Tide-well Systems II: The Frequency Response of a Linear Tide-well System[1]

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ABSTRACT

Recent uses to which tide records have been put have exposed limitations of the non-linear tide well with an orifice. By replacing the orifice with a long horizontal pipe connection, the system becomes essentially linear and the limitations of the conventional system no longer apply. The analysis in this paper shows that unsteady effects in the pipe flow produce the main features of the frequency response of the system; the hydrodynamic filtering of waves due to the depth of the pipe connection and the effect of the inertia of the water in the well are relatively unimportant. The response is governed by a tide-well parameter N, the optimum system having $N = 1/3$. For values of N less than $1/3$ the system amplifies certain wave components, and for values of $N \geq 5$ the response is similar to that given by a quasi-steady approximation. The linearity of the system depends on the pipe flow being laminar and criteria to assure this are given. Experimental results for some tide-well models show remarkable agreement with theoretical predictions. Three such systems installed in Australia before the development of this theory are critically discussed.

1. Introduction. Most tide gauges measure the level of water inside a circular well, the well being connected to the sea through an orifice. The disadvantages of this conventional type of tide well have been considered by Lennon (1967), Noye (1968, 1970, 1972, 1974), Cross (1968) and Halliwell and Perry (1969). Lennon discussed the recording of the water level in the well, the response of the system and the effect of the environment on the results obtained from the well. Halliwell and Perry were specifically concerned with the set-down of the well level due to tidal streams; Cross and Noye were concerned with the nature of the response of the conventional tide-well system.

The most important characteristic of the response of the conventional tide-well system is that it is non-linear; the water level in the well oscillates with the frequencies of the oscillations in the sea level, plus higher harmonics and

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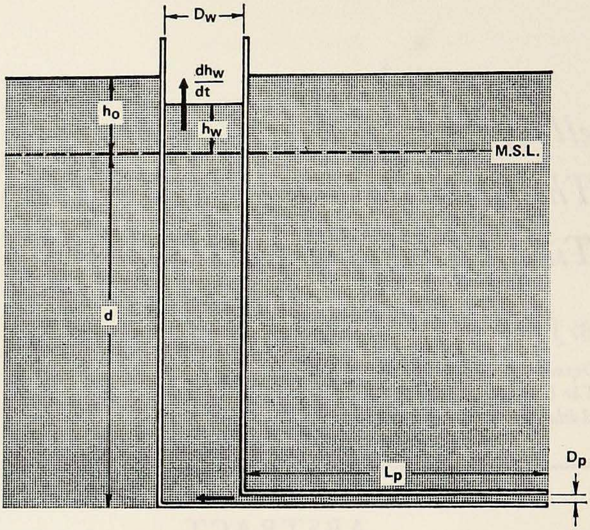

Figure 1. Notation used for the tide-well system with a long horizontal pipe connection near the sea floor.

oscillations at sum and difference frequencies, and the amplitudes of the resulting well oscillations are non-linear functions of the amplitudes of the sea-level oscillations. This non-linearity creates difficulties when measurements of harbour oscillations, excited by tsunamis or other causes, are to be read from tide-well records. These relatively high frequency oscillations are superimposed on the periodic rise and fall of the tide.

Problems also arise when records from a conventional tide-well system are analysed to determine tidal components. With an input which consists of superposed waves of different frequencies, such as the tidal components, it is not clear how much the non-linearity of the system contributes to the amplitudes of harmonics of these components, or to oscillations which occur at sum and difference frequencies of these components. It is likely that some of the energy attributed to shallow water components may not in fact be due to influences from outside the well, rather arising from non-linear effects at the orifice.

The tide well with an orifice also gives readings which are too low when wind waves and swell are present in addition to the tides. Noye (1974) used the method of Cross (1968) to show that the mean level in the well may be up to 15 cm lower than that in the surrounding sea in such a case. It is therefore not feasible to use records from such a system to find mean sea-level for land surveying.

The response of a tide well with a long horizontal pipe connection near the sea floor (see Fig. 1) is very different from that of the conventional tide well. Provided certain criteria are satisfied, the flow in the pipe is laminar and the pipe acts as a linear filter. The response function of such a system is independent of the incident wave amplitude and the principle of superposition of solution holds.

One estimate of the response function of such a system was made by O'Brien (1950) and similar results were derived by Noye (1968). In both papers, it was assumed that the Poiseuille formula for fully developed laminar flow in a pipe under a steady pressure gradient was a reasonable approximation for oscillatory laminar pipe flow with a given instantaneous pressure gradient; thus this theory may be described as quasi-steady. By this method the amplitude response α_1, the ratio of the amplitude of the well response to the amplitude of an incident wave of circular frequency ω, is found to be

$$\alpha_1 = (1 + \beta_2^2)^{-1/2} \tag{1.1}$$

and the phase lag at this frequency is

$$\varepsilon_1 = \arctan \beta_2 \tag{1.2}$$

where β_2 is a dimensionless frequency given by

$$\beta_2 = \left(\frac{32 v L_p D_w^2}{g D_p^4}\right)\omega. \tag{1.3}$$

Here v is the kinematic coefficient of viscosity of seawater ($1.2 \times 10^{-3}\text{cm}^2\text{-sec}^{-1}$ at sea temperature), L_p is the length of the horizontal pipe connection, D_p is the uniform diameter of this connection, D_w the uniform diameter of the well and g the gravitational acceleration.

In obtaining this estimate, no account was taken of attenuation of wave pressure due to the depth of the pipe beneath the sea surface, of the effect of unsteadiness on the flow through the pipe and of the effect of the inertia of the water in the well. These three effects have now been investigated and the findings are reported in this paper.

The hydrodynamic filtering due to depth reduces the amplitude response below the value given in equation (1.1) but leaves the phase lag unaffected. The effect of unsteadiness in the pipe flow is to increase the phase lag given in equation (1.2) at all frequencies and the amplitude response is higher at low frequencies and lower at high frequencies than the estimate (1.1). For large values of a parameter N, which is a function only of well and pipe dimensions (see equation (2.25)), the values of the amplitude response and phase lag are similar to the quasi-steady estimates α_1 and ε_1. For small values of N, the amplitude response can be greater than unity at low frequencies. Finally, at low frequencies, the effect of the inertia of the water in the well is to slightly increase the amplitude response if the phase lag is less than $\pi/2$ and decrease this response if the lag is greater than $\pi/2$.

The first and last of these effects are not important until high frequencies are reached, when the amplitude response is already very small. On the other hand, for moderate values of N the unsteadiness in the pipe flow has significant effects at all frequencies and this is the major conclusion of the present paper.

Criteria for the transition from laminar to turbulent flow in a long pipe under unsteady conditions under which the system remains linear have been established. Furthermore, it is shown that, unlike the conventional tide-well system, there is no difference in mean level inside and outside the well in the presence of wind waves.

Using these results, three different models were designed and constructed for testing in the laboratory. Experimentally determined estimates of the response functions of these models were found by simulating a long ocean wave of known amplitude and frequency outside the well and measuring the response of the water level in the well. All experimental results agreed exceptionally well with the theoretical values.

Three tide-well systems with a pipe connection have been in use in Australia for some years. All three have peculiar characteristics in their frequency responses which could have been predicted if the results of the present paper had been available at the time they were installed. One is a well and pipe system used to measure water level oscillations in lagoons in the south-east part of South Australia. The other two are used to measure tides, at Macquarie Island in the Southern Ocean, and at Cairns in Queensland.

