Covariance Equations for a Linear Sea[1]

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ABSTRACT

The equations governing the second-order covariances between surface gravity-wave field variables are derived for a linear sea. These equations are solved formally for two cases: the case of stationary homogeneous statistics and the case of quasistationary quasihomogeneous statistics. The first case defines the directional spectra and directional cross spectra for the field variables and establishes the relationship between these spectra; the second case defines the corresponding local spectra for these variables and leads to Hasselmann's (1960) equation for the development of a linear sea.

I. *Introduction.* A complete description of the state of the sea is contained in the field variable $\zeta(\boldsymbol{x},t)$, the surface elevation at horizontal position $\boldsymbol{x}$ and time t. This variable and others are dynamically related by the equations of motion.

A complete statistical description of the state of the sea is contained in the covariances

$$C_{\zeta^n} \equiv <\zeta(\boldsymbol{x}_1,t_1)\zeta(\boldsymbol{x}_2,t_2)\ldots\zeta(\boldsymbol{x}_n,t_n)>, \; n=2,\, 3,\ldots,\, \infty,$$

where the symbols $< >$ denote an ensemble average. These covariances and others formed by taking ensemble averages of mixed products of field variables are dynamically related by a set of covariance equations derivable from the equations of motion. Because the equations of motion are nonlinear, no finite subset of the resulting covariance equations is closed, a situation similar to that which obtains in the theory of turbulence. This difficulty is removed if the equations of motion are linearized (Eckart 1953a, 1953b). Because surface gravity waves are only weakly nonlinear, it is therefore relevant to examine the covariance equations under the assumption of a linear sea.

Two cases are considered: (1) the case of stationary homogeneous statistics, and (2) the case of quasistationary quasihomogeneous statistics. In both cases

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only the dynamics of the second-order covariances are examined. Partial justification for this restriction lies in the observed quasi-Gaussian character of surface gravity-wave statistics. (The statistics of a strictly Gaussian sea would be totally parameterized by the second-order covariances.)

II. *Development of the Covariance Equations.* Let $\zeta(\boldsymbol{x},t)$ be the surface elevation at the horizontal position $\boldsymbol{x}$ at time t, and let $P(\boldsymbol{x},z,t)$ and $\varphi(\boldsymbol{x},z,t)$ be the corresponding pressure and velocity potential at the point $(\boldsymbol{x},z)$. (The z axis is taken to be positive upward, with the mean water surface at $z = 0$). Let $Q(\boldsymbol{x},t)$ be the atmospheric pressure field at the mean surface (analytically continued where necessary). Then for small-amplitude deep-water gravity waves the equations of motion are

$$\nabla^2\varphi + \frac{\partial^2}{\partial z^2}\varphi = 0,$$

$$\frac{P}{\varrho} - \frac{\partial}{\partial t}\varphi = 0,$$

with

$$\frac{\partial}{\partial z}\varphi = 0, \quad \text{for} \quad z = -\infty,$$

$$\frac{\partial}{\partial t}\zeta + \frac{\partial}{\partial z}\varphi = 0, \quad \text{for} \quad z = 0,$$

and

$$\frac{\partial}{\partial t}\varphi - g\zeta = \frac{Q}{\varrho}, \quad \text{for} \quad z = 0.$$

From the fields ζ, φ, P, and Q, the second-order covariances,

$$C_{\zeta^2} \equiv <\zeta(\boldsymbol{x},t)\zeta(\boldsymbol{x}+\xi,t+\tau)>,$$

$$C_{\zeta\varphi} \equiv <\zeta(\boldsymbol{x},t)\varphi(\boldsymbol{x}+\xi,z_2,t+\tau)>,$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$C_{Q^2} \equiv <Q(\boldsymbol{x},t)Q(\boldsymbol{x}+\boldsymbol{\xi},t+\tau)>,$$

are defined. If we right-multiply and left-multiply the equations of motion by ζ, φ, P, and Q, and apply an ensemble average, the following equations are obtained:

