Mass Transport Induced by Wave Motion[1]

Norden E. Huang[2]

Department of Oceanography
University of Washington
Seattle, Washington 98105

ABSTRACT

The problem of mass transport induced by waves is re-examined. Under the usual boundary-layer assumptions, the whole field is divided into three regions: surface-boundary layers, bottom-boundary layers, and the inviscid interior. Then, by considering Reynolds stress in the boundary layers, an expression of the mass-transport velocity in the viscous fluid is derived.

Two extreme surface conditions have been considered. The first case is clean water without any surface stress. The result is similar to that of Longuet-Higgins (1953) except near the free surface. It is found, to the second-order approximation, that the surface velocity tends asymptotically to the Stokes' classical expansion as the relative depth, kd, increases; yet Longuet-Higgins' solution becomes unbounded.

The second case considered is when the surface is covered with a film that is incompressible to the tangential stress. Again the result is bounded for large kd and is similar to the first case except when $kd \ll 1$. However, when $kd \gg 1$, the influence of the film is an additional term to the surface velocity of the magnitude of one-fourth of the Stokes' classical expansion.

A comparison has been made with experimental observations conducted by Russell and Osorio (1957). The experimental results give strong support to the theoretical prediction.

1. *Introduction.* Mass transport in wave motion has long been recognized. Stokes (1847) first found that a nonzero mean velocity existed even in an inviscid irrotational wave field; he calculated, theoretically, the magnitude to the second order:

$$\bar{U}_l = \frac{\sigma a^2 k \cosh 2k(z-d)}{2 \sinh^2 kd}, \tag{1}$$

where σ = the frequency, a = the amplitude of the wave, k = the wave number, d = the depth of the water, z = the vertical position, and U_l = the

1. Contribution 512 from the Department of Oceanography, University of Washington. This research was supported by AEC Contract AT(45-1)-1725 to the Department of Oceanography, University of Washington (Ref: RLO-1725-116).

Accepted for publication and submitted to press 9 October 1969.

2. Present address: Department of Geosciences, North Carolina State University, Raleigh, N.C. 27607.

Stokes' drift. Since then, various experiments (Caligny 1878, U.S. Beach Erosion Board 1941, Bagnold 1947) confirmed the existence of the nonzero mean velocity, but the velocity profiles were not of exactly the same form as that predicted by Stokes' inviscid irrotational model, unless $kd \gg 1$. For kd comparable to one, a strong forward velocity just outside the bottom-boundary layer was consistently observed. The discrepancy obviously lies in the existence of finite viscosity, no matter how small, in all real fluid where the influence of viscosity prevents the fluid slipping on the solid boundary. This viscous effect dominates inside the boundary layers and, by diffusion, might influence the whole fluid, especially in shallow water. Simple as the physics is, a full analysis requires a nonlinear equation with a moving-wave boundary on the surface – a problem too difficult to be solved.

Various authors have sought solutions to this problem by approximation. Among the trials, the most successful has been reported by Longuet-Higgins (1953). In that solution, the whole flow field was divided into three regions: the interior, the surface, and the bottom-boundary layers. The boundary conditions were no-slip at the bottom, zero-stress on the surface, and zero-net-mass-transport in the whole field. Longuet-Higgins used a perturbation analysis for each region and matched the solutions at the boundaries. Through complicated algebra and various approximations, he found that the final result of mass-transport velocity in a progressive wave field, with surface displacement ζ given as

$$\zeta = a \cos (k \cdot x - \sigma t), \tag{2}$$

was

$$\left. \begin{aligned} U = \frac{\sigma a^2 k}{4 \sinh^2 kd} \Big\{ & 2 \cosh [2kd(1-\mu)] + 3 + kd(3\mu^2 - 4\mu + 1) \\ & \sinh 2kd + 3\left(\frac{\sinh 2kd}{2kd} + \frac{3}{2}\right)(\mu^2 - 1)\Big\}, \end{aligned} \right\} \tag{3}$$

where $\mu = z/d$.

From this formula, Longuet-Higgins succeeded in explaining the phenomenon of forward mass-transport velocity outside the bottom-boundary layer for shallow-water waves. In general the whole profile of mass-transport velocity agreed reasonably well with the experimental results of Russell and Osorio (1957) for shallow water with $0.7 \leq kd \leq 1.5$. On careful scrutiny, however, Longuet-Higgins' solution shows that the velocity on the free surface is a function of the depth and increases linearly with the depth when $kd \gg 1$. By setting $\mu = 0$ in (3), we have

$$\left. \begin{aligned} U_{z=0} &= \frac{\sigma a^2 k}{4 \sinh^2 kd}\left[2 \cosh 2kd - \frac{3}{2} + \left(kd - \frac{3}{2kd}\right)\sinh 2kd\right] \\ &\rightarrow \sigma a^2 k \left(1 + \frac{kd}{2} - \frac{3}{4kd}\right) \end{aligned} \right\} \tag{4}$$

as $kd \gg 1$.

The unboundedness of surface velocity with increasing depth contradicts not only physical intuition but also observed data.