2. Theoretical Frequency Response. Consider a tide well with a long horizontal pipe connection close to the ocean floor (Fig. 1). The mean sea level will be taken as datum for all measurements, the pipe being a depth d below this level. Consider the response h_w of the water in the well to an oscillation, the real part of

$$h_0 = X_0 e^{i\omega t} \tag{2.1}$$

of the sea level about the mean, t being the time and ω the angular frequency of the oscillation. Since the system is linear, we may write

$$h_w = Z_w e^{i\omega t}, \tag{2.2}$$

where $|Z_w|/|X_0|$ is the amplitude response and $\arg(X_0/Z_w)$ is the phase lag of h_w relative to h_0.

For long period waves, such as harbour oscillations and tides, the pressure p at the ocean end of the pipe is

$$p_0 = p_a + \varrho g d + \varrho g k_0 h_0, \tag{2.3}$$

where p_a is the atmospheric pressure, ϱ is the density of sea-water and k_0 is the pressure transmission factor introduced because of the hydrodynamic filtering occurring through the depth d. This factor is given by

$$k_0 = \operatorname{sech}(\varkappa d), \tag{2.4}$$

(e.g. Kinsman 1965 p. 143) where $\varkappa$ is the wave number. Equation (2.4) is only an approximation for wind waves which are considered in detail in Section 4.

In order to estimate the pressure p_w at the tide-well end of the pipe, assume that at any instant the velocity field in the well is constant throughout and is equal to that of the surface, dh_w/dt. Application of Bernoulli's theorem for unsteady flow with velocity potential xdh_w/dt, x being the vertical co-ordinate above the pipe, yields

$$x\frac{d^2h_w}{dt^2} + \frac{1}{2}\left(\frac{dh_w}{dt}\right)^2 + gx + p/\varrho = F(t) \tag{2.5}$$

throughout the well. Equating values at the water surface and at the pipe opening gives

$$p_w = p_a + \varrho g d + \varrho g\left(h_w + \frac{d}{g}\cdot\frac{d^2h_w}{dt^2}\right), \tag{2.6}$$

assuming that the amplitude of the incident waves is much smaller than the mean depth of the sea, i.e. $h_w \ll d$.

The pressure gradient along the pipe into the well is

$$P = (p_w - p_0)/L_p, \tag{2.7}$$

which, on substitution for p_w, p_0, h_0 and h_w from (2.6), (2.3), (2.1) and (2.2), respectively, becomes

$$\left.\begin{aligned} P &= \varrho g L_p^{-1}\left\{h_w + \frac{d}{g}\cdot\frac{d^2h_w}{dt^2} - k_0 h_0\right\} \\ &= \varrho g L_p^{-1}\{k_w Z_w - k_0 X_0\}\, e^{i\omega t}. \end{aligned}\right\} \tag{2.8}$$

The factor

$$k_w = 1 - \omega^2 d/g \tag{2.9}$$

incorporates the correction for the inertia of the water in the tide well.

Since the pressure gradient in (2.8) has the form

$$P = Ae^{i\omega t}, \tag{2.10}$$

where

$$A = \varrho g L_p^{-1}\{k_w Z_w - k_0 X_0\}, \tag{2.11}$$

an exact solution for the unsteady laminar flow of an incompressible viscous fluid in a pipe, due to a pressure gradient sinusoidal in time, will now be used.

The result for the velocity profile due to the pressure gradient (2.10) can be written (e.g. Lambossy, 1952)

$$v(r,t) = \frac{iA}{\varrho\omega}\left\{1 - \frac{J_0(i^{3/2}\sigma r/R_p)}{J_0(i^{3/2}\sigma)}\right\} e^{i\omega t}, \tag{2.12}$$

where r is the radial distance from the centre line of the pipe, R_p the pipe radius, and

$$\sigma^2 = \left(\frac{D_p^2}{4v}\right)\omega. \tag{2.13}$$

Then the instantaneous flux

$$Q(t) = 2\pi \int_0^{R_p} v(r,t) r dr, \tag{2.14}$$

yields, on using the relation $\int x J_0(x) dx = x J_1(x)$,

$$Q(t) = \frac{\pi D_p^4}{16 v \varrho} \cdot \frac{Ai}{\sigma^2}\left\{1 - \frac{2 J_1(\sigma i^{3/2})}{\sigma i^{3/2} J_0(\sigma i^{3/2})}\right\} e^{i\omega t}. \tag{2.15}$$

The steady Poiseuille result is obtained in the limit $\sigma \to 0$.

However, by continuity,

$$\frac{\pi D_w^2}{4}\frac{dh_w}{dt} = Q. \tag{2.16}$$

Combining (2.2), (2.11), (2.15) and (2.16) gives

$$Z_w = \frac{R(\sigma)}{i\beta_2}(k_w Z_w - k_0 X_0) \tag{2.17}$$

where

$$R(\sigma) = \frac{8i}{\sigma^2}\left\{1 - \frac{2 J_1(\sigma i^{3/2})}{\sigma i^{3/2} J_0(\sigma i^{3/2})}\right\}. \tag{2.18}$$

It follows that

$$\frac{Z_w}{X_0} = \frac{k_0}{k_w - i\beta_2 R^{-1}}. \tag{2.19}$$

Therefore a better estimate of the amplitude response for this tide-well system than that given by (1.1) is

$$\alpha_2 = k_0 |k_w - i\beta_2 R^{-1}|^{-1}, \tag{2.20}$$

with corresponding phase lag ε_2 in $[0,\pi]$ given by

$$\varepsilon_2 = \arg(k_w - i\beta_2 R^{-1}). \tag{2.21}$$

a. *The Effect of Unsteadiness in the Pipe Flow.* If $\omega^2 d/g \ll 1$, it is clear that $k_0 \simeq 1$ and $k_w \simeq 1$; this is the usual condition in practice, e.g. if the pipe is less than 10 metres below mean sea level, periods greater than one minute give $\omega^2 d/g < 0.01$. The response function then becomes

$$Z_w/X_0 = (1 - i\beta_2 R^{-1})^{-1},$$

yielding an amplitude response of

$$\alpha_2 = |1 - i\beta_2 R^{-1}|^{-1}, \tag{2.22}$$

and a phase lag of

$$\varepsilon_2 = \arg(1 - i\beta_2 R^{-1}). \tag{2.23}$$

However

$$\sigma = (\beta_2/N)^{1/2}, \tag{2.24}$$

where

$$N = \frac{128 v^2 L_p D_w^2}{g D_p^6}, \tag{2.25}$$

so that $R(\sigma) = R(\beta_2^{1/2} N^{-1/2})$ is a function of the dimensionless frequency β_2 and the dimensionless tide-well parameter N. Hence α_2 and ε_2 are also functions of these parameters.

The approximations for small β_2 and fixed N are

$$\alpha_2 = 1 - \frac{\beta_2}{2}\left(1 - \frac{1}{3N}\right) + O(\beta_2^4), \tag{2.26}$$

and

$$\varepsilon_2 = \beta_2 + O(\beta_2^3), \tag{2.27}$$

from which it is clear that for low frequencies the amplitude response exceeds unity if $N < \frac{1}{3}$ and tends to unity as $\beta_2 \to 0$.

The asymptotic expansions for large β_2 and fixed N are

$$\alpha_2 = \frac{8N}{\beta_2^2}\left\{1 - \left(\frac{2N}{\beta_2}\right)^{1/2}\right\} + O(\beta_2^{-3}), \tag{2.28}$$

with

$$\varepsilon_2 = \pi - \left(\frac{2N}{\beta_2}\right)^{1/2} + O(\beta_2^{-1}). \tag{2.29}$$

The amplitude response thus diminishes rapidly at high frequencies while the phase lag tends to π. The smaller is the value of N, the sharper will be the cut-off in the amplitude response α_2.

