Some Exact Solutions to the Equations Describing an Ideal-fluid Thermocline[1]

Pierre Welander

Oceanographic Institution
University of Gothenburg
Gothenburg 4, Sweden

ABSTRACT

The equations that describe a steady ideal-fluid motion have a first integral that expresses a functional relationship between the potential vorticity P, the density ϱ, and the Bernoulli function B: $P = F(\varrho, B)$. By using this relationship, solutions to the ideal-fluid-thermocline problem have been sought. An exact solution is obtained when $P = a\varrho + bB + c$. When it is fitted to observed surface data, it gives a realistic meridional-density field: an inflection point in the density profile, a sharp thermocline at finite depth at the equator, and a reversal of the meridional-density gradient at a subtropical latitude. A more general case that can also be solved exactly is $P = F(a\varrho + bB + c)$, with F arbitrary. Without a direct estimate of the diffusive scale depth in the oceans, the idea of an ideal-fluid thermocline must be taken seriously and this model ought to be explored further.

1. *The Basic Equations.* Consider the steady motion of an ideal and incompressible fluid in a uniformly rotating system. The governing equations are:

$$\varrho\left(\frac{d\vec{v}}{dt} + 2\vec{\Omega}\times\vec{v}\right) = -\nabla p - \varrho\nabla\Phi, \tag{1}$$

$$\nabla\cdot\vec{v} = 0, \tag{2}$$

$$\frac{d\varrho}{dt} = 0, \tag{3}$$

where p, ϱ, $\vec{v}$ are pressure, density, and velocity, $\vec{\Omega}$ is the angular velocity of the system, and Φ is the potential of gravity (including the basic centrifugal force of the rotating system); d/dt stands for the advective operator $(\vec{v}\cdot\nabla)$. The above equations give material conservation of the potential vorticity $P = (2\vec{\Omega}$

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$+\text{curl}\,\vec{v})\cdot\nabla\varrho$, as a special case of the theorem by Ertel (1942); furthermore, the Bernoulli function, $B = p + \varrho\Phi + {}^1/_2|\vec{v}|^2$, is conserved. If the streamlines are represented by the intersection of the surfaces, ψ = constant, χ = constant, we can write $P = P(\psi,\chi)$, $B = B(\psi,\chi)$, and $\varrho = \varrho(\psi,\chi)$. Elimination of ψ,χ then gives a functional relationship between P, B, and ϱ, say

$$P = F(\varrho, B). \tag{4}$$

This represents a general first integral of the equations.

2. *Specialization in the Oceanic Case.* In the open ocean, the Rossby number, $V/L\Omega$, is small; L represents a horizontal scale and V a characteristic horizontal speed. Eq. (1) can then be replaced with the geostrophic-hydrostatic-balance equation:

$$\varrho(2\vec{\Omega}\times\vec{v}) = -\nabla p - \varrho\nabla\Phi. \tag{1a}$$

The corresponding expressions for the potential vorticity and Bernoulli function are $P = 2\vec{\Omega}\cdot\nabla\varrho$, $B = p + \varrho\Phi$. Further, the ocean is confined to a thin and nearly spherical layer. This permits us to drop certain Coriolis accelerations and geometric terms in the equations. The radius, r, can be replaced with a standard value, R, when it is undifferentiated; and the gravity force can be made constant over the depth. Since the relative-density variations in the ocean are only a few per mille, a (partial) Boussinesq approximation can be employed, replacing the density with a standard value when it multiplies the acceleration terms. These approximations are standard in most oceanic models. If we introduce longitude λ, latitude φ, and vertical distance z (counted positive upward) as coordinates, (1a), (2), (3) assume a component form:

$$-2\Omega\sin\varphi\,\varrho v = -\frac{\partial p}{R\cos\varphi\,\partial\lambda}, \tag{5}$$

$$2\Omega\sin\varphi\,\varrho u = -\frac{\partial p}{R\partial\varphi}, \tag{6}$$

$$0 = -\frac{\partial p}{\partial z} - g\varrho, \tag{7}$$

$$\frac{1}{R\cos\varphi}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\varphi}(v\cos\varphi)\right] + \frac{\partial w}{\partial z} = 0, \tag{8}$$

$$u\frac{\partial\varrho}{R\cos\varphi\,\partial\lambda} + v\frac{\partial\varrho}{R\partial\varphi} + w\frac{\partial\varrho}{\partial z} = 0. \tag{9}$$

