A linear theory of stationary coastal upwelling in a continuously stratified ocean with an unstratified shelf area[1]

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ABSTRACT

The circulation in a coastal upwelling region with a shelf area is calculated. It is assumed that the circulation on the shelf is independent of the deep-sea circulation and consists of a surface Ekman layer outflow, a coastal boundary layer upwelling and a bottom Ekman layer inflow. The resultant velocity distribution is used as a boundary condition for the deep-sea circulation. In the homogeneous ocean upwelling occurs on the shelf only, bottom inflow being fed by the transport of water within the frictional boundary layer extending to infinite depth. Introduction of stratification into the model produces, in addition to the expected weakening of the boundary layer and inflow into the shelf area from the oceanic interior, a secondary upwelling region above the shelf edge of very shallow extent.

1. Definition of the problem

The problem of the circulation in a region of coastal upwelling has been treated, during recent years, by various authors. A basic result of several theories is the fact that a realistic picture of the upwelling process cannot be obtained with a homogeneous model (for details, see Tomczak (1972)). Inclusion of stratification, which necessarily leads to a model which takes into account diffusion at least in the vertical, results in a finite depth of the upwelling regime and, in some cases, in a weak poleward undercurrent. Despite these encouraging results, there are still large discrepancies between observation and theory. Some of these can be attributed to the fact that most of the theoretical work did not take into account a shelf area. For example, the maximum upwelling intensity usually is found off-shore above the shelf edge, and the upwelling undercurrent also seems to be linked with the shelf edge. Although the location of the maximum intensity can be understood by "natural rea-

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soning", the theories to explain it qualitatively are not at all frequent and mostly based on a numerical procedure of solving the hydrodynamic equations (O'Brien and Hurlburt, 1972; Killworth, 1973; Hurlburt and Thompson, 1973).

The main problem for a theory of coastal upwelling which includes a shelf area consists in adapting the physical situation to the methods of solution on hand. The problem of the semi-infinite stratified ocean with a straight boundary is quite straight-forward today: Fourier transform, and boundary layer technique in the wavenumber domain. For the treatment of the semi-infinite ocean with a boundary of a given shape we have either to find another method of solution or to reduce the problem to the problem of the ocean with a straight boundary with modified boundary conditions. It is this second way which is done in the present paper.

Consider an ocean as sketched in figure 1. Adjacent to an oceanic region which extends infinitely in x and z and is bounded by the surface at $z = 0$ and by the shelf edge at $x = 0$ for all $z > H$, there is a shelf region of depth H extending towards negative values of x. The ocean is continuously stratified with constant Brunt-Väisälä frequency N, and stratification is weak, so that the shelf area can be treated as homogeneous. The wind is blowing parallel to the coastline and uniform in space. There are no barriers along the coast which allows us to assume that all along-shore gradients vanish. Under these assumptions the only possible circulation pattern on the shelf is the one shown in the figure (Verstraete, 1970): The mass transport directed off-shore is restricted to the surface Ekman layer and compensated by a flow entering the shelf through the bottom Ekman layer and then rising towards the surface within the coastal boundary layer. Hence, the problem that remains is the determination of the circulation pattern in the region $x > 0$, $z > 0$. For this purpose, the influence of the shelf can be expressed in terms of a boundary condition on the horizontal current components at $x = 0$.

The treatment of the problem as described above is in a certain sense inverse to the problem treated by Hsueh and O'Brien (1971) who investigated the effect of a given oceanic current on the circulation within the shelf area. However, the model of Hsueh and O'Brien did not take into account friction and thus could not include any Ekman layers, a procedure which might be inadequate in shallow waters such as shelf areas.

2. Equations and boundary conditions

The linearized equations in their Boussinesq form appropriate to the present problem have been derived in an earlier paper (Tomczak, 1970) which will henceforth be referred to as MT:

$$-f\mathring{v} + \mathring{P}_{\mathring{x}} - \mu^H \mathring{u}_{\mathring{x}\mathring{x}} - \mu^V \mathring{u}_{\mathring{z}\mathring{z}} = 0 \qquad (1)$$

$$f\mathring{u} - \mu^H \mathring{v}_{\mathring{x}\mathring{x}} - \mu^V \mathring{v}_{\mathring{z}\mathring{z}} = 0 \tag{2}$$

$$\mathring{P}_{\mathring{z}} - g\mathring{R} = 0 \tag{3}$$

$$N^2 \mathring{w} - g\gamma \mathring{R}_{\mathring{z}\mathring{z}} = 0 \tag{4}$$

$$\mathring{u}_{\mathring{x}} + \mathring{w}_{\mathring{z}} = 0 \tag{5}$$

Apart from usual notations, μ^H and μ^V denote the coefficients of virtual horizontal and vertical Austausch, γ the coefficient of virtual vertical diffusion, and N the Brunt-Väisälä frequency which is assumed to be constant in the model.