Also, for the values of the frequency which are of interest in tidal work, equation (2.24) shows that the larger is the value of N, the more nearly correct is the quasi-steady approximation ($\sigma = 0$) when the amplitude response and the phase lag will be given by the estimates (1.1) and (1.2).

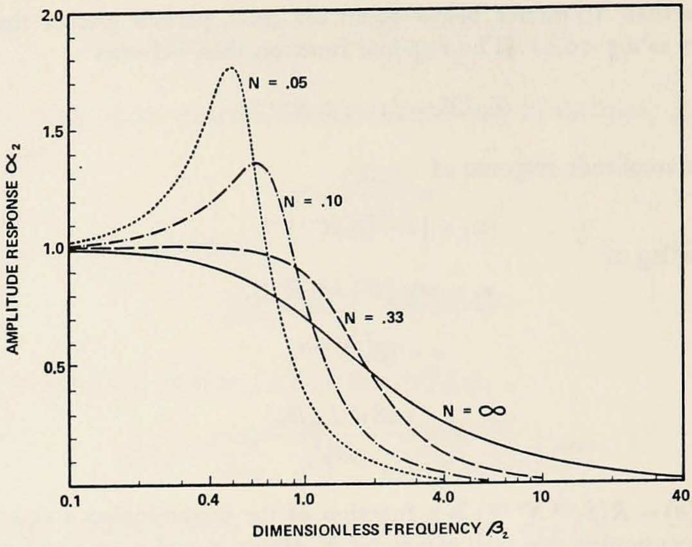

Figure 2. The amplitude response due to pipe effects, allowing for unsteadiness in the flow.

The values of the real and imaginary parts of $R(\sigma)$ are obtained in terms of Kelvin functions (see Jahnke, Emde and Lösch, 1960), from which α_2 and ε_2 are found using

$$\alpha_2 = \left\{ 1 + \beta_2^2 q^{-2} - 2\beta_2 q^{-1} \sin\theta \right\}^{-1/2} \tag{2.30}$$

and

$$\tan \varepsilon_2 = \cos\theta \,(q\beta_2^{-1} - \sin\theta)^{-1}, \tag{2.31}$$

where

$$R = qe^{i\theta}. \tag{2.32}$$

Graphs of α_2 and ε_2 against β_2 (Figs. 2 and 3) illustrate clearly the properties of the response functions referred to earlier. For $N < \frac{1}{3}$, the amplitude response is greater than unity at low frequencies; the smaller is the value of N, the higher is the peak in the amplitude response. For all values of N, the phase lag is almost zero at low frequencies and tends to π at high frequencies. For large values of N, both the amplitude response and the phase lag are close to the quasi-steady estimates, provided β_2 is not too large.

With regard to amplitude response, the optimum value of N is $\frac{1}{3}$. For such a system Fig. 2 shows that there is no amplification of waves, unit response is retained for the greatest range of β_2 and the cut-off is the sharpest. The value of α_2 remains near unity until $\beta_2 = 1$, when $\alpha_2 = 0.95$, then falls rapidly to $\alpha_2 = 0.1$ when $\beta_2 = 4$. A suitable choice of well and pipe dimensions will produce near unit response up to any desired frequency. For instance, with $N = \frac{1}{3}$, a requirement that tides be recorded with no attenuation for all periods greater than one hour implies that $\beta_2 = 1$ corresponds to a period of approximately

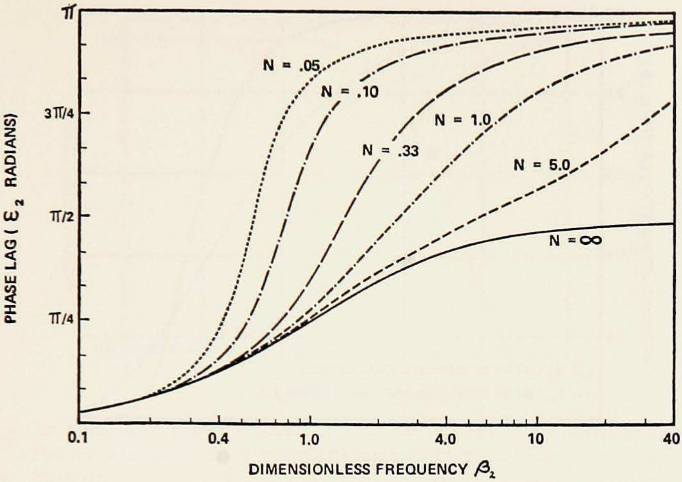

Figure 3. The phase lag due to pipe effects, allowing for unsteadiness in the flow.

30 minutes. In such a case $\beta_2 = 4$ corresponds to a period of 7½ minutes, so that as the period decreases from 30 to 7½ minutes the amplitude response falls from a value of 0.95 to 0.1.

From Fig. 3, it is clear that some phase lag occurs for most values of β_2. In the example just given, a wave with a period of one hour corresponds to $\beta_2 = 0.5$, so that the phase-lag is nearly $\pi/6$ radians and the time lag is nearly 6 minutes. The same time lag applies for both semi-diurnal and diurnal tides. For low frequencies (2.27) shows that the phase-lag approximates β_2 radians, irrespective of the value of N. For tidal frequencies the time lag is therefore

$$\tau = \omega^{-1}\beta_2, \tag{2.33}$$

which yields on application of (1.3),

$$\tau = \frac{32\nu L_p D_w^2}{g D_p^4}, \tag{2.34}$$

i.e. the time lag depends only on the tide-well dimensions. As the system is linear, the same lag applies through the complete cycle for any tidal component. This is in contrast to the conventional tide-well system in which the lag increases with increasing wave amplitudes and varies through the wave cycle (Noye, 1968). With the linear system, accurate corrections for phase lag can be applied to give the correct time of occurrence of tidal extremes and the correct phases of tidal constituents.

b. *Hydrodynamic Filtering due to Depth and Effect of Inertia of Water in the Well.* If $\omega^2 d/g$ is not small, then the effect of k_0 and k_w on the response function can no longer be neglected.

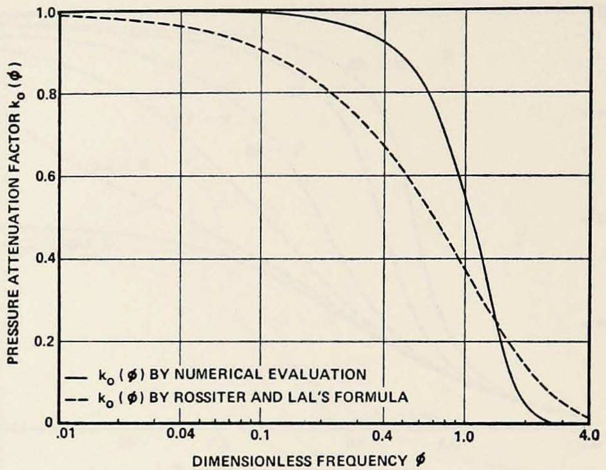

Figure 4. The pressure attenuation factor.

The pressure transmission factor k_0 has no effect on the phase lag ε_2; it simply multiplies the amplitude response α_2. The dispersion relation

$$\omega^2 = g\varkappa \tanh(\varkappa d) \tag{2.35}$$

leads to

$$(\varkappa d) \tanh(\varkappa d) = \varphi^2, \tag{2.36}$$

where φ is a dimensionless frequency given by

$$\varphi = \omega(d/g)^{1/2}. \tag{2.37}$$

Combining (2.4) with (2.36), we can express k_0 as a function of φ.

