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*On Free Modes of Oscillation of a Hemispherical Basin Centered on the Equator*¹

Niels Christensen, Jr.

*Hawaii Institute of Geophysics
University of Hawaii
Honolulu, Hawaii 96822*

ABSTRACT

Five barotropic modes of oscillation are computed for a hemispherical basin centered on the equator of a rotating earth. The method of Longuet-Higgins and Pond (1970), which takes full account of horizontal divergence and the sphericity of the earth, is used. Two of the modes are planetary modes and three are gravity modes. Periods of oscillation for the five modes range from one day to over four days. The surface waves exhibit different characteristics; two have the largest amplitudes on the boundaries, one has almost no oscillation on the boundaries, and the other two have nearly equal amplitude everywhere. These solutions agree in period and wave form with those of Mofjeld and Rattray (1971) for a similar β -plane basin.

Introduction. Barotropic modes of water oscillations in an enclosed basin on a rotating earth are computed. The method of Longuet-Higgins and Pond (1970) is used to compute five solutions for a 4-km-deep hemispherical basin centered on the equator. A solution consists of a surface wave and a period of oscillation.

Longuet-Higgins (1966) and Longuet-Higgins and Pond (1970) studied the behavior of free oscillations on a hemisphere bounded by meridians of longitude. In the first paper, complete solutions for the spectrum of planetary seiches were calculated, neglecting horizontal divergence. In the second paper, frequencies were calculated for planetary and gravity waves, taking full account of horizontal divergence, over a large range of depth.

Longuet-Higgins and Pond examined asymptotic forms, for both large and small depth. For large depth, the frequencies asymptotically approach those of the nondivergent case. Frequencies for intermediate depths can be selected, using their curves of nondimensional frequency versus a nondimensional depth parameter.

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Although Longuet-Higgins and Pond did not compute sea-surface topographies for the various solutions, the formulations to do so are available in their paper. Here, several solutions are computed, using their formulas, using numerical techniques similar to theirs, and using techniques developed by Christensen (1972) specifically for the purpose of computing sea-surface topographies and solutions at particular depths.

Mofjeld and Rattray (1971) have calculated the planetary and gravity waves for ocean-sized β -plane basins placed symmetrically about the equator. Their planetary modes resemble those of Rattray and Charnell (1966). The gravity modes are dynamically similar to either Kelvin waves or Poincaré waves. Their β -plane solutions can be compared with the solutions in this paper, where full account is taken of the sphericity of the earth.

Method of Calculation. The following equations, extracted from Longuet-Higgins and Pond (1970), are initially credited to Proudman (1916). They are presented here without derivation.

For ocean basins of arbitrary shape on a rotating globe, sea surface, ζ , can be expanded in the form

$$\zeta = - \sum_{r=1}^{\infty} \mu_r p_r \varphi_r, \quad (1)$$

where

$$\varphi_r = C_n^m P_n^m (\cos \theta) \cos m \varphi \quad (2)$$

for

$$n = 0, 1, 2, \dots,$$

$$m = 0, 1, 2, \dots, n,$$

and

$$\mu_r = n(n+1)h.$$

Here, m and n refer to the spherical harmonics, φ_r , of (2), and with each suffix, r , a pair of suffixes, $(\begin{smallmatrix} m \\ n \end{smallmatrix})$, are associated; h is the water depth; θ and φ are colatitude and east longitude, respectively. The coefficients, p_r , as well as the nondimensional frequency, λ , are determined by the equations of motion and the boundary conditions [Longuet-Higgins and Pond 1970: eq (4.4)]. To convert from λ to wave period, T , in hours, divide twelve by λ : $T = 12/\lambda$.