Recently, Chang (1969) also analyzed the same problem by using Lagrangian equations of motion *ab initio* and by considering the full viscous effect of the fluid in a random wave field. Unfortunately, Chang solved the problem under the assumption of deep-water conditions; therefore the influence of the bottom-boundary layer and the evolution of the profile of the mass-transport velocity from shallow to deep water are entirely missing. For deep water, even with viscosity, Chang found that the mass-transport velocity was essentially the same as Stokes' inviscid irrotational solution. This conclusion was confirmed by extensive laboratory observations reported in the same paper. However, an apparent mathematical slip led to the conclusion that the mean velocity gradient near the free surface was, as reported by Longuet-Higgins (1953, 1960), exactly twice the value of Stokes' expression. Notwithstanding the analytical results, all the experimental findings of Russell and Osorio and those of Chang have clearly indicated that neither Stokes' nor Longuet-Higgins' results were entirely satisfactory. Longuet-Higgins' analysis was closer to reality in the shallow-water case, but Stokes' inviscid irrotational model definitely represented the asymptotic value as $kd \to \infty$. A link is necessary to fill the gap between the ranges of validity of kd for Stokes' and Longuet-Higgins' solutions. A solution is needed that will be valid uniformly for all values of kd, and the present analysis is directed toward this goal.

This analysis follows the same approaches as those of Longuet-Higgins' (1953, 1958) and Phillips' (1966) for the interior region and the bottom-boundary layer, but it employs a different method for the surface-boundary layer. The final result uniformly covers the whole range of cases from $kd \ll 1$ to $kd \to \infty$. In the case of $kd \ll 1$, the result indicates the important feature of finite forward velocity near the bottom; then, as kd increases, the profile changes continuously and eventually approaches Stokes' solution as $kd \to \infty$ for a stress-free clean surface.

A special case of contaminated surface is also considered. The important influence of surface contamination is amply illustrated in § 5, particularly for the shallow-water case.

2. *Equations of Motion for the Interior.* Consider a two-dimensional flow, with x horizontal in the direction of wave propagation and z downward. Assume that the motion in the Eulerian sense is periodic in time and that the perturbation method is applicable in the whole motion. This motion is easily realized by a wave train with the surface displacement,

$$\zeta = a \cos (k \cdot x - \sigma t).$$

In this two-dimensional model, a stream function can be defined and expanded in a power series with respect to a small perturbation parameter, ε, which is of the order of the steepness of the wave, ak. Thus we have

$$\psi = \varepsilon\psi_1 + \varepsilon^2\psi_2 + \ldots, \tag{5}$$

and

$$(u, w) = \varepsilon(u_1, w_1) + \varepsilon^2(u_2, w_2) + \ldots, \tag{6}$$

where u, w are the horizontal and vertical components of the velocity and

$$(u_i, w_i) = \left(\frac{\partial\psi_i}{\partial z}, -\frac{\partial\psi_i}{\partial x}\right), \quad i = 1, 2, \ldots. \tag{7}$$

The equation of continuity is automatically satisfied.

All of the above analysis has been in the Eulerian sense, but the mass-transport velocity is a Lagrangian property; that is, the mean velocity following the particles. Hence, we have to transform the Eulerian solution into the Lagrangian form. Let U denote the Lagrangian velocity; then

$$U = \varepsilon U_1 + \varepsilon^2 U_2 + \ldots \tag{8}$$

A formal relationship between u and U (see, for example, Phillips 1966), correct to the second order of ε, is given by

$$U(x_0, t) = u(x_0, t) + \left(\int_0^t u(x_0, t')\, dt'\right) \cdot \nabla u(x_0, t), \tag{9}$$

where x_0 is the position of the particle at $t = 0$. On substituting (6), (8), in (9), we have

$$\left.\begin{aligned} U_1 &= u_1, \\ U_2 &= u_2 + \left(\int_0^t u_1\, dt'\right) \cdot \nabla u_1; \end{aligned}\right\} \tag{10}$$

and taking the mean of (10) and denoting the mean value by a bar, we have

$$\left.\begin{aligned} \bar{U}_1 &= 0, \\ \bar{U}_2 &= \bar{u}_2 + \overline{\left(\int_0^t u_1\, dt'\right) \cdot \nabla u_1}. \end{aligned}\right\} \tag{11}$$

Longuet-Higgins (1953) showed that

$$\left.\begin{aligned} \bar{U}_2 &= \bar{u}_2 + \overline{\int_0^t \frac{\partial\psi_1}{\partial z} dt \frac{\partial^2\psi_1}{\partial x\partial z}} - \overline{\int_0^t \frac{\partial\psi_1}{\partial x} dt \frac{\partial^2\psi_1}{\partial z^2}} = \frac{\partial\Psi}{\partial z}, \\ \bar{W}_2 &= \bar{w}_2 + \overline{\int_0^t \frac{\partial\psi_1}{\partial x} dt \frac{\partial^2\psi_1}{\partial x\partial z}} - \overline{\int_0^t \frac{\partial\psi_1}{\partial z} dt \frac{\partial^2\psi_1}{\partial x^2}} = -\frac{\partial\Psi}{\partial x}, \end{aligned}\right\} \tag{12}$$

where

$$\Psi = \bar{\psi}_2 + \overline{\int_0^t \frac{\partial\psi_1}{\partial z} dt \frac{\partial\psi_1}{\partial x}} \tag{13}$$

is the stream function of the mass-transport velocity. In order to evaluate Ψ to the second order, we must obtain a second-order Eulerian solution.