Rossiter and Lal (1963) developed an approximate formula for the attenuation of pressure with depth, viz.

$$k_0 = \exp(-\varphi), \tag{2.38}$$

by substituting the shallow water approximation for (2.35) in the deep water approximation for the decay of wave pressure with depth. In Fig. 4, this approximation is compared with the exact evaluation of (2.4) with (2.36). Large differences occur for intermediate values of φ; for waves with period 6.3 seconds in 10 metres of water the pressure transmission factor is 0.36 by the exponential approximation and 0.55 by the exact evaluation. A rapid decrease in the value of the pressure transmission as φ increases through unity is clear from Fig. 4; this causes the amplitude response for large values of β_2 to decrease even more rapidly than shown in Fig. 2.

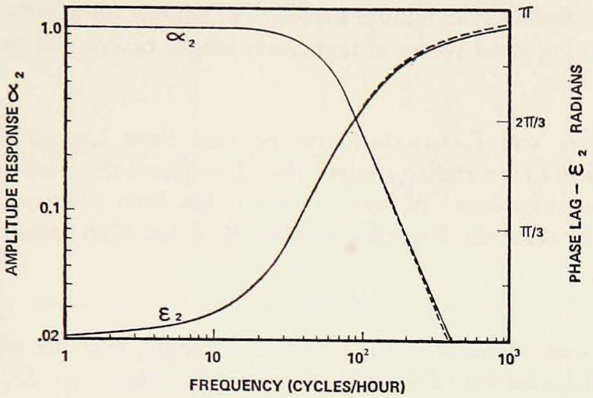

Figure 5. An "optimum" tide-well system with $N = {}^1/_3$ for use in ten metres of water. The solid line indicates the contribution of the pipe to the response, the dashed line is the complete response.

The effect of the inertia term k_w, given by

$$k_w = 1 - \varphi^2, \tag{2.39}$$

is to change both the phase lag ε_2 and the amplitude response α_2. The complete response, given by equations (2.20) and (2.21), becomes

$$\alpha_2 = k_0 \{k_w^2 + \beta_2^2 q^{-2} - 2 k_w \beta_2 q^{-1} \sin \theta\}^{-1/2} \tag{2.40}$$

and

$$\tan \varepsilon_2 = \cos \theta (k_w q \beta_2^{-1} - \sin \theta)^{-1}, \quad \varepsilon_2 \text{ in } [0,\pi]. \tag{2.41}$$

However, it must be realised that the approximations used in deriving equation (2.6) imply that $\omega^2 d/g \ll 1$. Consideration of (2.40) and (2.41) shows that, for values of φ which are small compared to unity, the effect of k_w is to slightly increase α_2 when the value of the frequency is small enough that $\varepsilon_2 < \pi/2$ and to decrease α_2 when $\varepsilon_2 > \pi/2$.

Therefore, at low frequencies, the effect of the inclusion of both k_0 and k_w in the amplitude response α_2 is very small, particularly since one tends to compensate for the other. At higher frequencies, they combine to cause a reduction in the amplitude response. This is seen in Fig. 5 where, for the tide-well in question, the complete amplitude response is shown as the dashed line, whereas the result of ignoring the pressure attenuation and the inertia of the water in the well is shown by the solid line. This situation corresponds to the extreme case when the pipe is deep, approximately ten metres below mean sea-level, and the effects of pressure attenuation and inertia of water in the well are even more pronounced than is to be expected normally. The tide well was designed to permit unattenuated linear recording of tides with maximum range of 2 metres and suspected four minute harbour oscillations up to 20 cm in height, while suppressing wind waves of maximum height 60 cm and greatest period 8 seconds.

3. Conditions for Laminar Flow in the Pipe Connection. It has been assumed in the preceding analysis that throughout the whole pipe connection the flow is laminar. In other words it has been assumed, firstly, that throughout the pipe, the Reynolds number Re is less than some critical value Re^*, where

$$Re = VD_p v^{-1}, \tag{3.1}$$

V being the mean velocity in the pipe and, secondly, that the end effects are negligible. Minimisation of the end effects requires that $L_p \gg D_p$, the usually accepted condition being (Duncan et.al., 1960, p. 397)

$$L_p > 100\, D_p. \tag{3.2}$$

Little is known about the stability of time dependent laminar flow in pipes, although it has been demonstrated by Combs and Gilbrech (1964) and others, that modulation of steady flow in a pipe generally has a stabilising effect. The only measurements of critical Reynolds numbers for oscillatory pipe flow about a zero mean velocity appear to be those of Binnie (1945), Ury (1962) and Sergeev (1966). Binnie's results cover only a very limited range of the frequency parameter and, further, the Reynolds number used was not defined. The results obtained by Ury are not directly applicable, because a U-tube was used rather than a long straight pipe and the oscillations were damped and hence not exactly periodic. Therefore Sergeev's results for forced oscillatory flow in a vertical pipe are the most appropriate. Fig. 6 is a plot of his values of Re^* against the parameter σ in the case of undamped oscillations.

Ury (1962) also used a semi-empirical approach to determine the critical Reynolds number for oscillatory flow in a U-tube, by finding the "kinetic" Reynolds number R_k at which the friction factors for oscillatory laminar flow and turbulent flow are equal. He introduced the concept of a "kinetic" Reynolds number, in order to define an equivalent Reynolds number for oscillatory flow which is equal to that for a steady flow with the same average kinetic energy. It was defined by the relation

$$R_k = \delta\, Re, \tag{3.3}$$

where $\delta^2 > 1$ is an apparent-mass factor for the liquid in the U-tube. This factor occurs because the distribution of flow velocities over the cross-section of a tube leads to a total amount of kinetic energy which is always greater than that computed on the assumption of constant velocity, equal to the arithmetic mean, over the entire cross-section.

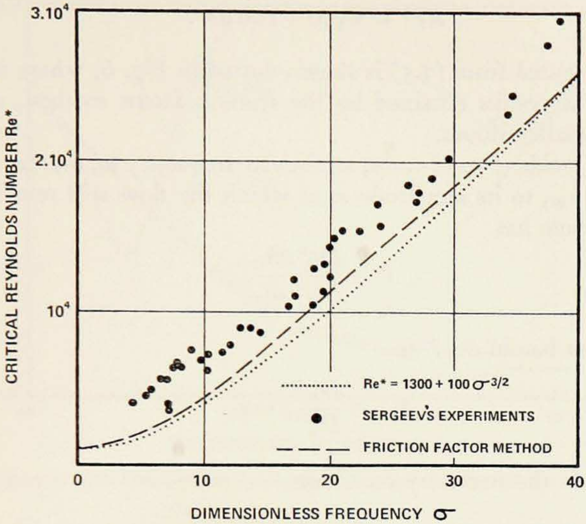

Figure 6. The critical Reynolds number for oscillatory flow in a long pipe.

Ury's method can be applied to oscillatory laminar flow in a long pipe by using

$$\delta^2 = \left(2\int_0^{R_p} v^3\,dr\right)\Big/V^3 R_p^{\,2}, \tag{3.4}$$

where v is the velocity in the pipe, given by (2.12). The corresponding critical Reynolds numbers, for different values of σ computed in this way, are shown by a dashed line in Figure 6. This estimate is slightly lower than Sergeev's experimental values, applicable for $\sigma \geq 4$, and lower than the accepted empirical value $Re^* \simeq 2000$ for steady pipe flow. For steady flow, the friction factor method yields $Re^* \simeq 1330$.