Combining (3) and (4) into

$$N^2 \mathring{w} - \gamma \mathring{P}_{\mathring{z}\mathring{z}\mathring{z}} = 0 \tag{6}$$

and introducing dimensionless variables

$$z = \frac{\mathring{z}}{D} \quad x = \frac{\mathring{x}}{\alpha D} \quad u = \frac{\mathring{u}}{u_r} \quad v = \frac{\mathring{v}}{u_r} \quad w = \frac{\alpha \mathring{w}}{u_r}$$

$$\overset{*}{P} = \frac{\mathring{P}}{\alpha u_r^2} \quad \overset{*}{R} = \frac{gD}{\alpha u_r^2} \cdot \mathring{R} \quad \vec{\tau} = \frac{\vec{\mathring{\tau}}}{u_r^2}$$

with the notations

$$D^2 = \frac{2\mu^V}{f} \quad \alpha^2 = \frac{\mu^H}{\mu^V} \quad u_r^2 = 2\mu^V f,$$

a new set of dimensionless equations is obtained:

$$-v + \overset{*}{P}_x - \frac{1}{2} u_{xx} - \frac{1}{2} u_{zz} = 0 \tag{7}$$

$$u - \frac{1}{2} v_{xx} - \frac{1}{2} v_{zz} = 0 \tag{8}$$

$$2\sigma S w - \overset{*}{P}_{zzz} = 0 \tag{9}$$

$$u_x + w_z = 0 \tag{10}$$

The only parameter in this set of equations is σS, composed of the vertical Prandtl number $\sigma = \mu^V/\gamma$ and a stability parameter $S = 1/\alpha^2 \cdot N^2/f^2$. Under natural conditions, $\sigma S \ll 1$ in the ocean which is the basis for the present boundary layer approximation (as the shelf area is taken as homogeneous, the second condition $\sigma S \ll H^{-4}$ must be satisfied simultaneously). The horizontal Prandtl number has been taken as zero from the very beginning and thus does not occur as a second parameter. The implications of this assumption have been discussed by Tomczak (1973).

The boundary conditions in dimensionless form are

$$u, v < \infty \quad \text{at} \quad x \to \infty \tag{11}$$

$$u = u_E, \quad v = v_E \quad \text{at} \quad x = 0 \tag{12}$$

$$-\frac{1}{2} u_z = \tau^x - \frac{1}{2} v_z = \tau^y \quad \text{at} \quad z = 0 \tag{13}$$

$$u = v = w = 0 \quad \text{at} \quad z \to \infty \tag{14}$$

$$R_z = w = 0 \quad \text{at} \quad z = 0 \tag{15}$$

$$R = 0 \quad \text{at} \quad z \to \infty \tag{16}$$

Here, τ^x and τ^y denote the components of the wind stress $\vec{\tau}$, and u_E and v_E the components of the Ekman current within the surface and bottom boundary layer on the shelf, which, in the case $H \gg 1$ to which the present study is restricted, are:

$$\left.\begin{aligned} u_E = {} & \exp(-z)\left[(\tau^x + \tau^y)\cos z - (\tau^x - \tau^y)\sin z\right] \\ & - \exp(z - H)\left[(\tau^x + \tau^y)\cos(H - z) - (\tau^x - \tau^y)\sin(H - z)\right] \\ & \text{at} \quad z \leqslant H, \quad u_E = 0 \quad \text{at} \quad z > H \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} v_E = {} & -\exp(-z)\left[(\tau^x - \tau^y)\cos z + (\tau^x + \tau^y)\sin z\right] \\ & + \exp(z - H)\left[(\tau^x - \tau^y)\cos(H - z) + (\tau^x + \tau^y)\sin(H - z)\right] \\ & \text{at} \quad z \leqslant H, \quad v_E = 0 \quad \text{at} \quad z > H \end{aligned}\right\} \tag{18}$$

u_E is shown in figure 1 for the case $\tau^x = 0$. For simplicity, the inflow at the bottom has been assumed to be the inverted Ekman spiral. In general, this compares as well with a realistic circulation on a shelf area as a boundary layer flow satisfying $u = v = 0$ or $u_z = 0$ at the boundary, and it simplifies the mathematics.

