110–114). Therefore the Bernoulli function and potential vorticity are conserved on each streamline.

In the model explored here, the simplest possible choice for $B(\psi)$ is made, namely

$$B = P_0\psi + \frac{f_c}{2P_0},$$

where P_0 is a constant and f_c is the value of the dimensionless Coriolis parameter at some latitude $y = y_c$. The potential vorticity therefore is constant everywhere and equal to P_0. Such a simple choice may seem to be a severe restriction on the class of available solutions. There seems to be no reason to choose a more complicated function $B(\psi)$ (see Section 11). In fact, the analysis of even this simple case is moderately difficult. The essential features of the circulation discussed here can be easily extended to a wide class of function $B(\psi)$ satisfying certain restrictions. These restrictions will be stated during the analysis of the simplest case, and the relation of this work to previous efforts involving more complicated choices for $B(\psi)$ will be discussed. Furthermore, no generality is lost by choosing $P_0 = 1$, for the dependence on P_0 can be removed by scaling U, V and D by $\left(\frac{1}{P_0}\right)$, ψ by $\left(\frac{1}{P_0^2}\right)$ and ε by P_0.

The equations for the conservation of Bernoulli energy and potential vorticity now become

$$\frac{\varepsilon}{2}(U^2 + V^2) + D = \psi + \frac{f_c}{2}, \tag{2.11}$$

and

$$\varepsilon(V_x - U_y) + f = D. \tag{2.12}$$

For typical oceanic values of the parameters, the Rossby number ε is on the order of 10^{-4}. In the following analysis, we develop the asymptotic solution for the circulation pattern in the limit of small ε.

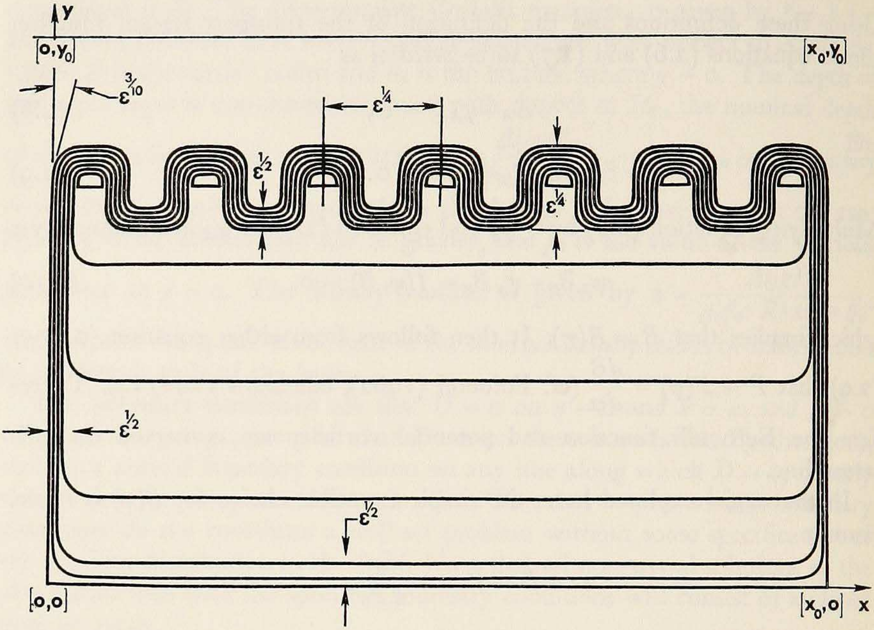

Figure 1. Schematic representation of the circulation pattern. In the case illustrated, the flow in the boundary layer along $y = 0$ is to the west (see Section 10).

3. Synopsis of the analysis

The analysis we present involves the derivation and solution of various asymptotic approximations to the equations of Section 2. These approximations are valid in the limit $\varepsilon \to 0$. Many different regions are distinguished, each of which is characterized by a specific ε-dependence for both the dependent and independent variables. The purpose of this section is to describe how the various regions fit together, and to present an overview of the circulation pattern.

In the interior of the basin, the flow is a slow westward drift, and the depth of the moving layer increases linearly with latitude. This flow encounters the meridional boundary and is deflected to the north as an intense, narrow inertial boundary current. The width of the current is $O(\varepsilon^{1/2})$ and the northward velocity is $O(\varepsilon^{-1/2})$. The depth of the moving layer on the western coast decreases to the north, and the interface surfaces near an apparent separation latitude $f = f_c$. At this latitude, the jet leaves this coast but remains coherent with cross-stream width of $O(\varepsilon^{1/2})$. It leaves the coast going nearly parallel to it, with an initial deflection through an angle which is $O(\varepsilon^{3/10})$. The free jet meanders to the east, while oscillating about the latitude $f = f_c$ with an $O(\varepsilon^{1/4})$ amplitude and $O(\varepsilon^{1/4})$ downstream wavelength. We present solutions that are symmetric about the mid-longitude of the basin. The overall circulation pattern is presented schematically in Figure 1.

The major part of the analysis is concerned with the details of the transition from a western boundary current to a free inertial meander, the so-called separation process. In this model, separation occurs when the depth of the moving layer right at the boundary goes to zero, and the $\psi = 0$ streamline leaves the coast. Much of the analysis is concerned with determining the behavior of the layer depth at the coast as a function of latitude. This analysis begins by noting that near $f = f_c$, the solution for the flow in the western boundary current is singular. In this solution, the layer depth at the coast goes to zero like $\sqrt{f_c - f}$, so that D_y is singular. The U component of flow becomes large, and the scaling arguments used to derive the approximate solutions for the western boundary current are not valid near $f = f_c$.

The appropriate rescaling there is then derived. The relevant length scale in the y direction near $f = f_c$ is $O(\varepsilon^{2/5})$, and the corresponding velocity U is $O(\varepsilon^{-1/5})$. This choice makes the dimensional U_{yy} comparable to β $(= f_0 R^{-1} \cot \theta_0)$. The northward velocity component V remains $O(\varepsilon^{-1/2})$. In this region, the current is deflected uniformly through an angle which is $O(\varepsilon^{3/10})$, except in a very narrow region $(O(\varepsilon^{7/10}))$ near the coast. In this narrow region, the depth D of the moving layer is small $(O(\varepsilon^{1/5}))$. Again, the layer depth at the coast goes to zero, but this time with a finite slope (D_y is no longer singular). However, a detailed examination of the validity of these solutions reveals that the scaling approximations used to derive them again become invalid before the interface actually surfaces.

To describe the mechanism for the actual separation, a further rescaling is found to be necessary. The relevant new x and y scales are $O(\varepsilon^{11/10})$ and $O(\varepsilon^{4/5})$ respectively. The scales of U and V are $O(\varepsilon^{-1/5})$ and $O(\varepsilon^{-1/2})$ as before, but D is $O(\varepsilon^{3/5})$. The equations for the flow in this region are exactly analogous to the non-linear shallow water equations for one-dimensional time dependent flow over a uniformly sloping beach, with a seawall at $x = 0$. The physical mechanism for separation is completely revealed by this analogy.

Figure 2 is a detailed schematic representation of the various regions just described, in which the transition from western boundary current to free inertial jet is accomplished.

4. The geostrophic interior

In the interior region of the ocean away from longitudinal boundaries, the non-dimensional equations are assumed to be correctly scaled as written. All fields are taken as $O(1)$ quantities and expanded in power series in ε. The interior solution will be denoted by a subscript I. It is

$$V_I = O(\varepsilon), \tag{4.1}$$

$$D_I = f + O(\varepsilon), \tag{4.2}$$

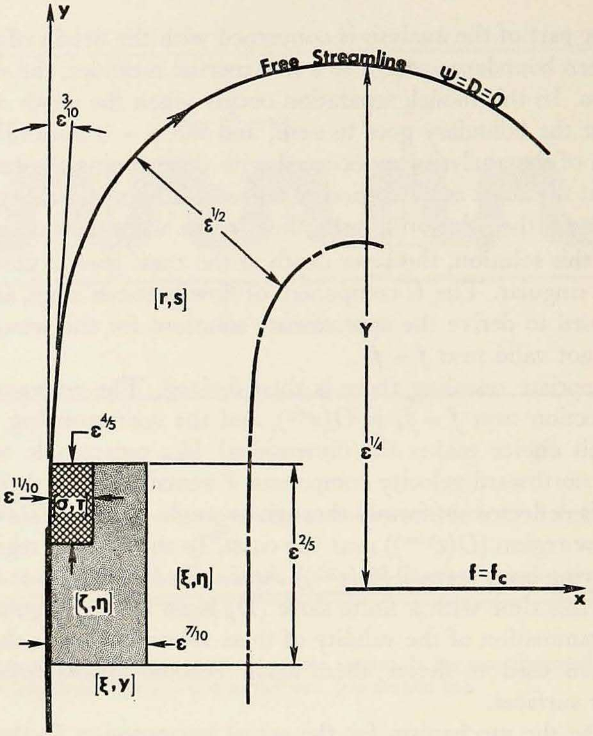

Figure 2. Schematic representation of the various regions involved in the transition from the western boundary current to the free inertial meander. The definitions of the various independent variables for the different regions are found in the text.

$$\psi_1 = \left[f - \frac{f_c}{2}\right] + O(\varepsilon), \tag{4.3}$$

and

$$U_1 = \frac{-1}{f} + O(\varepsilon). \tag{4.4}$$

Since $f = 1 + y$, the interior depth is linear in y, and so is the transport stream function. The zonal velocity, U_1, is decreasing to the north like $1/f$, so the interior transport per unit width is constant and to the west.