$$\left(\nabla^2_{\xi} + \frac{\partial^2}{\partial z_2^2}\right)\begin{bmatrix} C_{\zeta\varphi} \\ C_{\varphi^2} \\ C_{P\varphi} \\ C_{Q\varphi} \end{bmatrix} = 0, \tag{1}$$

$$\left(\left(\nabla_x - \nabla_\xi\right)^2 + \frac{\partial^2}{\partial z_1^2}\right) \begin{bmatrix} C_{\varphi\zeta} \\ C_{\varphi^2} \\ C_{\varphi P} \\ C_{\varphi Q} \end{bmatrix} = 0, \tag{2}$$

$$\frac{1}{\varrho} \begin{bmatrix} C_{\zeta P} \\ C_{\varphi P} \\ C_{P^2} \\ C_{QP} \end{bmatrix} - \frac{\partial}{\partial \tau} \begin{bmatrix} C_{\zeta\varphi} \\ C_{\varphi^2} \\ C_{P\varphi} \\ C_{Q\varphi} \end{bmatrix} = 0, \tag{3}$$

$$\frac{1}{\varrho} \begin{bmatrix} C_{P\zeta} \\ C_{P\varphi} \\ C_{P^2} \\ C_{PQ} \end{bmatrix} - \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \tau}\right) \begin{bmatrix} C_{\varphi\zeta} \\ C_{\varphi^2} \\ C_{\varphi P} \\ C_{\varphi Q} \end{bmatrix} = 0, \tag{4}$$

$$\frac{\partial}{\partial z_2} \begin{bmatrix} C_{\zeta\varphi} \\ C_{\varphi^2} \\ C_{P\varphi} \\ C_{Q\varphi} \end{bmatrix} = 0,\ z_2 = -\infty, \tag{5}$$

$$\frac{\partial}{\partial z_1} \begin{bmatrix} C_{\varphi\zeta} \\ C_{\varphi^2} \\ C_{\varphi P} \\ C_{\varphi Q} \end{bmatrix} = 0,\ z_1 = -\infty, \tag{6}$$

$$\frac{\partial}{\partial \tau} \begin{bmatrix} C_{\zeta^2} \\ C_{\varphi\zeta} \\ C_{P\zeta} \\ C_{Q\zeta} \end{bmatrix} + \frac{\partial}{\partial z_2} \begin{bmatrix} C_{\zeta\varphi} \\ C_{\varphi^2} \\ C_{P\varphi} \\ C_{Q\varphi} \end{bmatrix} = 0,\ z_2 = 0, \tag{7}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \tau}\right) \begin{bmatrix} C_{\zeta^2} \\ C_{\zeta\varphi} \\ C_{\zeta P} \\ C_{\zeta Q} \end{bmatrix} + \frac{\partial}{\partial z_1} \begin{bmatrix} C_{\varphi\zeta} \\ C_{\varphi^2} \\ C_{\varphi P} \\ C_{\varphi Q} \end{bmatrix} = 0,\ z_1 = 0, \tag{8}$$

$$\frac{\partial}{\partial \tau} \begin{bmatrix} C_{\zeta\varphi} \\ C_{\varphi^2} \\ C_{P\varphi} \\ C_{Q\varphi} \end{bmatrix} - g \begin{bmatrix} C_{\zeta^2} \\ C_{\varphi\zeta} \\ C_{P\zeta} \\ C_{Q\zeta} \end{bmatrix} = \frac{1}{\varrho} \begin{bmatrix} C_{\zeta Q} \\ C_{\varphi Q} \\ C_{PQ} \\ C_{Q^2} \end{bmatrix},\ z_2 = 0, \tag{9}$$

and

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \tau}\right) \begin{bmatrix} C_{\varphi\zeta} \\ C_{\varphi^2} \\ C_{\varphi P} \\ C_{\varphi Q} \end{bmatrix} - g \begin{bmatrix} C_{\zeta^2} \\ C_{\zeta\varphi} \\ C_{\zeta P} \\ C_{\zeta Q} \end{bmatrix} = \frac{1}{\varrho} \begin{bmatrix} C_{Q\zeta} \\ C_{Q\varphi} \\ C_{QP} \\ C_{Q^2} \end{bmatrix},\ z_1 = 0. \tag{10}$$