3. Method of solution

Equations (7)–(10) subject to boundary conditions (11)–(16) are solved with the aid of a double Fourier transform. If we define the transform

$$(\tilde{U}, \tilde{V})(\chi, z) = \int_0^\infty (u, v)(x, z) \sin \chi x dx \tag{19}$$

$$(\tilde{T}^x, \tilde{T}^y)(\chi) = \int_0^\infty (\tau^x, \tau^y)(x) \sin \chi x dx \tag{20}$$

$$(\tilde{R}, \tilde{P}, \tilde{W})(\chi, z) = \int_0^\infty (\overset{*}{R}, \overset{*}{P}, w)(x, z) \cos \chi x dx \tag{21}$$

and on the defined functions, define another transform

$$\left.(U,V,P)(\chi,\zeta)=\int_0^\infty(\tilde{U},\tilde{V},\tilde{P})(\chi,z)\cos\zeta z\,dz\right\}\quad(22)$$

$$\left.(R,W)(\chi,\zeta)=\int_0^\infty(\tilde{R},\tilde{W})(\chi,z)\sin\zeta z\,dz\right\}\quad(23)$$

we obtain from (7)–(10), by appropriate integration, the algebraic set of equations

$$\left.-V-\chi P+\frac{1}{2}(\chi^2+\zeta^2)U=\tilde{T}^x+\frac{1}{2}\chi U_E\right\}\quad(24)$$

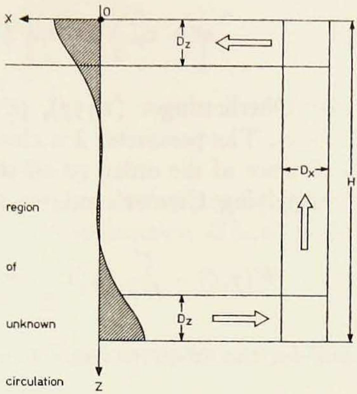

Figure 1. Theoretical model of coastal upwelling used in the present study. O: origin of coordinate axes. D_x, D_z: horizontal and vertical Ekman layer width, respectively. The black curve is the x component of the Ekman current prescribed at the "boundary" of the oceanic region, $x = 0$, and the arrows indicate the direction of the integrated Ekman layer mass transport.

$$U+\frac{1}{2}(\chi^2+\zeta^2)V=\tilde{T}^y+\frac{1}{2}\chi V_E \quad (25)$$

$$2\sigma SW-\zeta^3P=\zeta\tilde{R}_0 \quad (26)$$

$$\chi U+\zeta W=U_E \quad (27)$$

where U, V, W, P are unknown functions of χ and ζ, $\tilde{T}^x$ and $\tilde{T}^y$ are known functions of χ, U_E and V_E are known functions of ζ given in appendix A, and $\tilde{R}_0$ is an unknown function of χ which has to be determined by applying the only boundary condition not used in the derivation of (24)–(27). This procedure has been described in MT and led to the result that the influence of the thermodynamic boundary conditions is of the order of α^{-2} which in the ocean is of the order of 10^{-6}. We anticipate here that the same result is true in the present problem and facilitate the presentation of the computations by neglecting $\tilde{R}_0$ in (26).

If we now restrict our attention to the case of a uniform windstress directed parallel to the coastline, we have $\tau^x = 0$, $\tau^y = 1$. As, for applying Fourier transform techniques, all functions have to vanish at large x, we replace this condition by

$$\tau^x = 0, \quad \tau^y = e^{-\lambda x} \quad \text{or} \quad \tilde{T}^x = 0, \quad \tilde{T}^y = \frac{\chi}{\chi^2 + \lambda^2} \tag{28}$$

(see Oberhettinger (1957), p. 121), and we replace (11) by $u = v = 0$ at $x \to \infty$. The parameter λ is chosen in a way that τ^y is essentially constant over a distance of the order 10 off-shore but, apart from this condition, arbitrarily.

