# *Tide-well Systems I: Some Non-linear Effects of the Conventional Tide Well*[1]

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## ABSTRACT

Recent uses to which tide records have been put has revealed that modern analytical techniques have outstripped the quality of available data. Lack of knowledge of the response of the conventional type tide well has been a serious handicap to workers in this field. The analysis in this paper shows that non-linearities, caused by the flow through the orifice, produce the main features of the response of the system; the hydrodynamic filtering of waves, due to the depth of the orifice and the effect of the inertia of the water in the well, are relatively unimportant. It shows that the quasi-steady model, used in some previous work, is a realistic one under most conditions and that results obtained using the quasi-steady theory still apply. Such previous theoretical findings agree well with some experimental results obtained using tide-well models. The most important feature of the response of the conventional tide well is that it is non-linear; the water level in the well oscillates with the frequencies of the oscillations in the sea level, plus higher harmonics and oscillations at sum and difference frequencies, and the amplitudes of the resulting well oscillations are non-linear functions of the amplitudes of the sea-level oscillations. This makes it almost impossible to correctly interpret small contributions in the results of a spectral analysis of a tide record. Finally, Cross's method of finding the lowering of mean level in a tide well in the presence of wind waves is improved and estimates of the "set-down" are computed.

1. Introduction. Lennon (1967) has pointed out a fundamental weakness in recent research involving tidal data, namely, that the analytical techniques used are much more sophisticated than is warranted by the quality of the data. Advances in methods of numerical analysis, which have been made since the introduction of the electronic digital computer, have not been matched by improvement in instrumentation. As a result, the marine scientist must ask himself to what extent are certain features, apparent in tide records, real or to what extent are they due to instrumental distortion of the true tide. Two such

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phenomena which must be critically examined from this point of view are the cusps of energy in the vicinity of the major lines in a tidal spectrum and the large number of small harmonic constituents obtained from the analysis of a distorted shallow water tidal profile. In particular, ignorance of the nature of the response of tide gauges has contributed to this uncertainty.

There has also been some recent interest in the response of tide gauges to sea-level oscillations with different frequencies and amplitudes, for other reasons. The introduction of digital tide recording has necessitated the choice of a suitable time interval between successive readings from the gauge. This choice must be based upon the frequency response of the tide gauge to prevent aliasing, which results in spurious components appearing in a Fourier analysis of the records. The same choice must be made when tidal components are to be determined by analysing tide heights digitised from a continuous record on a tide chart. Depending on the nature of the response, the components obtained may also require correction for attenuation at higher frequencies.

Readings from tide gauges are also used to investigate harbour oscillations with periods varying between one and six minutes and tsunamis which have a period between 15 and 60 minutes. The record of these sea-level oscillations is superposed on the much longer period tidal oscillations. A knowledge of the frequency response of a tide gauge will indicate whether the readings of such waves, taken from a tide record, must be corrected for attenuation.

The mean sea level calculated from the tide records is often used as a datum for land surveying. For this work, it has been assumed that tide gauges always read tide heights correctly, particularly in the presence of large wind waves, a false assumption for certain types of wells.

A recent survey by Easton, 1968, of tide gauges on the Australian coast revealed that these consist of recorders which measure sea level inside a circular well, the water inside being connected to the ocean outside in several ways. In nearly every case, the connection is through a circular orifice near the bottom of the well.

A search of the literature on tide gauges revealed little information on the response of the conventional tide-well system (with an orifice connection) to sea-level oscillations. In 1950, O'Brien published some preliminary studies of the lag and reduction of range for this type of tide well. Keulegan (1967) produced a variation of O'Brien's method of approximate solution of the differential equation which models the conventional tide-well system, in his work on tidal effects in harbour basins; graphs of Keulegan's results were published by Cross (1968) in his studies of the standard tide gauge used by the United States Coast and Geodetic Survey. A numerical method for estimating the response of the conventional system was devised by Shipley (1963); an alternative numerical method, which improved Shipley's results, was produced by Noye (1968).

Theoretical studies of this tide-well system are described in Noye (1968, 1970, 1972), here-after referred to as N1, N2 and N3. By a quasi-steady theory, it is shown that a model for this system is given by

$$dH_w/dt + C_1 |H_w|^{1/2} sgn(H_w) = dh_0/dt, \qquad (1.1)$$

where $H_w(t)$ is the head response of the tide well to the fluctuating sea level $h_0(t)$, for tide wells with a sharp edged orifice connecting the water in the well with the sea outside; $t$ is the time and $C_1$ a constant depending only on the dimensions of the tide well.

In Section 2, the analysis of the conventional tide-well system given in N1 is expanded and extended. Unsteady effects in the flow are considered, as well as the hydrodynamic filtering of waves due to the depth of the orifice connection and the effect of the inertia of the water in the well. The results give a clear indication of the limitations which must be placed on the system, if (1.1) is to be a realistic mathematical model.

In N1 and N2, estimates of the well response are determined for the conventional tide well by considering the steady-state solutions of (1.1) with an input of circular frequency $\omega$, i.e., $h_0 = a \sin(\omega t)$. It is shown that such a tide well is a non-linear device, with a response which depends on the amplitude of the incident oscillation as well as the frequency.

In contrast to the case of a linear system, there is no unique way in which one may define the response function of this non-linear tide-well system. The response to an input which is constant in amplitude and sinusoidal in time, involves an oscillation at the same frequency as the input and odd harmonics of this fundamental. There are two ways of describing this response which are of use in practice. Either the amplitude ratio and phase lag of the fundamental and harmonics of the output relative to the input may be of prime importance; this is the case when spectral analysis of a tsunami or tide record is taken from such a system. Alternatively, the behaviour of the response as a whole may be of primary importance. Then one is concerned with the lag of the well response behind the input at their turning points, as well as the ratio of the maxima of the periodic output and the sinusoidal input. For example, if we have a tide record on which is superposed the record of a tsunami, then the lag at the turning point yields the exact time of arrival of the tsunami and the ratio of the maximum value of the output to the amplitude of the input yields the true amplitude of the incident wave.

No exact solution to (1.1) with the sinusoidal input $h_0 = a \sin(\omega t)$ has been found. The well response at the fundamental frequency and its harmonics is determined for small frequencies of the input from an asymptotic solution and for higher frequencies from an exact solution for an "almost-sinusoidal" input. The overall response, estimated by a numerical method, is compared with Shipley's results which proved to be in error on two counts: firstly, his values for the amplitude response are too low at high frequencies and, secondly, his results do not yield the different lag between input and output which occurs at the zero-crossings and the turning points.

In Section 3, these theoretical results, for the well response at the fundamental frequencies and its harmonics and for the overall well response, are found to agree closely with a set of experimental values.

In N3, the method leading to an asymptotic solution is applied to the steady-state solution of (1.1) for any input composed of oscillations of either small amplitude or frequency. In particular, an input consisting of two waves of reasonably close frequency is considered. It is found that the well response includes oscillations of reduced amplitude at the input frequencies, together with oscillations at frequencies which are their sum, difference and other linear combinations of these frequencies. In Section 4, a series of experiments is described, which gave results agreeing closely with the theoretical findings. This points towards a major problem arising in the spectral analysis of records from a tide-well with an orifice; since there is no simple relation between the Fourier components of the input and output of such a system, it is not possible to transform the energy-density spectrum of the tide-well record into that of the sea-level fluctuations. In particular, one cannot be sure whether small peaks in the spectrum are direct contributions from sea-level oscillations or due to non-linear effects of the orifice.

Finally, in Section 5, the effect of short-period wind waves on the mean level in the well is investigated. Cross (1968) has shown that a tide-well with an orifice gives readings which are up to 30 cm too low when wind waves or swell are superposed on the tides. It is therefore not advisable to use records from such a system to compute mean sea levels for land surveying. An improved second-order expression for the mean wave pressure at a given depth is derived and applied using Cross's method. In this manner, a better estimate of the "set-down" in the tide-well is obtained: for an orifice near the sea surface in deep water, the results are similar, but when the orifice is close to the sea floor or the depth to wavelength ratio is small, the "set-down" is nearly halved.

