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Asymptotic Extreme-value Distributions of Wave Heights in the Open Ocean[1]

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ABSTRACT

Application of the Fréchet extreme-value distribution to extreme wave heights is discussed. Annual extreme significant wave-height data for 12 Ocean Station Vessels have been fitted by the distribution. These distributions have been adjusted to extreme wave-height distributions by applying a scale transformation to the significant wave-height distributions based on previous theoretical and empirical studies of wave-height observations.

Introduction. Although there have been many nonreal time studies of ocean-wave heights, most of the work in recent years has been directed toward real time studies. In the latter work, attempts have been made to characterize the ocean in specific real time states by using spectral analysis, often with a view to making a real time or conditional prediction in terms of either expected values or quantiles, the latter being values associated with their probabilities of occurrence. Pierson and others have carried the spectral approach to real time predictions to a high state of development, much of which has been discussed by Neumann and Pierson (1966).

Some attempts have been made to interpret the real time results as nonreal time predictions, but the interpretations do not seem to be altogether clear; in such a procedure, the analysis begins with the wave spectrum (a real time function) and then proceeds to some extreme wave interpretation. In an analysis of wind speeds for engineering-design purposes, the reverse procedure is followed; i.e., an extreme-value distribution on some variable is first determined, then a spectrum is superimposed on it to obtain the shorter time-extreme distribution; this form of analysis is applied here to waves in the open ocean. Note that all results apply to only the open ocean or to pelagic waves. The data analyzed are the annual significant wave heights observed by the Ocean Station Vessels (OSV's; ships stationed on the airlanes).

The Fréchet Distribution. Although there are three Fisher-Tippett asymptotic extreme-value distributions, the Type-II or Fréchet distribution has a number of physical advantages in studying the wave-height problem. The wave heights have a zero lower bound, as does the Fréchet distribution. At first this may not seem important, but when probabilities for negative wave heights are encountered after fitting the Type-I distribution, often erroneously called the Gumbel distribution, the problem of a lower bound becomes more important. The most important property of the Fréchet distribution is that the transformation between the variate and largest value is a scale change. This is the way bounded variates behave in geophysics. They do not transform by translations, as do unbounded variates. The bounded variates transform by scale changes at levels below the extremes. For example, on the average the precipitation at one station is a factor greater than one times the precipitation at another station that averages less precipitation. The wind-gust speed is a factor greater than one times a wind speed observed over a longer period where the sampling is the choice of an extreme value out of the longer period. Likewise, the extreme wave height has a scale relationship to the significant wave height. On the other hand, variates like temperature and pressure, which behave like unbounded variates, transform by translation, scale change being a relatively conservative property. For example, the standard deviation of the January average temperature is the same, 3.5°, on the coast of northern Maine and across Florida at the latitude of Lake Okeechobee, but the normal January temperatures differ by 45°. The effect of the ocean has kept the scale of temperature the same, and all of the variation is in translation.

Significant Wave-height Distributions. The Fréchet distribution function (cumulative distribution) for maximum values is expressed by

$$F(h) = \exp\left[-(h/\beta_2)^{-\gamma}\right]. \tag{1}$$

It is related to the Fisher-Tippett Type-I distribution for maximum values, given by

$$F(x) = \exp\left[-e^{-(x-\alpha)/\beta_1}\right], \tag{2}$$

by a logarithmic transformation. Thus

$$x = \ln h, \tag{3}$$

from which it follows that

$$\alpha = \ln \beta_2 \quad \text{or} \quad \beta_2 = e^{\alpha} \tag{4}$$

and

$$\beta_1 = \frac{1}{\gamma} \quad \text{or} \quad \gamma = \frac{1}{\beta_1}. \tag{5}$$

Table I.