Now, the two-dimensional vorticity equation for a viscous incompressible fluid is

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+w\frac{\partial}{\partial z}-\nu\nabla^2\right)\nabla^2\psi=0. \tag{14}$$

Substituting (5) and (6) into (14), we have the first-order equation as

$$\left(\frac{\partial}{\partial t}+\nu\nabla^2\right)\nabla^2\psi_1=0. \tag{15}$$

Hence,

$$\nabla^2\psi_1=\nu\int\nabla^4\psi_1\,dt. \tag{16}$$

Taking the time average of (14), we substitute (5) and (6) into the averaged equation. We then have

$$\overline{\left(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z}\right)\nabla^2\psi_1}-\nu\nabla^2\bar{\psi}_2=0. \tag{17}$$

Combining (16) and (17), we obtain

$$\nabla^4\bar{\psi}_2=\overline{\left(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z}\right)\int\nabla^4\psi_1\,dt}. \tag{18}$$

We can now write the field equation for the stream function of the mass-transport velocity to the second order in terms of ψ_1 by substituting (18) in (13);

$$\nabla^4\Psi=\nabla^4\overline{\left(\frac{\partial\psi_1}{\partial x}\int\frac{\partial\psi_1}{\partial z}\,dt\right)}+\overline{\left(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z}\right)\int\nabla^4\psi_1\,dt}. \tag{19}$$

This equation determines only the motion in the interior, subject to the boundary conditions to be specified later.

The actual solution can be obtained by using the classical matching principles of standard boundary-layer problem; that is, the limiting value of the boundary-layer solution equals the boundary value of the interior solution. The boundary-layer solutions are derived in the following section.

3. *Boundary Layers.* Assume, to the first order, that the motion is given by a wave train with surface displacement

$$\zeta=a\cos(k\cdot x-\sigma t) \tag{20}$$

and that the associated stream function is

$$\psi = -\frac{\sigma a}{k}\frac{\sinh k(z-d)}{\sinh kd}\cos(k\cdot x - \sigma t). \tag{21}$$

Under this condition, the boundary-layer flow near the bottom has been obtained by Longuet-Higgins (1953) and Phillips (1966). The same results were derived by both authors; that is, $u_2(\eta)$, the Eulerian velocity inside the boundary layer, correct to the second order, is given as:

$$\left.\begin{aligned}\overline{u_2(\eta)} = \frac{a^2 k\sigma}{4\sinh^2 kd}[3 - 2(\beta\eta + 2)e^{-\beta\eta}\cos\beta\eta - 2(\beta\eta - 1)e^{-\beta\eta} \\ \sin\beta\eta + e^{-2\beta\eta}],\end{aligned}\right\} \tag{22}$$

where $\eta = z - d$ and $\beta = (\sigma/2\nu)^{1/2}$, which is inversely proportional to the thickness of the boundary layer δ.

Just outside the boundary layer, $\beta\eta \rightarrow \infty$; then

$$\overline{u_2(\eta)} = \frac{3a^2 k\sigma}{4\sinh^2 kd}. \tag{23}$$

This nonzero second-order Eulerian velocity, combined with the Lagrangian drift resulting from the first-order irrotational motion, gives the total mass-transport velocity just outside the boundary layer:

$$U = \frac{5a^2 k\sigma}{4\sinh^2 kd}. \tag{24}$$

This furnishes a boundary condition at the bottom for the interior flow.

The free surface was treated by Longuet-Higgins as a moving boundary, consequently complications arose and various approximations had to be made in order to reduce the equation to a manageable form. However, the free surface can be changed into a stationary-wave boundary by a proper coordinate transformation defined as

$$\left.\begin{aligned}\xi &= x - \frac{a\cosh k(z+d)}{\sinh kd}\sin kx, \\ \eta &= z - \frac{a\sinh k(z+d)}{\sinh kd}\cos kx,\end{aligned}\right\} \tag{25}$$

where x, z are Eulerian positions in a coordinate system moving at speed c, and z is measured upward. It can be shown that the transformation is orthogonal and to the first order; $\eta = 0$ gives the free surface.

The Jacobian of transformation, to the first order, is

$$J = \frac{\partial(\xi,\eta)}{\partial(x,z)} = 1 + \frac{2ak\cosh k(\eta+d)}{\sinh kd}\cos k\xi. \tag{26}$$

Assume that the motion consists of a first-order irrotational wave motion and a viscous perturbation. In this curvilinear coordinate, the irrotational part becomes simply $-c\eta$. Let the viscous perturbation be given by ψ'. We have

$$\psi = -c\eta + \psi'. \tag{27}$$

Furthermore, with the coordinates defined in (25), the vorticity equation, to the first order, becomes

$$-c\frac{\partial \omega}{\partial \xi} = \nu \frac{\partial^2 \omega}{\partial \eta^2}, \tag{28}$$

where $\omega = -\mathcal{J}(\psi'_{\xi\xi} + \psi'_{\eta\eta}) \simeq -\psi'_{\eta\eta}$ to the first order. Now, since the surface stress is zero,

$$\tau_s = \nu[(\mathcal{J}\psi_\eta)_\eta - (\mathcal{J}\psi_\xi)_\xi] = 0, \quad \text{at} \quad \eta = 0. \tag{29}$$

This requires that

$$\omega = -\psi'_{\eta\eta} = -2ak\sigma \cos k\xi, \quad \text{at} \quad \eta = 0. \tag{30}$$