In (2), the P_n^m are associated Legendre polynomials:

$$P_n^m(Z) = \frac{(1-Z^2)^{1/2m}}{2^n n!} \frac{d^{n+m}}{dZ^{n+m}} (Z^2-1)^n. \quad (3)$$

The C_n^m are normalizing constants, which are chosen so that

$$\frac{1}{a^2} \iint h \left[\left(\frac{\partial \varphi_r}{\partial \theta} \right)^2 + \left(\frac{1}{\sin \theta} \frac{\partial \varphi_r}{\partial \varphi} \right)^2 \right] dA = 1$$

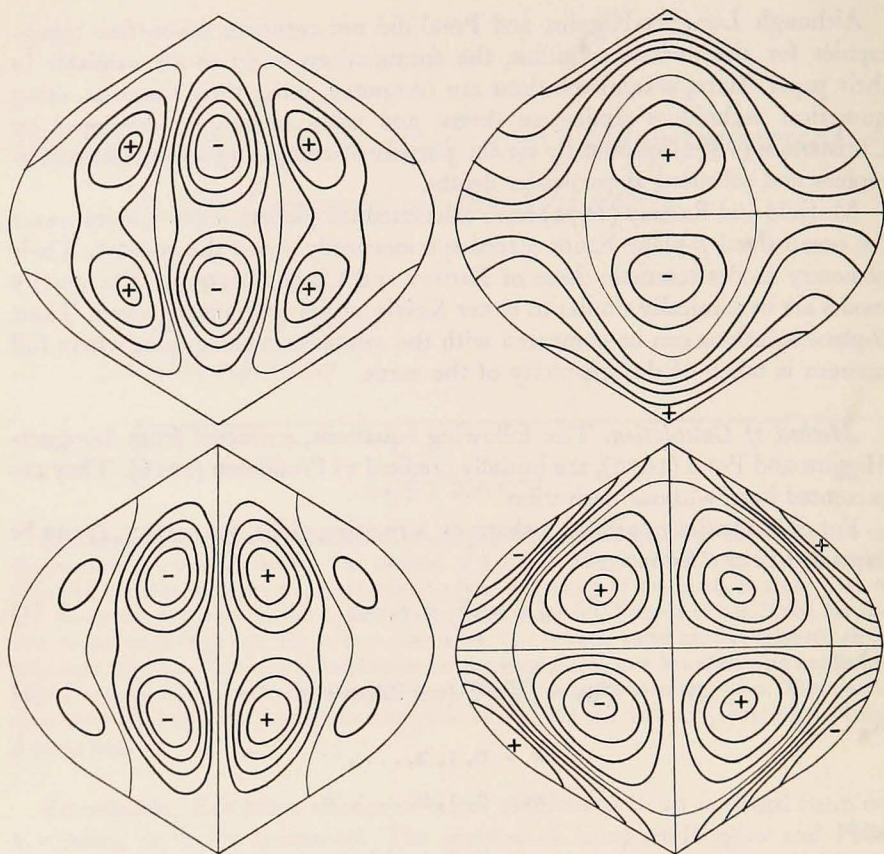


Figure 1 (left). Planetary oscillation $(1, 2, p)$ with period = 4.43 days for a hemisphere with $h = 4$ km.
 Figure 2 (right). Planetary oscillation $(1, 1, p)$ with period = 2.01 days for a hemisphere with $h = 4$ km.
 Above: $\text{Re}(\zeta)$. Below $\text{Im}(\zeta)$.

over the basin, where a is the radius of the earth. This implies that

$$C_n^m = \alpha_{n,m} h^{-1/2}, \quad (4)$$

where

$$\alpha_{n,m} = \begin{cases} \left[\frac{1}{\pi} \frac{2n+1}{n(n+1)} \frac{(n-m)!}{(n+m)!} \right]^{1/2} & (m > 0) \\ \left[\frac{1}{\pi} \frac{n+1/2}{n(n+1)} \right]^{1/2} & (m = 0), \end{cases} \quad (5)$$

excluding the case where $n = m = 0$.

In practice, the infinite series of coefficients, p_r , is truncated, and multipliers of the elements of p_r are arranged in a matrix. The series is limited to terms

