On the Estimation of Surface Gravity-Wave Fields in the Vicinity of an Array of Wave Recorders[1]

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ABSTRACT

The optimum estimator for a surface-wave field in the vicinity of an array of wave recorders is determined for a linear sea. Examples for a single-component and a four-component array are presented.

I. *Introduction.* The estimation of one surface-wave field from a second field measured as a function of time at the same horizontal position has been investigated by Groves (1960). The present paper describes a method of estimating surface-wave fields anywhere in the vicinity of an array of wave recorders. The method departs from that of Groves through the introduction of a statistical description of the wave field. The basis of the method is similar to Wiener's (1949) linear predictor scheme for a random function of a single variable.

In a companion paper (Snyder 1973), the relationship between various second-order covariance fields for a linear sea is discussed for the case of stationary homogeneous and quasistationary quasihomogeneous statistics. Let $\chi(\boldsymbol{x},z,t)$ and $\psi(\boldsymbol{x},z,t)$ be any two field variables, and let

$$C_{\chi\psi} \equiv < \chi(\boldsymbol{x},z_1,t)\,\psi(\boldsymbol{x}+\boldsymbol{\xi},z_2,t+\tau) >$$

be the second-order covariance for the two fields. The symbols < > denote an ensemble average. $C_{\chi\psi}$ is a strong function of $\boldsymbol{\xi}$, τ, z_1, and z_2. For stationary homogeneous statistics, $C_{\chi\psi}$ is independent of $\boldsymbol{x}$ and t; for quasistationary

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quasihomogeneous statistics, it is a weak function of these variables. In either case, it can be shown that, for a linear sea,

$$C_{\chi\psi} \simeq \mathrm{Re}\left\{ \int_{-\infty}^{\infty} d^2k F_{\chi\psi} e^{i(\boldsymbol{k}\cdot\boldsymbol{\xi}-\omega(k)\tau)} \right\},$$

where $F_{\chi\psi}(z_1, z_2, \boldsymbol{k})$ is the (local) directional cross-spectrum for the two variables. For stationary homogeneous statistics, this relationship is exact; for quasistationary quasihomogeneous statistics, it is approximate, and both $C_{\chi\psi}$ and $F_{\chi\psi}$ are weak functions of $\boldsymbol{x}$ and t. In either case, $F_{\chi\psi}$ is proportional to the (local) directional spectrum for the surface elevation,

$$F_{\chi\psi} = R_{\chi\psi} F_{\zeta^2},$$

where $R_{\chi\psi}(z_1, z_2, \boldsymbol{k})$ is a known function of its arguments.

II. *Determination of the Optimum Linear Estimator for the Case of One Wave Recorder.* Suppose that a field variable $\psi(\boldsymbol{x}_1, z_1, t)$ is known for all t at some point $(\boldsymbol{x}_1, z_1)$. Let $\chi(\boldsymbol{x}, z, t)$ be the value of a second field variable at the point $(\boldsymbol{x}, z)$ at time t. We seek an optimum linear estimator for χ of the form

$$\chi'(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} d\tau K_{\chi\psi}(t-\tau)\psi(\boldsymbol{x}_1, \tau),$$

where, for simplicity of notation, we suppress the dependence of $K_{\chi\psi}$ on $\boldsymbol{x}$ and $\boldsymbol{x}_1$ and the dependence of $K_{\chi\psi}$, χ, and ψ on z and z_1. (In the following, the z dependence of all fields and the $\boldsymbol{x}, t$ dependence of all covariance fields also will be suppressed.)

To determine $K_{\chi\psi}$ we demand that the variance

$$\begin{aligned} V &\equiv \langle(\chi'-\chi)^2\rangle \\ &= \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 K_{\chi\psi}(t-\tau_1) K_{\chi\psi}(t-\tau_2) C_{\psi^2}(\boldsymbol{o}, \tau_2-\tau_1) \\ &\quad - 2\int_{-\infty}^{\infty} d\tau K_{\chi\psi}(t-\tau) C_{\chi\psi}(\boldsymbol{x}-\boldsymbol{x}_1, t-\tau) + C_{\chi^2}(\boldsymbol{o}, o) \end{aligned}$$

be a minimum. We replace $K_{\chi\psi}$ with $K_{\chi\psi} + \varepsilon\eta_{\chi\psi}$, where $\eta_{\chi\psi}$ is an arbitrary function, and set $(\partial/\partial\varepsilon) V|_{\varepsilon=0} = 0$. This procedure leads to the integral equation

$$\int_{-\infty}^{\infty} d\tau K_{\chi\psi}(t-\tau) C_{\psi^2}(\boldsymbol{o}, \tau) - C_{\chi\psi}(\boldsymbol{x}-\boldsymbol{x}_1, t) = 0.$$

