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# *Oceanic Flow over Varying Bottom Topography*<sup>1</sup>

R. A. Clarke and N. P. Fofonoff<sup>2</sup>

*Institute of Oceanography  
University of British Columbia  
Vancouver 8, B.C., Canada*

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## ABSTRACT

A study has been made of steady barotropic flows on the  $\beta$  plane over a bottom topography that varies in a direction inclined to the circles of latitude. The solutions obtained, starting with both the Eulerian and the Lagrangian systems of equations, are shown to be identical in the case of flow over a single-depth discontinuity.

Within a restricted range of depth variation, the amplitudes of the stationary planetary wave solutions obtained continue to change downstream from the initial depth variation, even in a region of constant depth. Larger depth variations lead to cellular flow patterns aligned parallel to the lines of constant depth. The effects on the solutions of Ekman bottom friction and of wind stress have been investigated numerically, using the Lagrangian equations.

*Introduction.* Observations of ocean currents have shown that their paths are affected by bottom topography; the most notable examples are (i) the meandering of the Gulf Stream after it leaves the continental shelf (Fuglister and Worthington 1951) and (ii) the stream's turning to the south as it passes by the southeastern Newfoundland Ridge (Mann 1967). Early theoretical studies of ocean currents have usually assumed a level bottom topography. Warren (1963), using numerical techniques, showed that the Gulf Stream's meanders could be accounted for by the observed topography. Porter and Rattray (1964) obtained analytical solutions for the problem of steady barotropic zonal flow over a bottom that was assumed to vary zonally. These solutions took the form of large-amplitude planetary waves for flow over one or more depth discontinuities, corresponding to oceanic flow over continental slopes, ridges, and valleys. Their solutions agreed qualitatively with the deflection of the Antarctic Circumpolar Current where it passes over several ridges.

In this paper, the problem of an initially zonal and barotropic flow entering a region where depth is changing in some direction other than north-south

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2. Woods Hole Oceanographic Institution, Woods Hole, Mass.



is treated on the  $\beta$  plane. For the simplest case—that of flow over a single depth discontinuity aligned in any direction, solutions are obtained by using both the Lagrangian and the Eulerian equations. In general, these two systems yield the pathlines and the streamlines, respectively, and it is difficult to move from one representation of the flow to the other. For steady flow, the two representations are identical, as is shown for the case of the single step.

Knowing that identical solutions result from both sets of equations in the simplest case, it is then possible to use either set to look at a more complex bottom topography or to include the effects of friction or wind stress. By using the Eulerian equations, solutions may be obtained for uniform flow over a wide range of depth variations, provided the depth can be expressed as a function of a single co-ordinate in some direction other than north-south. Solutions are calculated for flows over one and two depth discontinuities, representing, in the ocean, uniform barotropic flow over ridges, valleys, and continental slopes.

On the other hand, the Lagrangian set of equations leads to a set of ordinary differential equations, even when constant curl wind stress and Ekman bottom friction are included in the model. Such a set of ordinary differential equations plus an initial set of conditions is well suited for study on an analogue computer. For this reason the problem of uniform barotropic flow over a single step with wind stress and bottom friction has been investigated by using the Lagrangian set of equations and an analogue computer.

This model only crudely approximates the real ocean. In particular, real ocean flows are baroclinic, nonuniform, and time-dependent. Since the model is none of these, quantitative agreement between its solutions and what is actually observed in the real ocean cannot be expected. Porter and Rattray (1964) have argued that a barotropic model will show a larger response to a given bathymetry than to the baroclinic flow.

Although the model does not reproduce oceanic flow quantitatively, the analytical solutions obtained show a mechanism for the generation of large-amplitude planetary waves that change their amplitude even over a level bottom. This simple model provides some understanding of the mechanism that causes currents, such as the Gulf Stream, to develop meanders of increasing amplitude after flowing over bathymetric features such as the continental slope.