| OSV | Location | | Record | | Missing |
	Lat.	Long.	3 Hourly	Hourly	months
A	62°00′N	33°00′W	I/49-VIII/54	---	IX/54-XII/54
B	56°30′N	51°00′W	I/54- XII/61	I/62-XII/66	
C	52°45′N	35°30′W	I/54- VI/64	VII/64-XII/66	
D	44°00′N	41°00′W	I/54- VI/64	VII/64-XII/66	
E	35°00′N	48°00′W	I/54- VI/64	VII/64-XII/66	
I	59°00′N	19°00′W	I/49- VI/61	VII/61-XII/62	
J	52°30′N	20°00′W	I/50- VI/61	VII/61-XII/62	
K	45°00′N	16°00′W	I/51- VI/61	VII/61-XII/65	
M	66°00′N	02°00′E	I/54- XII/63	---	XII/58
N	30°00′N	140°00′W	I/54- XII/61	I/62-XII/66	
P	50°00′N	145°00′W	I/51- XII/66	---	
V	34°00′N	164°00′E	I/54- IX/64	X/64-XII/66	

Note that the location, α, of the Type-I distribution is transformed into the scale β_2 of the Fréchet distribution, and the Type-I scale goes into the shape, γ, of the Fréchet distribution.

Since optimum methods of estimating the parameters by order statistics are available in Lieblein (1954) and Mann (1967) for the Type-I distribution, the best estimates of the Fréchet parameters are most easily obtained by fitting ln h. Mann's tables were employed for this purpose; they give statistics a and b_1, which are estimates of α and β_1. These are easily converted to b_2 and g, the estimates of β_2 and γ through use of (4) and (5).

The Type-I distribution was fitted to the extreme annual logarithms of the visually observed significant wave-height series for 12 OSV's. Table I gives the locations, the periods of record employed, the observation frequency, and the missing months for each station.

The estimates of the parameters a and b are shown in Table II. These estimates, converted to Fréchet distribution estimates b_2 and g, are also shown in Table II.

Although all estimated distributions are good fits at the 0.90 level of the Kolmogorov-Smirnov test, Figs. 1–4 are presented in order to give the reader some idea of the appearance of the fits of the distribution function; the two best appearing fits are shown in Figs. 1 and 2, the two poorest fits in Figs. 3 and 4. Given on the lower scale are the probabilities; given on the upper scale is the mean recurrence interval, which is defined as

$$R = 1/[1 - F(h)]. \qquad (6)$$

For purposes of interpolation, the recurrence interval begins at 1.01, this being the nearest convenient value to 1.00; by (6), 1.00 does not appear, since the variable is unbounded below.
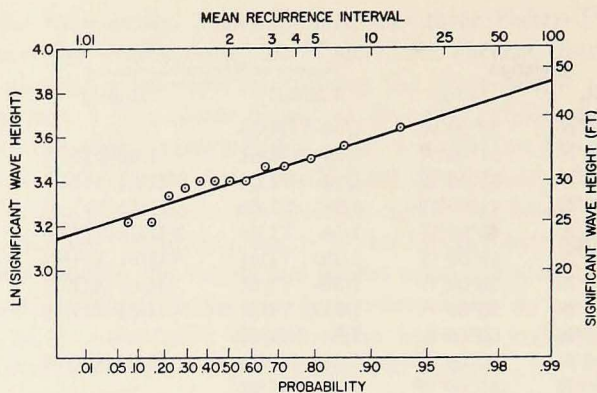
Figure 1. A better-fitting Fréchet extreme-value distribution function of significant wave heights for OSV St. E.