It is clear from equation (4.3) that $\psi_1 = 0$ on $y = 0$ only if $f_c \equiv 2$. It will be assumed not that $f_c \equiv 2$, but that $f_c = 2 + o(1)$. Just how big this $o(1)$ correction to f_c is, and the necessity for it, will become clear in Section 10. The important point to note is that regardless of the value of f_c, the geostrophic interior solution does not satisfy the boundary condition that $U = 0$ on $x = 0$ and $x = x_0$. The assumption that the original scaling of the equations is correct must be modified near the meridional boundaries.

5. The western boundary current

The slow interior flow encounters the boundary at $x = 0$ and is deflected to form a current along the western boundary. Clearly, the non-dimensional length and velocity scales associated with this deflection are not $O(1)$. This was recognized by Charney (1955). He showed that the natural width for this boundary current is the radius of deformation, and also determined the corresponding velocity scale. In the notation of this paper, the horizontal scale of this current is $\varepsilon^{1/2}$. The x-coordinate is rescaled by

$$x = \varepsilon^{1/2}\xi,$$

and the northward velocity V by

$$V = \varepsilon^{-1/2} v_2.$$

In this boundary layer region, all fields are denoted by a subscript 2.

Assuming that all other scales are unchanged, the equations are

$$v_{2\xi} - \varepsilon U_{2y} + f = D_2, \tag{5.1}$$

$$\varepsilon(U_2 U_{2\xi} + v_2 U_{2y}) - f v_2 + D_{2\xi} = 0, \tag{5.2}$$

$$\frac{\varepsilon}{2} U_2^2 + \frac{1}{2} v_2^2 + D_2 = \psi_2 + \frac{f_c}{2}, \tag{5.3}$$

$$U_2 D_2 = -\psi_{2y}, \tag{5.4}$$

and

$$v_2 D_2 = \psi_{2\xi}. \tag{5.5}$$

The boundary conditions for the solution of equations (5.1)–(5.5) are that $U_2 = \psi_2 = 0$ on $\xi = 0$, and $v_2 \to 0$, $\psi_2 \to \psi_1$, and $U_2 \to U_1$ as $\xi \to \infty$.

All dependent variables are expanded in power series in ε, e.g., $v_2(\xi,y) = v_{20}(\xi,y) + \varepsilon v_{21}(\xi,y) + \varepsilon^2 v_{22}(\xi,y) + O(\varepsilon^3)$. To leading order equations, (5.1)–(5.5) become

$$v_{20\xi} + f = D_{20}, \tag{5.6}$$

$$-f v_{20} + D_{20\xi} = 0, \tag{5.7}$$

$$\frac{1}{2} v_{20}^2 + D_{20} = \psi_{20} + \frac{f_c}{2}, \tag{5.8}$$

$$U_{20} D_{20} = -\psi_{20y}, \tag{5.9}$$

and

$$v_{20} D_{20} = \psi_{20\xi}. \tag{5.10}$$

Is is seen that the boundary current described by this system is an intense flow to the north, which is in geostrophic balance in the cross-stream direction.

In the current, the relative vorticity is comparable to the planetary vorticity, and the inertial acceleration terms in the north-south momentum equation (not written here) are comparable with the Coriolis force and pressure gradient. Therefore, this kind of boundary layer is called "inertial".

The explicit solution is obtained by differentiating equation (5.6) with respect to ξ and substituting into equation (5.7). There results an equation for v_{20}, which is

$$v_{20\xi\xi} = f v_{20}. \tag{5.11}$$

The solution for v_{20} which satisfies the boundary conditions as $\xi \rightarrow \infty$ is

$$v_{20} = A_2(y) e^{-\sqrt{f}\xi}, \tag{5.12}$$

where $A_2(y)$ is to be determined from the conditions at $\xi = 0$.

From equation (5.6) it follows that

$$D_{20} = f - \sqrt{f} A_2(y) e^{-\sqrt{f}\xi}. \tag{5.13}$$

Evaluating v_{20} and D_{20} on $\xi = 0$ and substituting into the Bernoulli equation (5.8) with $\psi_{20} = 0$ gives a quadratic equation for A_2. The root of this equation which corresponds to $D_{20} > 0$ on $\xi = 0$ is

$$A_2 = \sqrt{f} - \sqrt{f_c - f}. \tag{5.14}$$

The problem just presented was first discussed by Stommel (Deacon, *et al.*, 1955), who chose $A_2 = \sqrt{f}$, which corresponds to $D_{20}(\xi = 0, y) = 0$ for all y. Robinson (1963) noted that Stommel's solution was associated with an unbounded U_{20} at $\xi = 0$ in such a way that $\xi = 0$ was not a streamline. Robinson gave the correct solution corresponding to the boundary condition $\psi_{20} = 0$ on $\xi = 0$.

It can now be seen that the depth D_{20} on the wall $\xi = 0$ is given by

$$D_{20}(0, y) = \sqrt{f(f_c - f)}, \tag{5.15}$$

and the moving layer apparently surfaces as $f \rightarrow f_c$. We now calculate both ψ_{20} and U_{20}, in order to determine the actual magnitude of the terms omitted in deriving equations (5.6) – (5.10) from (5.1) – (5.5). From this we will show that the actual surfacing of the warm layer cannot be accurately described by equations (5.6) – (5.10). We will determine the range of y for which these equations are valid. The ψ_{20} field may be obtained directly from equation (5.8), or just as easily by multiplying equation (5.7) by D_{20} and integrating in ξ to find that the quantity $\left(f\psi_{20} - \frac{D_{20}^2}{2}\right)$ is independent of ξ. Therefore, by matching to the interior as $\xi \rightarrow \infty$, ψ_{20} is found to be

$$\psi_{20} = \frac{1}{2f}[D_{20}^2 - f(f_c - f)]. \tag{5.16}$$

The east-west velocity U_{20} is now computed by substituting ψ_{20} and D_{20} into equation (5.9). The result may be written in the form

$$\left.\begin{aligned} U_{20} = \frac{2f-f_c}{2D_{20}\sqrt{f(f_c-f)}}\,Z(1-Z) \\ -\frac{1}{D_{20}}\left\{\sqrt{f(f_c-f)}\frac{\partial}{\partial y}(Z-Z^2)+\frac{1}{2}\frac{\partial}{\partial y}[f(1-Z)^2+(f_c-f)(Z^2-1)]\right\}, \end{aligned}\right\} \tag{5.17}$$

where $Z = e^{-\sqrt{f}\xi}$.

The terms on the second line are well behaved, for all values of ξ and $f \le f_c$, but the term on the first line becomes infinite as $f \to f_c$ for all $\xi > 0$. Thus, it is clear that the assumption that U remains $O(1)$ breaks down as $f \to f_c$, and the solutions obtained here become invalid as the flow approaches this latitude. The latitude $f = f_c$ will be referred to as the apparent separation latitude, since D apparently vanishes on $\xi = 0$ as $f \to f_c$. The breakdown of the scaling assumptions of (5.1)–(5.5) near this latitude will now be discussed in detail.