Let $H_{\chi\psi}(\omega)$ be the Fourier transform of $K_{\chi\psi}(\tau)$. Then it follows from the convolution theorem that

$$H_{\chi\psi}(\omega) = \frac{1}{2\pi} \frac{D_{\chi\psi}(\boldsymbol{x} - \boldsymbol{x}_1, \omega)}{D_{\psi^2}(\boldsymbol{0}, \omega)},$$

where

$$\begin{aligned} D_{\chi\psi}(\xi, \omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau\, C_{\chi\psi}(\xi, \tau)\, e^{-i\omega\tau}, \\ &= \frac{1}{2} \int_{-\infty}^{\infty} d^2k\, F_{\chi\psi}{}^{*}\, \delta(\omega - \omega(k))\, e^{-i\boldsymbol{k}\cdot\xi}, \omega > 0, \\ &= \frac{1}{2} \int_{-\infty}^{\infty} d^2k\, F_{\chi\psi}\, \delta(\omega + \omega(k))\, e^{i\boldsymbol{k}\cdot\xi}, \omega < 0. \end{aligned}$$

We note that, by employing the change of variables,

$$\Omega = (gk)^{1/2}$$

and

$$\theta = \arg(\boldsymbol{k});$$

$D_{\chi\psi}$ can be brought into the form

$$\left.\begin{aligned} D_{\chi\psi} &= \frac{\omega^3}{g} \int_0^{2\pi} d\theta\, R_{\chi\psi}{}^{*}\, F_{\zeta^2}\, e^{-i\boldsymbol{k}\cdot\xi}, \omega > 0, \\ &= \frac{|\omega|^3}{g} \int_0^{2\pi} d\theta\, R_{\chi\psi}\, F_{\zeta^2}\, e^{i\boldsymbol{k}\cdot\xi}, \omega < 0, \end{aligned}\right\} \tag{1}$$

with

$$\boldsymbol{k}(\omega, \theta) = \left(\frac{\omega^2}{g}\cos\theta,\ \frac{\omega^2}{g}\sin\theta\right).$$

The resulting variance V may be reduced to the form

$$V = \int_0^{\infty} d\omega \left(D_{\chi^2}(\boldsymbol{0}, \omega) - \frac{|D_{\chi\psi}(\boldsymbol{x} - \boldsymbol{x}_1, \omega)|^2}{D_{\psi^2}(\boldsymbol{0}, \omega)} \right). \tag{2}$$

Note that, for $\boldsymbol{x} \neq \boldsymbol{x}_1$, this variance vanishes only for the singular case of $F_{\zeta^2}(\boldsymbol{k})$ concentrated along a line (unidirectional spectrum). For $\boldsymbol{x} = \boldsymbol{x}_1$, on the other hand, there exist sets of fields for which the variance associated with any pair from the set vanishes regardless of the character of $F_{\zeta^2}(\boldsymbol{k})$. One such set is the set ζ, φ, P, and all fields derived therefrom through successive and mixed application of the operators $\partial/\partial t$, $\partial/\partial z$, and ∇^2. To illustrate this result, consider as an example the pair ζ and φ. We have from (1), using table I in Snyder (1973),

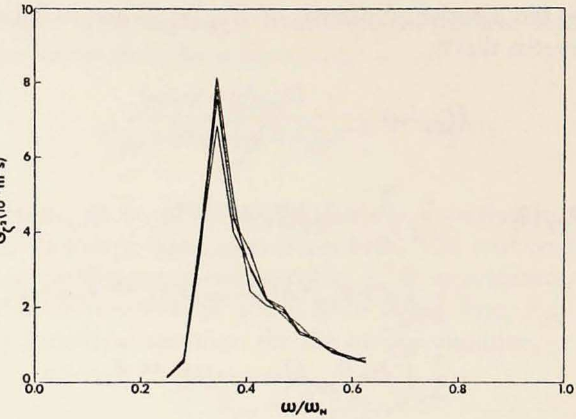

Figure 1. Spectrum of surface elevation. The spectrum plotted is $G_{\zeta^2}(\omega) = \int_0^{2\pi} d\theta\, E_{\zeta^2}(\omega, \theta)$. The estimates are Bartlett (1950) estimates with 256 degrees of freedom. The Nyquist frequency, ω_N, is $2\pi\, s^{-1}$(1 *Hz*).