Recognizing the considerable difficulty involved in making a significant wave-height observation, it is difficult to understand why no observation of some estimate of the actual wave height was recorded when 30.5 ft. (9.5 m) was entered, as required by instructions. Of the 152 observations analyzed, 33 (over 20%) were for 30.5 ft. (9.5 m). Some other wave heights also have inflated frequencies; e.g., see Fig. 4. For nonreal time analysis, from which all planning and design information and much of the real-time-methodology development must come, real time observations that are carelessly recorded are much less useful than carefully recorded observations. Nonetheless, the results obtained from these observations, though variable, seem reasonable relative to the geographic position and climate.
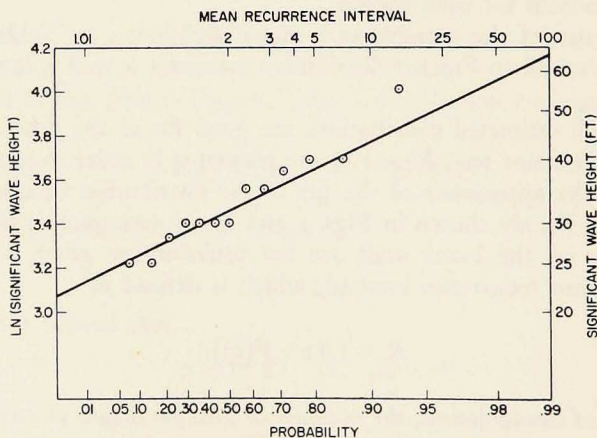

Figure 2. A better-fitting Fréchet extreme-value distribution function of significant wave heights for OSV St. V.
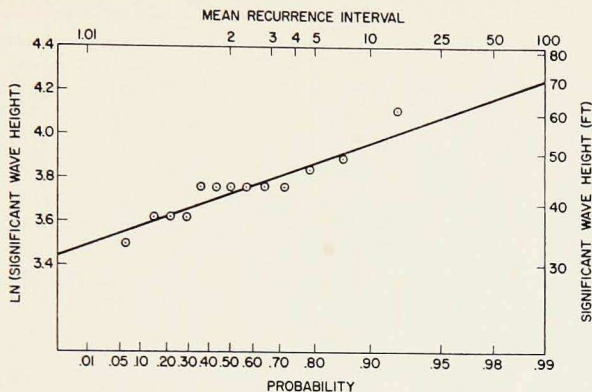
Figure 3. A poorer-fitting Fréchet extreme-value distribution function of significant wave heights for OSV St. B.

To obtain quantiles, we employed an inversion of (1), expressed by

$$h = \exp\left[\ln b_2 - \ln(1/F)\right]. \qquad (7)$$

In Table II, these quantiles are shown for various probabilities for extreme significant waves. Note that the probabilities of exceeding the wave heights listed are one minus those listed at the heading of the tables. In Fig. 5 the 0.98 quantiles, 50-year mean recurrence-interval values, are shown plotted as
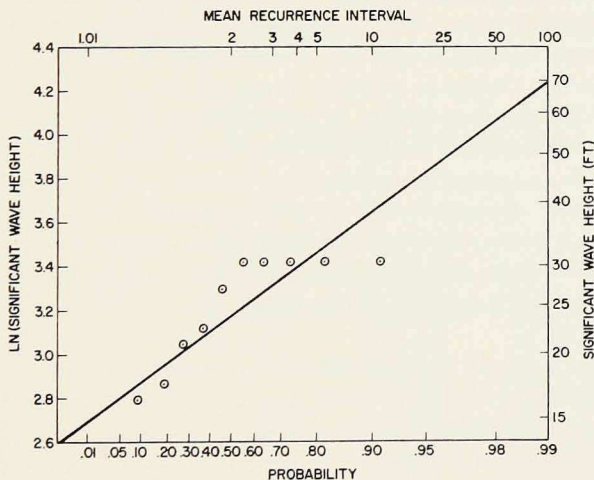