*Vorticity Equation.* For the case of steady barotropic flow in an homogeneous ocean with an applied wind stress at the surface and with frictional stress at the bottom, the equations of motion and continuity can be integrated over the depth of the ocean to give

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fv = -g \frac{\partial \eta}{\partial x} - R(u - v) + \frac{\tau_x}{h + \eta}, \quad (1)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu = -g \frac{\partial \eta}{\partial y} - R(u+v) + \frac{\tau_y}{h+\eta}, \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{u}{h+\eta} \frac{\partial}{\partial x} (h+\eta) - \frac{v}{h+\eta} \frac{\partial}{\partial y} (h+\eta). \quad (3)$$

Here the various symbols are defined as:

- $u, v$  eastward and northward components of velocity,  
 $g$  acceleration of gravity,  
 $f$  Coriolis parameter,  $f = \alpha + \beta y$ ,  
 $\eta$  surface elevation,  
 $h$  undisturbed water depth,  
 $R$  frictional coefficient,  $R = (Af/2h^2)^{1/2}$ ,  
 $A$  coefficient of eddy viscosity,  
 $\tau_x, \tau_y$  eastward and northward components of wind stress.

In the usual manner, we obtain the vorticity equation through cross-differentiation of (1) and (2) followed by substitution from (3):

$$(h+\eta) \left\{ u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right\} \frac{\xi+f}{h+\eta} = -R\xi + \text{curl} \frac{\tau}{h+\eta} + \frac{R}{h+\eta} \left. \begin{array}{l} \right\} \quad (4) \\ \left\{ u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right\} (h+\eta), \end{array}$$

where  $\xi$ , [ $\xi = (\partial v/\partial x) - (\partial u/\partial y)$ ] is the vertical component of relative vorticity.

*Eulerian Solution.* In this section we treat the problem of an oceanic flow having a uniform and zonal initial velocity,  $U$ , over a level bottom of depth  $H$  entering into a region where the depth  $h$  varies along a coordinate  $X$  inclined at an angle  $\theta$  to the  $x$  axis. Wind stress and friction are neglected; the effect of both will be reported later following an investigation using the Lagrangian solutions. Since the surface elevation is of the order of a meter, it will be neglected relative to the total depth,  $h$ , which is of the order of several kilometers. The problem is treated on the  $\beta$  plane; therefore, solutions in which  $y$  (the distance from the latitude around which the  $\beta$  approximation is taken) becomes of the order of the earth's radius should no longer correspond to actual ocean flows. Under the above approximations, (4) becomes

$$\left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \left( \frac{\xi+f}{h} \right) = 0, \quad (5)$$



subject to the upstream condition that  $u = U$ ,  $v = 0$  in the initial level bottom region. Since (5) is invariant under co-ordinate rotation, it is convenient to transform to a system in which the axes are parallel and perpendicular to the depth variations. This new co-ordinate system is given by the transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (6)$$

where  $\theta$  is given by the condition  $h = h(X)$ .

Since the transport field is nondivergent, a transport streamfunction,  $\Psi$ , is defined by the relationships

$$\frac{\partial \Psi}{\partial X} = v' h, \quad \frac{\partial \Psi}{\partial Y} = -u' h, \quad (7)$$

where  $u'$  and  $v'$  are the velocity components along the  $X$  and  $Y$  directions, respectively. In terms of the transport stream function, (5) becomes

$$\mathcal{J} \left( \frac{\nabla \cdot \left( \frac{1}{h} \nabla \Psi \right) + f}{h}, \Psi \right) = 0. \quad (8)$$

This can be integrated immediately to give

$$\frac{\nabla \cdot \left( \frac{1}{h} \nabla \Psi \right) + f}{h} = K(\Psi), \quad (9)$$

where  $K(\Psi)$  is an arbitrary integration function that is to be determined from the upstream condition. For a uniform zonal flow of constant velocity,  $U$ , and constant depth,  $H$ , the transport stream function is

$$\Psi = -UH(Y \cos \theta - X \sin \theta). \quad (10)$$

Substituting (10) into the integrated vorticity equation (9), we determine  $K(\Psi) = (\alpha/H) - (\beta\Psi/UH)$ , where the  $\beta$  approximation gives  $f = \alpha + \beta(Y \cos \theta - X \sin \theta)$ . Recalling that  $h = h(X)$ , (9) is now rewritten as

$$h \nabla^2 \Psi - \frac{\partial h}{\partial X} \frac{\partial \Psi}{\partial X} + \frac{\beta h^3}{UH} = \alpha h^2 \left( \frac{h}{H} - 1 \right) + \beta h^2 (X \sin \theta - Y \cos \theta). \quad (11)$$