Because the covariances are real,

$$\begin{bmatrix} C_{\zeta^2} \\ C_{\zeta\varphi} \\ \cdot \\ \cdot \\ \cdot \\ C_{Q^2} \end{bmatrix}^* = \begin{bmatrix} C_{\zeta^2} \\ C_{\varphi\zeta} \\ \cdot \\ \cdot \\ \cdot \\ C_{Q^2} \end{bmatrix}. \tag{11}$$

An additional symmetry condition is implied by the definition of the covariances

$$\begin{bmatrix} C_{\zeta^2}(\boldsymbol{x},t,-\boldsymbol{\xi},-\tau) \\ C_{\zeta\varphi}(\boldsymbol{x},t,-\boldsymbol{\xi},-\tau) \\ \cdot \\ \cdot \\ \cdot \\ C_{Q^2}(\boldsymbol{x},t,-\boldsymbol{\xi},-\tau) \end{bmatrix} = \begin{bmatrix} C_{\zeta^2}(\boldsymbol{x}-\boldsymbol{\xi},t-\tau,\boldsymbol{\xi},\tau) \\ C_{\varphi\zeta}(\boldsymbol{x}-\boldsymbol{\xi},t-\tau,\boldsymbol{\xi},\tau) \\ \cdot \\ \cdot \\ \cdot \\ C_{Q^2}(\boldsymbol{x}-\boldsymbol{\xi},t-\tau,,\boldsymbol{\xi}\tau) \end{bmatrix}. \tag{12}$$

The above equations constitute the covariance equations for a linear sea.

III. *Solution of the Covariance Equations for Stationary Homogeneous Statistics.* The assumption of stationary homogeneous statistics implies that the second-order covariances are independent of $\boldsymbol{x}$ and t. The solution of the resulting covariance equations is analogous to the solution of the linearized field equations governing the propagation of an initial disturbance of limited extent, the well-known Cauchy-Poisson problem. Because of the stationary homogeneous statistics, the field variables themselves are not Fourier-integrable; no such restriction, however, is implied for the covariances. Accordingly, these may be expanded as Fourier integrals in $\boldsymbol{\xi}$. Applying the covariance eqs. (1), (2), (5), and (6) gives the representations

$$\begin{bmatrix} C_{\zeta^2} \\ C_{\zeta\varphi} \\ C_{\varphi\zeta} \\ C_{\varphi^2} \end{bmatrix} = \int_{-\infty}^{\infty} d^2k \begin{bmatrix} D_{\zeta^2} \\ D_{\zeta\varphi}\, e^{kz_2} \\ D_{\varphi\zeta}\, e^{kz_1} \\ D_{\varphi^2}\, e^{k(z_1+z_2)} \end{bmatrix} e^{i\boldsymbol{k}\cdot\boldsymbol{\xi}},$$

where the D's are functions of $(\boldsymbol{k},\tau)$.

It follows that, for $Q = 0$,

$$\frac{\partial}{\partial\tau}\begin{bmatrix} D_{\zeta^2} \\ D_{\varphi\zeta} \end{bmatrix} + k\begin{bmatrix} D_{\zeta\varphi} \\ D_{\varphi^2} \end{bmatrix} = 0,$$

$$-\frac{\partial}{\partial\tau}\begin{bmatrix} D_{\zeta^2} \\ D_{\zeta\varphi} \end{bmatrix} + k\begin{bmatrix} D_{\varphi\zeta} \\ D_{\varphi^2} \end{bmatrix} = 0,$$

$$\frac{\partial}{\partial \tau}\begin{bmatrix} D_{\zeta\varphi} \\ D_{\varphi^2} \end{bmatrix} - g \begin{bmatrix} D_{\zeta^2} \\ D_{\varphi\zeta} \end{bmatrix} = 0,$$

and

$$-\frac{\partial}{\partial \tau}\begin{bmatrix} D_{\varphi\zeta} \\ D_{\varphi^2} \end{bmatrix} - g \begin{bmatrix} D_{\zeta^2} \\ D_{\zeta\varphi} \end{bmatrix} = 0,$$