Applying Cramer's rule to (24)–(27), we obtain for the vertical velocity

$$W(\chi,\zeta) = \frac{\zeta^3}{DN}[4(1-\cos\zeta H) - \zeta(\zeta^2+\chi^2+2)\sin\zeta H - 4\chi\tilde{T}^y] \tag{29}$$

and for the off-shore component of the horizontal current

$$\left.\begin{aligned} U(\chi,\zeta) = \frac{4}{DN}\Bigg[&\frac{1}{\zeta^4+4}(1-\cos\zeta H)(8\chi\zeta^6+4\chi^3\zeta^4+16(\zeta^2+\chi^2)\chi\sigma S) \\ &-\frac{1}{\zeta^4+4}\chi\zeta\sin\zeta H(\zeta^2\chi^2+2\chi^2+\zeta^4+4\zeta^2-4) \\ &-\frac{4}{\zeta^3(\zeta^4+4)}(\zeta^2+2)(\zeta^2+\chi^2)\chi\sigma S\sin\zeta H+\tilde{T}^y\Bigg] \end{aligned}\right\} \tag{30}$$

with

$$DN = (\zeta^8 + 2\chi^2\zeta^6 + (\chi^4+4)\zeta^4 + 4\sigma S\chi^2\zeta^2 + 4\sigma S\chi^4) \tag{31}$$

This solution can be transformed back into the x, z-domain. The details of the procedure yielding $\tilde{W}(\chi,z)$ are given in appendix B. The result is

$$\left.\begin{aligned} \tilde{W}(\chi,z) = &\frac{1}{4(\chi^4+4)}[2[\exp(-\varphi^+z)(2\cos\varphi^-z+\chi^2\sin\varphi^-z) \\ &-2\cos\psi z\exp(-\psi z)](\tilde{T}^y\chi-1)-\frac{1}{2}\exp(-\varphi^+(z-H)\operatorname{sign}(z-H)) \\ &[[\varphi^+(\chi^2-2)+\varphi^-(\chi^2+2)-2\chi^2\operatorname{sign}(z-H)]\sin\varphi^-(z-H)\operatorname{sign}(z-H) \\ &-[4\operatorname{sign}(z-H)-(\chi^2+2)\varphi^++(\chi^2-2)\varphi^-]\cos\varphi^-(z-H)] \\ &-\frac{1}{2}\exp(-\psi(z+H))[(4+\psi(\chi^2+2))\cos\psi(z+H)+\psi(\chi^2+2)\sin\psi \\ &(z+H)]-\frac{1}{2}\exp(-\psi(z-H)\operatorname{sign}(z-H))[(4\operatorname{sign}(z-H) \\ &-\psi(\chi^2+2))\cos\psi(z-H)-\psi(\chi^2+2)\operatorname{sign}(z-H)\sin\psi(z-H)]] \end{aligned}\right\} \tag{32}$$

φ^+, φ^- and ψ are functions of χ and are defined as

$$\varphi^+ = \left[\frac{(\chi^4+4)^{1/2}+\chi^2}{2}\right]^{1/2} \qquad \varphi^- = \left[\frac{(\chi^4+4)^{1/2}-\chi^2}{2}\right]^{1/2} \tag{33}$$

$$\psi = \left[\frac{\sigma S \chi^4}{\chi^4+4}\right]^{1/4} \tag{34}$$

The transformation of $\tilde{W}(\chi,z)$ into $w(x,z)$ is done numerically and the result is presented in the form of streamlines. The streamfunction $G(x,z)$ is defined by

$$G_x = u, \quad G_z = -w \tag{35}$$

The boundary conditions for φ are obtained in the following way: By definition, $G = \int w dx + F(z)$ where $F(z) = G(x = 0, z)$.