Then the solution of (28), subject to the boundary condition of (30) and the condition that ω remains finite as $-\beta\eta \to \infty$, is

$$\omega = -2ak\sigma e^{\beta\eta} \cos(k\xi - \beta\eta). \tag{31}$$

But, in the boundary layer, $\omega \simeq \partial u'/\partial \eta$, hence

$$u' = \frac{ak\sigma}{\beta} e^{\beta\eta}[-\cos(k\xi - \beta\eta) + \sin(k\xi - \beta\eta)] + f(\xi). \tag{32}$$

By using the continuity equation and the boundary conditions that $w' = 0$ at $\eta = 0$ and that w' is finite just outside the boundary layer, we have

$$w' = \int_0^\eta -\frac{\partial u'}{\partial \xi} d\eta = -\frac{ak^2\sigma}{\beta^2}[e^{\beta\eta}\cos(k\xi - \beta\eta) - \cos k\xi + f'(\xi)\beta\eta. \tag{33}$$

But the requirement that w' be finite implies that $f'(\xi) = 0$; hence, inside the boundary layer the total velocity is

$$\left.\begin{aligned} u &= c + \frac{ak\sigma}{\beta} e^{\beta\eta}[\sin(k\xi - \beta\eta) - \cos(k\xi - \beta\eta)] + A, \\ w &= \frac{ak^2\sigma}{\beta^2}[\cos k\xi - e^{\beta\eta}\cos(k\xi - \beta\eta)]. \end{aligned}\right\} \tag{34}$$

Having obtained the velocity, we can calculate the induced motion by balancing the Reynolds stress with the shear stress of the second-order Eulerian mean flow in the boundary, which is governed by the equation

$$\nu \frac{\partial^2 \bar{u}}{\partial \eta^2} = \frac{\partial}{\partial \eta} \overline{uw}. \tag{35}$$

Then, by integration,

$$\nu \frac{\partial \bar{u}}{\partial \eta} - \nu \frac{\partial \bar{u}}{\partial \eta}\bigg|_{\eta=0} = \overline{uw} - \overline{uw}\bigg|_{\eta=0}. \tag{36}$$

But the total stress at the surface is zero, so we have

$$\left.\begin{aligned} \nu \frac{\partial \bar{u}}{\partial \eta} &= \overline{uw} \\ &= \overline{\left\{c + \frac{ak\sigma}{\beta} e^{\beta\eta} [\sin(k\xi - \beta\eta) - \cos(k\xi - \beta\eta)] + A\right\} \cdot} \\ &\quad \overline{\cdot \left\{\frac{ak^2\sigma}{\beta^2} [\cos k\xi - e^{\beta\eta} \cos(k\xi - \beta\eta)]\right\}}, \end{aligned}\right\} \tag{37}$$

where the average is over a wavelength. Hence

$$\nu \frac{\partial \bar{u}}{\partial \eta} = \frac{a^2 k^3 \sigma^2}{2\beta^2} \left\{e^{\beta\eta}(-\cos\beta\eta - \sin\beta\eta) + e^{2\beta\eta}\right\}. \tag{38}$$

As $-\beta\eta \to \infty$, the right-hand side approaches zero exponentially. Thus

$$\nu \frac{\delta \bar{u}}{\partial \eta} \to 0, \qquad -\beta\eta \gg 1. \tag{39}$$

Therefore, to this order of approximation, the viscous perturbation velocity contributes nothing to the second-order Eulerian mean velocity gradient outside the surface-boundary layer. This gives another boundary condition to the interior flow. Including the condition of zero net mass transport, we are now ready to solve eq. (19) for the interior field.

4. *Interior Solution.* Assume that the first-order solution is

$$\psi_1 = -\frac{a\sigma}{k} \frac{\sinh k(z-d)}{\sinh kd} \cos(kx - \sigma t). \tag{40}$$

Then, from (19), we have

$$\nabla^4 \Psi = \nabla^4 \frac{a^2\sigma}{4} \frac{\sinh 2k(z-d)}{\sinh^2 kd}. \tag{41}$$

Assume a solution of the form

$$\Psi = \frac{a^2\sigma}{4\sinh^2 kd} [\sinh 2k(z-d) + P(z)]. \tag{42}$$

Then $P(z)$ must satisfy

$$\frac{d^4 P(z)}{dz^4} = 0, \tag{43}$$

with the boundary conditions

$$\left.\frac{\partial P(z)}{\partial z}\right|_{z=d} = 3k, \tag{44}$$

$$\left.\frac{\partial^2 P(z)}{\partial z^2}\right|_{z=0} = 0, \tag{45}$$

$$P(z)\,|_{z=d} = 0, \tag{46}$$

$$P(z)\,|_{z=0} = \sinh 2kd. \tag{47}$$

The solution can be easily found and the final result is

$$U = \frac{\sigma a^2 k}{4 \sinh^2 kd}\left\{2 \cosh\left[2kd(1-\mu)\right] - \frac{3}{2}\left[1-\mu^2\right]\frac{\sinh 2kd}{kd} + \frac{9}{2}\mu^2 - \frac{3}{2}\right\}, \tag{48}$$

where $\mu = z/d$.

This mass-transport velocity profile is different from that of Longuet-Higgins (3). On term-by-term comparison, it is easily seen that the only difference is the extra term

$$\frac{\sigma a^2 k}{4 \sinh^2 kd}\, kd\,(3\mu^2 - 4\mu + 1) \sinh 2kd, \tag{49}$$

which appeared in Longuet-Higgins' result but not in the present solution. This term represents the flow caused by an extra stress induced by the boundary-layer flow on the surface. But in the present analysis it is shown that the stress induced by viscous perturbation is of second order at most and that it dies off exponentially on approaching the value zero just outside the boundary layer.