$$D_{\zeta^2}(\mathbf{o},\omega) = \frac{|\omega|^3}{g^2}\int_0^{2\pi} d\theta\, F_{\zeta^2}(\boldsymbol{k}),$$

$$D_{\varphi^2}(\mathbf{o},\omega) = \frac{|\omega|^3}{g^2}\frac{g}{k} e^{\frac{2\omega^2}{g}z}\int_0^{2\pi} d\theta\, F_{\zeta^2}(\boldsymbol{k}),$$

and

$$D_{\zeta\varphi}(\mathbf{o},\omega) = i\frac{|\omega|^3}{g^2}\left(\frac{g}{k}\right)^{1/2} e^{\frac{2\omega^2}{g}z}\int_0^{2\pi} d\theta\, F_{\zeta^2}(\boldsymbol{k}),$$

from which

$$D_{\zeta^2}D_{\varphi^2} - |D_{\zeta\varphi}|^2 = 0;$$

hence

$$V = 0.$$

For $\boldsymbol{x} = \boldsymbol{x}_1$, the integrand of V is of the form

$$D_{\chi^2}(1 - \mu_{\chi\psi}),$$

where

$$\mu_{\chi\psi} \equiv \frac{|D_{\chi\psi}|^2}{D_{\chi^2}D_{\psi^2}}$$

is the (square) coherence of the signals χ and ψ. The vanishing of this integrand thus expresses the fact that the signals are coherent and that the resulting linear estimator is exact. This is the case discussed by Groves (1960).

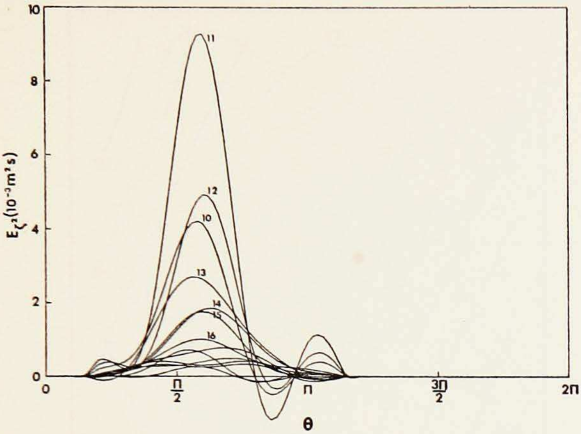

Figure 2. Directional spectrum $E_{\zeta^2}(\omega,\theta)$. The spectrum is plotted for frequency bands 8–20. (Band 32 is the Nyquist band.) θ is the direction of travel of the component counterclockwise from east.

III. *Determination of the Optimum Linear Estimator for the Case of a Recorder Array.* We next suppose that the field variable $\psi(\boldsymbol{x},z,t)$ is known for all t at the N points $(\boldsymbol{x}_n,z)$. Records at differing elevations z_n can be normalized to a single elevation by convolution (as implied in § 2). Let $\chi(\boldsymbol{x},z,t)$ be the value of a second field variable at the point $(\boldsymbol{x},z)$ at time t. We seek an optimum linear estimator for χ of the form

$$\chi'(\boldsymbol{x},t) = \sum_{n}^{N} \int_{-\infty}^{\infty} d\tau\, {}_nK_{\chi\psi}(t-\tau)\,\psi(\boldsymbol{x}_n,\tau),$$

where, for simplicity of notation, we suppress the dependence of ${}_nK_{\chi\psi}$ on $\boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ and the dependence of ${}_nK_{\chi\psi}$, χ and ψ on z. To determine the ${}_nK_{\chi\psi}$, we again demand that the variance

$$V = <(\chi'-\chi)^2>$$

be a minimum. This demand leads to the simultaneous integral equations

$$\sum_{m}^{N} \int_{-\infty}^{\infty} dt\, {}_mK_{\chi\psi}(t-\tau)\, C_{\psi^2}(\boldsymbol{x}_n - \boldsymbol{x}_m, \tau)$$

$$-\, C_{\chi\psi}(\boldsymbol{x}-\boldsymbol{x}_n,t) = 0, \quad n = 1, 2, \ldots, N.$$