These are the governing equations for an ideal-fluid thermocline. Further, $P = 2\Omega \sin\varphi\, \partial\varrho/\partial z$, $B = p + \varrho g z$, and the first integral (4) becomes

$$\sin\varphi \frac{\partial\varrho}{\partial z} = F(\varrho, p + \varrho g z), \tag{10}$$

where the constant factor, 2Ω, has been absorbed in the function. Eqs. (7), (10) form a pair of first-order differential equations for p and ϱ; these equations can be conveniently used to produce solutions. If the function $F(\varrho, B)$ is given and if the distributions $p_0(\lambda,\varphi)$, $\varrho_0(\lambda,\varphi)$ are specified at a surface z = constant, then the equations can be integrated in the z direction to obtain the three-dimensional pressure and density fields (excluding certain singular cases). The horizontal velocity is then obtained from the geostrophic eqs. (5), (6), the vertical velocity from the density-conservation equation (9).

3. *The Case* $P = a\varrho + bB + c$. In this case we have

$$\sin\varphi \frac{\partial\varrho}{\partial z} = a\varrho + b(p + \varrho g z) + c. \tag{11}$$

By taking a z derivative and by using the hydrostatic equation, we obtain

$$\sin\varphi \frac{\partial^2\varrho}{\partial z^2} = (a + bgz)\frac{\partial\varrho}{\partial z};$$

with two integrations,

$$\frac{\partial\varrho}{\partial z} = C_1(\lambda,\varphi)\, e^{\frac{az + {}^1/_2\, bgz^2}{\sin\varphi}},$$

and

$$\varrho = \varrho_0(\lambda,\varphi) + C(\lambda,\varphi)\int_0^z e^{-\left(\frac{\zeta + z_0}{D}\right)^2 \frac{1}{\sin\varphi}}\, d\zeta, \tag{12}$$

where the following notations are used:

$$C = C_1 e^{-\frac{1}{2}\frac{a^2}{bg}}, \qquad z_0 = \frac{a}{bg}, \qquad D = \left(-\frac{2}{bg}\right)^{1/2}. \tag{13}$$

The functions $\varrho_0(\lambda,\varphi)$ and $C(\lambda,\varphi)$ are arbitrary. The constant, b, must be taken negative to make the solution decay at great depths. If the constant a is positive, the profile is exponential-like for negative z; it is assumed that the ocean occu-

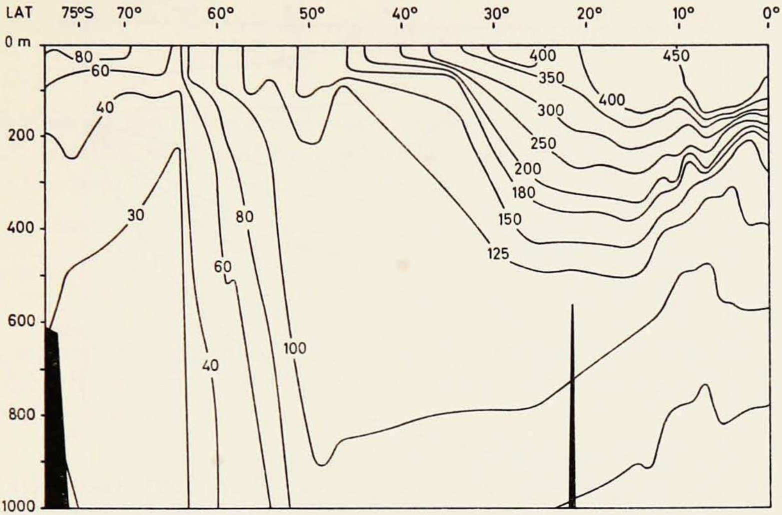

Figure 1. The distribution of thermosteric anomaly in the South Pacific along 160°W according to Reid (1965). The thermosteric anomaly can be translated to an equivalent σ_t (see Reid 1965: 41).