6. The validity of the boundary current approximation near the apparent separation latitude

Let $y' = f_c - f$ measure distance to the south of the apparent separation latitude. Then for $\xi = O(1)$ and y' small, the v_{20}, D_{20}, and U_{20} fields from the previous section may be written as

$$v_{20} = (\sqrt{f_c} - \sqrt{y'})e^{-\sqrt{f_c}\xi} + O(y'), \tag{6.1}$$

$$D_{20} = f_c(1 - e^{-\sqrt{f_c}\xi}) + \sqrt{f_c y'}\, e^{-\sqrt{f_c}\xi} + O(y'), \tag{6.2}$$

and

$$U_{20} = \frac{e^{-\sqrt{f_c}\xi}}{2\sqrt{f_c y'}} + O(1). \tag{6.3}$$

Note that the largest term in U_{20} is simply obtained from $f_c U_{20} = -D_{20_y}$, which means that it is determined geostrophically; this is because $O(y'^{-1/2})$ terms in $U_{20}v_{20_\xi} + v_{20}v_{20_y}$ cancel exactly.

We now compare the $-\varepsilon U_y$ term, which was dropped from equation (5.1) to obtain equation (5.6), with typical retained terms. Comparing

$$-\varepsilon U_{20_y} = \varepsilon U_{20_{y'}} = O(\varepsilon y'^{-3/2})$$

to a typical $O(1)$ term (e.g., $f \simeq f_c$), we see that if y' is $O(\varepsilon^{2/3})$, the neglected term is $O(1)$. If y' is $O(\varepsilon^{2/3})$, U_{20} is $O(\varepsilon^{-1/3})$, which suggests rescaling y' and U

according to these scales. However, when this is done, we find by integrating the rescaled equations that the ψ, V, and D fields are independent of latitude to leading order. Then we conclude from the rescaled form of equation (5.1) that the $O(\varepsilon^{-1/3})$ piece of the U field is at most linear in latitude, and will not match the U_{20} field given by equation (6.3). Therefore, this scaling doesn't work, in that the solution to the resulting equations will not match the upcoming fields.

We note in passing that a U scale of $O(\varepsilon^{-1/2})$, which would make the $\varepsilon U^2/2$ term in equation (5.3) an $O(1)$ term, would imply (from equation (6.3)) a y' scale of $O(\varepsilon)$. In that case, the leading term in the rescaled form of equation (5.1) would imply U is again independent of latitude. Therefore, this scaling does not work either. In fact, there is no rescaling consistent with equation (6.3) for which terms multipled by ε in equations (5.1)–(5.3) are comparable to the $O(1)$ terms in those equations.

The next possibility is that εU_y, is comparable to an $O(y'^{1/2})$ term (e.g., the $\sqrt{f_c y'}\, e^{-\sqrt{f_c}\xi}$ in the term v_{20_ξ}). Again from equation (6.3) this implies that $y' = O(\varepsilon^{1/2})$ and $U = O(\varepsilon^{-1/4})$. This rescaling was done and the resulting equations were solved to two orders in $\varepsilon^{1/4}$. The leading piece of the U field is found in the second order calculation, but will not match the form of equation (6.3).

We have now presented, without demonstrating the details, three possible scalings that do not work. The algebra is straight-forward but laborious, and serves no useful purpose by itself. We simply wished to demonstrate how we arrived at the rescaling which produces the first cogent set of equations for the flow near the separation latitude. We proceed to the correct rescaling by making $\varepsilon U_{y'}$, in equation (5.1) comparable to y' (e.g., the y' term in $f = f_c - y'$). In dimensional terms, this means that the first breakdown of the solution in the western boundary current happens when U_{yy} is comparable to β. From equation (6.3), we see that $\varepsilon U_{y'}$, comparable to y' means that $y' = O(\varepsilon^{2/5})$ and $U = O(\varepsilon^{-1/5})$.

7. The first correction to the western boundary current

a. *The Correction in the Region $\xi = O(1)$.*

Let

$$f_c - f = \varepsilon^{2/5}\eta$$

and

$$U = \varepsilon^{-1/5}u_3$$

define the rescaled variable measuring distance south from the apparent separation latitude and the rescaled eastward velocity component. The dependent variables in the region for ξ and η both $O(1)$ are denoted by the subscript 3. With these rescalings, equations (5.1)–(5.4) become

$$v_{3\xi}+\varepsilon^{2/5}u_{3\eta}+f_c-\varepsilon^{2/5}\eta = D_3, \tag{7.1}$$

$$\varepsilon^{3/5}u_3u_{3\xi}-\varepsilon^{2/5}v_3u_{3\eta}-(f_c-\varepsilon^{2/5}\eta)v_3+D_{3\xi}=0, \tag{7.2}$$

$$\varepsilon^{3/5}\frac{u_3^2}{2}+\frac{v_3^2}{2}+D_3=\psi_3+\frac{f_c}{2}, \tag{7.3}$$

and

$$\varepsilon^{1/5}u_3D_3=\psi_{3\eta}. \tag{7.4}$$

The variables u_3, v_3, D_3, and ψ_3 are all expanded in powers of $\varepsilon^{1/5}$, e.g.,

$$D_3(\xi,\eta) = D_{30}(\xi,\eta)+\varepsilon^{1/5}D_{31}(\xi,\eta)+\varepsilon^{2/5}D_{32}(\xi,\eta)+O(\varepsilon^{3/5}).$$

The zeroth order equations are

$$v_{30\xi}+f_c = D_{30}, \tag{7.5}$$

$$-f_cv_{30}+D_{30\xi}=0, \tag{7.6}$$

$$\frac{v_{30}^2}{2}+D_{30}=\psi_{30}+\frac{f_c}{2}, \tag{7.7}$$

and

$$0=\psi_{30\eta}. \tag{7.8}$$

The boundary conditions on the fields in region 3 are obtained by matching the fields to the upcoming solution from region 2 and the interior solution (region 1), as well as making $\psi_3=0$ on $\xi=0$. To zeroth order, this means $\lim_{\eta\to\infty}\psi_{30}=\lim_{y'\to 0}\psi_{20}$, etc. But equation (7.8) shows that ψ_{30} is independent of η, and therefore D_{30} and v_{30} are also η independent. The solutions are

$$v_{30}=\sqrt{f_c}\,Z_c, \tag{7.9}$$

$$D_{30}=f_c(1-Z_c), \tag{7.10}$$

and

$$\psi_{30}=\frac{f_c}{2}(1-Z_c)^2, \tag{7.11}$$

where

$$Z_c=e^{-\sqrt{f_c}\xi}.$$

The equations for the first order problem are

$$v_{31\xi}=D_{31}, \tag{7.12}$$

$$-f_cv_{31}+D_{31\xi}=0, \tag{7.13}$$

$$v_{30}v_{31}+D_{31}=\psi_{31}, \tag{7.14}$$

and

$$u_{30}D_{30} = \psi_{31_\eta}. \tag{7.15}$$

From equations (7.12) and (7.13) we find that v_{31} satisfies

$$v_{31_{\xi\xi}} = f_c v_{31}. \tag{7.16}$$

Therefore

$$v_{31} = -A_3(\eta)Z_c, \tag{7.17}$$

$$D_{31} = \sqrt{f_c}A_3(\eta)Z_c, \tag{7.18}$$

and

$$\psi_{31} = \sqrt{f_c}A_3(\eta)Z_c(1 - Z_c), \tag{7.19}$$

where $A_3(\eta)$ is a function to be determined. Note that, since $Z_c = 1$ on $\xi = 0$, ψ_{31} satisfies the boundary condition $\psi_{31} = 0$ on $\xi = 0$ for any choice of $A_3(\eta)$. This is because ψ_{31} is determined from equation (7.14) directly, without using any boundary condition on $\xi = 0$.

From equation (7.15) we find

$$u_{30} = \frac{A_{3\eta}}{\sqrt{f_c}} Z_c. \tag{7.20}$$

For $\xi = O(1)$, this u_{30} field will match the U_{20} field given by equation (6.3) if

$$A_{3_\eta} \sim \frac{1}{2\sqrt{\eta}} \text{ as } \eta \to \infty.$$

We note that the above solutions are only valid for $\xi = O(1)$, and are not valid as $\xi \to 0$. In the limit $\xi \to 0$, we can write

$$\left.\begin{aligned} D_3(\xi,\eta) &= D_{30}(\xi,\eta) + \varepsilon^{1/5}D_{31}(\xi,\eta) + O(\varepsilon^{2/5}) \\ &= f_c^{3/2}\xi + O(\xi^2) + \varepsilon^{1/5}\sqrt{f_c}A_3(\eta)(1 - \sqrt{f_c}\xi) + O(\xi^2) + O(\varepsilon^{2/5}). \end{aligned}\right\} \tag{7.21}$$

Thus when $\xi = O(\varepsilon^{1/5})$ the D_{30} and $\varepsilon^{1/5}D_{31}$ terms are both $O(\varepsilon^{1/5})$. Therefore, the assumption that $D_3 = O(1)$ is not valid for $\xi = O(\varepsilon^{1/5})$ or smaller, and another rescaling is necessary.