At the shelf edge $x = 0$ there is no rigid boundary at all depths $z < H$, and $F(z)$ is given by $G(x = 0, z) = -\int u dz|_{x=0}$, which, from (17) and (18), results in

$$\left.\begin{aligned} -F(z) &= \tau^x \exp(-z)\sin z - \tau^y \exp(-z)\cos z \\ &\quad + \exp(z-H)(\tau^x \sin(H-z) - \tau^y \cos(H-z)) \\ &\quad \text{at} \quad z<H, \quad -F(z) = -1 \quad \text{at} \quad z \geq H \end{aligned}\right\} \tag{36}$$

In the special case $\tau^x = 0$, $\tau^y = \exp(-\lambda x)$ treated here, we have

$$\left.\begin{aligned} F(z) &= \exp(-z)\cos z + \exp(z-H)\cos(H-z) \\ &\quad \text{at} \quad z<H, \quad F(z) = 1 \quad \text{at} \quad z \geq H \end{aligned}\right\} \tag{37}$$

4. Results and discussion

Before the results are given in the form of streamlines, some comments on the solution help to understand the physics of the circulation. The first term of eq. (32) takes into account the variation of the wind stress $\tilde{T}^y$, it describes the form of the surface Ekman layer in response to a wind which varies in space. In the present example where $\tilde{T}^y$ is given by (28), this term is very small as compared with the other two groups of terms and for $\lambda \to 0$ in fact tends to zero.

The second term or rather group of terms is important in a corner region only where a horizontal and a vertical frictional boundary layer merge. The role of this type of terms has been elucidated in MT. In the present case these terms are important only in the area of small x and values of z which come close to $z = H$, i.e. near the shelf edge. It can be shown that they contribute to the solution only for values $z \leq H$.

The last group of terms (involving ψ) is the only one which exhibits a

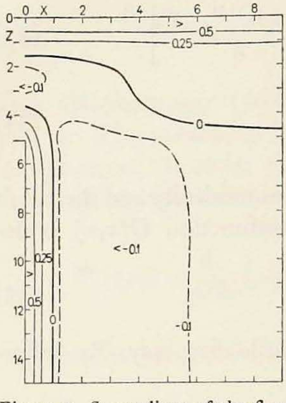

Figure 2. Streamlines of the flow produced by $\tau^x = 0$, $\tau^y = \exp(-0.001\,x)$ in a homogeneous ocean ($\sigma S = 0$), given in units of G as defined by (35) and (37).

dependence on stratification. The general type of solution has again been described in MT. The special form which these terms attain in the present solution corresponds to the boundary conditions: In the homogeneous case the terms vanish for all $z \leq H$; at $z < H$ they result, for x large enough, in a constant value of -1 counterbalancing the value of $F = 1$ given in (37)

Streamlines of the flow perpendicular to the coast are shown in figures 2–4. The circulation on the shelf (which is prescribed as a boundary condition) is not shown. As can be seen from fig. 2, the effect of a shelf, which is sufficiently deep in order to display a simple boundary layer circulation with an interior at rest, on the circulation of a homogeneous ocean is simply the combination of two circulations, both of which display a flow restricted to the frictional boundary layers, into a single one where again all transport occurs in the boundary layers. Thus, at the surface upwelling is present only at the inner end of the shelf (the coastline), while subsurface upward motion occurs along the coast and along the shelf edge. Above the shelf edge there is no upwelling at the surface; the wind-driven Ekman flow is completely uniform (the shape of the zero-streamline at large x is not very significant since for large x the stream function is close to zero at all depths except within the Ekman layer).

The circulation in a stratified ocean is given by fig. 3 and 4. It can be seen that the bottom inflow into the shelf area is no longer fed by a frictional boundary layer along the shelf edge; appearance of the diffusive scale ψ causes the inflow to be fed to a large part from the oceanic interior. This result is well known from MT. More important is the fact that above the shelf edge vertical flow occurs very close to the surface which is accompanied by an increase of off-shore velocity. The circulation established here can be described as a secondary center of upwelling bounded on its seaward side by a region of diffuse downwelling. The effect is essentially restricted to the Ekman layer; "upwelling" above the shelf edge in the present model occurs as a decrease of Ekman layer thickness rather than additional transport into the Ekman layer. This of course is mainly due to the boundary conditions: It has been assumed that the Ekman layer transport generated on the shelf enters the ocean completely. As a consequence additional inflow into the Ekman layer in the oceanic region is impossible because of mass continuity.