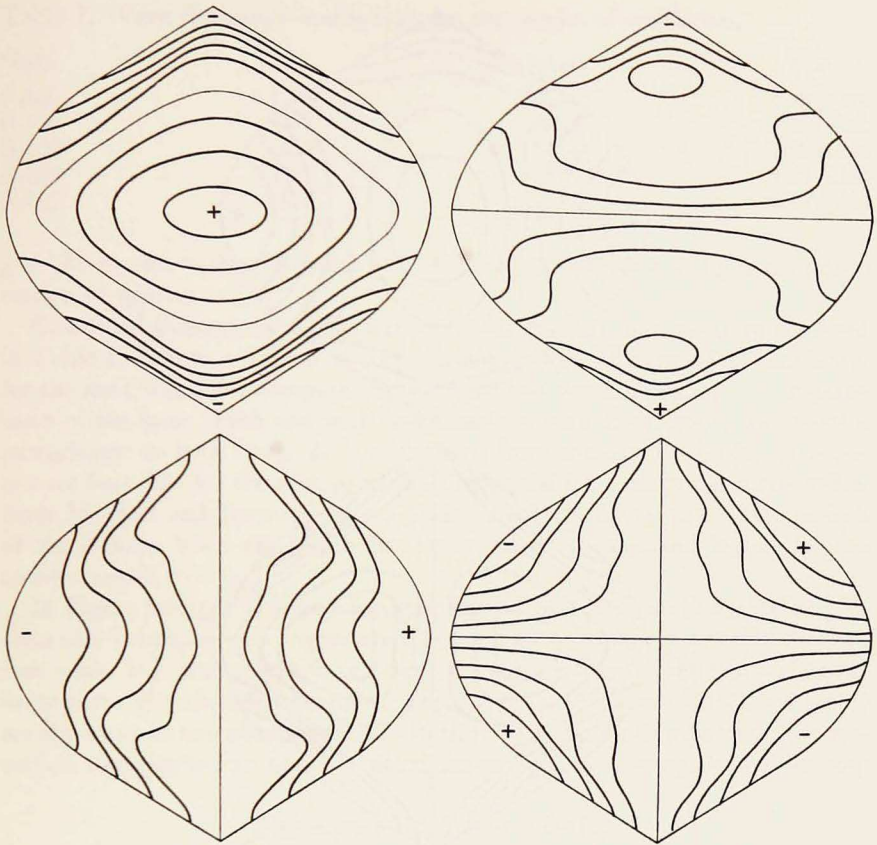


Figure 3 (left). Gravity oscillation $(1,1,g)$ with period = 1.87 days for a hemisphere with $h = 4$ km.

Figure 4 (right). Gravity oscillation $(0,1,g)$ with period = 1.28 days for a hemisphere with $h = 4$ km.

Above: $\text{Re}(\zeta)$. Below $\text{Im}(\zeta)$.

with m and n equal to, or less than, ten. This gives results that are accurate to the fourth significant figure. These multipliers are determined by the equations governing the motion and the boundary conditions. This becomes a standard eigenvalue problem, with p_r as the eigenvector and λ (which enters the equations, by assuming periodic motion) as the eigenvalue. Once the coefficients, p_r , are determined for a given frequency, the sea-surface topography can be generated by use of (1) through (5).

Modes of Oscillation. Five solutions were computed:

$(1,2,p)$ = first symmetric planetary mode;

$(1,1,p)$ = first antisymmetric planetary mode;

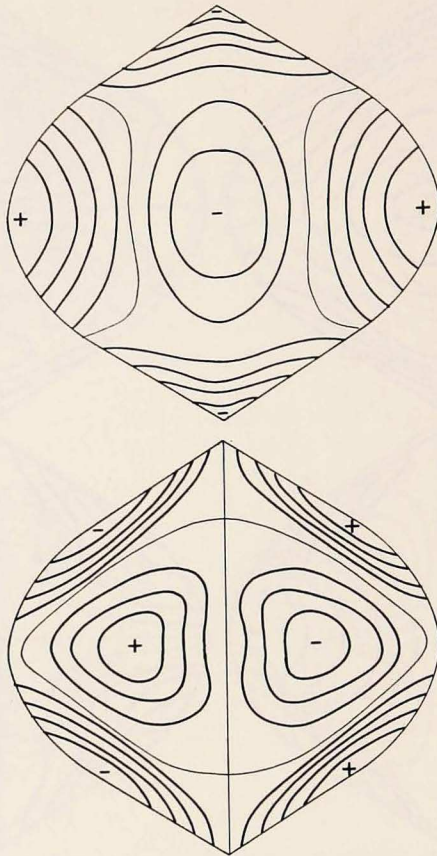


Figure 5. Gravity oscillation $(2,2,g)$ with period = 0.99 days for a hemisphere with $h = 4$ km.
Above: $\text{Re}(\xi)$. Below $\text{Im}(\xi)$.