2. The Tide-Well Equation. Let $h_w(t)$ denote the height of the water level inside the tide well and $h_0(t)$ the external level, both referred to the mean sea level (M.S.L.) (Figure 1). Denote the depth of the ocean floor and of the orifice beneath M.S.L. by $d$ and $y$, respectively.

Inside the well, application of Bernoulli's Theorem for unsteady flow with velocity potential $\Phi$ and velocity $\underset{\sim}{v}$, at a height $x$ above the orifice, yields

$$d\Phi/dt + 1/2|\underset{\sim}{v}|^2 + gx + p/\varrho = F(t), \tag{2.1}$$

where $p$ is the pressure at $x$, $\varrho$ the density of sea water, $g$ the acceleration of gravity and $F(t)$ a function of $t$ only. Equating values of $F(t)$ at the water surface $S$ and at the orifice $O$, one obtains

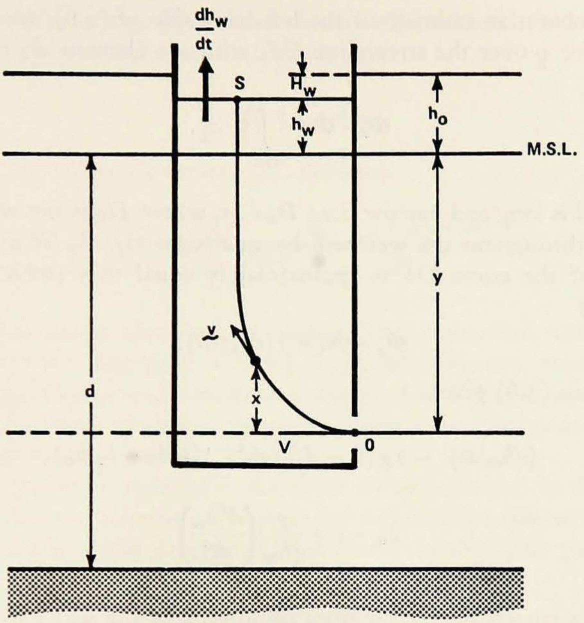


Figure 1. Notation used for the conventional tide-well system in the presence of linear waves.

$$d\Phi/dt|_S + 1/2(dh_w/dt)^2 + g(d + h_w) + p_a/\varrho = d\Phi/dt|_O + 1/2 V^2 + p_0/\varrho, \quad (2.2)$$

where $V$ is the velocity of influx at the orifice, $p_a$ the atmospheric pressure at $S$ and $p_0$ the wave pressure at the orifice. Introducing the pressure transmission factor (Kinsman, 1965, p. 143)

$$k_0 = \text{sech}\,(\varkappa d) \cosh \varkappa(d - y), \quad (2.3)$$

where $\varkappa$ is the wave number, one has for $\varkappa h_0 \ll 1$ and $h_0 \ll d$, the wave pressure

$$p_0 = p_a + \varrho g d + \varrho g k_0 h_0. \quad (2.4)$$

Using the continuity condition

$$A_w dh_w/dt = A_e V, \quad (2.5)$$

where $A_w$ is the area of cross section of the well and $A_e$ is the effective area of the inlet, and substituting for $p_0$ and $V$ from (2.4) and (2.5) into (2.2), one finds

$$d\Phi/dt|_S - d\Phi/dt|_O = g(k_0 h_0 - h_w) + 1/2(dh_w/dt)^2(A_w^2/A_e^2 - 1). \quad (2.6)$$

In order to obtain an estimate of the left-hand side of (2.6), we integrate the velocity vector $\underset{\sim}{v}$ over the streamline $OS$, with arc element $d\underset{\sim}{c}$, to obtain

$$\Phi_s - \Phi_0 = \int_{OS} \underset{\sim}{v} \cdot d\underset{\sim}{c}. \tag{2.7}$$

If the well is long and narrow (i.e., $D_w \langle\langle y$, where $D_w$ is the well diameter), the velocity throughout the well will be approximately $dh_w/dt$ upwards and if the length of the curve $OS$ is approximately equal to $y$ (which is true for $h_w \langle\langle y$), then

$$\Phi_s - \Phi_0 = y\,(dh_w/dt). \tag{2.8}$$

Substitution in (2.6) gives

$$(dh_w/dt)^2 = 2g\,(1 - A_w^2/A_e^2)^{-1}(k_0 h_0 - k_w h_w), \tag{2.9}$$

where

$$k_w = 1 + \frac{y}{gh_w}\left(\frac{d^2 h_w}{dt^2}\right) \tag{2.10}$$

incorporates a correction arising from the inertia of the water in the well.

For long period waves with circular frequency $\omega$, the response of the well is approximately sinusoidal with the same circular frequency, so that

$$k_w = 1 - \omega^2 y g^{-1}. \tag{2.11}$$

Furthermore, one has $\omega \langle\langle 1$ for long waves, so that $k_w$ is nearly unity; for example, if the orifice is less than 10 metres below mean sea level, periods greater than one minute give $\omega^2 y g^{-1} < .01$.

With $k_0 = k_w = 1$ for long waves, Equation (2.9) simplifies to

$$\frac{d}{dt}(h_0 - H_w) = \frac{A_e}{A_w}\left\{1 - \frac{A_e^2}{A_w^2}\right\}^{-1/2}(2g)^{1/2}\,|H_w|^{1/2}\,sgn\,(H_w), \tag{2.12}$$

where

$$H_w = h_0 - h_w \tag{2.13}$$

is the head response to the input $h_0$.

Rearrangement of (2.12) yields

$$dH_w/dt + C_1\,|H_w|^{1/2}\,sgn\,(H_w) = dh_0/dt, \tag{2.14}$$

where

$$C_1 = \frac{A_e}{A_w}\left\{1 - \frac{A_e^2}{A_w^2}\right\}^{-1/2}(2g)^{1/2}. \tag{2.15}$$

Since the effective area $A_e$ of the orifice may be written in terms of its actual area $A_p$ and its contraction coefficient $C_c$, viz.

$$A_e = C_c A_p,$$

the value of the tide-well constant $C_1$ becomes

$$C_1 = \frac{C_c A_p}{A_w}(2g)^{1/2}\left\{1 - \frac{C_c^2 A_p^2}{A_w^2}\right\}^{-1/2}. \qquad (2.16)$$

For flow through a sharp edged orifice, the theoretical value for $C_c$ is $\pi(\pi+2)^{-1} = 0.61$; assuming a small frictional loss, this value is close to $C_c = 0.6$ (e.g., Dodge and Thompson, 1937). For a Borda mouthpiece, one has $C_c = 0.5$, while, for inlets through a surge cone, the value of the influx is approximately 0.8 of that of the efflux.

Equation (2.14) is essentially the same as that derived in N1 using the quasi-steady theory. However, an improved value of the tide-well constant $C_1$ is given by (2.16), which is approximated closely by the value given in N1 for $A_p \ll A_w$,

$$\text{i.e.,} \quad C_1 \simeq 0.6(2g)^{1/2} A_p/A_w. \qquad (2.17)$$

In order that (2.14) be a reasonable mathematical model, the following assumptions have been made:

(a) the motion is laminar,
(b) $h_w \ll y$, i.e., the amplitude of the oscillations in the well are much smaller than the depth of the orifice below mean sea level,
(c) $D_w \ll y$, i.e., the well is relatively deep and narrow,
(d) $\omega \ll (g/d)^{1/2}$, $\varkappa d \ll 1$, which is true for most waves with periods greater than one minute.