with the solution

$$D_{\zeta^2} = \frac{1}{2}\left(F_{\zeta^2} e^{-i\omega(k)\tau} + G_{\zeta^2} e^{i\omega(k)\tau}\right),$$

$$D_{\zeta\varphi} = -D_{\varphi\zeta} = \frac{i\omega(k)}{2k}\left(F_{\zeta^2} e^{-i\omega(k)\tau} - G_{\zeta^2} e^{i\omega(k)\tau}\right),$$

and

$$D_{\varphi^2} = \frac{\omega(k)^2}{2k^2}\left(F_{\zeta^2} e^{-i\omega(k)\tau} + G_{\zeta^2} e^{i\omega(k)\tau}\right),$$

where

$$\omega(k) \equiv (gk)^{1/2}$$

and F_{ζ^2} and G_{ζ^2} are arbitrary functions of $\boldsymbol{k}$, determined by the initial conditions.

The reality and symmetry conditions (11) and (12) imply that

$$F_{\zeta^2}(-\boldsymbol{k}) = G_{\zeta^2}(\boldsymbol{k})$$

and

$$F_{\zeta^2}(\boldsymbol{k})^* = F_{\zeta^2}(\boldsymbol{k}).$$

Thus

$$C_{\zeta^2} = \int_{-\infty}^{\infty} d^2k\, F_{\zeta^2} \cos(\boldsymbol{k}\cdot\boldsymbol{\xi} - \omega(k)\tau),$$

$$C_{\zeta\varphi} = \int_{-\infty}^{\infty} d^2k \left(\frac{g}{k}\right)^{1/2} e^{kz_2} F_{\zeta^2} \sin(\boldsymbol{k}\cdot\boldsymbol{\xi} - \omega(k)\tau),$$

$$C_{\varphi\zeta} = -\int_{-\infty}^{\infty} d^2k \left(\frac{g}{k}\right)^{1/2} e^{kz_1} F_{\zeta^2} \sin(\boldsymbol{k}\cdot\boldsymbol{\xi} - \omega(k)\tau),$$

and

$$C_{\varphi^2} = \int_{-\infty}^{\infty} d^2k \left(\frac{g}{k}\right) e^{k(z_1+z_2)} F_{\zeta^2} \cos(\boldsymbol{k}\cdot\boldsymbol{\xi} - \omega(k)\tau).$$

Similar expressions may be derived for the covariances C_{P^2}, $C_{\zeta P}$, $C_{P\zeta}$, $C_{\varphi P}$, and $C_{P\varphi}$. Characteristically, if χ and ψ are any of the fields ζ, φ, P or any of their derivatives, the covariance $C_{\chi\psi}$ may be expressed in the form

$$C_{\chi\psi} = Re\left\{\int_{-\infty}^{\infty} d^2k\, F_{\chi\psi} e^{i(\boldsymbol{k}\cdot\boldsymbol{\xi} - \omega(k)\tau)}\right\},$$

with

$$F_{\chi\psi} = R_{\chi\psi} F_{\zeta^2},$$

where $R_{\chi\psi}$ is some transfer function.

The functions $F_{\chi\psi}$ constitute the directional cross spectra for the field variables. It is seen that these spectra are linearly related to one another and to the directional spectrum for the surface elevation F_{ζ^2}. The ratios between these spectra and F_{ζ^2} are functions of the vertical coordinates z_1 and z_2 and the propagation vector $\boldsymbol{k}$. These ratios (transfer functions) are tabulated for the fields ζ, φ, and P in Table I. Extension of the tabulated results to derivative fields is accomplished by multiplication by $-i\omega$ (time derivative) or $i\boldsymbol{k}$ (horizontal gradient). The formalism may be extended to include the wave-coherent part of atmospheric fields above the water surface.

Table I. $R_{\chi\psi}$ for the fields ζ, φ, and P.