pies the region $z \leq 0$. If a is negative, making z_0 positive, an inflection point appears in the density profile. This case is now considered. The density profile is represented by two scales: the depth of the inflection point and the width of the thermocline. The first depth, z_0, is constant, and the second depth, $D(\sin\varphi)^{1/2}$, varies with latitude. This two-scale representation allows a realistic solution. In particular, we obtain a sharp thermocline at finite depth at the equator. It can be shown further that the meridional density gradient has a reversal of sign at some midlatitude, at depths below the inflection point, when the surface density increases monotonically poleward.

For example, the solution (12) is calculated with $\varrho_0(\lambda,\varphi)$ fitted to the surface density given by Reid (1965) for a meridional section of the South Pacific and with $C(\lambda,\varphi)$ chosen such that the deep density takes on a constant value. Reid's field is shown in Fig. 1, the theoretical field in Fig. 2.

The pressure field corresponding to the density solution (12) is

$$p = p_0(\lambda,\varphi) - gz\varrho_0(\lambda,\varphi) - gC(\lambda,\varphi)\int_0^z\int_0^\zeta e^{-\left(\frac{\zeta^1+z_0}{D}\right)^2\frac{1}{\sin\varphi}}d\zeta^1 d\zeta. \quad (14)$$

However, the function $p_0(\lambda,\varphi)$ is not independent of the two functions $\varrho_0(\lambda,\varphi)$ and $B(\lambda,\varphi)$. Inserting (12), (14) into (11) and setting $z = 0$, we have

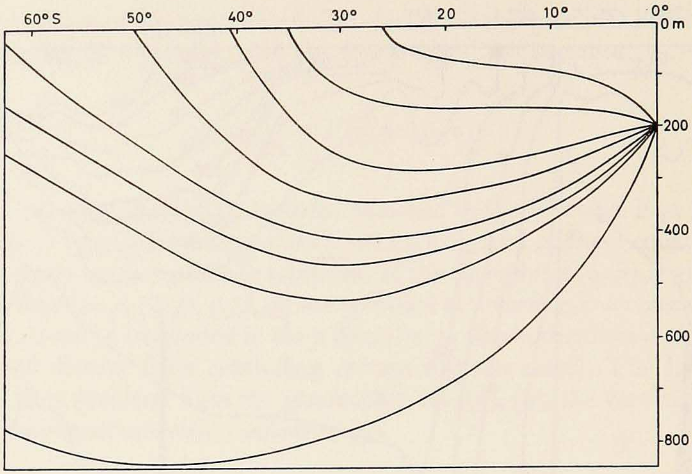

Figure 2. The theoretical density distribution obtained from solution (12). The function $\varrho_0(\lambda, \varphi)$ is fitted to Reid's surface values; the function $C(\lambda, \varphi)$ is determined by the condition that the deep densities approach a given constant value.

$$\sin\varphi \, e^{\frac{1}{2}\frac{a^2}{bg}} \cdot C = a\varrho_0 + bp_0 + c. \tag{15}$$

Therefore, by prescribing the density at the surface and at great depths, both the pressure and the velocity field are uniquely determined.

The special case where P depends only on ϱ has been discussed (Welander 1959). Eq. (11) reduces to $\sin\varphi \, \partial\varrho/\partial z = a\varrho + c$; the interpretation of this case as a first integral has been given by Phillips (1963). The solution is exponential,

$$\varrho = \varrho_D + [\varrho_0(\lambda, \varphi) - \varrho_D] \, e^{\frac{az}{\sin\varphi}}, \tag{12a}$$

where $\varrho_0(\lambda, \varphi)$ is an arbitrary surface density and ϱ_D is a constant representing the deep density. In this case there is a single scale depth for the thermocline that varies in proportion to $\sin\varphi$.