- $(0,1,g)$ = first antisymmetric gravity (anti-Kelvin) mode;
- $(1,1,g)$ = first symmetric gravity (Kelvin) mode;
- $(2,2,g)$ = second symmetric (Kelvin) mode.

The two numerical indices refer to indices of spherical harmonics [the first to m and the second to n in eq. (2)] that dominate the solution as depth becomes very large, as noted in Longuet-Higgins and Pond (1970: figs. 1, 2). Symbol p or g indicates a planetary or gravity wave, respectively. The symmetry refers to the configuration of ξ about the equator, and first and second refer to the ordering according to frequency.

The following values were used for the approximate mean values for the earth and its oceans: $h = 4$ km; $a = 6.37 \times 10^8$ cm; $\omega = 7.292 \times 10^{-5} \text{sec}^{-1}$;

Table I. Wave frequency and period for the modes of oscillation.

Mode	λ	T (hrs)	T (days)
(1,2, p)	0.11278	106.40	4.43 (4.20)
(1,1, p)	.2492	48.15	2.01 (2.60)
(1,1, g)	.2669	44.96	1.87 (1.60)
(0,1, g)	.3901	30.76	1.28 (1.33)
(2,2, g)	.5059	23.72	0.99 (.83)

$g = 981$ cm sec⁻²; here ω is the rate of the earth's rotation and g is the acceleration of gravity.

Computed frequencies, λ , and wave periods, T , of the five solutions are listed in Table I. Periods are given in both hours and days. The periods of oscillation for the same solutions, computed by Mofjeld and Rattray (1971) for a β -plane basin of the same depth and with comparable surface dimensions, are given in parentheses. In both cases, the periods range from one day for the (2,2, g) mode to over four days for the (1,2, p) mode. The period for a particular mode differs from Mofjeld and Rattray's values by less than 5% up to 30%. The periods of the β -plane basin are lower for the symmetric modes and higher for the antisymmetric modes.

In Figs. 1 through 5, surface topographies are contoured on sinusoidal projections of a hemisphere. Two plots are necessary to adequately describe the surface wave. It is convenient to represent the surface wave as the real and imaginary parts of a complex quantity, $\text{Re}(\zeta)$ and $\text{Im}(\zeta)$. Then $\text{Re}(\zeta)$ and $\text{Im}(\zeta)$ are the wave at two phases, different in time by a quarter of a period. The sea-surface topography can be constructed for any phase, ν , using the relationship

$$\zeta(\nu) = \cos \nu \text{Re}(\zeta) + \sin \nu \text{Im}(\zeta).$$

Planetary solutions (Figs. 1 and 2) are seen to be a series of highs and lows in ζ , forming at the eastern boundary and moving westward across the basin. On the other hand, for the gravity modes (Figs. 3 through 5), the highs and lows move eastward along the equator, either symmetric about it or as anti-symmetric pairs on either side of it. They move poleward along the eastern boundary and return to the equator along the western boundary, thus completing the circuit.

In the case of the (0,1, g) mode, the amplitude of the wave is greatly increased near the poles. To a lesser extent, this is also true of the (1,1, p) mode, the anti-symmetric planetary mode. The oscillations are most intense along the boundaries and are a maximum at the poles. This is not true for the symmetric modes. There are no appreciable fluctuations in ζ at the boundaries for (1,1, p). Modes (1,1, g) and (2,2, g) have equally large oscillations at the poles, on the boundaries, and in the central basin. In general, these waves behave much like the waves for the β -plane basin of Mofjeld and Rattray.

Conclusions. Five solutions were computed according to Longuet-Higgins and Pond (1970). The two planetary modes, one of each symmetry, have different characteristics. The $(1,1,p)$ mode has its largest amplitude along the boundaries whereas the $(1,2,p)$ mode exhibits essentially no motion on the boundaries. Amplitude of the $(0,1,g)$ mode is greatest near the poles. The two symmetric gravity modes, $(1,1,g)$ and $(2,2,g)$, are nearly equal in amplitude everywhere. There is good agreement between these solutions and those of Mofjeld and Rattray (1971) for an ocean-sized β -plane basin symmetric about the equator.

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