The origin of the co-ordinates is chosen such that  $h(X) = H$  for  $X \leq 0$ . Boundary conditions along  $X = 0$ , given by the requirement that the transports be continuous, are therefore

$$\left. \frac{\partial \Psi}{\partial X} \right|_{X=0} = UH \sin \theta; \quad \left. \frac{\partial \Psi}{\partial Y} \right|_{X=0} = -UH \cos \theta. \quad (12)$$

Equation (11) can be separated and simplified through the transformations

$$\Psi(X, Y) = F(s) + YG(s); \quad s(X) = \frac{q}{H} \int_0^X h(t) dt; \quad q = \beta/U. \quad (13)$$

The form of the first of these transformations is suggested by the form of the upstream transport stream function to which it must be matched. The second, transforming  $X$  to  $s$ , eliminates the first derivatives of  $h$  and  $\Psi$ , thereby simplifying the equations. Applying these transformations, (11) becomes

$$G''(s) + G(s) = -\frac{UH^2}{h} \cos \theta, \quad (14)$$

$$F''(s) + F(s) = \frac{\alpha UH}{\beta} \left(1 - \frac{H}{h}\right) + \frac{UH^2}{h} X \sin \theta, \quad (15)$$

subject to the conditions

$$\left. \begin{aligned} F(0) = 0, & & F'(0) = \frac{UH \sin \theta}{qr_1}, \\ G(0) = -UH \cos \theta, & & G'(0) = 0, \end{aligned} \right\} \quad (16)$$

where

$$r_1 = \lim_{X \rightarrow 0^+} \frac{h(X)}{H},$$

allowing depth to be discontinuous at  $X = 0$ . These have as solutions

$$G(s) = -UH \cos \theta \left[ \cos s + H \int_0^s \frac{\sin(s-t)}{h(t)} dt \right], \quad (17)$$

$$\left. \begin{aligned} F(s) = \frac{\alpha UH}{\beta} \left[ 1 - \cos s - H \int_0^s \frac{\sin(s-t)}{h(t)} dt \right] + \\ + UH \sin \theta \left[ \frac{\sin s}{r_1 q} + H \int_0^s \frac{X(t) \sin(s-t)}{h(t)} dt \right]. \end{aligned} \right\} \quad (18)$$

Using this general form of the solution, we can calculate the stream function, at least numerically, for a variety of depth profiles. In this study, however, only two simple cases are worked out analytically. The first is that of flow over a single depth discontinuity from  $H$  to  $r_1 H$  at  $X = 0$ . Here the transport stream function, as determined from (13), (17), and (18), is



$$\Psi = \left. \begin{aligned} & \frac{UH \sin \theta}{qr_1} [qr_1 X - (1 - r_1) \sin(qr_1 X)] - \\ & - \frac{UH \cos \theta}{r_1} [1 - (1 - r_1) \cos(qr_1 X)] Y - \\ & - \frac{\alpha UH}{\beta} \frac{1 - r_1}{r_1} [1 - \cos(qr_1 X)]. \end{aligned} \right\} \quad (19)$$

By holding  $\Psi$  fixed, (19) gives  $Y$  as a function of  $X$ , thus describing the position of the given streamline. So that  $Y$  will remain bounded for finite values of  $X$  and  $\Psi$ , the coefficient of  $Y$  in (19) must remain nonzero for all  $X$ . Such solutions, occurring for  $0 < r_1 < 2$ , will be called stable solutions; they exhibit a wave-like periodicity. For  $r_1 > 2$ ,  $Y$  is unbounded for some finite  $X$ , and the solutions consist of a series of cells similar to those described by Porter and Rattray (1964). Since the  $\beta$  approximation is valid for only a limited  $y$ , only the stable solutions can be considered to have physical significance. The single-step model corresponds crudely to flow over a continental slope; a simple extension with a second step from  $r_1 H$  to  $r_2 H$  at  $X = a$  corresponds to flow over a ridge or a valley of width  $a$ . For  $X < a$ , the solution is again given by (19), but for  $X > a$ ,