5. *Solution with Surface Film.* It is well known in experimental work on surface waves that the surface condition is extremely critical. Any contamination on the surface changes the surface-boundary condition and hence the interior flow pattern. The characteristics of surface contamination vary widely. In one extreme case, the surface is covered with a densely packed layer of film that is incompressible to tangential stress set up by the waves; a film of highly viscous oil is a practical approximation to this case. Phillips (1966) has discussed briefly some important consequences of the surface contamination. By using the same coordinate transformation as that in § 3, Phillips showed that the velocity perturbation in the boundary layer is

$$u' = -\sigma a \coth kd e^{\beta\eta} \cos(k\xi - \beta\eta). \tag{50}$$

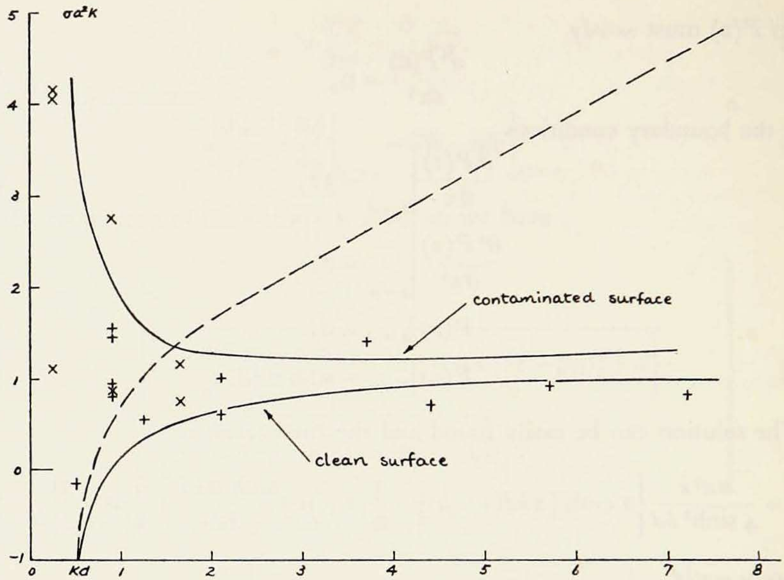

Figure 1. Comparison of the surface velocities in analytical and experimental results: dashed line, Longuet-Higgins; solid line, present analysis; plus sign, experiments with flat channel bed; X, experiments with sloped channel bed (1:20).

By employing the continuity equation and the condition $w' = 0$ at $\eta = 0$, it follows that

$$w' = \frac{a k \sigma}{2\beta} \coth kd \{[\sin (k\xi - \beta\eta) + \cos (k\xi - \beta\eta)] e^{\beta\eta} - \sin k\xi - \cos k\xi\}. \quad (51)$$

Then the total velocity is

$$u = c + u', \qquad w = w'. \quad (52)$$

The Reynolds stress inside the boundary layer is therefore

$$\left.\begin{aligned} \overline{uw} &= \overline{cw'} + \overline{u'w'} \\ &= -\frac{a^2 k\sigma}{4\beta} \coth kd [e^{2\beta\eta} - e^{\beta\eta} (\cos \beta\eta + \sin \beta\eta)]. \end{aligned}\right\} \quad (53)$$

On balancing the shear stress and the Reynolds stress and integrating, we obtain the second-order Eulerian velocity $\bar{u}$:

$$\bar{u} = \frac{a^2 k\sigma}{4} \coth^2 kd (1 - e^{2\beta\eta} + 2 e^{\beta\eta} \sin \beta\eta). \quad (54)$$

Just outside the boundary layer,

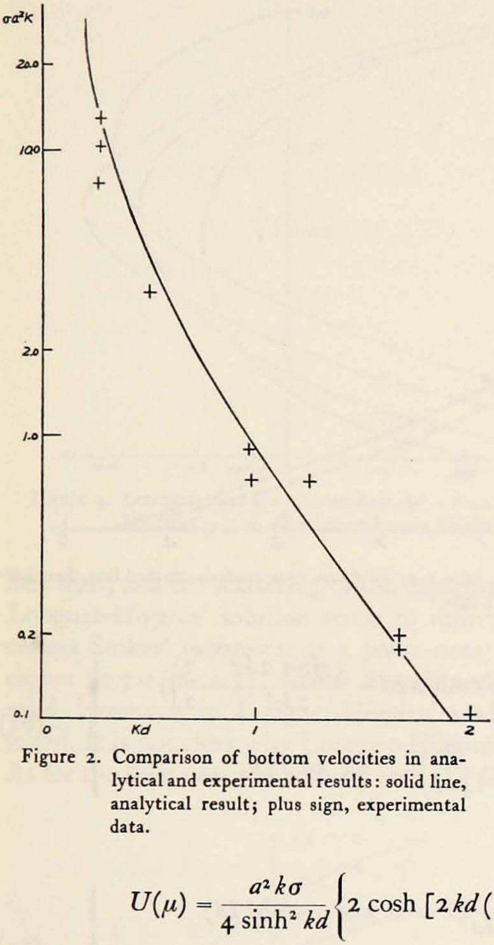

Figure 2. Comparison of bottom velocities in analytical and experimental results: solid line, analytical result; plus sign, experimental data.

$$\bar{u} \to \frac{a^2 k\sigma}{4} \coth^2 kd, \qquad (55)$$

which is exactly one-fourth of the Stokes classical expansion.