However, before we do this rescaling, we compute the solutions in region 3 to one more order. These fields will be needed to find the zeroth order solution for all the fields which is uniformly valid in ξ (η fixed), and to determine $A_3(\eta)$.

The second order equations are

$$v_{32_\xi} + u_{30_\eta} - \eta = D_{32}, \tag{7.22}$$

$$-f_c v_{32} + D_{32_\xi} = v_{30}(u_{30_\eta} - \eta), \tag{7.23}$$

and

$$v_{30}v_{32}+\frac{v_{31}^2}{2}+D_{32}=\psi_{32}. \tag{7.24}$$

Note from equation (7.18) and (7.20) that u_{30} is in geostrophic balance. Also, from equations (7.6) and (7.13), we find v_{30} and v_{31} are in geostrophic balance. Therefore, as the flow approaches the apparent separation latitude, it is geostrophic to leading order in both directions. The first quasi-geostrophic effects are due to the $u_{30_\eta}-\eta$ terms in equations (7.22) and (7.23).

From equations (7.22) and (7.23) we find

$$v_{32_{\xi\xi}}-f_c v_{32}=v_{30}(u_{30_\eta}-\eta)-u_{30_{\xi\eta}}. \tag{7.25}$$

The solution of equation (7.25) is

$$v_{32}=C_3(\eta)Z_c+\frac{A_{3\eta\eta}}{3f_c}Z_c^2+\frac{1}{2}\left(\eta-\frac{A_{3\eta\eta}}{\sqrt{f_c}}\right)\xi Z_c, \tag{7.26}$$

where $C_3(\eta)$ is a function to be determined. Then we find from equation (7.22)

$$\left.\begin{aligned} D_{32}=-\eta-\sqrt{f_c}C_3(\eta)Z_c+\left[\frac{A_{3\eta\eta}}{2\sqrt{f_c}}+\frac{\eta}{2}-\frac{\xi}{2}(\sqrt{f_c}\eta-A_{3\eta\eta})\right]Z_c \\ -\frac{2}{3\sqrt{f_c}}A_{3\eta\eta}Z_c^2, \end{aligned}\right\} \tag{7.27}$$

and from equation (7.24),

$$\left.\begin{aligned} \psi_{32}=+\sqrt{f_c}C_3(\eta)Z_c(1-Z_c) \\ +\frac{A_{3\eta\eta}}{\sqrt{f_c}}\left[\frac{Z_c}{2}-\frac{2Z_c^2}{3}+\frac{Z_c^3}{3}-\frac{\xi\sqrt{f_c}}{2}(Z_c^2-Z_c)\right]+\frac{A_3^2}{2}Z_c^2 \\ -\eta\left[1-\frac{Z_c}{2}+\frac{\xi\sqrt{f_c}}{2}(Z_c-Z_c^2)\right]. \end{aligned}\right\} \tag{7.28}$$

b. *The Correction in the Region* $\xi=O(\varepsilon^{1/5})$. In the last section, we showed that when ξ is $O(\varepsilon^{1/5})$ the expression for D_3 is $O(\varepsilon^{1/5})$. Also, since $\psi_3\sim D_3^2/2f_c$, it is $O(\varepsilon^{2/5})$ when ξ is $O(\varepsilon^{1/5})$.

Therefore, let

$$\xi=\varepsilon^{1/5}\zeta,$$

$$D=\varepsilon^{1/5}d_4,$$

and

$$\psi=\varepsilon^{2/5}\varphi_4.$$

The region for which $\zeta=O(1)$ will be denoted by a subscript 4. With all other variables scaled as in region 3, the equations for region 4 are

$$v_{4\zeta} + \varepsilon^{3/5} u_{4\eta} + \varepsilon^{1/5} f_c - \varepsilon^{3/5}\eta = \varepsilon^{2/5} d_4, \tag{7.29}$$

$$\frac{v_4^2}{2} + \varepsilon^{3/5}\frac{u_4^2}{2} + \varepsilon^{1/5} d_4 = \varepsilon^{2/5}\varphi_4 + \frac{f_c}{2}, \tag{7.30}$$

$$u_4 d_4 = \varphi_{4\eta}, \tag{7.31}$$

and

$$v_4 d_4 = \varphi_{4\zeta}. \tag{7.32}$$

The dependent variables are again expanded in powers of $\varepsilon^{1/5}$, for example

$$v_4(\zeta,\eta) = v_{40}(\zeta,\eta) + \varepsilon^{1/5} v_{41}(\zeta,\eta) + O(\varepsilon^{2/5}).$$

We only require the zeroth order fields for u_4, d_4, and φ_4, and the zeroth and first order fields for v_4. From the zeroth order contributions of equations (7.29) and (7.30) we see that

$$v_{40} = \sqrt{f_c}. \tag{7.33}$$

From the first order contributions of (7.29), it follows that

$$v_{41} = -A_4(\eta) - f_c\zeta, \tag{7.34}$$

where $A_4(\eta)$ is to be determined by matching to region 3. From the first order contribution of equation (7.30) we obtain

$$d_{40} = -v_{40}v_{41} = \sqrt{f_c}\,[A_4(\eta) + f_c\zeta]. \tag{7.35}$$

By integrating equation (7.32) with the boundary condition $\varphi_{40} = 0$ on $\zeta = 0$, we find

$$\varphi_{40} = f_c\left[A_4(\eta)\,\zeta + \frac{f_c\zeta^2}{2}\right]. \tag{7.36}$$

Then equation (7.31) gives

$$u_{40} = \sqrt{f_c}\,\frac{A_{4\eta}\,\zeta}{A_4(\eta) + f_c\zeta}, \tag{7.37}$$

and we see that $u_{40} = 0$ on $\zeta = 0$ as long as $A_4(\eta) \neq 0$.

The proper matching of the solutions in region 3 and region 4 can be accomplished by using an intermediate scale variable, but in this case the functions are sufficiently simple that we proceed by writing the region 3 stream function in terms of ζ through $O(\varepsilon^{2/5})$ and match by inspection. We find that

$$\left.\begin{aligned} &\psi_{30}(\varepsilon^{1/5}\zeta,\eta) + \varepsilon^{1/5}\psi_{31}(\varepsilon^{1/5}\zeta,\eta) + \varepsilon^{2/5}\psi_{32}(\varepsilon^{1/5}\zeta,\eta) \\ &= \varepsilon^{2/5}\frac{f_c^2\zeta^2}{2} + \varepsilon^{2/5} f_c A_3(\eta)\,\zeta + \frac{\varepsilon^{2/5}}{2}\left[\frac{A_{3\eta\eta}}{3\sqrt{f_c}} + A_3^2 - \eta\right] + O(\varepsilon^{3/5}). \end{aligned}\right\} \tag{7.38}$$

This should match $\varepsilon^{2/5}\varphi_{40}$. Comparing equations (7.36) and (7.38), we see that the ζ^2 terms are identical, the ζ terms match if $A_4(\eta) = A_3(\eta)$, and the constant term gives

$$\frac{A_{3\eta\eta}}{3\sqrt{f_c}} + A_3^2 = \eta. \tag{7.39}$$

c. *The behavior of the layer depth on the western boundary.* The solution of equation (7.39) determines the unknown function A_3 (= A_4). From equation (7.35) we see that A_4 is proportional to the depth of the warm water along the western boundary ($\xi = \zeta = 0$). Equation (5.15) for the depth of the warm layer on the boundary according to the region 2 solution may be written in terms on η as

$$D_{20}(0, y) = \varepsilon^{1/5}\sqrt{f_c\eta} + o(\varepsilon^{2/5}). \tag{7.40}$$