χ/ψ	ζ	φ	P
ζ	1	$i\left(\frac{g}{k}\right)^{1/2} e^{kz_2}$	$\varrho g e^{kz_2}$
φ	$-i\left(\frac{g}{k}\right)^{1/2} e^{kz_1}$	$\frac{g}{k} e^{k(z_1+z_2)}$	$-i\varrho\left(\frac{g^3}{k}\right)^{1/2} e^{k(z_1+z_2)}$
P	$-\varrho g e^{kz_1}$	$i\varrho\left(\frac{g^3}{k}\right)^{1/2} e^{k(z_1+z_2)}$	$(\varrho g)^2 e^{k(z_1+z_2)}$

IV. *Solution of the Covariance Equations for Quasistationary Quasihomogeneous Statistics.* With quasistationary quasihomogeneous statistics, the second-order covariances are no longer independent of $\boldsymbol{x}$ and t. However, the scales of the variation with respect to $\boldsymbol{x}$ and t are large compared with those of the variation with respect to $\boldsymbol{\xi}$ and τ. This difference in scale allows some later approximations. Q is taken as the sum of a turbulent and a wave-induced part

$$Q = q(\boldsymbol{x},t) + \int_{-\infty}^{\infty} d^2\xi\, d\tau\, \lambda(\boldsymbol{x},t,\boldsymbol{\xi},\tau)\,\zeta(\boldsymbol{x}+\boldsymbol{\xi},t+\tau) + \ldots$$

Higher-order terms in the functional-power series expansion for the wave-induced part are neglected. Because of the space-time invariance of the system (the mean atmospheric fields being assumed independent of $\boldsymbol{x}$ and t),

$$\lambda = \lambda(\boldsymbol{\xi},\tau).$$

λ is next expanded as a Fourier integral,

$$\lambda(\boldsymbol{\xi},\tau) = \frac{\varrho g}{(2\pi)^3}\int_{-\infty}^{\infty} d^2k\, d\omega\, \Lambda(\boldsymbol{k},\omega)\, e^{i(\boldsymbol{k}\cdot\boldsymbol{\xi}+\omega\tau)},$$

and $\Lambda(\boldsymbol{k},\omega)$ is assumed small:

$$|\Lambda(\boldsymbol{k},\omega)| \ll 1.$$

Because λ is real,

$$\Lambda(-\boldsymbol{k},-\omega) = \Lambda(\boldsymbol{k},\omega)^*.$$

The covariances are assumed expandable as Fourier integrals in x, t, ξ, and τ. (The reader who is disturbed by this assumption may consider the following as schematic of a more rigorous demonstration). Application of (1), (2), (5), and (6) gives the corresponding representations

$$\begin{bmatrix} C_{\zeta^2} \\ C_{\zeta\varphi} \\ C_{\zeta q} \\ C_{\varphi\zeta} \\ C_{\varphi^2} \\ C_{\varphi q} \\ C_{q\zeta} \\ C_{q\varphi} \\ C_{q^2} \end{bmatrix} = \int_{-\infty}^{\infty} d^2K\,d\Omega\,d^2k\,d\omega \begin{bmatrix} D_{\zeta^2} \\ D_{\zeta\varphi}\,e^{kz_2} \\ D_{\zeta q} \\ D_{\varphi\zeta}\,e^{|\boldsymbol{k}-\boldsymbol{K}|z_1} \\ D_{\varphi^2}\,e^{|\boldsymbol{k}-\boldsymbol{K}|z_1+kz_2} \\ D_{\varphi q}\,e^{|\boldsymbol{k}-\boldsymbol{K}|z_1} \\ D_{q\zeta} \\ D_{q\varphi}\,e^{kz_2} \\ D_{q^2} \end{bmatrix} e^{i(\boldsymbol{K}\cdot\boldsymbol{x}+\Omega t+\boldsymbol{k}\cdot\boldsymbol{\xi}+\omega\tau)},$$

where the D's are functions of $(\boldsymbol{K},\Omega,\boldsymbol{k},\omega)$.