Using (55) as the surface-boundary condition in combination with (24) and zero net-mass transport, we are again ready for the solution of the interior field. The procedure is exactly the same as that in § 4, except that the boundary conditions on $P(z)$ are now

$$\left.\frac{\partial P}{\partial z}\right|_{z=0} = k \cosh^2 kd, \qquad (56)$$

$$\left.\frac{\partial P}{\partial z}\right|_{z=d} = 3k, \qquad (57)$$

$$P\,|_{z=0} = \sinh 2kd, \qquad (58)$$

$$P\,|_{z=d} = 0. \qquad (59)$$

The final result is

$$\left.\begin{aligned} U(\mu) = \frac{a^2 k\sigma}{4 \sinh^2 kd}\Big\{ & 2 \cosh\left[2kd(1-\mu)\right] + (9\mu^2 - 6\mu) + \\ & + (3\mu^2 - 4\mu + 1)\cosh^2 kd + \frac{6}{kd}(\mu^2 - \mu)\sinh 2kd \Big\}. \end{aligned}\right\} \qquad (60)$$

6. *Comparison with Experimental Data.* So far the most comprehensive set of experimental data concerning mass transport in wave motion is that of Russell and Osorio (1957). In those experiments, they covered a wide variety of cases of different wave heights and water depth, even with a sloped bottom. But experimental results have shown that the effect of bottom slope (at 20:1) is not substantial. A detailed comparison of the present results and those of Longuet-Higgins with the experimental data are illustrative. In the following discussion, all the experimental data referred to are those of Russell and Osorio.

First let us examine the surface velocity. By setting $\mu = 0$ in (48) and (60), we get

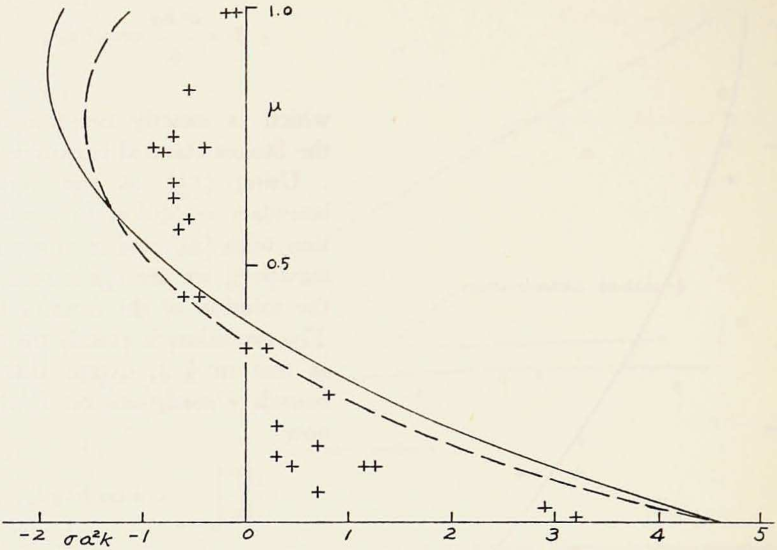

Figure 3. Comparison of the interior flow, $kd = 0.5$: solid line, new analysis; dashed line, Longuet-Higgins; plus sign, experimental data.

$$\left.\begin{aligned} U_{z=0} &= \frac{a^2 k\sigma}{4\sinh^2 kd}\left(2\cosh 2kd - \frac{3\sinh 2kd}{2kd} - \frac{3}{2}\right), \\ &\rightarrow a^2 k\sigma\left(1 - \frac{3}{4kd}\right), \quad \text{as } kd \gg 1 \end{aligned}\right\} \tag{61}$$

for the clean surface and

$$\left.\begin{aligned} U_{z=0} &= \frac{a^2 k\sigma}{4\sinh^2 kd}(2\cosh 2kd + \cosh^2 kd) \\ &\rightarrow \frac{5}{4} a^2 k\sigma, \quad \text{as } kd \gg 1 \end{aligned}\right\} \tag{62}$$

for the contamined surface.

In order to see the comparison more clearly, eqs. (4), (61), and (62) are plotted in Fig. 1 against Russell's observed data. Some interesting points should be mentioned. First, both the present results and those of Longuet-Higgins predict a backward surface flow in clean shallow water arising as a consequence of zero net-mass transport and a definite forward velocity near the bottom. But the existence of this backward surface flow was not observed consistently by Russell. A possible explanation is that the influence of the surface film becomes increasingly important in shallow-water waves, as indicated by eq. (60). Therefore, any surface contamination might cause a drastic change in the sur-

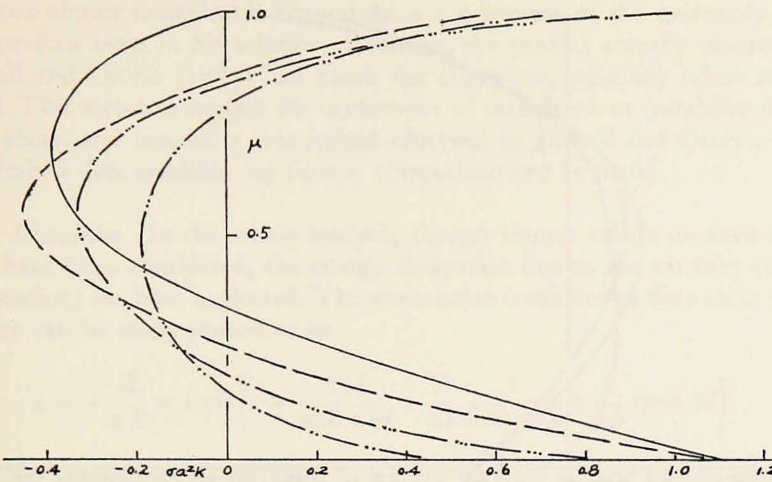

Figure 4. Comparison of the interior flow, $kd = 0.92$: solid line, new analysis; dashed line, Longuet-Higgins; — ··· — experimental wave height 8 in.; — ·· — wave height 20 in.