4. *The Case* $P = F(a\varrho + bB + c)$. In this case the governing equation is

$$\sin\varphi \frac{\partial\varrho}{\partial z} = F[a\varrho + b(p + \varrho g z) + c], \tag{16}$$

with F an arbitrary function. By introducing a new coordinate, $z^* = z + a/bg$, and a corresponding Bernoulli function, B^*, we can write

$$\sin\varphi \frac{\partial \varrho}{\partial z^*} = F(bB^* + c) = G(B^*).$$

This equation can be integrated exactly, as was pointed out to me by Professor Louis Howard. With $\partial B^*/\partial z = gz^*(\partial\varrho/\partial z^*)$, the equation can be integrated once to obtain

$$\frac{g}{2\sin\varphi}\left[z_1^{*2}(\lambda,\varphi) - z^{*2}\right] = I(B^*), \quad \text{where} \quad I(B^*) = \int_{B^*_1}^{B^*} \frac{dx}{G(x)}. \tag{17}$$

B_1^*, a standard value of the Bernoulli function, applies at a surface $z^* = z_1^*(\lambda,\varphi)$. By prescribing the pressure, $p_0(\lambda,\varphi)$, and the density, $\varrho_0(\lambda,\varphi)$, at a level $z^* = c$, we obtain

$$z_1^*(\lambda,\varphi) = \left[c^2 + \frac{2\sin\varphi}{g} I(p_0 + \varrho_0 gc)\right]^{1/2}. \tag{18}$$

If we invert (17), the Bernoulli function is obtained:

$$B^* = I^{-1}\left[\frac{g}{\sin\varphi}\left(z_1^{*2} - z^{*2}\right)\right]. \tag{19}$$

But B can also be written in the form $-z^{*2}[\partial(P/z^*)/\partial z^*]$. Using this expression, we can integrate (18) to obtain the pressure:

$$p = p_0(\lambda,\varphi)\frac{z^*}{c} - z^* \int_c^{z^*} \frac{1}{\zeta^2} I^{-1}\left[\frac{g}{\sin\varphi}\left(z_1^{*2} - \zeta^2\right)\right] d\zeta. \tag{20}$$

Finally, the density is obtained from the hydrostatic equation:

$$\left.\begin{aligned} \varrho = -\frac{p_0(\lambda,\varphi)}{gc} + \frac{1}{gz^*} I^{-1}\left[\frac{g}{\sin\varphi}\left(z_1^{*2} - z^{*2}\right)\right] + \\ + \frac{1}{g}\int_c^{z^*} \frac{1}{\zeta^2} I^{-1}\left[\frac{g}{\sin\varphi}\left(z_1^{*2} - \zeta^2\right)\right] d\zeta \end{aligned}\right\} \tag{21}$$

with z_1^* given by (18). We can verify that this expression reduces to $\varrho_0(\lambda,\varphi)$ for $z^* = c$. This is the most general solution found so far for the ideal-fluid-thermocline equations.

A special case, which is simple to integrate, occurs when P depends on only $\varrho, P = F(\varrho)$. The governing equation is in ϱ alone, $\sin\varphi\, \partial\varrho/\partial z = F(\varrho)$; this can be integrated to obtain

$$\varrho = I^{-1}\left\{\frac{z}{\sin\varphi} + I[\varrho_0(\lambda,\varphi)]\right\}, \quad I(\varrho) = \int_{\varrho_1}^{\varrho} \frac{dx}{F(x)}, \tag{21a}$$

where $\varrho_0(\lambda,\varphi)$ is the density at $z = 0$. Verification is obtained from the solution $\varrho_{z=0} = I^{-1}\{I(\varrho_0)\} = \varrho_0$. In this case the density is prescribed by a single profile form. The profile is translated vertically, depending on λ,φ, and it has a variation in scale depth that is proportional to $\sin\varphi$.

5. *Significance of the Ideal-fluid Solutions.* The previous solutions contain arbitrary functions, but they are not general enough to satisfy boundary conditions of the form $\varrho = \varrho_0(\lambda,\varphi)$, $w = w_0(\lambda,\varphi)$ at a top surface ($z = 0$), and $v_n = 0$ at a bottom surface ($z = -H(\lambda,\varphi)$). In the most general solution the function $F(\varrho,B)$ must be arbitrary, but no way has been found to integrate this case exactly.