As in N1, it is now convenient to rewrite equation (2.14) in terms of the dimensionless input $X = h_0/a$, the dimensionless head response $Y = H_w/a$ and the dimensionless frequency $\beta_1 = \omega\sqrt{a}/C_1$, where $X$ and $Y$ are functions of $\tau = \omega t$, and $a$ and $\omega$, respectively, are the characteristic amplitude and circular frequency of the sea-level oscillations. Equation (2.14) then becomes

$$\beta_1(dY/d\tau) + |Y|^{1/2} sgn(Y) = \beta_1(dX/d\tau). \qquad (2.18)$$

The head response $Y$ for a given input $X$ must be found from this equation; the dimensionless well response, $Z = h_w/a$, is then given by

$$Z = X - Y. \qquad (2.19)$$

No exact solution to (2.18) has been found for the linear wave input

$$h_0 = a \sin(\omega t),$$

i.e.,

$$X = \sin \tau. \tag{2.20}$$

Estimates of the well response to this wave are given in Section 3, where they are compared with corresponding values determined experimentally. The overall response of the tide well to such a sinusoidal fluctuation of the sea level, found using a numerical method in which the sine wave is approximated by a series of steps, agrees closely with the experimental values. The well response at the fundamental frequency and its harmonics, found for small $\beta_1$ using an asymptotic expansion and for large $\beta_1$ using an "almost sinusoidal" input, are given. These results are compared with some derived from the approximate solution of Keulegan (1967) to the same differential equation, as well as some experimental results. In particular, the ratio of the amplitude of the third harmonic to that of the fundamental, which can be used as a measure of the distortion of the response, is found by each method and compared.

In Section 4, the response of the system to sea-level fluctuations, consisting of the sum of two sinusoidal oscillations with either small amplitudes or frequencies, is listed. In particular, the response at the frequencies of the incident oscillations, their sums, differences and certain harmonics are examined in detail and compared with some experimental results.

3. Response to the Linear Wave $X = \sin \tau$. a. *Theoretical results:* The response to an input, which is constant in amplitude and sinusoidal in time, involves an oscillation at the same frequency as the input and odd harmonics of this fundamental.

If the behaviour of the response as a whole is of primary importance, then one is concerned with the lag of the well response behind the input at their turning points, as well as the ratio of the maxima of the periodic output and the sinusoidal input. Because (2.18) has an analytic solution for a constant input, the tide-well response for the input $X = \sin \tau$ can be found by replacing the sinusoid by a series of steps. This method is described in N1 and justification for the superposition of step responses in such a non-linear system is given in N2. The graph of the overall amplitude response is reproduced in Figure 2, with some values computed by Shipley (1963) using a variation of the Runge-Kutta method to solve the same differential equation. His values at $\beta_1 = 2.5$ and 5 are low and his results do not indicate a rather distinctive feature of the response, namely, that the lag at the zero-crossings is greater than that at the turning points. These lags are compared with Shipley's results in Figure 3. Agreement is good between some experimental results, determined by the method described later in this section and plotted on the same graphs, and the values computed by the step method.

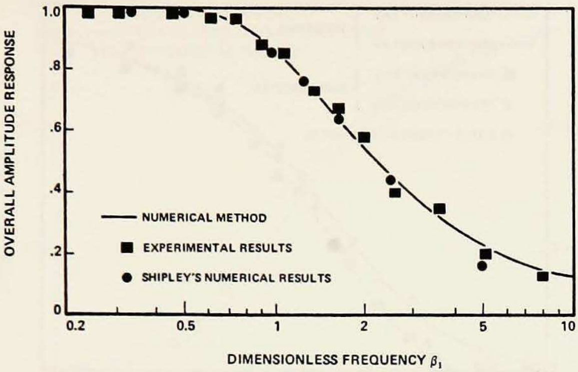


Figure 2. Estimates of the overall amplitude response of the conventional tide well to a sinusoidal input.

These results are particularly useful in tsunami studies. If we have a tide record on which is superposed the record of a tsunami, then the lag at the turning points is used to determine the time of arrival of the crest, and the overall amplitude response is used to give the true amplitude of the incident wave.

If the amplitude ratio and phase lag of the fundamental and harmonics of the output relative to the input are of prime importance, the results obtained in N3 can be used. With the input $X = \sin \tau$, a perturbation solution correct to $O(\beta_1^6)$, for small $\beta_1$, gives the well response

$$\left.\begin{aligned} Z = \alpha_f^{(1)} \sin(\tau - \theta_f^{(1)}) - \beta_1^2 \left\{ \beta_1^2 \sin 3\tau + \frac{8}{15\pi}(1 - 29\beta_1^4)\cos 3\tau \right\} + \\ + \sum_{m=3}^{\infty} (C_m \beta_1^2 + D_m \beta_1^4) \cos(2m-1)\tau + O(\beta_1^8), \end{aligned}\right\} \quad (3.1)$$

where $C_m$ and $D_m$ are constants. The amplitude response at the fundamental frequency is

$$\alpha_f^{(1)} = 1 - 0.64\,\beta_1^4 + O(\beta_1^8), \quad (3.2)$$

and the corresponding phase lag is

$$\theta_f^{(1)} = \frac{8\beta_1^2}{3\pi} + O(\beta_1^6). \quad (3.3)$$

For small $\beta_1$, the values of $\alpha_f^{(1)}$ may be compared with the values of the approximation by Keulegan (1967) to the solution of (2.18). He finds a solution in the form

$$Y = \sum_{n=1}^{\infty} A_{2n-1} \sin(2n-1)\varphi + \sum_{n=1}^{\infty} B_{2n-1}[\cos(2n-1)\varphi - \cos(2n+1)\varphi], \quad (3.4)$$

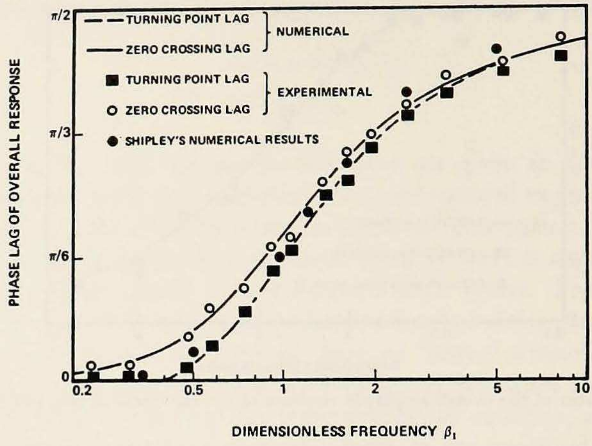


Figure 3. Estimates of the phase lag of the overall response of the conventional tide well to a sinusoidal input.

where $\varphi = \tau + \psi$, and $\psi$ is a zero of $\Upsilon(\tau) = 0$. The coefficients $A_1, A_3$ and $B_1$ are evaluated by truncation of the series for $\Upsilon$ and a process of iteration, using approximate values of these three coefficients in the given differential equation to obtain more accurate values. Besides the complete omission of all harmonics of order greater than three, there is partial omission of the contribution of the third harmonic since this involves the coefficient $B_3$. This observation explains why, when using Keulegan's results in the range $0 < \beta_1 < 0.5$, it is predicted erroneously that the amplitude response of the fundamental oscillation in the output exceeds unity (Figure 4).

Using an exact solution for an "almost sinusoidal" input to this system,

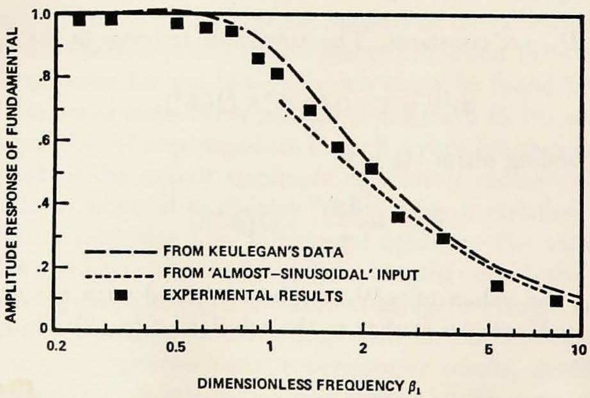


Figure 4. Estimates of the amplitude response, at the input frequency, of the conventional tide well to a sinusoidal input.