Figure 4. A poorer-fitting Fréchet extreme-value distribution function of significant wave heights for OSV St. M.
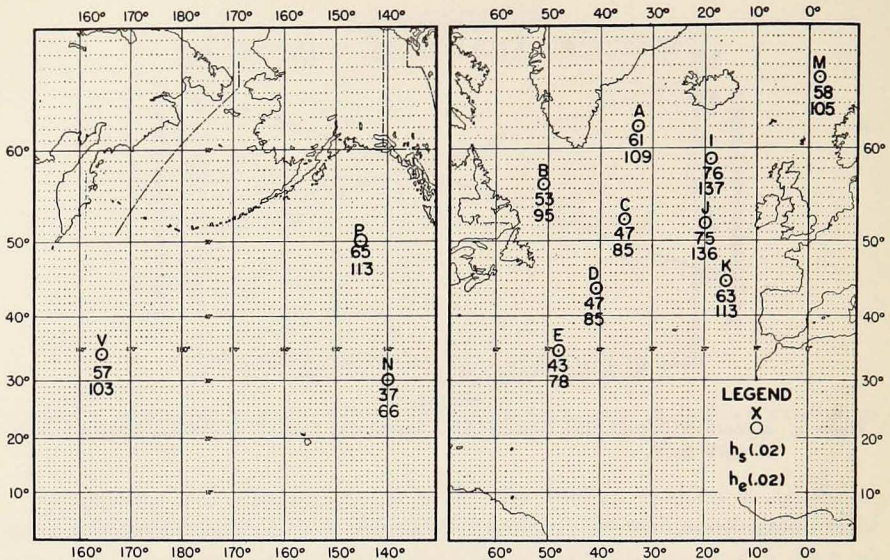
Figure 5. 0.02 quantiles (50-year mean-recurrence values) for significant waves and extreme waves for all OSV's.

the upper value at each OSV location. The values seem to follow a fairly logical variation with geographic position and known wind conditions. Relating them to wind distributions will be the object of our next study.

Extreme Wave-height Distributions. The relationship between extreme wave heights and significant wave heights has been investigated through use of both the empirical and theoretical methods. These studies by various oceanographers were made for what might be considered specific, though perhaps different, sea conditions. With the introduction of generalized harmonic analysis of waves by Seiwell and Wadsworth (1949) and with the subsequent shift in the depiction of waves from harmonic oscillations to spectra of oscillations, the way was opened to a more rational treatment of wave-height observations. Longuet-Higgins (1952), in the key paper in this area, developed theoretical relationships between various observed-height functions and compared them with empirical estimates. Among the most interesting comparisons were those by Wiegel (1949).

Wiegel's fig. 6 shows, for a 46-day period, the ratio of the maximum wave height each day to the average of the highest one-third of the waves for the day. This shows that the ratio of the maximum to the highest one-third remains constant at 1.85, within the sampling error, for the lowest waves to the highest waves observed. The highest wave gives a ratio close to 1.75. Also, all three of Wiegel's comparisons of wave measurements transform from one height to

Table II. Maximum significant wave-height statistics. Quantiles in feet and meters.

OSV	Type-I (Logarithms a, b₁)		Fréchet (b₂, g)		n	Probabilities 0.50	0.90	0.96	0.98
	a,	b₁	b₂,	g		Wave height			
A	3.50	.156	33.02	6.39	6	35 (10.7)	47 (14.3)	54 (16.5)	61 (18.6)
B	3.47	.125	32.29	8.01	13	34 (10.4)	43 (13.1)	48 (14.6)	53 (16.2)
C	3.51	.090	33.31	11.09	13	34 (10.4)	41 (12.5)	44 (13.4)	47 (14.3)
D	3.50	.090	33.15	11.07	13	34 (10.4)	41 (12.5)	44 (13.4)	47 (14.3)
E	3.35	.106	28.55	9.40	13	30 (9.1)	36 (11.0)	40 (12.2)	43 (13.1)
I	3.62	.182	37.37	5.50	14	40 (12.2)	56 (17.1)	67 (20.4)	76 (23.2)
J	3.62	.181	37.21	5.53	13	40 (12.2)	56 (17.1)	66 (20.1)	75 (22.9)
K	3.54	.154	34.46	6.48	15	36 (11.0)	49 (14.9)	56 (17.1)	63 (19.2)
M	3.08	.252	21.81	3.98	10	24 (7.3)	38 (11.6)	49 (14.9)	58 (17.7)
N	2.91	.177	18.34	5.65	13	20 (6.1)	27 (8.2)	32 (9.8)	37 (11.3)
P	3.51	.171	33.52	5.85	16	36 (11.0)	49 (14.9)	58 (17.7)	65 (19.8)
V	3.39	.168	29.80	5.95	13	32 (9.8)	44 (13.4)	51 (15.5)	57 (17.4)