The boundary conditions for equation (7.39) are obtained by matching $\varepsilon^{1/5} d_4$ on the western boundary to equation (7.40). This means

$$A_3 \sim \sqrt{\eta} \text{ as } \eta \to \infty. \tag{7.41}$$

We also recall from Section 7a that matching u_{20} and u_{30} for $\xi = O(1)$, $\eta \to \infty$ implied

$$A_{3\eta} \sim \frac{1}{2\sqrt{\eta}} \text{ as } \eta \to \infty. \tag{7.42}$$

It is clear from equation (7.39) that the conditions (7.41) and (7.42) are consistent with a solution for η large of the form

$$A_3 \sim \eta^{1/2} + \frac{1}{24\sqrt{f_c}}\eta^{-2} + O(\eta^{-9/2}). \tag{7.43}$$

The general solution of equation (7.39) is called a first Painlevé Transcendent (see Ince, p. 345). For our purposes, it suffices to note that the solution is not obtainable by quadrature, and numerical integration is necessary. The dependence of A_3 on f_c can be removed by setting $A_3' = (3\sqrt{f_c})^{1/3} A_3$ and $\eta' = (3\sqrt{f_c})^{2/3}\eta$. Then equation (7.39) becomes

$$A'_{3\eta'\eta'} + A_3'^2 = \eta', \tag{7.44}$$

and the asymptotic form (7.43) becomes

$$A_3' \sim \eta'^{1/2} + \frac{1}{8\eta'^2} + O(\eta'^{-9/2}). \tag{7.45}$$

Equation (7.44) has been integrated numerically using the asymptotic solution give by equation (7.45) and its first derivative evaluated at $\eta' = 20$ to

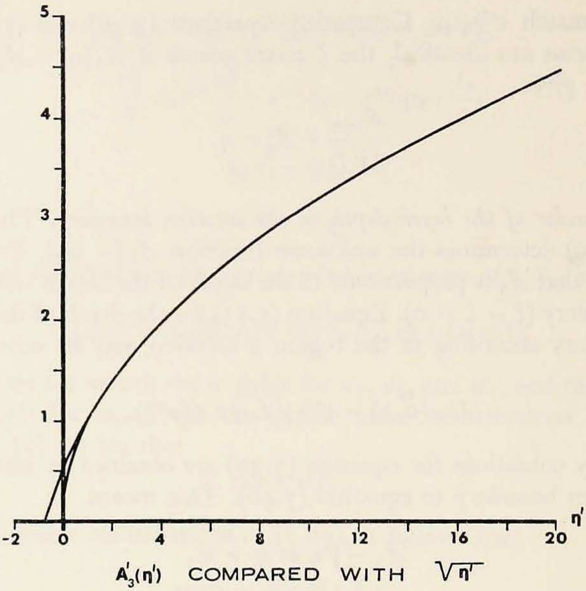

Figure 3. The solution of equation (7.44) for $A'_3(\eta')$, compared with $\sqrt{\eta'}$.

provide initial conditions for A' and $dA'/d\eta'$. The solution is shown in Figure 3. For comparison, $\sqrt{\eta'}$ is also shown. We see that A' is nearly indistinguishable for $\sqrt{\eta'}$ for $\eta' > 2$. The important property of A' is that it goes to zero at $\eta' = \eta'_0 \simeq -.715$ with a finite slope $A'_{\eta'} \simeq .959$. Therefore, the $1/\sqrt{y'}$ singularity in equation (6.3) for u_{20} has been removed, and the apparent separation latitude is a small distance (larger, however, than the boundary current width) north of $f = f_c$.

It is interesting to investigate the direction of the streamlines near the separation latitude. The direction of a streamline is given by

$$\theta = \tan^{-1}\left[\left(\frac{dX}{dY}\right)_{\psi=\text{const}}\right] = \tan^{-1}\left(\frac{U}{V}\right), \tag{7.46}$$

where θ is measured clockwise from due north. In region 3, we find from equations (7.9) and (7.20) that

$$\theta_3 \simeq \tan^{-1}\left(\frac{\varepsilon^{-1/5}\dfrac{A_{3\eta}}{\sqrt{f_c}}Z_c}{\varepsilon^{-1/2}\sqrt{f_c}\,Z_c}\right) \sim \varepsilon^{3/10}\frac{A_{3\eta}}{f_c} \tag{7.47}$$

independent of ξ. Thus to leading order for $\xi = O(1)$ the streamlines are parallel, and θ depends only on latitude. The bulk of the western boundary current veers away from the coast at a very small angle.

Near the coast, in region 4, we find from equations (7.33) and (7.37) that

$$\theta_4 \sim \frac{\varepsilon^{3/10} A_{4\eta}\zeta}{A_4(\eta)+f_c\zeta}. \tag{7.48}$$

Therefore, $\theta_4 \to \theta_3$ as $\zeta \to \infty$, but the streamline on the coast $\zeta = 0$ still goes straight north as long as $A_4(\eta) \neq 0$. However, near $\eta = \eta_0$, where $A_4(\eta) \to 0$, the expression (7.48) is not analytic. The value of θ_4 at $\zeta = 0$, $\eta = \eta_0$ is undefined. We have not yet uncovered a mechanism for the separation of the $\psi = 0$ streamline from the coast. To do so, we must examine the validity of the scaling in region 4 as $\eta \to \eta_0$.

From equation (7.33)–(7.35) and (7.37), we can compute the behavior of each term in equation (7.29) near $\zeta = 0$, $\eta = \eta_0$. The quantities $v_{4\zeta}$ and d_4 are regular in ζ and η, but

$$\left.\begin{aligned} \varepsilon^{3/5} u_{40_\eta} &= -\sqrt{f_c}\,\frac{\varepsilon^{3/5} A_{4\eta}^2 \zeta}{(A_4+f_c\zeta)^2} + \sqrt{f_c}\,\frac{\varepsilon^{3/5} A_{4\eta\eta}\zeta}{A_4+f_c\zeta} \\ &\sim -\sqrt{f_c}\,\frac{\varepsilon^{3/5} A_{4\eta}^2\zeta}{(A_4+f_c\zeta)^2} \end{aligned}\right\} \tag{7.49}$$

is singular at $\zeta = 0$, $\eta = \eta_0$. Near $\eta = \eta_0$, A_4 is approximately $\alpha(\eta-\eta_0)$ where

$$\alpha = \left.\frac{dA_4}{d\eta}\right|_{\eta=\eta_0}.$$

We see that if ζ and $\eta-\eta_0$ are both $O(\varepsilon^{2/5})$, $\varepsilon^{3/5}u_{40_\eta}$ is $O(\varepsilon^{1/5})$ and is comparable to $\varepsilon^{1/5}f_c$, which suggests rescaling $\eta-\eta_0$ and ζ with respect to $\varepsilon^{2/5}$. Rescaling $\eta-\eta_0$ with respect to any higher power of ε would give u independent of that rescaled latitude to leading order and therefore the fields could not match those in region 4.

8. The second correction to the western boundary current

a. *The equations for the separation of the boundary streamline.* The depth of the interface on the western boundary in region 4 is given by $D = \varepsilon^{1/5} A_4(\eta)$, which goes to zero at $\eta = \eta_0$, with slope

$$\alpha = \left.\frac{dA_4(\eta)}{d\eta}\right|_{\eta=\eta_0}.$$

The apparent separation latitude is now at $f = f_c - \varepsilon^{2/5}\eta_0$, which is north of f_c since η_0 is negative and η was measured south from f_c. To discuss the separation of the boundary streamline $\psi = 0$, we introduce the following rescaled variables.