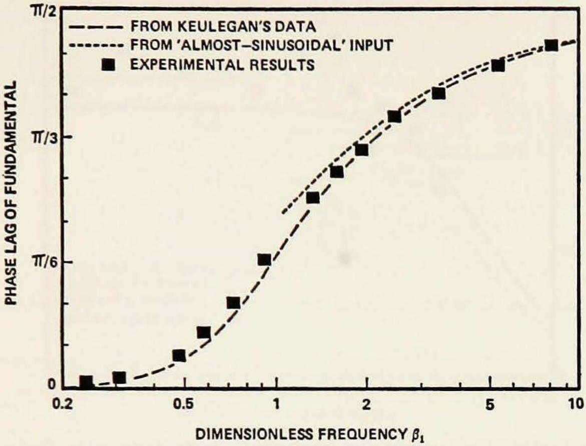


Figure 5. A comparison of estimates of the phase lag of the fundamental oscillation in the output behind the input, for the conventional tide well.

estimates of the amplitude response and phase lag of the fundamental oscillation in the output can be found for large $\beta_1$. In N1, Equation (2.18) is solved assuming the head difference is a sinusoid, viz., $Y = \sin \tau$. The Fourier series representation of the corresponding input is

$$\left.\begin{aligned} X &= (1 + 1.24\beta_1^{-2})^{1/2} \sin (\tau - \arctan 1.113\beta_1^{-1}) + \\ &\quad + \beta_1^{-2}(0.053 \cos 3\tau + 0.014 \cos 5\tau + \ldots). \end{aligned}\right\} \quad (3.5)$$

In the limit as $\beta_1 \to \infty$, $X \to \sin \tau$. The well response to this input $X$ is

$$\left.\begin{aligned} Z &= X - Y \\ &= -\beta^{-1}(1.113 \cos \tau + 0.053 \cos 3\tau + 0.014 \cos 5\tau + \ldots). \end{aligned}\right\} \quad (3.6)$$

The resulting estimate for the amplitude response to a sinusoidal input is, for large $\beta_1$,

$$\alpha_f^{(2)} = (1 + 0.81\,\beta_1^2)^{-1/2}, \quad (3.7)$$

with corresponding phase lag

$$\theta_f^{(2)} = \arctan (0.90\,\beta_1). \quad (3.8)$$

Graphs of these estimates are displayed in Figures 4 and 5 for $\beta_1 > 1$, where they are compared with values obtained using Keulegan's data and some experimental results.

The ratio of the amplitude of the third harmonic to the amplitude of the fundamental in the output is a measure of the distortion of the output. By (3.1), the amplitude of this harmonic is, for small $\beta_1$,

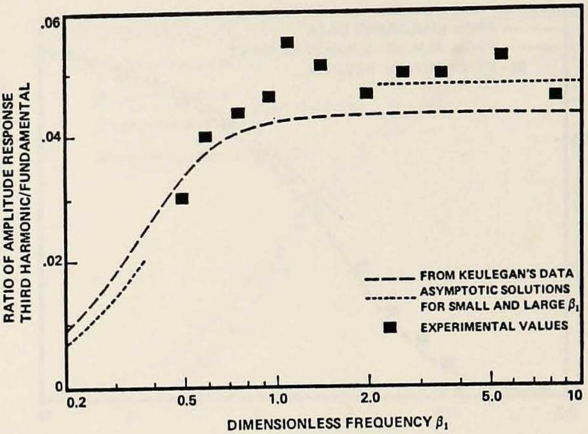


Figure 6. The ratio of amplitudes of the third harmonic to the fundamental oscillation in the output, for a conventional tide well.

$$\alpha_1^* = \frac{8\beta_1^2}{15\pi}\left\{1 + \left(\frac{225\pi^2}{128} - 29\right)\beta_1^4\right\} + O(\beta_1^8), \tag{3.9}$$

and the required ratio is

$$\frac{\alpha_1^*}{\alpha_f^{(1)}} = 0.17\beta_1^2 + O(\beta_1^6). \tag{3.10}$$

Comparing this result over the range $0 < \beta_1 < 0.4$, with the same ratio calculated from Keulegan's approximate data, it is found that the latter value is approximately 25 per cent larger throughout (Figure 6). For large $\beta_1$, we find from (3.6) that

$$\alpha_2^* = 0.053\beta_1^{-1}, \tag{3.11}$$

and the corresponding measure of distortion is

$$\frac{\alpha_2^*}{\alpha_f^{(2)}} \sim \frac{0.053\beta_1^{-1}}{(1 + 0.81\,\beta_1^2)^{-1/2}} \sim 0.048. \tag{3.12}$$

The same ratio calculated from Keulegan's data is 0.042. Figure 6 shows how this relative distortion of the output increases from a negligible value for very low values of $\beta_1$ to a constant value of nearly 5%.

These results clearly show the presence of higher harmonics in the response of a conventional tide well to a sinusoidal input. Therefore care must be taken in the interpretation of the results of Fourier analysis of tide records. Some or the components found may not exist in the fluctuations of the sea level outside the well; they may appear in the oscillations of the water inside the well simply because of the non-linearity of the tide-well with an orifice.

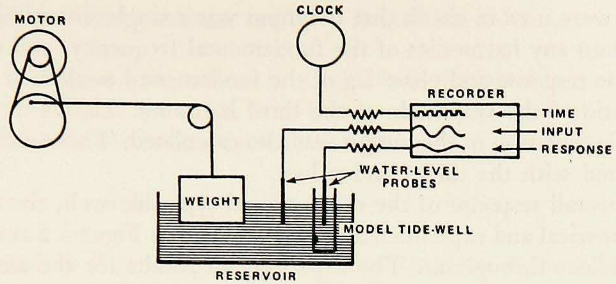


Figure 7. Schematic diagram of the apparatus used in the experimental work with a sinusoidal input.

b. *Experimental Results:* Estimates of the amplitude response and phase lag of model tide wells have been determined experimentally and compared with the theoretical results.

The apparatus (Figure 7) consisted of a large reservoir of water in which were placed model wells, with an internal diameter of 10 cm and with sharp edged circular orifices of various sizes. The surface level in the reservoir was varied sinusoidally in time with different amplitudes and frequencies, by means of a partially submerged weight of uniform cross section, which oscillated vertically in simple harmonic motion. The level of the water in both the reservoir and the well were recorded on a paper chart by means of capacitance-type wave probes. Since the wave heights to be measured were small, errors due to the meniscus effect on the probe became relatively important. In contrast to the results on p. 29 of Kinsman (1964), the distortion consisted of a lag in the recording with cutting off of the troughs and crests. This defect was finally overcome by putting a layer of kerosene on the surface of the water. It was found that there was then no change in the shape of the meniscus at either the water-kerosene or the kerosene-air interface, during a complete period of the simulated wave motion in the apparatus, even at the highest frequency used.

The experiments were carried out in the following manner. Having fixed the new frequency and amplitude of the oscillation of the water level in the reservoir, the amplifier gain for each probe was adjusted to give full width recording on the chart. The recorder was then calibrated by noting the chart readings for 2 mm increments in water level over the whole chart width. The apparatus was then run for several minutes to eliminate starting transients. Twenty oscillations of the reservoir and well response were then recorded. From this record, the overall amplitude response namely the ratio of the maximum values of the output and input amplitudes, was determined and also the lag of the output behind the input at both the zero-crossings and the turning points.

In addition, the records of both input and output were digitised, using an automatic digital curve reader capable of chart reading to 0.05 mm. (Norrie et.al., 1965), calibrated and Fourier analyzed on a CDC 6400 digital computer.

The results were used to check that the input was a single sinusoid in time and did not contain any harmonics of the fundamental frequency, and to calculate the amplitude response and phase lag of the fundamental oscillation in the output. The ratio of the amplitude of the third harmonic relative to that of the fundamental oscillation in the output was also calculated. These estimates were then compared with the theoretical values.

For the overall response of the conventional type tide well, the comparison between numerical and experimental values are seen in Figures 2 and 3. Agreement is excellent throughout. The experimental results for the amplitude response and phase lag at the fundamental frequency are shown in Figures 4 and 5, where they are compared with the theoretical results of Keulegan (1967) and of the method employing the "almost sinusoidal" input. Experimental values for the amplitude ratio of the third harmonic to that of the fundamental are shown on Figure 6. Agreement is good, even though relatively small amplitudes of oscillation were involved.