Smith (1960) and other authors consider that turbulent flow may commence in the inlet region and be carried into the pipe. Although little is known about inlet conditions for unsteady flow, a theoretical study of steady flow by Tatsumi (1952) yielded a stability limit with a minimum critical Reynolds number of 9700 in the entrance region. However, disturbed conditions at the entry do not imply turbulent flow throughout a pipe; unless the magnitude of a disturbance is very large, it will die out as it passes along the pipe. Patel and Head (1968) quote many cases when reverse transitions may occur, i.e., when turbulent flow reverts to laminar flow as a result of favourable pressure gradients.

Basing the criteria for laminar pipe flow on the results of Sergeev's experiments and the results of the friction factor method, one may suggest for the critical Reynolds number, in the range $0 < \sigma < 40$, the empirical formula

$$Re^* = 1300 + 100\,\sigma^{3/2}. \tag{3.5}$$

The curve computed from (3.5) is shown dotted in Fig. 6, where it is seen to be similar to the results obtained by the friction factor method, although it takes slightly smaller values.

For a single incident wave-train, at circular frequency ω, one can determine an upper limit a_m, to its amplitude a, at which the flow will remain laminar. By continuity, one has

$$V = \frac{D_w^2}{D_p^2}\frac{dh_w}{dt}, \tag{3.6}$$

where the upper bound on V is

$$V_m = \frac{D_w^2}{D_p^2}\,a\,\omega\,\alpha_2. \tag{3.7}$$

Substitution in the necessary condition

$$Re \leq Re^* \tag{3.8}$$

yields, by (3.1),

$$a \leq \frac{D_p v}{D_w^2}\cdot\frac{Re^*}{\omega\alpha_2}. \tag{3.9}$$

The right-hand side of this inequality gives

$$a_m = \left(\frac{32\,v^2 L_p}{g D_p^3}\right)\frac{Re^*}{\beta_2\,\alpha_2}. \tag{3.10}$$

Fig. 7 shows graphs of the dimensionless amplitude

$$a_m^* = a_m\left(\frac{g D_p^3}{32\,v^2 L_p}\right) \tag{3.11}$$

plotted against β_2, for different values of N.

The upper bound for the permissible recorded amplitude is given by the right-hand side of the relation

$$a\,\alpha_2 \leq \frac{D_p v}{D_w^2}\frac{Re^*}{\omega}. \tag{3.12}$$

When n wave trains of amplitude a_r and circular frequency ω_r are superposed, then the maximum value of dh_w/dt is

$$\left(\frac{dh_w}{dt}\right)_{\max} \leq \sum_{r=1}^{n} a_r\,\omega_r\,\alpha_2(\omega_r). \tag{3.13}$$

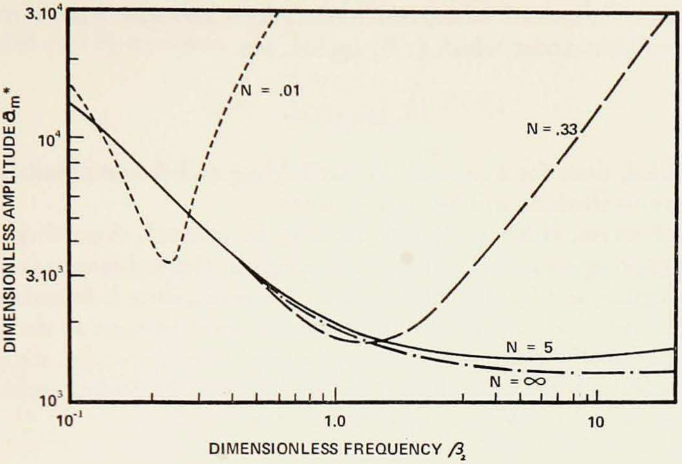

Figure 7. The dimensionless amplitude $a_m{}^*$ graphed against β_2 for various N.

It follows that V_m, the maximum value of the mean pipe velocity, satisfies the inequality

$$V_m \leq \frac{D_w{}^2}{D_p{}^2} \sum_{r=1}^{n} a_r \omega_r \alpha_2(\omega_r). \tag{3.14}$$

The maximum possible Reynolds number for the pipe flow in the system is therefore

$$Re_{\max} = \frac{D_w{}^2}{v D_p} \sum_{r=1}^{n} a_r \omega_r \alpha_2(\omega_r). \tag{3.15}$$

If

$$Re_{\max} < 1300, \tag{3.16}$$

laminar pipe flow will be assured. This criterion for laminar flow is, in fact, rather severe, since in (3.14) V_m is usually much less than the right-hand side. For applications to normal tide wells, consideration of only dominant tidal components, harbour oscillations and wind waves in (3.15), will give a good estimate of $Re_{\max}$.

4. The Effect of Wind Waves. Formulae (2.3) and (2.4), for the pressure at the sea end of the pipe, imply that the wave amplitude $|h_0|_{\max}$ satisfies the two relations

$$\varkappa |h_0|_{\max} \ll 1 \tag{4.1}$$

and

$$|h_0|_{\max} \ll d. \tag{4.2}$$

For long period waves such as tides, tsunamis, and harbour oscillations, when amplitudes are much smaller than wave-lengths, (4.1) is satisfied; if the pipe lies in sufficient depth of water, condition (4.2) is also met. Hence it follows that the condition under which (2.6) applies, viz.

$$|h_w|_{\max} \ll d, \tag{4.3}$$

is also fulfilled, since for a normal tide well $N \geq \frac{1}{8}$ and the amplitude response to the above oscillations will be at most unity.

For wind waves, either (4.1) or (4.2) may be violated. Assuming that the amplitude response of a well to waves is negligible, the oscillations in the well will be negligible, and (4.3) will be satisfied. However, since infinitesimal wave theory no longer applies, expression (2.3) for the wave pressure at the sea floor has additional small terms added to the right hand side. Typically, the pressure at a depth y beneath the mean wave level can be given by Stokes' second order wave theory (see, e.g., Wiegel, 1964 p. 31), viz.

$$p = \varrho g(y + C_1 \cos \omega t + C_2 \cos 2\omega t + C_3), \tag{4.4}$$

for a wave of circular frequency ω. C_1, C_2, and C_3, are coefficients dependent upon wave height H, wave number $\varkappa$, and the depths d and y, e.g.

$$C_3 = -\frac{\varkappa H^2}{8} \operatorname{cosech}(2\varkappa d) \cosh 2\varkappa(d-y). \tag{4.5}$$

The tide well will have negligible response to the harmonic of frequency 2ω, but the term C_3, being a non-zero constant term present so long as the waves persist, will have an apparent effect of lowering the mean level in the well relative to the mean level outside. Cross (1968) used this second order wave theory to show that the difference in mean levels is large in the case of the non-linear conventional tide-well system.

However, it must be remembered that (4.4) is an expansion only to second order and that further terms in the expansion may change the nature of the term C_3. As an alternative we may use the result of Longuet-Higgins and Stewart (1964), who derived the exact expression

$$\bar{p} = \varrho g y - \varrho \overline{w^2}, \tag{4.6}$$

where p is the pressure and w is the vertical velocity due to the waves at the depth y. The bar implies time-averaging.