Let ${}_mH_{\chi\psi}(\omega)$ be the Fourier transform of ${}_mK_{\chi\psi}$. Then

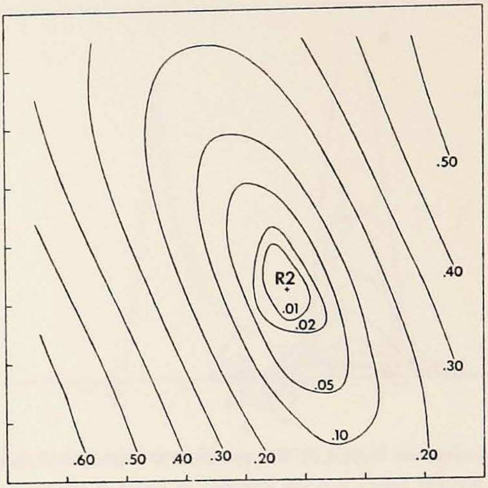

Figure 3. Proportional variance P for a single-component array. The location of the recorder ($R2$) is shown by the +. Scale is 1 m per scale division. North is at the top.

$$\sum_{m}^{N} {}_mH_{\chi\psi}(\omega)\, D_{\psi^2}(\boldsymbol{x}_n - \boldsymbol{x}_m, \omega) = \frac{1}{2\pi} D_{\chi\psi}(\boldsymbol{x} - \boldsymbol{x}_n, \omega)\, n = 1, 2, \ldots, N,$$

which can be numerically inverted to give ${}_mH_{\chi\psi}(\omega)$.

The resulting variance V is of the form

$$V = \int_0^\infty d\omega\, [D_{\chi^2}(\mathbf{o}, \omega) - 2\pi \sum_{m}^{N} {}_mH_{\chi\psi}(\omega)\, D_{\chi\psi}(\boldsymbol{x} - \boldsymbol{x}_m, -\omega)]. \qquad (3)$$

For $\chi = \psi$, V vanishes at the points $\boldsymbol{x}_n$ (as would be anticipated). V also vanishes at $\boldsymbol{x}_n$ if χ and ψ are related through successive mixed application of the operators $\partial/\partial t$, $\partial/\partial z$, and ∇^2.

IV. *Example.* The following example is taken from data obtained at an experimental site in the Bight of Abaco, Bahamas, in connection with a study of whitecap formation. (The technique described in this paper will be used to linearly reconstruct surface-wave fields in the neighborhood of whitecaps, in an attempt to determine a threshold criterion for their formation.) An array of four wave recorders (Snodgrass Mark X) was configured in a Mercedes star with arms of approximately 3 m (see Fig. 4). The pressure ports on the gauges were approximately 0.5 m below the mean surface. A 34-min 8-sec section of record was analyzed for each gauge. The resulting spectrum of surface elevation is shown in Fig. 1. A directional spectrum analysis was made for these records, using the truncated expansion

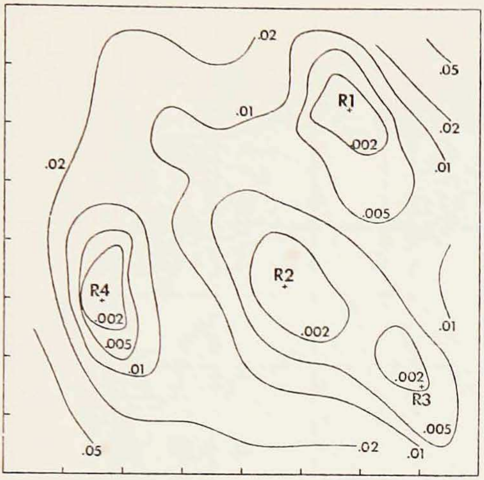

Figure 4. Proportional variance P for a four-component array. The locations of the four recorders ($R1$, $R2$, $R3$, and $R4$) are shown by the +'s. Scale is 1 m per scale division. North is at the top.