Let

$$\zeta = \varepsilon^{2/5}\alpha^2 f_c^{-5/2}\sigma,$$
$$\eta_0 - \eta = \varepsilon^{2/5}\alpha f_c^{-3/2}\tau,$$
$$U = \varepsilon^{-1/5}\alpha f_c^{-1/2}u_5,$$
$$V = \varepsilon^{-1/2}[f_c^{1/2} + \varepsilon^{3/5}\alpha^2 f_c^{-3/2}v_5],$$
$$D = \varepsilon^{3/5}\alpha^2 f_c^{-1}\mathscr{D}_5,$$

and

$$\psi = \varepsilon^{6/5}\alpha^4 f_c^{-3}\chi_5.$$

The region where σ and τ are $O(1)$ is denoted by a subscript 5. The various powers of α and f_c that appear in the rescaling are simply there for later convenience, to avoid yet another re-definition of variables. The potential vorticity equation in terms of the scaled variables is

$$v_{5_\sigma} - u_{5_\tau} + 1 = \varepsilon^{2/5} f_c^{-1}\eta_0 - \varepsilon^{4/5}\alpha f_c^{-5/2}\tau + \varepsilon^{3/5}\alpha^2 f_c^{-2}\mathscr{D}_5, \tag{8.1}$$

where all the small terms have been collected on the right hand side of the equation. In the Bernoulli equation, the $O(1)$ balance is between the $f_c/2$ on the right hand side and the leading term of $\dfrac{\varepsilon V^2}{2} = \left(\dfrac{f_c}{2} + O(\varepsilon^{3/5})\right)$ on the left hand side. The residual Bernoulli equation is

$$\frac{u_5^2}{2} + v_5 + \mathscr{D}_5 = \varepsilon^{3/5}\alpha^2 f_c^{-2}\left[\chi_5 - \frac{v_5^2}{2}\right]. \tag{8.2}$$

The equations for the transport stream function are

$$u_5\mathscr{D}_5 = -\chi_{5_\tau}, \tag{8.3}$$

and

$$\mathscr{D}_5[1 + \varepsilon^{3/5}\alpha^2 f_c^{-2}v_5] = \chi_{5_\sigma}. \tag{8.4}$$

The dependent variables are expanded in power series in $\varepsilon^{1/5}$. The zeroth order equations are

$$v_{50_\sigma} - u_{50_\tau} + 1 = 0, \tag{8.5}$$

$$\frac{u_{50}^2}{2} + v_{50} + \mathscr{D}_{50} = 0, \tag{8.6}$$

$$u_{50}\mathscr{D}_{50} = -\chi_{50_\tau}, \tag{8.7}$$

and

$$\mathscr{D}_{50} = \chi_{50_\sigma}. \tag{8.8}$$

It is convenient to use equations which involve u_{50} and $\mathscr{D}_{50}$ alone. Substituting for v_{50} from equation (8.6) into (8.8) gives

$$u_{50_\tau} + u_{50}u_{50_\sigma} - 1 + \mathscr{D}_{50_\sigma} = 0, \tag{8.9}$$

and eliminating χ_{50} between equations (8.7) and (8.8) gives

$$\mathscr{D}_{50_\tau} + (u_{50}\mathscr{D}_{50})_\sigma = 0. \tag{8.10}$$

Equations (8.9) and (8.10) are the first momentum equation and the continuity equation. Since V is a constant to leading order, northward advection becomes time-like $\left(V\dfrac{\partial}{\partial y} \rightarrow \dfrac{\partial}{\partial \tau}\right)$, and the Coriolis term ($-fV \rightarrow -1$ in equation (8.9)) is a uniform force which accelerates the fluid seaward.

The matching conditions from region 4 to region 5, obtained from equations (7.35) and (7.37) are that

$$\mathscr{D}_{50} \sim \sigma - \tau \tag{8.11}$$

as $\tau \rightarrow -\infty$ for all σ, and that

$$u_{50} \sim \frac{\sigma}{\sigma - \tau} \tag{8.12}$$

as $\tau \rightarrow -\infty$ for all σ.

There exists an exact analogy between this problem and a problem in non-linear, non-rotating shallow water theory. Let

$$\mathscr{D}_{50} = h_{50} + \sigma.$$

Then equations (8.9) and (8.10) become

$$u_{50_\tau} + u_{50}u_{50_\sigma} + h_{50_\sigma} = 0, \tag{8.13}$$

and

$$h_{50_\tau} + (u_{50}(h_{50} + \sigma))_\sigma = 0. \tag{8.14}$$

These are the equations which govern one-dimensional shallow water motions over a uniformly sloping beach. The boundary condition that $u_{50} = 0$ at $\sigma = 0$ if $h_{50} > 0$ there can be interpreted as having a seawall at $\sigma = 0$. The quantity h_{50} is then the elevation of the free surface above the foot of the seawall, as illustrated in Figure 4. At an initial time $\tau = -\tau_0$ ($\tau_0 \gg 1$), the free surface is level with $h_{50} = \tau_0$, and the fluid is moving seaward with an initial velocity distribution $u_{50} = \dfrac{\sigma}{\sigma + \tau_0}$. We see from this problem that for large σ there is uniform outflow, $u_{50}(\sigma \rightarrow \infty, \tau) = 1$. Therefore, the water level at the seawall must decrease, and eventually the waterline recedes down the beach. When the water level decreases past the foot of the seawall, u_{50} at the waterline is the rate at which the waterline recedes away from the seawall. This waterline is given by $h_{50} = -\sigma$, which is $\mathscr{D}_{50} = 0$.

One exact solution of equations (8.9) and (8.10) is $u_{50} = 1$, $\mathscr{D}_{50} = \sigma - \tau + \tau_1$,

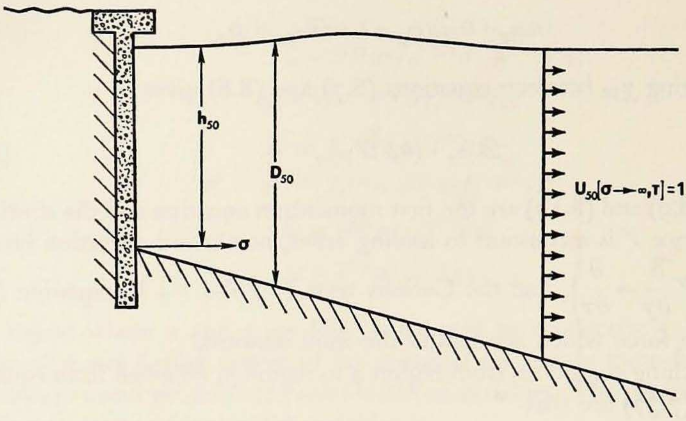

Figure 4. Schematic representation of the non-linear shallow water flow over a sloping beach with a seawall.

where τ_1 is a constant. The corresponding $h_{50} = \tau_1 - \tau$. In the shallow water analogy, this exact solution represents uniform outflow with the water level dropping at a uniform rate. In the separation problem, this solution shows that the separated streamline is straight and makes an angle with the coast which is equal to θ_3 as given in equation (7.47).

We assume that the solution of equations (8.9) and (8.10) with initial conditions given by equations (8.11) and (8.12) will be asymptotically equal to the above solution as $\tau \to \infty$, where the constant τ_1 defines a further refinement in the position of the separation latitude.

9. The free meandering jet

After the boundary current leaves the coast, it is useful to describe the flow in terms of a curvilinear coordinate system. Let $\varepsilon^{1/2}r$ measure distance normal to the streamline $\psi = D = 0$ (hereafter called the reference streamline), and let $\varepsilon^{1/4}s$ measure distance along this streamline. The motivation for choosing $\varepsilon^{1/4}$ for the downstream scale has been thoroughly discussed by Robinson and Niiler (1967). See their equations (5.2) and (5.3), and for the slope parameter S substitute

$$\frac{\beta H_0}{f_0} = \frac{H_0}{R \tan \theta_0},$$

since in their case the scale is topographically determined, but in our case it is determined by β. The $\varepsilon^{1/2}$ cross-stream scale reflects the fact that the entire western boundary current separates from the coast in a coherent fashion. The coordinate system is sketched in Figure 5. The quantity $\varepsilon^{1/4}Y_6 = (f - f_c)$ measures the distance of the reference streamline north from the apparent separation latitude. The velocity parallel to the reference streamline is denoted by

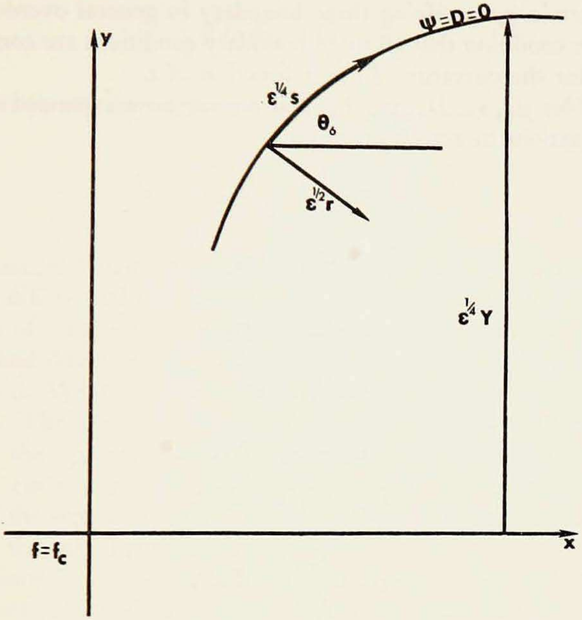

Figure 5. Schematic representation of the coordinate system for the free meander.