face flow, and the scattering of the experimental data is to be expected. Second, Longuet-Higgins' solution tends to infinity at both large and small *kd* and crosses Stokes' expansion at a point near $kd = 1$, which coincides with the region ($0.7 < kd < 1.5$), where Russell and Osorio claimed the solution was valid. However, in Longuet-Higgins' analysis, no restriction on *kd* could be found. It is not clear why Longuet-Higgins' solution fails beyond such a range. As for the flow near the bottom, both the present results and those of Longuet-

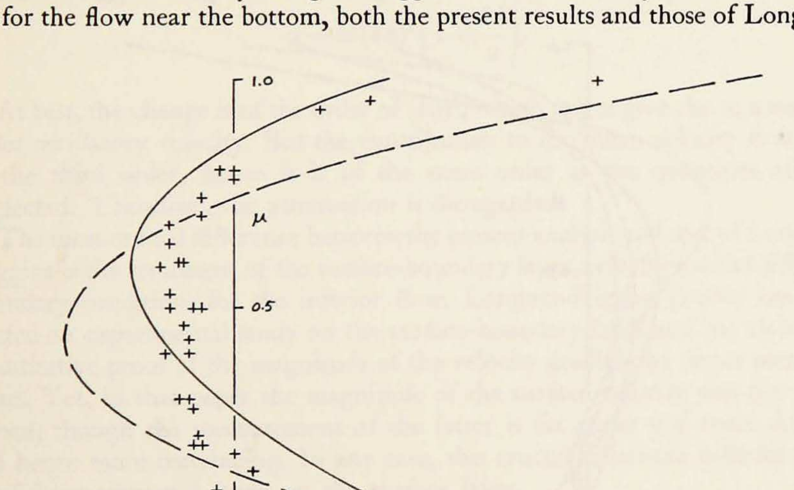

Figure 5. Comparison of the interior flow, $kd = 1.25$: solid line, new analysis; dashed line, Longuet-Higgins; plus sign, experimental data.

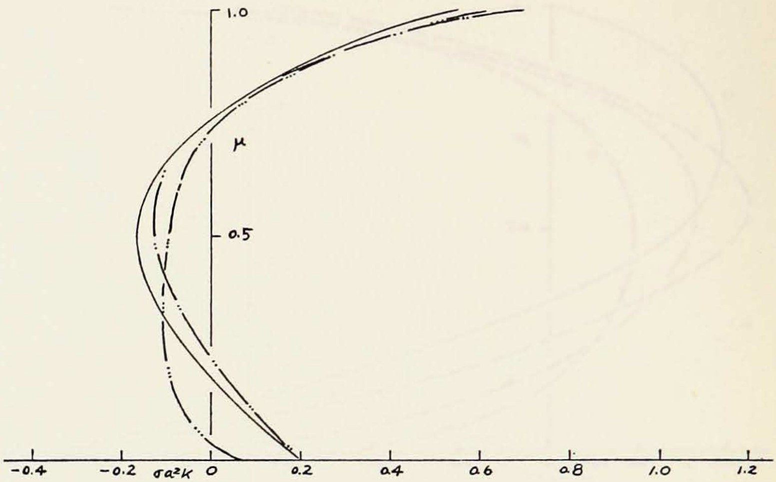

Figure 6. Comparison of the interior flow, kd = 1.67: solid line, new analysis; – ·· – experiments with upward slope 1:20; – ... – experiments with downward slope 1:20.

Higgins are exactly the same. Experimental data confirm the analytical result remarkably, as can be seen in Fig. 2.

For the interior region, Figs. 3–7 cover the cases of the kd values equal to 0.5, 0.92, 1.25, 1.67, 2.1, and 7.16, respectively. In most cases the discrepancy between the present result and the experimental points is fairly small, and the accuracy increases with increasing kd. But Longuet-Higgins' solution