$$E_{\zeta^2}(\omega,\theta) \equiv \frac{2\,\omega^3}{g^2} F(\boldsymbol{k}(\omega,\theta)) = \sum_n^N {}_nE_{\zeta^2}(\omega)\,\psi_n(\theta),$$

with

$$\begin{aligned}\psi_n(\theta) &\equiv \cos(\vartheta-\vartheta_W)\cos[(n-1)(\theta-\theta_W)], \quad \cos(\vartheta-\vartheta_W)>0 \quad \text{and} \quad n \text{ odd}\\ &\equiv \cos(\vartheta-\vartheta_W)\sin n(\theta-\theta_W), \quad \cos(\vartheta-\vartheta_W)>0 \quad \text{and} \quad n \text{ even}\\ &\equiv 0, \quad \cos(\vartheta-\vartheta_W)<0,\end{aligned}$$

where θ_W is the wind azimuth. This expansion, which does not allow wave components traveling against the wind, leads to a modified version of the Fourier-Bessel technique for directional spectrum analysis, as described by Gilchrist (1966). The resulting best-fit directional spectrum to the co- and quadrature spectra between the four wave recorders is displayed in Fig. 2.

The corresponding proportional variances

$$P = \frac{<(\zeta'-\zeta)^2>}{<\zeta^2>}$$

were calculated from (2) and (3) for a single-component array (the central recorder) and for the four-component array. These proportional variances are displayed in Figs. 3 and 4. Note that, in the second case, the maximum proportional variance in the vicinity of the array is of the order 0.01. The variance distribution clearly reflects the fact that downwind coherences are generally greater than crosswind coherences.

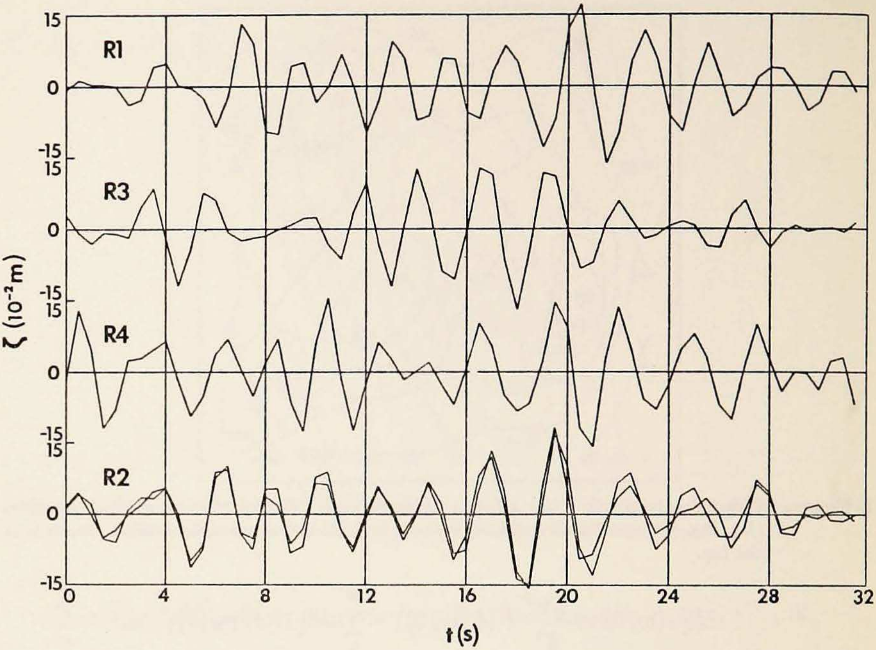

Figure 5. Comparison of a portion of the record for $R2$, with a record estimated from the three outer records.

An estimate of the surface elevation at the central recorder was made from the three outer recorders for a short section of the records. This estimate is compared in Fig. 5 with the recorded elevation. The variance of the difference (0.00052 m^2) compares favorably with that computed from the relation (3) (0.00057 m^2).

The surface elevation was estimated at 64 evenly spaced points in the vicinity of the wave-recorder array, using all four recorders; the resulting distribution was contoured in Fig. 6 at half-second intervals for a period of six sec. This distribution shows clearly the progression of the waves from the SSE to the NNW.