It must be noted that the numerical examples are not completely consistent with the basic assumptions $(\sigma)^{-1/4} \gg H \gg 1$. As $H = 5$, the second condition

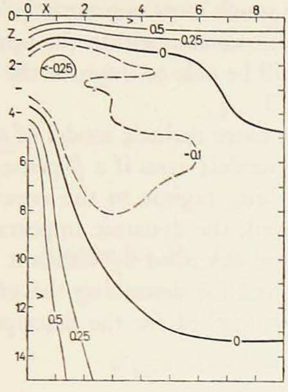

Figure 3. Streamlines of the flow produced by $\tau^x = 0$, $\tau^y = \exp(-0.001\,x)$ in a stratified ocean ($\sigma S = 0.002$).

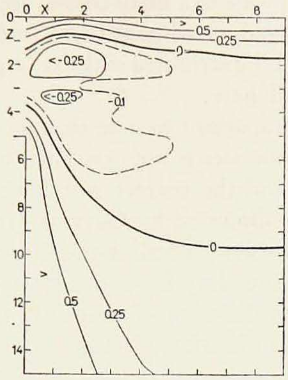

Figure 4. Streamlines of the flow produced by $\tau^x = 0$, $\tau^y = \exp(-0.001\,x)$ in a stratified ocean ($\sigma S = 0.02$).

is met to a certain extent. The first condition, however, strictly holds only for figure 2. As a consequence, the shelf area in figures 3 and 4 is weakly stratified. This is of importance for the upwelling circulation outside the shelf only with respect to the physical truth of the boundary conditions: For a stratified shelf area, (17) and (18) have to be replaced by a more exact circulation scheme which can be obtained from the solution presented in MT.

In the present calculation, however, it does not seem feasable to introduce the more complicated flow of a stratified shelf region as a boundary condition; too little is known about the velocity distribution on real shelf areas which might allow us to choose a truly realistic input into the system at the shelf edge. In order to demonstrate the principle influence of the circulation on the shelf and stratification in the deep ocean, the present solution is adequate.

The results shown in fig. 2–4 reaffirm some conclusions which have been drawn from other theoretical work on upwelling. They demonstrate in particular the limits of two-dimensional models: Consideration of stratification leads to a finite depth of upwelling, consideration of a shelf area to a weak secondary upwelling zone above the shelf edge. The fact that a reasonable strong upwelling undercurrent does not exist in the present solutions points to the need of the three-dimensional models. The fact that the secondary upwelling zone in reality is much stronger than indicated here and usually is accompanied by a frontal zone separating it from the oceanic regime, supports the idea expressed already some time ago (Tomczak, 1972) that the overall concept of modelling friction and diffusion should be changed. Instead of using constant eddy and diffusion coefficients (which, in order to come to reasonable results, have to be chosen rather large) within the whole ocean, restriction of the diffusive process to a small coastal area seems to be much more appropriate. Three-dimensional, quasi-geostrophic models with point sources within the upwelling region and distributed sinks at large distance will be able to improve the results presented here.

It is important to note that the problem of a more realistic model of the diffusive processes is not restricted to the f-plane model; even if a β-plane model is used and the correct matching of the upwelling regime to the oceanic interior as discussed by Garvine (1974) is achieved, the dynamic importance of mass diffusivity remains the same and cannot be described by constant coefficients. Boundary layer theory is an adequate tool for describing the effect of turbulent diffusion of momentum, it fails when applied for the description of turbulent diffusion of mass.

Appendix A. Fourier Transform of the Ekman Current

If we write equations (17) and (18) as

$$u_E = u_{ES} + u_{EB}, \quad v_E = v_{ES} + v_{EB}$$

denoting by the index S the surface boundary layer, by B the bottom boundary layer, we can easily evaluate (22) for u_{ES} and v_{ES} (Oberhettinger (1957), p. 18/20) and find

$$U_{ES} = \frac{2\tilde{T}^x\zeta^2 + 4\tilde{T}^y}{\zeta^4 + 4} \qquad V_{ES} = -\frac{4\tilde{T}^x - 2\tilde{T}^y\zeta^2}{\zeta^4 + 4}$$