Following the same reasoning as Cross, we consider an "equilibrium" time period, long compared with the wind wave period yet short enough to consider that the tide level remains constant. During this time the net flow into the tide-well is zero, i.e.

$$\overline{Q} = 0. \tag{4.7}$$

For the linear tide-well system, one may combine equations (2.7), (2.10), (2.15) and (2.18) to obtain

$$Q = \frac{\pi D_p^4 (p_w - p_0)}{128 \nu \varrho L_p} \cdot R(\sigma), \tag{4.8}$$

which on substitution into (4.7) yields

$$\bar{p}_0 - \bar{p}_w = 0. \tag{4.9}$$

When the system has practically no response to wind waves, one may write

$$\bar{p}_w = \varrho g \overline{H}_w, \tag{4.10}$$

where $\overline{H}_w$ is the mean depth inside the well over the "equilibrium" time period. If $\overline{H}_0$ is the corresponding mean level outside the well, use of (4.6) and (4.10) in (4.9) gives the difference between these mean levels as

$$\overline{H}_0 - \overline{H}_w = \overline{w^2}/g. \tag{4.11}$$

Provided the pipe is on the ocean floor, one has $w = 0$, and the mean levels are the same.

Application of this method to the conventional tide-well system, as in Noye (1974), is difficult because Q is a non-linear function of the pressure difference $(p_0 - p_w)$ and time averaging does not lead to a simple expression of the form (4.9).

5. Design Criteria. Using the results of the previous sections, tide-well dimensions can be chosen to give any desired response. The value of N selected defines the type of response (see Figs. 2 and 3) and the non-dimensional frequency β_2 which corresponds to the desired cut-off frequency f cycles/hour determines the pipe diameter D_p. Equations (1.3) and (2.25), together with

$$\omega = 2\pi f/3600, \tag{5.1}$$

give

$$D_p = \left\{\frac{7200 \nu \beta_2}{\pi N f}\right\}^{1/2}. \tag{5.2}$$

A suitable value of the pipe length L_p is then chosen so that (3.2), i.e. $L_p > 100 D_p$, is satisfied. The well diameter D_w is then calculated from a rearranged form of (2.25), viz.

$$D_w = \left\{\frac{g N D_p^6}{128 \nu^2 L_p}\right\}^{1/2}. \tag{5.3}$$

Larger values of L_p reduce end effects in the pipe flow as well as increasing the maximum permissible amplitude a_m of the incident oscillation, since from (3.2)

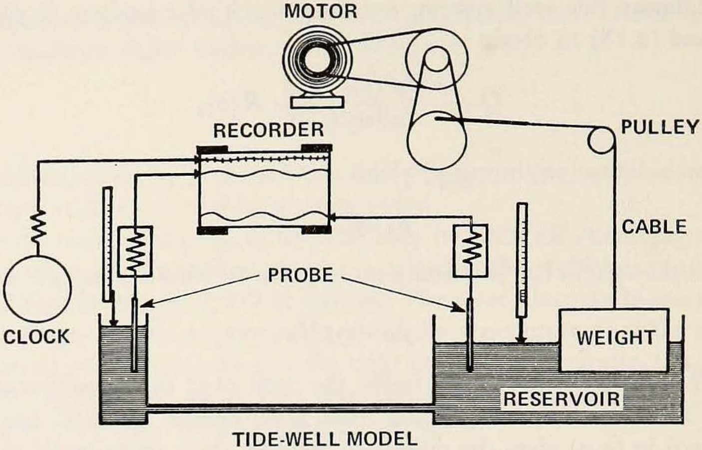

Figure 8. Schematic diagram of the apparatus used for experimental work.

we have $a_m \propto L_p$. However, the pipe length L_p cannot be increased indefinitely without the well diameter D_w becoming impractically small. Normally, D_w must be large enough to get a float into the well for attachment to a recorder, although other methods of sensing the water level in the well can be used.

Finally, the maximum permitted amplitude for linear recording must be compared with expected tidal amplitudes. In practice, the simplest way to finalise the decision on dimensions is to use a computer program which gives the complete frequency response for the chosen dimensions together with the values of the greatest incident amplitude allowed.

The above procedure was used to select the dimensions of the tide wells used in a series of experiments described in the next section.

6. Experimental Investigation. Amplitude responses and phase lags of three types of model tide wells with pipe connections were determined experimentally and compared with theoretically predicted values.

The model tide well was connected to a large reservoir of water (Fig. 8) and the method used was similar to that described fully in Noye (1974). As in the previous experimental work, the probes were recalibrated before each test, by comparing chart readings with 2 mm steps in water level. Again, after running the apparatus long enough to eliminate starting transients, at least ten oscillations of the input were recorded and the ratio of the output to input amplitudes and the lag of the output behind the input, measured.

In the choice of the dimensions of the models, attention was concentrated on responses in the frequency band 10-360 cycles/hour, since these were most conveniently produced in the laboratory. Furthermore, since the amplitudes to

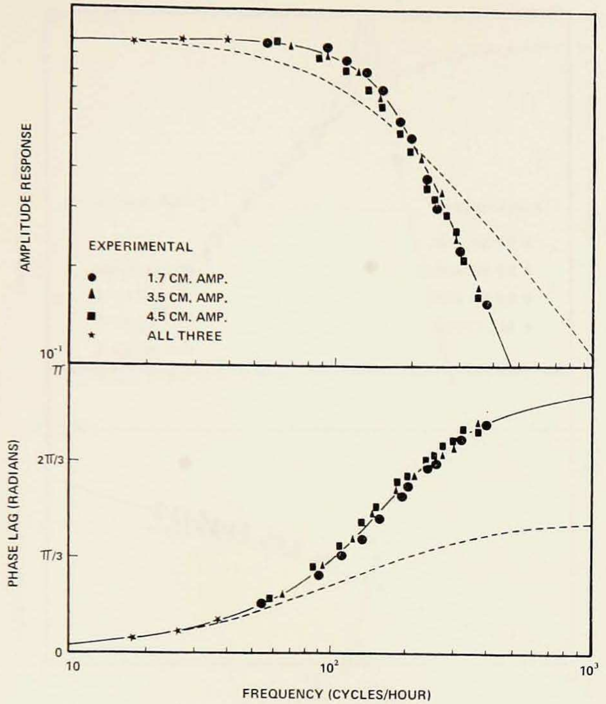

Figure 9. The theoretical amplitude response and phase lag (solid lines) compared with the experimental results (symbols) for the model with $N = {}^{1}/_{3}$. The dashed lines are the quasi-steady estimates.

be used in the experiments were to range from one to four centimetres, the models had to be linear for amplitudes up to approximately four centimetres.

For the first model tested, the selection $N = \frac{1}{3}$ was made. This is an optimum system; Fig. 2 shows that in this case a unit amplitude response is retained for the greatest range of the non-dimensional frequency β_2. With a pipe diameter 8.4 mm, pipe length 3.55 metres and well diameter 5.2 cm, the model had the theoretical amplitude response and phase lag shown by the solid line in Fig. 9. The experimental results are plotted on the same figure, three different symbols being used for the three different amplitudes of input. The results obtained between the frequencies 80 to 200 cycles/hour at the largest input amplitude are slightly lower than the others and in fact violated the criterion for laminar flow established in Section 3.