4. RESPONSE TO THE SUM OF SEVERAL LINEAR WAVES. a. *Theoretical Results:* If the input $h_0$ is a sum of sinusoidal functions with varying frequencies, viz.

$$h_0 = \sum_{n=1}^{N} a_n \sin(\omega_n t + \theta_n),$$

the position becomes more complicated. The non-linearity of the system does not permit the superposition of separate solutions for inputs

$$h_0^{(n)} = a_n \sin(\omega_n t + \theta_n).$$

The output of such systems might contain components with sums and differences of frequencies

$$\omega_i \pm \omega_j, \quad i,j = 1,2,3,\ldots,N, \quad i \neq j,$$

as well as the fundamental frequencies

$$\omega_n, \quad n = 1,2,3,\ldots,N,$$

and their harmonics

$$p\omega_n, \quad p,n = 1,2,3,\ldots.$$

The following questions can be asked: Does the tide well, because of its non-linear characteristics, give measurable oscillations in the tide record which do not exist in the sea-level fluctuations outside the well? Is it possible that under certain circumstances these spurious oscillations contribute to, or even are mistaken for, small constituents such as certain shallow water tidal components?

The solution to the particular sea level fluctuation

$$h_0 = a \sin \omega t - an(n+1)^{-1} \sin (1 + 1/n)\omega t, \qquad (4.1)$$

which gives the dimensionless input to (2.18)

$$X = \sin \tau - n(n+1)^{-1} \sin (1 + 1/n)\tau, \qquad (4.2)$$

has been found in N3 using the asymptotic expansion method for small $\beta_1$. The well response is found, correct to $O(\beta_1^6)$, in the form

$$Z = \beta_1^2 a_0 + \sum_{p=1}^{\infty} \alpha_p \sin (p\tau/n - \varepsilon_p) \qquad (4.3)$$

where $a_0$ is a constant. $\alpha_p$ and $\varepsilon_p$ are the amplitude response and phase lag of the oscillation at frequency $p\omega/n$, in the output to the system.

Figure 8 shows a typical set of contributions to the well response at different frequencies for a double sinusoidal input of this kind, for $\beta_1 = 0.4$ and $n = 3, 4, 5$. A tide well with an orifice-to-well diameter ratio of 1/20, in the presence of semi-diurnal tides of 2 metres amplitude, gives approximately this value for $\beta_1$.

There are significant contributions at side-bands to the frequencies of the incident oscillations. For instance, with $n = 3$, the frequencies of the incident oscillations are $\omega$ and $4\omega/3$. An amplitude of 2.89% of the incident amplitude at frequency $\omega$ occurs at the frequency $2\omega/3$ and an amplitude 6.48% of the incident amplitude at frequency $4\omega/3$ at the frequency $5\omega/3$. It is noteworthy that such a significant contribution as $\alpha_{n-1}$ occurs at a lower frequency than either of the incident waves, viz. $(1 - 1/n)\omega$.

The contributions to the output at harmonics of double the incident frequencies and the sum of these frequencies (see $\alpha_{2n}$, $\alpha_{2n+1}$ and $\alpha_{2n+2}$) are small compared with the contributions around three-times the incident frequencies. The third harmonics of the incident frequencies are given by $\alpha_{3n}$ and $\alpha_{3n+3}$. For $n = 3$, the third harmonic of the wave with frequency $\omega$ has an amplitude of 2.61% of the incident amplitude at that frequency and the third harmonic of the wave of frequency $4\omega/3$ has an amplitude of 4.59% of the corresponding incident amplitude. The most striking feature of all is the large contribution at frequencies given by doubling one incident frequency and adding the other; for all values of $n$ up to $n = 20$, $\alpha_{3n+1}$ and $\alpha_{3n+2}$ are greater than 7% of the incident amplitudes. In the past, contributions at such linear combinations of the frequencies of the incident waves have been attributed to effects outside the tide-well system, e.g., non-linear effects on the tides due to shallow water. Such harmonics may be wholly or partly due to the non-linear effects of the orifice. One would suspect that many of the 114 tidal constituents computed

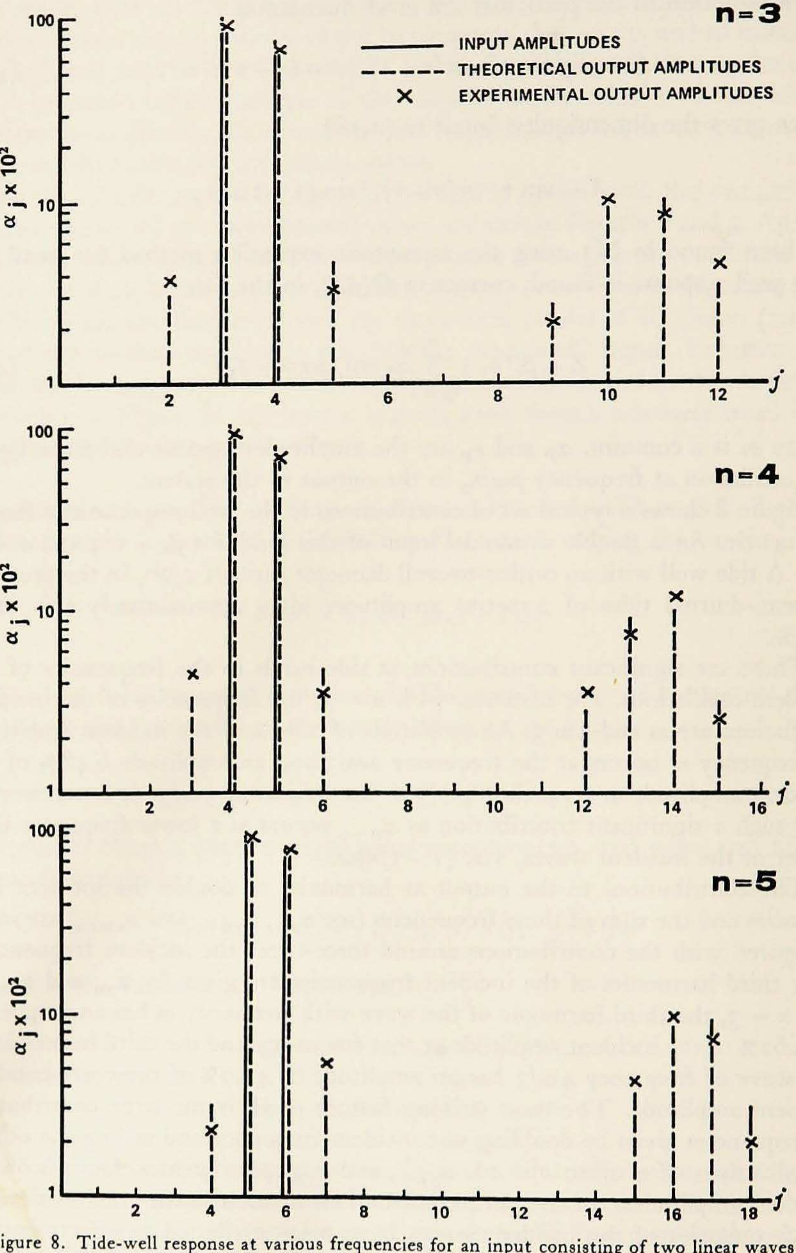


Figure 8. Tide-well response at various frequencies for an input consisting of two linear waves with $\beta_1 = 0.4$.

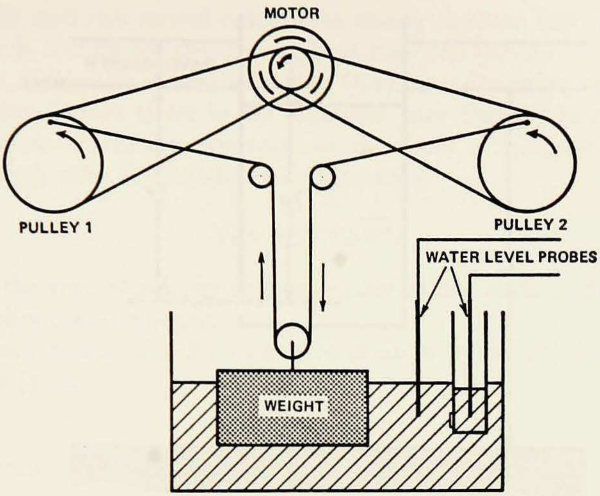


Figure 9. Pulley and weight system to model the input consisting of the sum of two linear waves. Pulley 1 has radius $(n + 1)$ units, Pulley 2 has radius $n$ units.

for Anchorage, Alaska, by Zetler and Cummings (1967), might include non-linear contributions from the orifice. The same contributions might be included in the tidal constituents computed for Southend on the Thames Estuary, by Rossiter and Lennon (1968). In this analysis, over 20 of the constituents had amplitudes less than 3 mm, compared with the M2 component with amplitude 2 metres.

b. *Experimental Results:* Estimates of the response of three models of the conventional tide-well to an input of the form (4.1) have been found experimentally and compared with the theoretical results.