another by scale change, as we proposed above. Longuet-Higgins (1952) obtained a theoretical value of 1.77 for Wiegel's data when he assumed an average period of 12 seconds. Wiegel stated that the period varied widely, so the average had little significance. Perhaps the comparison rests primarily on the fairly large number of waves: 300. Thus, in our study it appeared reasonable to apply the rounded ratio of 1.8 as a scale change to the Fréchet extreme-value distribution. This simply involves multiplying b_2 by 1.8 to obtain an estimate of the extreme-wave distribution. The results are given in Table III in the form of several quantiles. Under the OSV locations in Fig. 5, the 0.02 quantile, the 50-year mean recurrence interval wave, is also plotted as the lower value.

Table III. Extreme wave-height quantiles in feet and meters.

OSV	(1.8b₂, g)		Wave height F = 0.50	F = 0.90	F = 0.96	F = 0.98
A	59.44	6.39	63 (19.2)	84 (25.6)	98 (29.9)	109 (33.2)
B	58.12	8.01	61 (18.6)	77 (23.5)	87 (26.5)	95 (29.0)
C	59.96	11.09	62 (18.9)	73 (22.3)	80 (24.4)	85 (25.9)
D	59.67	11.07	62 (18.9)	73 (22.3)	80 (24.4)	85 (25.9)
E	51.39	9.40	53 (16.2)	65 (19.8)	72 (21.9)	78 (23.8)
I	67.27	5.50	72 (21.9)	101 (30.8)	120 (36.6)	137 (41.8)
J	66.98	5.53	72 (21.9)	101 (30.8)	119 (36.3)	136 (41.5)
K	62.03	6.48	66 (20.1)	88 (26.8)	102 (31.1)	113 (34.4)
M	39.26	3.98	43 (13.1)	69 (21.0)	88 (26.8)	105 (32.0)
N	33.01	5.65	35 (10.7)	49 (14.9)	58 (17.7)	66 (20.1)
P	60.34	5.85	64 (19.5)	89 (27.1)	104 (31.7)	118 (36.0)
V	53.64	5.95	57 (17.4)	78 (23.8)	92 (28.0)	103 (31.4)

Comparison with Unusual Observations. There have been many reports of huge waves in the open oceans, and these may be considered as waves of extreme height. Cornish (1934) has mentioned two of these: In October 1921, Captain Wilson of the Blue Funnel Line, while enroute from Yokohama, Japan, to Puget Sound, Washington, recorded waves that were higher than 70 ft. (21.3 m) in winds of hurricane force. The S.S. MAJESTIC's officers recorded waves of from 60 ft. (18.3 m) to 90 ft. (27.4 m) near 48°30'N, 21°5'W on December 29, 1922. Whitemarsh (1934) reported that the U.S.S. RAMAPO, a navy tanker, encountered a wave of 112 ft. (33.5 m) on February 7, 1933 between Manila, Philippines, and San Diego, California. More recently, the MICHELANGELO, during a North Atlantic crossing, was struck by a wave that collapsed superstructure and broke heavy windows at 81 ft. (24.6 m) above the water line (James 1966). James (1969) recently reported that a wave of close to 100 ft. (30.5 m) struck the oil-drilling rig SEDCO 135 F on October 22, 1968, while it was anchored in 450 ft. (137.2 m) of water off St. James Point, British Columbia. James does not mention an effect of water depth, perhaps because the type of storm indicates that the wave length was short, which implies that there should be little bottom effect. All of these observations seem to indicate the reasonableness of the distributions given above.

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