For the second model tested, the choice was $N = 5$. For $N \geq 5$, the theoretical response is similar to the quasi-steady approximation. This time the pipe diameter was 2.7 mm, the pipe length 27 cm and the well diameter 2.1 cm. The theoretical amplitude response and phase lag are shown in Fig. 10, together with the experimental results and the quasi-steady estimates. In this

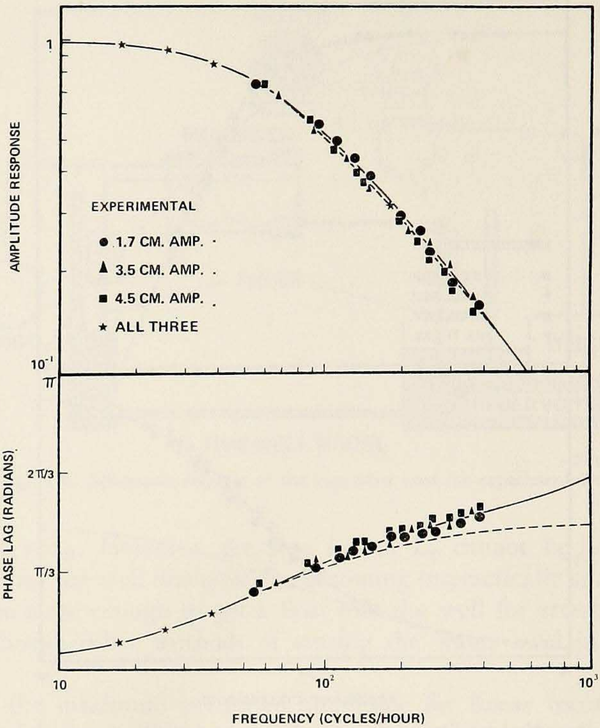

Figure 10. The theoretical amplitude response and phase lag (solid lines) compared with the experimental results (symbols) for the model with $N = 5$. The dashed lines are the quasi-steady estimates.

model, the system remained linear for all amplitudes up to 7 cm, irrespective of frequency.

The third model, with $N = 0.004$, had a peak amplitude response of 4.0. The dimensions were chosen such that this peak occurred at a frequency of 140 cycles/hour; the pipe diameter was 2.69 cm, the pipe length 15 metres and the well diameter 8.1 cm. The theoretical response curves and the experimental results are shown in Fig. 11. The larger amplitudes in the frequency range 120 to 180 cycles/hour violated the criterion for laminar flow, as well as the criterion (2.6) since $|h_w|_{\max}/d$ was of the order 0.3 at the frequency of greatest gain. The non-linearities introduced thereby may explain the fact that the experimental values are slightly low. For the two other models, the typical value of the ratio $|h_w|_{\max}/d$ was 0.05.

For all three models, the experimental results agree extra-ordinarily closely with the theoretical curve, provided the conditions for laminar flow were met. Even when pockets of turbulence were probably produced during part of a cycle, such as in the region of the peak response of Fig. 11, the

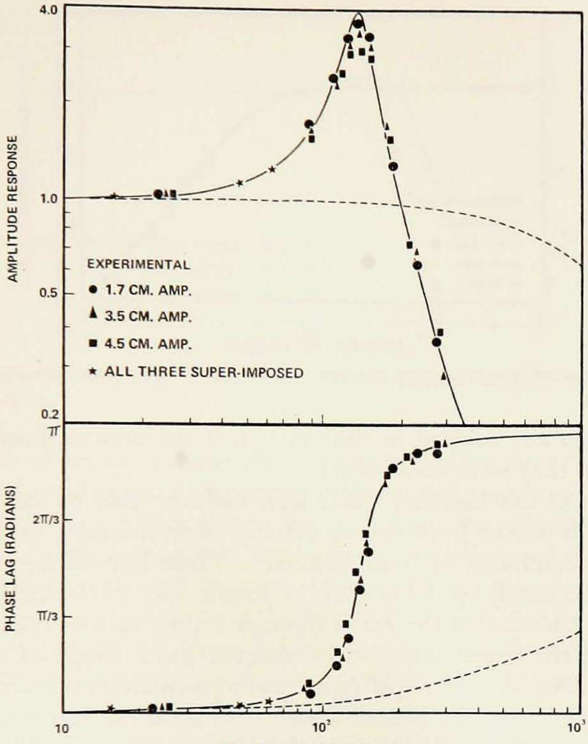

Figure 11. The theoretical amplitude response and phase lag (solid lines) compared with the experimental results (symbols) for the model with $N = .004$. The dashed lines are the quasi-steady estimates.

agreement is good. Perhaps the most remarkable feature of the experimental results is the verification of the gain of 4.0, predicted for the model with $N = 0.004$.

The linearity of the system is evident, since the amplitude response and phase lag were practically unchanged at a given frequency, when there were large changes in the input amplitude. The linearity was further tested by digitising long sections of the input and output records and subjecting the resulting time series to Fourier analysis on a CDC 6400 computer. In all cases, the results indicated that the input was essentially a single sinusoid in time and the output contained no significant higher harmonics of the fundamental frequency, except in those cases which exceeded the limits for laminar pipe flow.

7. Analysis of Some Actual Measuring Systems. Three tide wells of this kind have been in use in Australian waters for some time. After installation, each of these showed peculiar response characteristics which could have

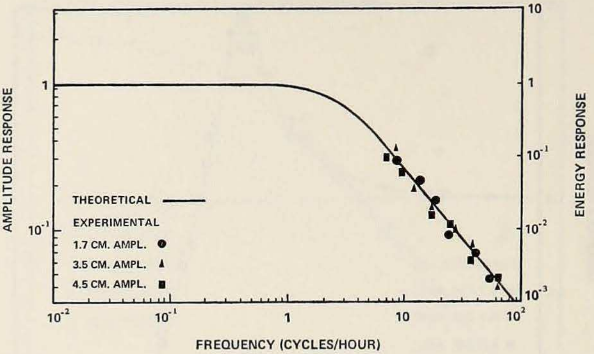

Figure 12. The amplitude and energy response of the water level indicator used at the Coorong.

been anticipated and avoided in their design, if the present theory had been available when they were constructed.

One system of this kind is a water level indicator used to measure surface oscillations with periods from twenty minutes to several days, in the Coorong lagoons in the south-east of South Australia. These lagoons are very shallow and are approximately 50 kilometres in length and 2 kilometres in width. Their only connection to the sea is through a very narrow channel. At instrument sites, the largest wind waves observed had a height of 20 cm. The well, with an internal diameter of 10 cm and pipe-connection 10 cm long with internal diameter 0.22 cm, located about 2 cm from the bottom, was placed into about 50 cm of water. It's tide-well parameter $N = 260$ and the theoretical amplitude response (Fig. 12) agrees essentially with the quasi-steady estimate (1.1). Some experimental estimates of the frequency response carried out in the laboratory for periods less than ten minutes, are also shown. Application

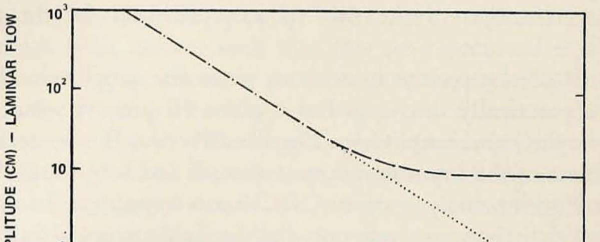

Figure 13. Maximum permissible amplitude, inside and outside the water level indicator used at the Coorong.

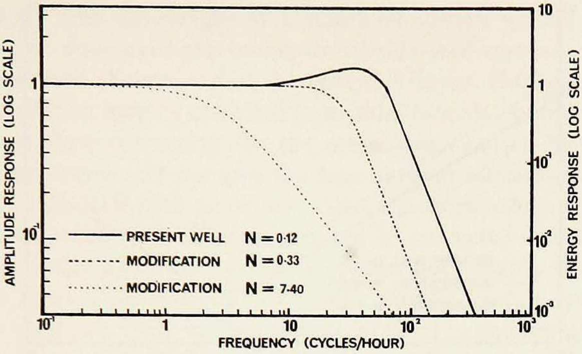

Figure 14. Amplitude and energy response for the Macquarie Island tide gauge and suggested modifications.

of the criteria of Section 3 shows that the system remains linear in the presence of water level oscillations of all periods, provided their amplitudes are less than 10 cm (Fig. 13). For periods greater than one hour, the permissible amplitudes may be much larger; for example, semi-diurnal oscillations can have amplitudes of up to 2 metres without turbulence occurring in the pipe flow.