The apparatus used (Figure 9) and the procedure adopted for recording input and output were similar to that described in Section 3. Pulleys 1 and 2, with diameters in the ratio of $n:(n+1)$, were driven off co-axial pulleys of equal radii. The connection between Pulleys 1 and 2 and the partly submerged weight produced the required oscillation in the water level of the reservoir containing the tide-well model. The motor speed was adjusted so that, using the appropriate values for the tide-well constant and the amplitude of oscillation, the value of $\beta_1 = 0.4$.

As in the previous experimental work, long sections of recorded input and output were digitised, using the automatic digital curve reader, calibrated and Fourier analysed. The results were used to calculate $\alpha_j$, $j = 1,20$, for $n = 3$, 4,5. For each value of $n$, the average of the results obtained from the three tide-well models was compared with the theoretical values (Figure 8). Excellent agreement between the two sets of results was found.

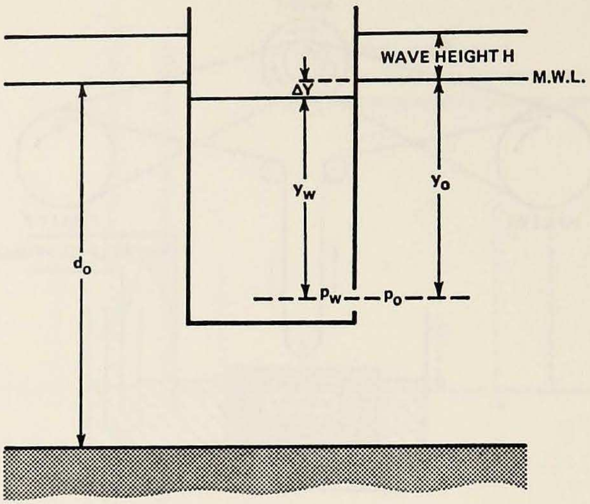


Figure 10. Notation for the conventional tide-well system in the presence of non-linear wind waves.

5. Response to Non-linear Wind Waves. For wind waves of significant amplitude, linear infinitesimal wave theory is no longer sufficiently accurate and the expression (2.4) for the wave pressure at the orifice should have additional small terms added to the righthand side.

The pressure at orifice depth $y_0$ beneath the mean wave level, which is $d_0$ above the sea floor (Figure 10), may be given by Stokes' second order theory (e.g., Wiegel 1964, p. 31) as

$$p_0 = \varrho g(y_0 + C_1 \cos \omega t + C_2 \cos 2\omega t + C_3) \qquad (5.1)$$

for a wave of circular frequency $\omega$. $C_1$, $C_2$ and $C_3$ are coefficients which depend on the wave height $H$, wave number $\varkappa$, and the depths $y_0$ and $d_0$:

$$\left.\begin{aligned} C_1 &= \frac{1}{2} H \operatorname{sech} \varkappa d_0 \cosh \varkappa(d_0 - y_0), \\ C_2 &= \frac{3\varkappa H^2}{8} \operatorname{cosech} (2\varkappa d_0) \left\{\frac{\cosh 2\varkappa(d_0 - y_0)}{\sinh^2 \varkappa d_0} - \frac{1}{3}\right\}, \\ C_3 &= -(\varkappa H^2/8) \operatorname{cosech} (2\varkappa d_0) \cosh 2\varkappa(d_0 - y_0). \end{aligned}\right\} \qquad (5.2)$$

If it is assumed that the tide-well system has negligible response to the oscillation of the pressure at frequency $\omega$, then it will also have negligible response to the harmonic of frequency $2\omega$. The term $C_3$, being a non-zero constant term present as long as the waves persist, will have the effect of lowering the mean level in the well, relative to the mean level outside, by a constant amount.

Cross (1968) used this second order wave theory to show that the difference in mean levels is large for the conventional tide-well system.

However, it must be remembered that (5.1) is an expansion only to second order and that further terms in the expansion may change the nature of the term $C_3$. As an alternative, one may use the result of Longuet-Higgins and Stewart (1964), who derived the exact expression

$$\bar{p}_0 = \varrho g y_0 - \varrho \overline{w^2}, \tag{5.3}$$

where $w$ is the vertical velocity at depth $y_0$ due to the waves at the orifice and the bar implies time-averaging.

An alternative expression for $C_3$ may be obtained from (5.3) in the following manner. By (5.1),

$$\bar{p}_0 = \varrho g (y_0 + C_3), \tag{5.4}$$

which, by comparison with (5.3), yields

$$C_3 = -\overline{w^2}/g. \tag{5.5}$$

Substituting the expression for the vertical velocity (e.g., Wiegel 1964, p. 31),

$$w = \frac{1}{2} \omega H \operatorname{cosech} (\varkappa d_0) \sinh \varkappa (d_0 - y_0) \sin \omega t + O(H^2) \tag{5.6}$$

in Equation (2.5), one obtains, after application of the dispersion relation

$$\omega^2 = g \varkappa \tanh (\varkappa d_0), \tag{5.7}$$

the expression

$$C_3 = -(\varkappa H^2/8) \operatorname{cosech} (2\varkappa d_0) \{\cosh 2\varkappa (d_0 - y_0) - 1\} + O(H^3). \tag{5.8}$$

This result differs from the expression for $C_3$ in (5.2) and to second order it appears to be more realistic; for example, it gives a time-averaged pressure on the ocean floor which is equivalent to hydrostatic pressure due to the mean wave level. For long waves ($\varkappa d_0 \ll 1$) or when the orifice is close to the sea floor ($y_0 \simeq d_0$), Equation (5.8) gives smaller values for $C_3$ than (5.2).

Following the reasoning which Cross used, we consider an "equilibrium" time period, long compared with the wind wave period yet short enough so that the tide level remains constant. During this time, the net flow into the tide well is zero, i.e.,

$$\bar{Q} = 0, \tag{5.9}$$

where $Q(t)$ is the flow rate into the well. Since it has been shown in Section 2 that under the given conditions one may assume that the steady flow relationship holds, then

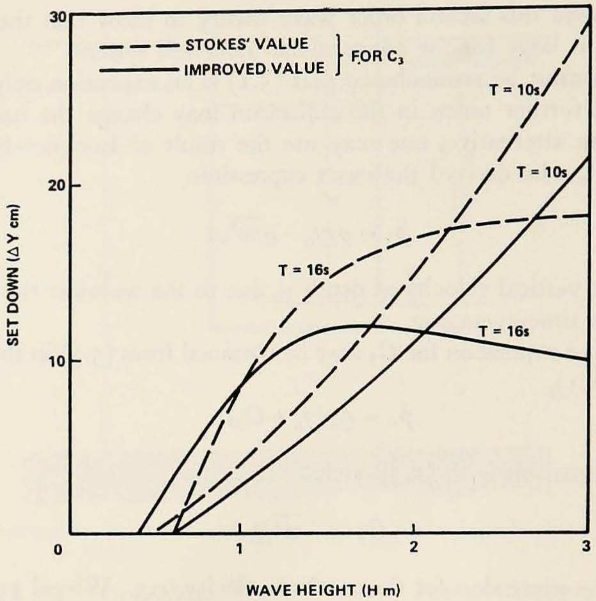


Figure 11. Comparison of set-down in a conventional tide-well using Stokes' formula for $C_3$ and the improved formula for $C_3$, with an orifice depth of 3 metres and sea depth 6 metres.

$$Q = C_c A_p \varrho^{-1} |2\Delta p|^{1/2} \operatorname{sgn}(\Delta p), \tag{5.10}$$

where $\Delta p$ is the differential pressure across the orifice.

$$\Delta p = p_0 - p_w, \tag{5.11}$$

where $p_0$ and $p_w$, respectively, are the pressures just outside and inside the orifice.