For the bottom boundary layer, the integral (22) has to be evaluated from 0 to H. By applying simple theorems for harmonic functions, we have

$$U_{EB} = -\frac{1}{2}\exp(-H)(\tilde{T}^x + \tilde{T}^y)$$

$$H\int_0^H \exp(z)\left[\cos(z(1+\zeta) - H) + \cos(z(1-\zeta) - H)\right]dz$$

$$-\frac{1}{2}\exp(-H)(\tilde{T}^x - \tilde{T}^y)$$

$$H\int_0^H \exp(z)\left[\sin(z(1+\zeta) - H) + \sin(z(1-\zeta) - H)\right]dz$$

This can be integrated (Gröbner and Hofreiter (1965), eq. 334.4 and 334.5) taking into account $H \gg 1$ and thus evaluating the integral at $z = 0$ only. The final result is

$$U_{EB} = -\tilde{T}^x \cdot \frac{\zeta^3\sin\zeta H + 2\zeta^2\cos\zeta H - 2\zeta\sin\zeta H}{\zeta^4 + 4}$$

$$-\tilde{T}^y \cdot \frac{\zeta^3\sin\zeta H + 2\zeta\sin\zeta H + 4\cos\zeta H}{\zeta^4 + 4}$$

and similarly

$$V_{EB} = \tilde{T}^x \cdot \frac{\zeta^3 \sin \zeta H + 2\zeta \sin \zeta H + 4 \cos \zeta H}{\zeta^4 + 4} - \tilde{T}^y \cdot \frac{\zeta^3 \sin \zeta H + 2\zeta^2 \cos \zeta H - 2\zeta \sin \zeta H}{\zeta^4 + 4}$$

Appendix B. Transformation of $W(\chi, \zeta)$ to $\tilde{W}(\chi, z)$

Since W is an odd function of ζ, we have

$$\tilde{W}(\chi, z) = \frac{2}{\pi}\int_0^{+\infty} W \sin \zeta z \, d\zeta = \frac{1}{\pi i}\int_{-\infty}^{+\infty} W \exp(i\zeta z)\, d\zeta$$

W has simple poles in the complex plane which for $\sigma S \ll 1$ have been given in MT:

$$\zeta_1 = -\zeta_5 = -\sqrt{\frac{\sqrt{\chi^4+4}-\chi^2}{2}} + i\sqrt{\frac{\sqrt{\chi^4+4}+\chi^2}{2}} = -\varphi^- + i\varphi^+$$

$$\zeta_2 = -\zeta_6 = +\sqrt{\frac{\sqrt{\chi^4+4}-\chi^2}{2}} + i\sqrt{\frac{\sqrt{\chi^4+4}+\chi^2}{2}} = \varphi^- + i\varphi^+$$

$$\zeta_3 = -\zeta_7 = (1+i)\sqrt[4]{\frac{\sigma S \chi^4}{\chi^4+4}} = (1+i)\psi \qquad \zeta_4 = -\zeta_8 = (-1+i)\sqrt[4]{\frac{\sigma S \chi^4}{\chi^4+4}} = (-1+i)\psi$$

Thus, we may write approximately

$$DN = (\zeta^2 + \chi^2 + 2i)(\zeta^2 + \chi^2 - 2i)(\zeta^2 + 2i\psi^2)(\zeta^2 - 2i\psi^2)$$

Writing the terms $\cos \zeta H$ and $\sin \zeta H$ as exponentials, the integrand reads

$$W \exp(i\zeta z) = A_0 \exp(i\zeta z) + A_1 \exp(i\zeta(z+H)) + A_2 \exp(i\zeta(z-H))$$

The path of integration in the complex plane consists of the path from $-\infty$ to $+\infty$ along the real axis and a semi circle in the upper half plane for the first two terms and the third term for $z - H > 0$. For $z - H < 0$ we take a semi circle in the lower half plane. The integral then is equal to $2\pi i$ times the sum of the residues in the upper half plane for positive imaginary parts in the arguments and $-2\pi i$ times the sum of residues in the lower half plane for negative imaginary parts. Thus we have

$$\tilde{W}(\chi, z) = \sum_{j=1}^{4} \frac{A_0(\zeta_j)\exp(i\zeta_j z) + A_1(\zeta_j)\exp(i\zeta_j(z+H))}{\zeta_j \prod_{m \neq j}(\zeta_j^2 - \zeta_m^2)} + \operatorname{sgn}(z-H) \cdot \sum_{j=1}^{4} \frac{A_2(\zeta_j)\exp(i\zeta_j(z-H))}{\zeta_j \prod_{m \neq j}(\zeta_j^2 - \zeta_m^2)}$$

which leads to (32).

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