$\varepsilon^{-1/2}v_6$, and normal to the reference streamline by $\varepsilon^{-1/4}\mu_6$. In the meandering jet region, all dependent variables are denoted by a subscript 6. The curvature of the reference streamline is denoted by

$$\varepsilon^{-1/4}K_6 = \frac{d\theta_6}{d(\varepsilon^{1/4}s)},$$

where θ_6 measures the direction of the reference streamline. The equations for the meandering jet are

$$v_{6_r} + \frac{1}{1+\varepsilon^{1/4}rK_6}\left[\varepsilon^{1/4}K_6 v_6 - \varepsilon^{1/2}\mu_{6_s}\right] + f_c + \varepsilon^{1/4}Y_6 - \varepsilon^{1/2}r\cos\theta_6 = D_6, \quad (9.1)$$

$$\frac{v_6^2}{2} + \varepsilon^{1/2}\frac{\mu_6^2}{2} + D_6 = \psi_6 + \frac{f_c}{2}, \quad (9.2)$$

$$\mu_6 D_6 = -\frac{\psi_{6s}}{1+\varepsilon^{1/4}rK_6}, \quad (9.3)$$

and

$$v_6 D_6 = \psi_{6r}. \quad (9.4)$$

The boundary conditions for the solution of Equations (9.1)–(9.4) are $\psi_6 = D_6 = 0$ on $r = 0$ and $\psi_6 \rightarrow \psi_1$ as $r \rightarrow \infty$. Since the differential equation for v_6

is second order in r, specifying three boundary in general overdetermines the problem. The condition that all three boundary conditions are compatible gives an equation for the curvature K_6 as a function of s.

The variables $\mu_6, \nu_6, D_6, \psi_6, \Upsilon_6$, and K_6 are now expanded in powers of $\varepsilon^{1/4}$. The equations to zeroth order are

$$\nu_{60_r} + fc = D_{60}, \tag{9.5}$$

$$\frac{\nu_{60}^2}{2} + D_{60} = \psi_{60} + \frac{fc}{2}, \tag{9.6}$$

$$\mu_{60} D_{60} = -\psi_{60_s}, \tag{9.7}$$

and

$$\nu_{60} D_{60} = \psi_{60_r}. \tag{9.8}$$

The solutions are

$$v_{60} = \sqrt{fc}\,\mathscr{L}_c, \tag{9.9}$$

$$D_{60} = fc(1 - \mathscr{L}_c), \tag{9.10}$$

and

$$\psi_{60} = \frac{fc}{2}(1 - \mathscr{L}_c)^2, \tag{9.11}$$

where

$$\mathscr{L}_c = \exp[-\sqrt{fc}\,r].$$

Since ψ_{60} is independent of s, $\mu_{60} = 0$, and the leading contribution to μ_6 is actually $O(\varepsilon^{1/4})$.

The equations for the first order fields are

$$\nu_{61_r} + K_{60}\nu_{60} + \Upsilon_{60} = D_{61}, \tag{9.12}$$

$$\nu_{61}\nu_{60} + D_{61} = \psi_{61}, \tag{9.13}$$

and

$$\nu_{61} D_{60} + \nu_{60} D_{61} = \psi_{61r}. \tag{9.14}$$

As usual, we eliminate D_{61} and ψ_{61} to obtain a linear equation for ν_{61} which contains an inhomogeneous term involving K_{60}, Υ_{60} and ν_{60}. The solution for ν_{61} which vanishes at $r = 0$ (as it must from equation (9.13) and the boundary conditions on ψ_{61} and D_{61}) and as $r \to \infty$ is

$$\nu_{61} = \frac{K_{60}}{3}[\mathscr{L}_c^2 - \mathscr{L}_c] - \frac{r}{2}[\Upsilon_{60} + \sqrt{fc}\,K_{60}]\mathscr{L}_c. \tag{9.15}$$

Substituting from equation (9.15) into equation (9.12) and evaluating the resulting expression at $r = 0$ gives an equation for K_{60} in terms of Υ_{60}, which is

$$\frac{1}{3}\sqrt{f_c}K_{60} + Y_{60} = 0. \tag{9.16}$$

In equation (9.16) the curvature K_{60} is

$$K_{60} = \frac{d^2 Y_{60}}{dX^2}\left(1 + \left(\frac{dY_{60}}{dX}\right)^2\right)^{-3/2},$$

where X measures longitude in units of $\varepsilon^{1/4}$. Therefore, equation (9.16) is a second order differential equation for the location of the reference streamline as a function of longitude. The initial conditions are given by specifying the position Y_{60} and direction dY_{60}/dX of the reference streamline as it leaves the coast at $X = 0$. We recall that Y_{60} measures distance north from $f = f_c$ in units of $\varepsilon^{1/4}$. The first correction (cf. Section 7) to the western boundary layer moved the apparent separation latitude from $f = f_c$ to $f = f_c - \varepsilon^{2/5}\eta_0$. The second correction (cf. Section 8) produced an additional $o(\varepsilon^{4/5})$ displacement of the separation point. On the scale of $\varepsilon^{1/4}$ both these displacements are small, so the initial position is $Y_{60} = 0$ at $X = 0$.

The solutions given in Sections 7 and 8 show that the entire stream leaves the coast coherently at a small angle from due north, $\theta_3 = O(\varepsilon^{3/10})$ (see equation (7.47)). Since X and Y_{60} in equation (9.16) are both measured on the same scale (i.e., $\varepsilon^{1/4}$), the initial direction is due north to leading order. This means $dX/dY_{60} = 0$ at $X = 0$. The solution of equation (9.16) with these initial conditions can be written in term of elliptic integrals. The position Y_{60} is a sinuous function of X, with maximum amplitude $\left(\frac{2}{3}\sqrt{f_c}\right)^{1/2}$ and wavelength $3.39\, f_c^{1/4}/\sqrt{3}$. Figure 6 shows the shape of the reference streamline for one wavelength of the meander.

The dynamics of this free meandering jet are identical to the baroclinic case of inertial jets discussed by Robinson and Niiler (1967). Equation (9.16) can be obtained from the cross-stream integral balance of vorticity flux in the jet, and is seen to represent a balance between the total advection of relative vorticity and planetary vorticity. In this formulation, correct to $O(\varepsilon^{1/4})$, Robinson and Niiler's (1967) equation (2.31) becomes

$$K_{60}\int_0^\infty v_{60}^2 D_{60}\,dr + Y_{60}\int_0^\infty v_{60} D_{60}\,dr = 0, \tag{9.17}$$

and upon substituting for v_{60} and D_{60} from equations (9.9) and (9.10), we again obtain equation (9.16).

There is a recirculation within the southern side of each meander. This steady pattern is simply the zonal geostrophic drift to the west given by the

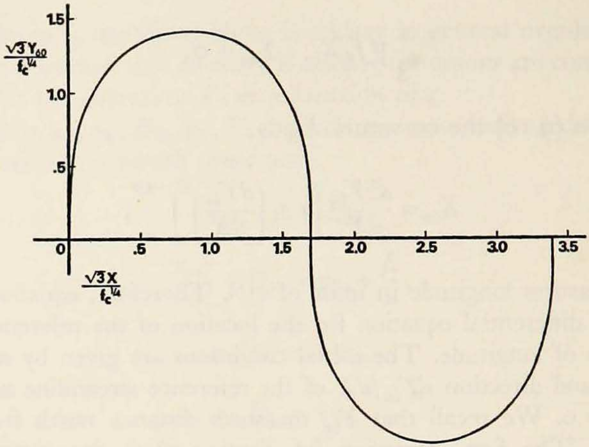

Figure 6. The solution to equation (9.16) for Y_{60} as a function of X.

original interior constant potential vorticity flow, and thus does not represent the "radiation of a geostrophic wave" by the meander suggested by Robinson and Niiler (1967).