A second system of this type is the tide-well installed at Macquarie Island, in September 1967, by the Flinders University of South Australia. In view of the prevailing rough coastal conditions a pipeline, about 76 metres long with internal diameter of 2.55 cm, was run in a trench from a well with an internal diameter of 15 cm about 50 metres inland, into water about 2 metres deep. Application of the theory developed in this paper shows that the tide-well parameter for this installation is $N = 0.12$ and that the amplitude response has a gain which is measurably greater than unity for the frequency range 8–80 cycles/hour (Fig. 14). Also, fifteen second swell has an amplitude response of approximately 0.1, so that these waves, which occur consistently with heights of at least one metre (Noye and Radok, 1966), produce 10 cm well oscillations of the same period. In practice these well oscillations blur the tide record very badly. For the dominant tidal oscillations and swell, this system gives a value $Re_{\max} \simeq 3000$, so that (3.16) is not satisfied and turbulent flow may be expected in the pipe at some time or other. Fig. 15 shows that a critical range of frequencies exists between 10 and 100 cycles/hour; for waves of height 8 cm or more, non-linear recording may occur.

An optimum system ($N = \frac{1}{3}$), with an amplitude response of one half at a frequency of 40 cycles/hour, could replace this system by using a well 10.8 cm in diameter attached to a pipe with a diameter of 2.13 cm and a length of 104 metres so its open end lies in water approximately 6 metres deep. In this case $Re_{\max} \simeq 1300$ and linearity of the system would be assured. The amplitude response to fifteen second swell would be 0.015 and no blurring of the tide records would occur (Fig. 14). The previous restriction on wave-heights

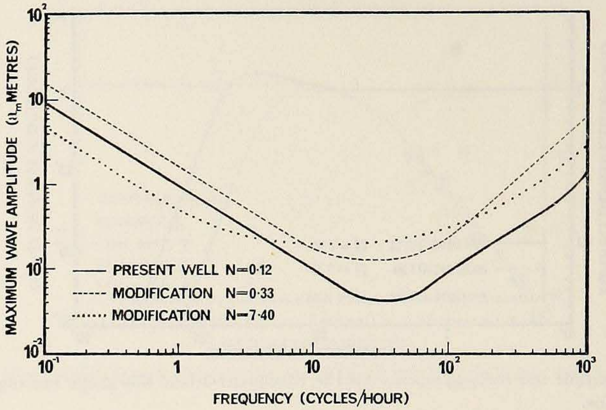

Figure 15. Maximum permissible wave amplitude for the Macquarie Island tide gauge and suggested modifications.

in the critical frequency range is raised to 32 cm (Fig. 15). Such a system would record, without attentuation, waves with a period greater than 4 minutes and would be admirable for the recording of tides and the unique 6 minute waves of several centimetres amplitude reported by Longuet-Higgins (1967).

If only tidal periods down to 1 hour are of interest, the present pipe may be replaced by one of half the diameter ($N = 7.4$, Fig. 14). Then $Re_{\max} \simeq 1250$ and the critical frequency band for laminar flow is 4–40 cycles/hour, when wave-heights must be restricted to 40 cm or less.

A third tide well of this kind, located at Cairns in Queensland, is described by Easton (1968). Since the sea floor is exposed occasionally at very low tides at the chosen site, a 6.7 metre long pipe with a diameter of 10 cm was run from the well, approximately 1 metre in diameter, to deeper water. This system has $N \simeq 0.002$; hence there is a large peak in the response, such that there is a gain of 7.0 for waves with a frequency of 60 cycles/hour, followed by a very sharp fall in response, until wind waves of 300 cycles/hour or higher are transmitted with less than one-twentieth of their incident amplitude. The records obtained from this well show no signs of blurring, verifying that the system does not respond to wind waves. Some occasional bumps in the record, reported to occur when rain squalls hit the harbour, are undoubtedly due to the amplification of the appropriate long wave components produced. This amplification will no doubt also cause non-linearity in the system; waves with amplitude greater than one and a half millimetres at a frequency of 60 cycles/hour might produce turbulence in the pipe flow. The non-linearity due to end effects will also be relatively large, since the condition (3.2) is not met. Use of the quasi-steady approximation gives misleading results for this system, as it does for the experimental model with $N = 0.004$ (Fig. 11); it gives an estimate of the amplitude response according to which waves in a wide frequency band, including wind waves, are transmitted into the well with negligible attenuation.

8. Conclusion. The advantages of a tide-well system with a pipe-connection near the sea floor over the conventional tide-well system are many-fold. The most important of these is that the system is linear. Consequently, records of harbour oscillation may be extracted from tidal records by removing the predicted tide and then correcting for the attenuation caused by the tide-well system. This process would not give the true harbour oscillations, if applied to records from a tide well with an orifice. Also, the amplitude and phase of tidal constituents computed from tidal records can be corrected directly for the attenuation and phase lag of the system, a process not possible after an analysis using records from a conventional tide well. Finally, such a system does not suffer from a set-down error in the presence of short period wind waves, in the way that the conventional tide well does. Records from the system can therefore be used for the determination of mean sea level for surveying purposes.

The excellent agreement between experiment and theory in this paper shows that unsteadiness in the pipe flow is a major factor in determining the nature of the response of this type of tide well. The nature of the response is determined by the value of the tide-well parameter N. With $N = \frac{1}{3}$, the system has a unit amplitude response and small phase lag for a large range of frequencies, followed by a sharp cut-off. With increasing values of N this cut-off becomes smoother, until with $N \geq 5$ the result is indistinguishable from the quasi-steady approximation found by assuming Poiseuille pipe flow. For values of N which are less than $\frac{1}{3}$, the system could be used to selectively amplify certain wave components. In order for the system to be linear the pipe-flow must remain laminar and this can be assured by choosing the dimensions of the system suitably.

Incorrect choice of dimensions, based on the quasisteady approximation, may result in accentuation of certain waves instead of attenuation, with small waves at some frequencies causing turbulence in the pipe flow. The Cairns tide-well system typifies this.

Most conventional tide-well installations can be converted to linear systems, with chosen response characteristics, by replacing the orifice with a suitable long horizontal pipe connection. This can be done quickly and simply and at little cost; the last feature becomes a major consideration when one realises that there are 77 tide-well installations operating along the coast of Australia alone.

The tide well with a long pipe connection also has an important physical advantage over the conventional system, because the tides can be sensed at a place some distance from the well and recorder. The Macquarie Island tide-gauge is an illustration, with the well located at a sheltered spot inshore and the tides measured as an exposed location away from the very rough coastline.

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NOTE: A computer program, written in Fortran IV, was used to calculate the response characteristics of the tide wells with a long pipe connection, given the well and pipe dimensions. The output listed both energy and amplitude response, phase and time lag and the maximum permissible wave amplitude over the frequency range 0–1000 cycles/hour. The calculations included the effect of pressure attenuation due to the pipe depth and the effect of the inertia of the water in the well, in addition to the effect of the unsteadiness in the pipe flow. A listing of this program is available from the author.

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