If the response to the wind waves is negligible, the pressure inside the orifice can be taken to be hydrostatic, i.e., $\varrho g y_w$, where $y_w$, the distance from the surface inside the well to the orifice, is assumed constant during the "equilibrium" time period. Substitution of (5.10) and (5.11) into (5.9) gives

$$\int_0^{2\pi/\omega} |S(t)|^{1/2} \operatorname{sgn}\{S(t)\}\, dt = 0, \tag{5.12}$$

where $S(t) = y_0 - y_w + C_1 \cos \omega t + C_2 \cos 2\omega t + C_3$. For several values of the orifice depth $y_0$, the water depth $d_0$, the wave height $H$ and the wave frequency $\omega$, Equation (5.12) has been solved numerically using successive approximations to $y_w$. The "set-down", i.e., the lowering of the mean level in the tide-well, is then given by

$$\Delta Y = y_0 - y_w. \tag{5.13}$$

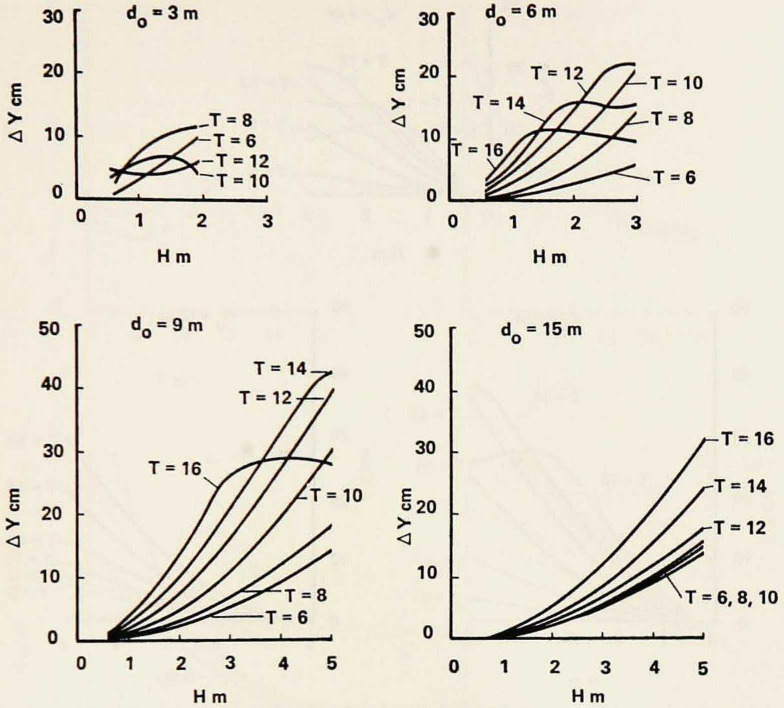


Figure 12. Set-down $\Delta \Upsilon$ cm plotted against wave-heights $H$ metres for an orifice depth $y_0 = 3$ metres and sea depths $d_0 = 3, 6, 9$ and 12 metres.

Figure 11 compares the results of Cross, using computations based on Stokes' value of $C_3$, and the improved values obtained using $C_3$ from (5.8). The latter gives a smaller set-down for large wave heights, particularly at higher periods. Note that, for the improved value of $C_3$, higher periods have a maximum set-down for rather low wave heights and that increasing the height then reduces this; for example, for $T = 16$ sec, the set-down is a maximum of 12 cm for a wave height of 1.5 m and it is smaller for bigger wave heights. Also, at low wave heights, the higher period waves produce a greater set-down in the well; for example, for $T = 16$ sec, the set-down is 8 cm for wave heights of 1 m and, for $T = 10$ sec, it is only 2.5 cm for the same wave height. For high waves, lower periods generate the greater set-down.

Figures 12 and 13 illustrate the set-down $\Delta \Upsilon$ in a tide well for different wave heights $H$, periods $T = 6, 8, 10, 12, 14, 16$ sec, water depths $d_0 = 3, 6, 9, 15$ m and orifice depths $y_0 = 3$ and 6 m. Figure 14 shows $H_{10}$ plotted against $T$ for various values of $y_0$ and $d_0$, where $H_{10}$ is the wave height which produces an error $\Delta \Upsilon = 10$ cm. Clearly, in shallow water, quite small wave heights can

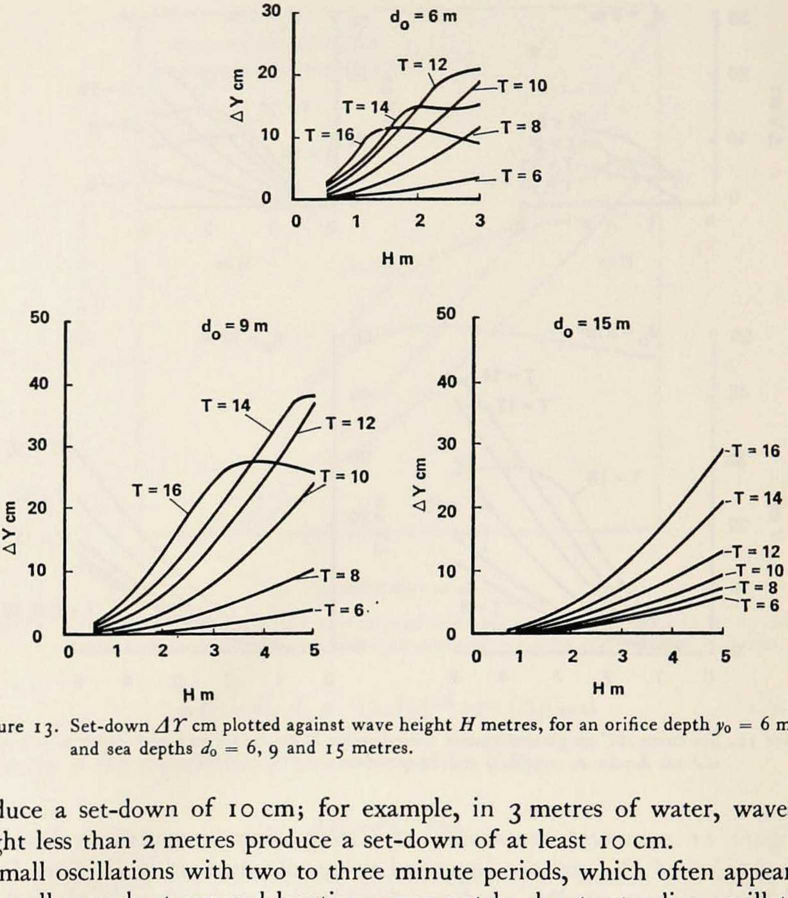


Figure 13. Set-down $\Delta Y$ cm plotted against wave height $H$ metres, for an orifice depth $y_0 = 6$ metres and sea depths $d_0 = 6$, 9 and 15 metres.

produce a set-down of 10 cm; for example, in 3 metres of water, waves of height less than 2 metres produce a set-down of at least 10 cm.

Small oscillations with two to three minute periods, which often appear on tide-well records at exposed locations, may not be due to standing oscillations of nearby waters, which has been the explanation in the past, but could have been caused by gradual variation of the set-down due to incident amplitude-modulated wind waves. Oscillations with periods of several minutes are evident in the records from the tide well at Point Lonsdale at the entrance to Port Phillip Bay, Victoria. Incident sea waves with a period of 15 seconds, arriving in groups of eight in each wave-envelope, would modulate the set-down with a period of two minutes, giving the impression that two minute standing oscillations were occurring in the sea state. The water in the bay to the North of the tide well, or on the continental shelf to the South, would oscillate naturally with periods greatly in excess of a few minutes, in fact with periods of the order of hours.