10. The completion of the circulation pattern

The equations and boundary conditions for the circulation, given in Section 2, admit solutions for which the stream function ψ is symmetric about the mid-longitude $x = x_0/2$ of the basin. The solution we have presented so far does not admit this symmetry unless the east-west wavelength of the free meandering stream can be adjusted so that a crest or a trough of the wave falls exactly at $x = x_0/2$. That is to say that an odd number of half wavelengths of the meander should just fit into the basin. This can always be done by changing f_c by an amount which is $O(\varepsilon^{1/4})$, and we recall here that we stated in Section 4 that such a $o(1)$ correction to f_c would indeed be made. The wavelength of the meander is approximately $1.96\ \varepsilon^{1/4} f_c^{1/4}$, and f_c was assumed to be $2 + o(1)$. Therefore, the number of waves across the basin is an $O(\varepsilon^{-1/4})$ quantity. To make the meandering wave fit in the basin so that the solution may be taken as symmetric about $x_0/2$ means that if we change f_c by an $O(\varepsilon^{1/4})$ amount, the wavelength of each wave will change by an $O(\varepsilon^{1/2})$ amount, and since there are $O(\varepsilon^{-1/4})$ waves in the basin, the total number of half waves between $x = 0$ and $x = x_0$ will change by $O(1)$.

Here is a specific example. Let $\varepsilon \ll 1$ and x_0 be given. Compute the quantity $x_0/.98\ (2\varepsilon)^{-1/4}$ and find the largest odd integer $(2N+1)$ which is less than or equal to this quantity, i.e.,

$$\frac{x_0}{.98}(2\varepsilon)^{-1/4} = 2N + 1 + \delta \qquad \text{where} \qquad 0 \leq \delta < 2. \tag{10.1}$$

Then we choose f_c to be

$$f_c = 2 + \frac{8}{x_0}(.98\,\delta)(2\varepsilon)^{1/4}. \qquad (10.2)$$

The $O(\varepsilon^{1/4})$ contribution to f_c then provides the wavelength correction necessary to just fit $2N+1$ half wavelengths into the basin. Of course, equation (10.2) is not the only possible choice for f_c which will satisfy symmetry about $x_0/2$. We could have chosen any odd integer $2N+1$ as long as δ in equation (10.1) remained an $O(1)$ quantity; in general, δ could be positive or negative. Equation (10.2) would still apply and $2N+1$ half wavelengths would complete the northern edge of the circulation pattern.

We recall from Section 4 that if $f_c \neq 2$, the interior stream function ψ_1 does not satisfy the boundary condition $\psi_1 = 0$ on $y = 0$. In fact, if $f_c = 2 + \Delta\varepsilon^{1/4}$, then we find from equation (4.3) that

$$\psi_1(x, y = 0) = -\frac{\Delta}{2}\,\varepsilon^{1/4} + O(\varepsilon).$$

Therefore, to make the total stream function equal to zero at $y = 0$, we must add a weak southern inertial boundary current to the circulation. Near $y = 0$, the solutions for U, V, D and ψ are easily found to be

$$U = -\frac{1}{f} + \frac{\Delta}{2}\,\varepsilon^{-1/4} e^{-\varepsilon^{-1/2} y}, \qquad (10.3)$$

$$V = 0, \qquad (10.4)$$

$$D = f + \frac{\Delta}{2}\,\varepsilon^{1/4} e^{-\varepsilon^{-1/2} y}, \qquad (10.5)$$

and

$$\psi = y - \frac{\Delta}{2}\,\varepsilon^{1/4}\left[1 - e^{-\varepsilon^{-1/2} y}\right]. \qquad (10.6)$$

For $\Delta > 0$, we see from equation (10.6) that $\psi = 0$ at $y = 0$ and again at $y = (\Delta/2)\,\varepsilon^{1/4}$ (i.e., at $f = f_c/2$). The southern boundary current in this case represents an eastward flow with $U = O(\varepsilon^{-1/4})$ in a region of width $O(\varepsilon^{1/2})$. The corresponding eastward transport is returned to the west by the interior westward drift south of $y = (\Delta/2)\,\varepsilon^{1/4}$. Thus there is a weak closed gyre between $y = 0$ and $y = (\Delta/2)\,\varepsilon^{1/4}$. Note from equation (5.13) and (5.14) that at $f = f_c/2$, $A_2 = 0$, $v_{20} = 0$, and D_{20} is independent of ξ. Thus the zero interior streamline at $y = (\Delta/2)\,\varepsilon^{1/4}$ extends straight to the meridional boundary. The westward interior drift south of $y = (\Delta/2)\,\varepsilon^{1/4}$ reaches the western boundary region and flows southward there. It turns from south to east in an $O(\varepsilon^{1/2}) \times O(\varepsilon^{1/2})$ corner region near $x = y = 0$, and flows eastward as an $O(\varepsilon^{1/2})$ wide current against the southern boundary, as shown by equations (10.3)–(10.6).

We have formulated the corner problem; the stream function correction in the corner obeys $\varepsilon\nabla^2\psi = \psi$, with $\psi = 0$ on $x = 0$ and $\psi = (\Delta/2)\,\varepsilon^{1/4}$ on $y = 0$. The solution to this problem has already been discussed by Moore (1968), and does provide a smooth turning of the boundary current.

If $\Delta < 0$, the southern boundary current flow is westward and this transport is turned northward in the southwest corner and flows into the western boundary current. The western boundary layer solution given in Section V is unchanged outside the corner region.

Fofonoff (1962) has postulated that the solution to this baroclinic circulation problem consists of a slow geostrophic interior flow to the west, inertial boundary currents along western and eastern boundaries, and a straight inertial eastward jet at the separation latitude which completes the pattern. See Figure 3 of his article. He does not discuss how the transition from western boundary current to straight eastward jet is accomplished. Our analysis shows that as long as the northern boundary of the basin is sufficiently far north, the correct circulation pattern for this model looks more like our Figure 1.

11. More general choices for $B(\psi)$

We recall from Section 2 that the Bernoulli function and potential vorticity are conserved on each streamline. Therefore, we write $B = B(\psi)$ and $P = P(\psi)$. But these conservation theorems derived from equation (2.10) do not imply that $B(\psi)$ and $P(\psi)$ are necessarily single valued functions of ψ. It is perfectly possible to have two or more different streamlines having the same value of ψ, and there is no reason to suppose B (and P) will be the same on these different streamlines. All we know for sure is that B and P are conserved on any given streamline. This apparently trivial remark is important for understanding the discussion of separation given by Charney (1955). In his model, Charney divides the flow into two regions, a slow geostrophic interior and an intense western boundary layer. The Bernoulli function and potential vorticity are conserved along streamlines in both regions. In the interior, he chooses a stream function distribution given by

$$\psi = \psi_0 - \gamma(y - y_0)^2,$$

which is symmetric about $y = y_0$. The corresponding interior D field in the notation of this paper is

$$D^2 = D_0^2 - 2\gamma(1 + y_0)(y - y_0)^2 - \frac{4\gamma}{3}(y - y_0)^3,$$

cf. Charney (1955), equation (17). This D field is not symmetric about $y = y_0$. Therefore, the Bernoulli function $B = D + \varepsilon\,\dfrac{u^2}{2} = D + \dfrac{\varepsilon}{2}\left(\dfrac{Dy}{f}\right)^2$ is not symmetric about $y = y_0$. This implies that B, considered as a function of ψ, is not single valued. Therefore, a streamline south of $y = y_0$ with a given value of the stream function $\psi = \psi_1$ cannot be the same streamline as the one an equal distance north of $y = y_0$ having the same stream function value. That is to say, a specification of the interior ψ field such as Charney made overspecifies the problem. A well set problem is obtained if B is specified as a single valued increasing analytic function of ψ, so that the potential vorticity $P = \frac{dB}{d\psi}$ is everywhere positive.

The constant potential vorticity distribution was chosen for this paper because the x dependence of the field in the western boundary current could be written down explicitly. It is well known (Charney 1955, Morgan 1956, Fofonoff 1962) that with more general choices for $B(\psi)$, the x dependence must be obtained by numerical quadrature. The use of explicit solutions in our example has facilitated the examination of the validity of the approximations involved. We have found that all the scaling arguments we have made can be consistently applied to more general choices for analytic $B(\psi)$ as long as $P(\psi) > 0$ and $P_\psi \geq 0$. The physics of the separation phenomenon is the same for these more general cases as for the one we have discussed in detail. The detailed analysis of the more general case has been carried out, but yields no new insights and is therefore not presented.

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