The D's satisfy the relationships

$$i\omega\begin{bmatrix} D_{\zeta^2} \\ D_{\varphi\zeta} \\ D_{q\zeta} \end{bmatrix} + k\begin{bmatrix} D_{\zeta\varphi} \\ D_{\varphi^2} \\ D_{q\varphi} \end{bmatrix} = 0,$$

$$-i(\omega-\Omega)\begin{bmatrix} D_{\zeta^2} \\ D_{\zeta\varphi} \\ D_{\zeta q} \end{bmatrix} + |\boldsymbol{k}-\boldsymbol{K}|\begin{bmatrix} D_{\varphi\zeta} \\ D_{\varphi^2} \\ D_{\varphi q} \end{bmatrix} = 0,$$

$$i\omega\begin{bmatrix} D_{\zeta\varphi} \\ D_{\varphi^2} \\ D_{q\varphi} \end{bmatrix} - g\begin{bmatrix} D_{\zeta^2} \\ D_{\varphi\zeta} \\ D_{q\zeta} \end{bmatrix} = \frac{1}{\varrho}\begin{bmatrix} D_{\zeta q} \\ D_{\varphi q} \\ D_{q^2} \end{bmatrix} + g\Lambda^*\begin{bmatrix} D_{\zeta^2} \\ D_{\varphi\zeta} \\ D_{q\zeta} \end{bmatrix},$$

and

$$-i(\omega-\Omega)\begin{bmatrix} D_{\varphi\zeta} \\ D_{\varphi^2} \\ D_{\varphi q} \end{bmatrix} - g\begin{bmatrix} D_{\zeta^2} \\ D_{\zeta\varphi} \\ D_{\zeta q} \end{bmatrix} = \frac{1}{\varrho}\begin{bmatrix} D_{q\zeta} \\ D_{q\varphi} \\ D_{q^2} \end{bmatrix} + g\Lambda(\boldsymbol{k}-\boldsymbol{K},\omega-\Omega)\begin{bmatrix} D_{\zeta^2} \\ D_{\zeta\varphi} \\ D_{\zeta q} \end{bmatrix},$$

from which

$$(\omega^2 - gk(1+\Lambda^*))\,D_{\zeta^2} = \frac{k}{\varrho}D_{\zeta q}$$

and

$$((\omega-\Omega)^2 - g|\boldsymbol{k}-\boldsymbol{K}|(1+\Lambda(\boldsymbol{k}-\boldsymbol{K},\omega-\Omega)))\,D_{\zeta q} = \frac{|\boldsymbol{k}-\boldsymbol{K}|}{\varrho}D_{q^2}.$$

It follows that

$$D_{\zeta^2} = \frac{k|\boldsymbol{k}-\boldsymbol{K}|}{\varrho^2} \frac{D_{q^2}}{(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)(\omega-\omega_4)}, \tag{13}$$

where, using the approximations

$$(1+\Lambda)^{1/2} \simeq 1 + i\frac{\Lambda_i}{2}$$

and

$$(1+\Lambda^*)^{1/2} \simeq 1 - i\frac{\Lambda_i}{2},$$

$$\omega_1 \simeq \omega(k)\left(1 - i\frac{\Lambda_i}{2}\right),$$

$$\omega_2 \simeq -\omega(k)\left(1 - i\frac{\Lambda_i}{2}\right),$$

$$\omega_3 \simeq \Omega + \omega(|\boldsymbol{k}-\boldsymbol{K}|)\left(1 + \frac{i\Lambda_i(\boldsymbol{k}-\boldsymbol{K},\omega-\Omega)}{2}\right),$$

and

$$\omega_4 \simeq \Omega - \omega(|\boldsymbol{k}-\boldsymbol{K}|)\left(1 + \frac{i\Lambda_i(\boldsymbol{k}-\boldsymbol{K},\omega-\Omega}{2}\right.$$

It is clear from (13) that, for given $\boldsymbol{k}$, D_{ζ^2} is large only in the neighborhood of $\pm\omega(k)$. (In accordance with the assumption of quasistationary quasihomogeneous statistics, Ω may be neglected relative to ω, and $\boldsymbol{K}$ relative to $\boldsymbol{k}$.) Furthermore, the symmetry condition (12) implies that

$$D_{\zeta^2}(\boldsymbol{K},\Omega,-\boldsymbol{k},-\omega) \simeq D_{\zeta^2}(\boldsymbol{K},\Omega,\boldsymbol{k},\omega).$$

($\boldsymbol{K}\cdot\nabla_{\boldsymbol{k}} D_{\zeta^2}$, $\Omega\frac{\partial}{\partial\omega}D_{\zeta^2}$, and higher-order terms may be neglected relative to D_{ζ^2}.)