6. Conclusions. The conventional tide well, consisting of a well with an orifice, has been shown to be a non-linear device with all the inherent dis-

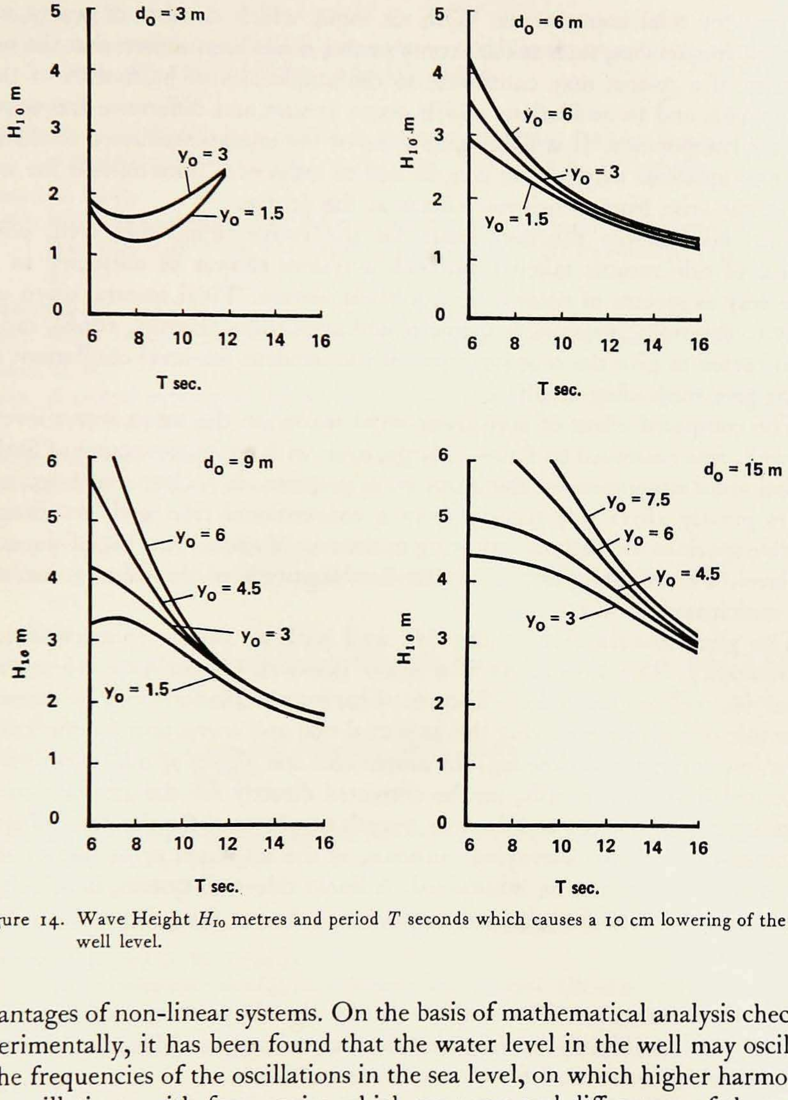


Figure 14. Wave Height $H_{10}$ metres and period $T$ seconds which causes a 10 cm lowering of the tide-well level.

advantages of non-linear systems. On the basis of mathematical analysis checked experimentally, it has been found that the water level in the well may oscillate at the frequencies of the oscillations in the sea level, on which higher harmonics and oscillations, with frequencies which are sums and differences of these frequencies, are superimposed. Furthermore, the amplitudes of the resulting well oscillations are non-linear functions of the amplitudes of the sea-level oscillations, a fact which creates many difficulties in the use of the tide-well records; for example, in the analysis of harbour oscillations. Pseudo-response functions give a rough idea of the attenuation and lag of a system, but they are difficult to use, because the response depends on the amplitude as well as the frequency.

Problems also arise when records from a conventional tide-well system are analysed for tidal components. With an input which consists of waves with different frequencies, such as tidal components, it has been shown that the non-linearity of a system may contribute to the amplitudes of harmonics of these components and to oscillations which occur at sum and difference frequencies of these components. It is likely that some of the energy attributed to shallow water components may not, in fact, be due to influences from outside the well, but rather arise from non-linear effects at the orifice.

Since no response function exists for the conventional tide well, power spectra of tide records taken from such a system cannot be corrected in the same way as spectra of records from a linear system. Tidal spectra, often used today to determine tsunami frequencies and amplitudes (Hatori, 1968), cannot be corrected to give the true spectrum of the incident sea-level oscillations and might give misleading results.

The computed effect of non-linear wind waves on the mean water level in the well, first examined by Cross, is large, even with an improvement of Stokes' second order expression for the mean wave pressure. It is clear that large wind waves greatly affect tide records from a conventional tide well, resulting in possible spurious oscillations appearing in the record and a lowering of the mean sea level. The results given show that the magnitude of the effect is less than was anticipated by Cross.

The great disadvantage of the tide well with an orifice connection is its non-linearity. The advantages of a linear tide-well system over the conventional tide well are many-fold. Records of harbour oscillations may be extracted from tide records by removing the predicted tide and correcting for the known reduction in range and time lag; the amplitudes and phases of tidal constituents, computed from tide records, can be corrected directly for the attenuation and phase lag of the system; and the tide records can be used for the determination of mean sea levels for surveying purposes, as the set down error due to short period wind waves can be minimized. A linear tide-well system, in which the orifice is replaced with a pipe connection near the sea floor, is presently being investigated.

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## REFERENCES

Cross, R. H.
1968. Tide gauge frequency response. J. WatWays Harb. Div. Am. Soc. civ. Engrs,: 317–329.

Dodge, R. A. and M. J. Thompson
1937. Fluid Mechanics. McGraw Hill, New York; 495 pp.

EASTON, A. K.
1968. A handbook of selected Australian tide gauges. Survey Paper No. 6, Horace Lamb Centre, Flinders University, Adelaide; 226 pp.

HATORI, T.
1968. Study on distant tsunamis along the coast of Japan. Part 2, Tsunamis of South American origin. Bull. Earthq. Res. Inst., Tokyo Univ., *46*: 345–359.

KEULEGAN, G. H.
1967. Tidal flow in entrances; water level fluctuations of basins in communication with seas. U.S. Army Corps of Engrs, Committee on Tidal Hydraulics, Tech. Bull. 14; 89 pp.

KINSMAN, B.
1965. Wind waves. Prentice-Hall, Inc., Englewood Cliffs, New Jersey. 676 pp.

LENNON, G. W.
1967. A critical examination of the conventional tide gauge. Proc. Symp. on Tides, Int. Hyd. Bur., Monaco, April 1967. UNESCO, Paris: 29–43.

LONGUET-HIGGINS, M. S. and R. W. STEWART
1964. Radiation stresses in water waves; a physical discussion with applications. Deep-Sea Res., *11*: 529–562.

NORRIE, D. H., I. WALL and A. E. R. WOOD
1965. An automatic digital curve reader. Journal of Scientific Instruments, Vol. 42: 356–357.

NOYE, B. J.
1968. The frequency response of a tide-well. Proc. 3rd Aust. Conf. Hyd. Fl. Mech., Sydney: 65–71.

NOYE, B. J.
1970. On a class of differential equations which model tide-well systems. Bull. Austral. Math. Soc., 3: 391–411.

NOYE, B. J.
1972. On the differential equation for the conventional tide-well system. Bull. Austral. Math. Soc., 7: 251–267.

O'BRIEN, M. P.
1950. The lag and reduction of range in tide gauge wells. Bull. Beach Eros. Bd., *4*: 24–40.

ROSSITER, J. R. and G. W. LENNON
1968. An intensive analysis of shallow-water tides. Geophys. J.R. Astr. Soc., *16*: 275–293.

SHIPLEY, A. M.
1963. On measuring long waves with a tide gauge. Dt. hydrogr. Z., *16*: 136–140.

WIEGEL, R. L.
1964. Oceanographical engineering. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 532 pp.

ZETLER, B. D. and R. A. CUMMINGS
1967. A harmonic method for predicting shallow-water tides. J. Mar. Res., *25*(1): 103–114.