Note that, in all circumstances, the functions $\varrho_0(\lambda,\varphi)$, $w_0(\lambda,\varphi)$ must meet certain essential requirements to satisfy an ideal-fluid solution for a closed oceanic basin. First, $\iint w_0 dA = 0$, $\iint \varrho_0 w_0 dA = 0$, taken over the entire top boundary, are required to satisfy the overall conditions of incompressibility and mass balance. However, this is not enough. Since the density is conserved, there must be detailed balancing of water at the top; water of a given density that moves downward must be replaced with upward-moving water of the same density. This may result in solutions that appear to be unrealistic. For example, if ϱ_0 is a function of only latitude, then all water that sinks at a given latitude must return to the surface at that latitude. Obviously an ideal-fluid regime can, at most, exist in parts of a real ocean. Near the top there must be a diffusive layer to permit transfer of atmospheric heat and momentum. Further, certain interior diffusive regions are also necessary. The equator and the western-boundary currents are likely to be such regions.

I feel that the ideal-fluid-thermocline model must be considered seriously, for it provides a description of the main meridional-density distribution in the oceans that is as good as existing diffusive-type models, in which a value of the vertical turbulent diffusion coefficient, $\varkappa$, is used. $\varkappa$ generally has been obtained by identifying the diffusive depth, $\delta = \varkappa/W$, with an observed scale depth for the main thermocline; here W is a characteristic vertical velocity. There have been no good direct measurements of $\varkappa$ in the thermocline region. However, some oceanographers feel that the values must be much lower than those that have been previously assumed. Some estimates by C. S. Cox and R. W. Stewart from temperature-fluctuation measurements (private communication) place $\varkappa$ in the range 0.01 to 0.1 $cm^2 sec^{-1}$. If we use the value $W = 10^{-5} cm\ sec^{-1}$ (that is, on the small side), δ then falls in the range 10–100 m. Of course there is the possibility that the real thermocline will be divided into an upper thin diffusive layer and a deeper broader ideal-fluid layer. It is possible to test this idea both theoretically and experimentally. In the theoretical approach we may learn much from numerical experiments carried out for small $\varkappa$ values; such experiments will eventually be carried out. In the experimental approach, direct measurements and determination of $\varkappa$ in the main thermocline is obviously the most important field work. However, we may also learn something from precise measurements of temperature and salinity. If the water sinks, ideally the T–S diagram should be conserved; strictly, potential temperature rather than σ_t should be used. Such a T–S conservation has already been demonstrated by Iselin (1936) for the western Atlantic and by Sverdrup et al. (1942) for other oceans. But a linear T–S relationship is conserved also in the mixing process. To test the ideal-fluid hypothesis it is necessary to study the conservation of nonlinearities in this relationship, and that calls for new and more precise measurements of temperature and salinity.

APPENDIX

A new form of the pressure equation for a diffusive thermocline.

The equation for the pressure used in the diffusive model described by Needler (1967) and Veronis (1969) can be rewritten in a form that displays the existence of a first integral in the limit $\varkappa = 0$. For a constant $\varkappa$ and with the notation $q = 2\Omega\bar{\varrho}R^2 \sin\varphi \cos\varphi$, the equation is

$$q\varkappa\left[\frac{\partial^4 p}{\partial z^4}\frac{\partial^2 p}{\partial z^2} - \left(\frac{\partial^3 p}{\partial z^3}\right)^2\right] = \frac{\partial\left(\sin\varphi\dfrac{\partial^2 p}{\partial z^2},\ p - z\dfrac{\partial p}{\partial z},\ \dfrac{\partial p}{\partial z}\right)}{\partial(\lambda,\varphi,z)}$$

When $\varkappa = 0$ the Jacobian in the right-hand side vanishes and there is a functional relationship between $\sin\varphi\, \partial^2 p/\partial z^2$, $p - z\,\partial p/\partial z$, and $\partial p/\partial z$. This can be directly identified with the first integral (10).

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