It follows that

$$C_{\zeta^2} \simeq \int_{-\infty}^{\infty} d^2k F_{\zeta^2} \cos(\boldsymbol{k}\cdot\boldsymbol{\xi} - \omega(k)\tau),$$

where

$$F_{\zeta^2} \equiv 2\int_{-\infty}^{\infty} d^2K \int_{-\infty}^{\infty} d\Omega \int_N d\omega D_{\zeta^2} e^{i(\boldsymbol{K}\cdot\boldsymbol{x}+\Omega t)} \tag{14}$$

and N is a suitably large neighborhood of $-\omega(k)$. $F_{\zeta^2}(\boldsymbol{x},t,\boldsymbol{k})$ is the local directional spectrum for quasistationary quasihomogeneous statistics and is analogous to $F_{\zeta^2}(\boldsymbol{k})$ for stationary homogeneous statistics. Local directional spectra and local directional cross spectra for the remaining field variables are defined in similar fashion. The ratios between these spectra and F_{ζ^2} are the same as for stationary homogeneous statistics (see Table I).

Integrating (13) with respect to ω over N gives

$$\int_N d\omega D_{\zeta^2} = \frac{k^2 D_{q^2}(\boldsymbol{K},\Omega,\boldsymbol{k},-\omega(k))}{4\varrho^2\omega(k)^2}\int_N d\omega \frac{1}{(\omega-\omega_2)(\omega-\omega_4)}.$$

The integral on the right may be evaluated approximately by extending the path of integration beyond the neighborhood of $-\omega(k)$, closing this path in either half-plane, and applying the calculus of residues. This procedure yields the result

$$\int_N d\omega \frac{1}{(\omega-\omega_2)(\omega-\omega_4)} \simeq \mp \frac{2\pi i}{(\Omega+(\omega(k)-\omega(|\boldsymbol{k}-\boldsymbol{K}|))-i\omega(k)\Lambda_i)},$$

where the sign of the right-hand side is opposite to that of Λ_i. Because

$$\omega(k)-\omega(|\boldsymbol{k}-\boldsymbol{K}|) \simeq \boldsymbol{K}\cdot\boldsymbol{V}(\boldsymbol{k}),$$

where $\boldsymbol{V}(\boldsymbol{k}) \equiv \nabla_{\boldsymbol{k}}\omega(k)$ is the group velocity associated with the component $\boldsymbol{k}$, the integration gives

$$\int_N d\omega(i\Omega+i\boldsymbol{K}\cdot\boldsymbol{V}(\boldsymbol{k})+\omega(k)\Lambda_i(\boldsymbol{k},-\omega(k)))D_{\zeta^2} = \pm\frac{\pi k^2}{2\varrho^2\omega(k)^2} D_{q^2}(\boldsymbol{K},\Omega,\boldsymbol{k},-\omega(k)),$$

where the sign of the right-hand side is the same as that of Λ_i. Multiplying by $2e^{i(\boldsymbol{K}\cdot\boldsymbol{x}+\Omega t)}$ and integrating with respect to $\boldsymbol{K}$ and Ω gives

$$\frac{\partial}{\partial t}F_{\zeta^2}+\boldsymbol{V}(\boldsymbol{k})\cdot\nabla_{\boldsymbol{x}}F_{\zeta^2} = \alpha+\beta F_{\zeta^2}, \qquad (15)$$

where

$$\alpha(\boldsymbol{x},t,\boldsymbol{k}) \equiv \pm\frac{\pi k^2}{\varrho^2\omega(k)^2}F_{q^2}(\boldsymbol{x},t,\boldsymbol{k},-\omega(k)),$$

$$\beta(\boldsymbol{k}) \equiv -\omega(k)\Lambda_i(\boldsymbol{k},-\omega(k)),$$

and

$$F_{q^2}(\boldsymbol{x},t,\boldsymbol{k},\omega) \equiv \int_{-\infty}^{\infty} d^2K \int_{-\infty}^{\infty} d\Omega D_{q^2}e^{i(\boldsymbol{K}\cdot\boldsymbol{x}+\Omega t)}.$$