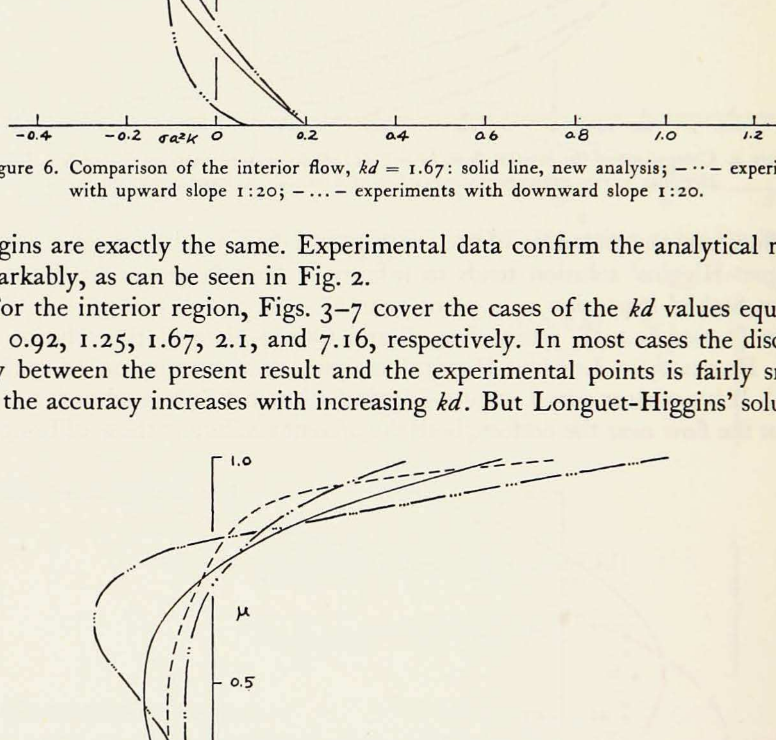

Figure 7. Comparison of the interior flow, kd = 2.1 and 7.16: solid line, analytical with kd = 2.1; – ·· – experiments with kd = 2.1 wave height 4.6 in.; – ··· – experiments with kd = 2.1 wave height 1.2 in.; dashed line, experiments with kd = 7.16 wave height 1.2 in.

becomes almost inapplicable beyond $kd = 1.5$ because of the extremely large surface-flow term in his solution. However, the profiles actually observed by Russell and Osorio (1957) had much flat curvature, especially when kd was small. This seems to suggest the occurrence of turbulence or instability due to high shear, and instability was indeed observed by Russell and Osorio. With the limited data available, no further comparison can be made.

7. *Discussion.* In the whole analysis, though viscous effects on mass transport have been considered, the energy dissipation due to the viscosity (or the attenuation) has been neglected. The attenuation coefficient α for a clean water surface can be easily proved to be

$$\alpha = -\frac{\dot{E}}{2E} = 2\nu k^2\left(1 + \frac{2kd}{\sinh 2kd} + \frac{\beta}{2k}\frac{1}{\sinh 2kd} + \frac{k}{2\beta}\tanh kd\right). \tag{63}$$

If the attenuation is converted to a space variable such as x (parallel to the direction of wave propagation), as realized in most experimental setups, then

$$\alpha = kx(k\delta)\left[\frac{1}{2\sinh 2kd} + (k\delta)\left(1 + \frac{2kd}{\sinh 2kd}\right) + \frac{(k\delta)^2}{2}\tanh kd\right]; \tag{64}$$

here the first term represents the contribution from the bottom-boundary layer, the second term is from the interior due to the irrotational strain, and the third term is due to the surface-boundary layer. As $kd \gg 1$,

$$\alpha \rightarrow kx(k\delta)^2\left(1 + \frac{k\delta}{2}\right).$$

At best, the change is of the order of $(k\delta)^2$, which might give rise to a second-order oscillatory velocity. But the contribution to the mean velocity is at least of the third order, hence it is of the same order as the quantities already neglected. Therefore, the attenuation is disregarded.

The most crucial difference between the present analysis and that of Longuet-Higgins is the treatment of the surface-boundary layer, which provides different boundary conditions for the interior flow. Longuet-Higgins (1960) has conducted an experimental study on the surface-boundary layer and has claimed a quantitative proof of the magnitude of the velocity gradient by direct measurement. Yet, in that paper the magnitude of the surface velocity was not mentioned, though the measurement of the latter is far easier and more definite and hence more convincing. In any case, this crucial difference calls for more careful experimental study on the surface layer.

Acknowledgment. I wish to thank Maurice Rattray, Jr. for suggesting the problem and for his valuable criticism.

REFERENCES

BAGNOLD, R. A.

1947. Sand movement by waves: Some small scale experiments with sand of very low density. J. Instn. Civil Engrs. (London), *27*: 447–469.

CALIGNY, A. F. H. DE

1878. Expériences sur les mouvements des molécules liquid des ondes courantes, considérées dans leur mode d'action sur la marche des navires. C. R. Acad. Sci., Paris, *87*: 1019–1023.

CHANG, M. S.

1969. Mass transport in deep-water long-crested random gravity waves. J. geophys. Res., *74*: 1515–1536.

LONGUET-HIGGINS, M. S.

1953. Mass transport in water waves. Phil. Trans., (A) *245*: 535–581.

1958. The mechanics of the boundary-layer near the bottom in a progressive wave, pp. 184–193. Proc. 6th Conf. coast. Engng., Council on Wave Research, University of California, Berkeley. 896 pp.

1960. Mass transport in the boundary layer at a free oscillating surface. J. fluid Mech., *8*: 293–315.

PHILLIPS, O. M.

1966. Dynamics of the upper ocean. Cambridge University Press, Cambridge. 261 pp.

RUSSELL, R. C. H., and J. D. C. OSORIO

1957. An experimental investigation of drift profiles in a closed channel, pp. 171–183. Proc. 6th Conf. coast. Engng., Council on Wave Research, University of California, Berkeley. 896 pp.

STOKES, G. G.

1847. On the theory of oscillatory waves. Trans. Cambridge phil. Soc., *8*: 441–455.

U.S. BEACH EROSION BOARD

1941. A study of progressive oscillatory waves in water. Tech. Report No. 1. U.S. Govt. Printing Office, Washington, D.C. 39 pp.