V. *Nonlinear Considerations*. The usefulness of the technique described in this paper is limited by its assumption of a linear sea. Although this assumption is generally considered to be a reasonable one, it is of interest to examine how the technique might be expanded in order to account for nonlinearities. A natural extension would attempt a bilinear representation of the form

$$\chi' = \sum_{n}^{N} \int d\tau_1 \, {}_nK_{\chi\psi}(t-\tau_1)\,\psi(\boldsymbol{x}_n, \tau_1)$$

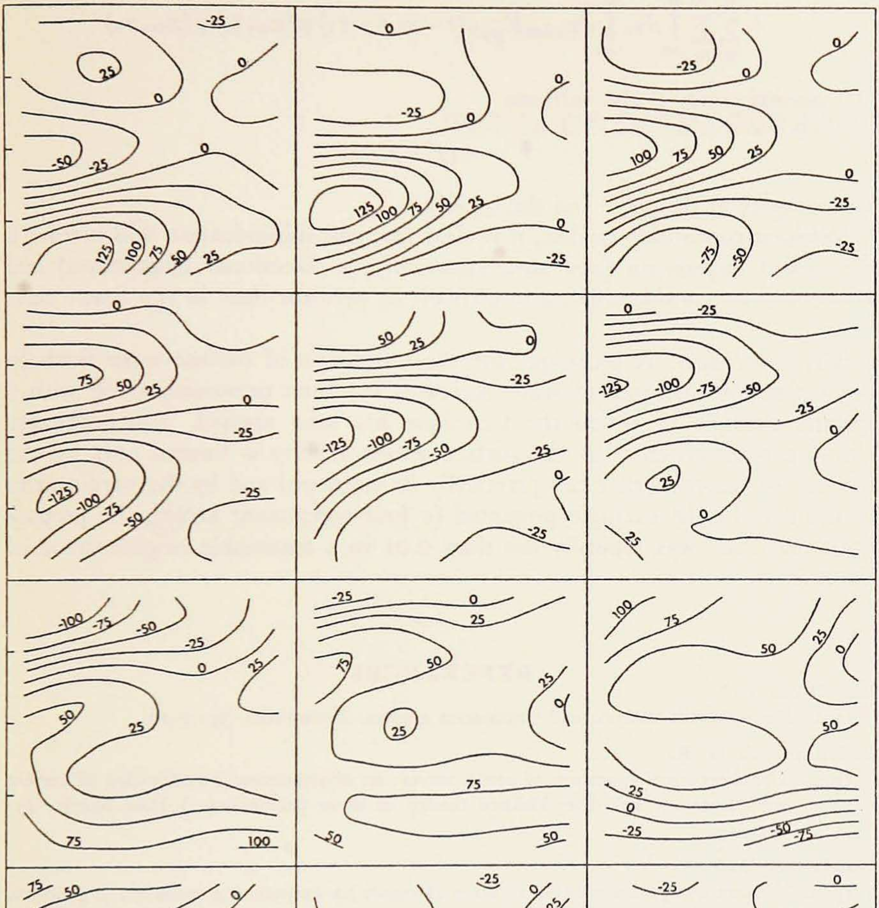

Figure 6. The surface elevation in the vicinity of the wave-recorder array as estimated from the four records. The elevation is contoured for each half second. The progression is from left to right, top to bottom. Each plot covers the same area as do Figs. 3 and 4. Contour values are in 10^{-2} m. Scale is 2 m per division. North is at the top.

$$+\sum_{n}^{N}\sum_{m}^{N}\int_{-\infty}^{\infty}d\tau_1\int_{-\infty}^{\infty}d\tau_2\,{}_{nm}K_{\chi\psi^2}(t-\tau_1,t-\tau_2)\,\psi(x_n,\tau_1)\,\psi(x_m,\tau_2)$$

and minimization of the variance

$$V = <(\chi'-\chi)^2>$$

with respect to the ${}_nK$'s and the ${}_{nm}K$'s.

Without continuing farther, it is clear that this minimization will involve a directional bispectrum (and also, apparently, a directional trispectrum) and hence will be considerably more difficult to perform than in the linear case.

VI. *Conclusions.* A technique for the estimation of surface-wave fields in the vicinity of an array of wave recorders has been presented, along with a specific example in which the technique has been applied. The technique allows a reconstruction of the surface whose fidelity is limited only by the number of recorders that can practically be employed and by the nonlinearity of the sea. In the example presented (a four-component array), the proportional variance was typically less than 0.01 in a reasonable neighborhood of the array.

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