Eq. (15) is Hasselmann's (1960) equation for the development of a linear sea, as derived from a consideration of the equations governing the covariances between field variables. This derivation contains Phillips' (1957) theory (the α term) explicitly and Miles' (1957) theory (the β term) in parametric form. (The dynamics of the atmospheric motion and the resulting form for λ are not discussed.)

The present derivation suggests a feature that is unexpected, paradoxical, and incorrect; the sign of the contribution of the turbulent pressure fluctuations to the rate of change of the spectrum is positive if the system is damped, negative if it is self-excited. The negative sign for the self-excited case is not apparent in integral derivations of the equation presented by either Phillips (1966) or Hasselmann (1962), and the resulting equation is physically unacceptable as a synoptic tool to treat the initial generation of waves starting from a state of rest. (The equation predicts a negative F_{ζ_2}.) An allied difficulty is that the sign of α in the neutral case ($\Lambda_i = 0$) is ambiguous. If we imagine the neutral case to be the limit of the damped case, then this sign is positive; if we imagine the neutral case to be the limit of the self-excited case, then this sign is negative. The crucial point appears to be the manner in which the integration in the neighborhood of $-\omega(k)$ passes by the poles at ω_2 and ω_4. Accordingly, the resolution of the difficulty would seem to involve specifying that the path N in the definition (14) for F_{ζ_2} departs from the real axis in such a way that it passes under the pole at ω_2 and over the pole at ω_4, regardless of Λ_i. This specification results in a positive α and F_{ζ_2} for all three cases and removes the ambiguity for the neutral case.

The assumption that

$$\lambda(\boldsymbol{x}, t, \boldsymbol{\xi}, \tau) = \lambda(\boldsymbol{\xi}, t)$$

has yielded a β that is independent of $\boldsymbol{x}$ and t. This result is somewhat less general than might be desired. It is intuitively clear that the appropriate generalization for λ slowly varying in $\boldsymbol{x}$ and t is

$$\beta = \beta(\boldsymbol{x}, t, \boldsymbol{k}) = -\omega(k)\Lambda_i(\boldsymbol{x}, t, \boldsymbol{k}, -\omega(k)),$$

where

$$\lambda = \frac{\varrho g}{(2\pi)^3} \int_{-\infty}^{\infty} d^2k d\omega\, \Lambda(\boldsymbol{x}, t, \boldsymbol{k}, \omega)\, e^{i(\boldsymbol{k}\cdot\boldsymbol{\xi}+\omega\tau)}.$$

However, the means of demonstrating this result, using the present formalism, is not clear to me. [The analogy to eq. (13) is, in this case, an integral equation].

V. *Conclusions.* The second-order covariance equations for a linear sea have been derived and solved formally for the case of stationary homogeneous and quasistationary quasihomogeneous statistics. These solutions lead to a definition of the (local) directional spectra and (local) directional cross spectra for the field variables and to Hasselmann's (1960) equation for the development of a linear sea.

REFERENCES

ECKART, CARL

1953a. Relation between time averages and ensemble averages in the statistical dynamics of continuous media. Phys. Rev., *91*(4): 784–790.

1953b. The generation of wind waves on a water surface. J. appl. Phys., *24*: 1485–1494.

HASSELMANN, KLAUS

1960. Grundgleichungen der Seegangsvoraussage. Shiffstechnik, *7*: 191–195.

1962. Über Zufallserregte Schwindungssysteme. Z. angew. Math. Mech., *42*(10–11): 465–476.

MILES, J. W.

1957. On the generation of surface waves by shear flows. Part I. J. fluid Mech., *3*: 185–204.

PHILLIPS, O. M.

1957. On the generation of waves by turbulent wind. J. fluid Mech., *2*: 417–445.

1966. The dynamics of the upper ocean. Cambridge Univ. Press. 261 pp.