$$\Psi = \left. \begin{aligned} & - \frac{\alpha UH}{\beta} \left[ \frac{1 - r_2}{r_2} + \left\{ \left( 1 - \frac{r_1}{r_2} \right) - (1 - r_1) \cos(r_1 qa) \right\} \frac{\cos r_2 q(X - a)}{r_1} - \right. \\ & \quad \left. - \left( 1 - \frac{1}{r_1} \right) \sin(r_1 qa) \sin r_2 q(X - a) \right] + \\ & + \frac{UH \sin \theta}{q} \left[ \frac{qX}{r_2} + \left\{ 1 - \left( \frac{r_1}{r_2} \right)^2 - (1 - r_1) \cos r_1 qa \right\} \frac{\sin r_2 q(X - a)}{r_1} + \right. \\ & \quad \left. + \left\{ qa \left( 1 - \frac{r_1}{r_2} \right) + \left( 1 - \frac{1}{r_1} \right) \sin r_1 qa \right\} \frac{\cos r_2 q(X - a)}{r_1^2} \right] - \\ & - UH \cos \theta Y \left[ \frac{1}{r_2} + \left\{ \left( 1 - \frac{r_1}{r_2} \right) - (1 - r_1) \cos r_1 qa \right\} \frac{\cos r_2 q(X - a)}{r_1} + \right. \\ & \quad \left. + (1 - r_1) \sin r_1 qa \frac{\sin r_2 q(X - a)}{r_1} \right]. \end{aligned} \right\} \quad (20)$$

There are now two stability criteria: (i)  $\cos(r_1 qX) \neq 1/(1 - r_1)$  for  $X \leq a$ , and (ii)

$$\left( \frac{r_1}{r_2} \right)^2 > (1 - r_1)^2 + \left( 1 - \frac{r_1}{r_2} \right)^2 - 2(1 - r_1) \left( 1 - \frac{r_1}{r_2} \right) \cos r_1 qa. \quad (21)$$

In the analysis leading to these results, the surface elevation,  $\eta'$ , has been small with respect to the total depth. In the case of flow over the single step, using Bernouilli's equation, we find the surface elevation given by

$$\eta = \frac{U^2}{2g} \left[ I - \left\{ \frac{I - (I - r_1) \cos r_1 q X}{r_1} \right\}^2 + (I - r_1) q^2 \left\{ \frac{\alpha}{\beta} + Y \cos \theta \right\}^2 \sin r_1 q X + \right. \\ \left. + \frac{2q(I - r_1) \sin \theta}{r_1} \sin r_1 q X \left\{ \frac{\alpha}{\beta} + Y \cos \theta \right\} \left\{ I - (I - r_1) \cos r_1 q X \right\} \right]. \quad (22)$$

Although this shows that  $\eta$  grows as  $(Y \cos \theta)^2$ , the assumption that  $\eta$  is small is consistent with the  $\beta$  approximation, since this also involves neglect of terms in  $y^2$ . Because both of these approximations break down for large  $y$ , we expect real ocean flow to diverge from these solutions in both the larger amplitude and the nonstable cases.

*Lagrangian Solution.* The Lagrangian solution is found analytically for only the simple case of an initially uniform current flowing over a single-depth discontinuity. It is possible, through matching of solutions, to treat any number of steps, but such a generalization only increases the complexity of the problem. If the surface elevation is again small compared with the total depth, (4) becomes, in a region of constant depth,  $r_1 H$ ,

$$\left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) (\xi + f) = -R\xi + \frac{\text{curl } \tau}{r_1 H}. \quad (23)$$

Since the flow is both steady and nondivergent, the pathlines of the fluid columns are given by the streamlines; therefore the position of a fluid column is described by the streamfunction  $\psi$  and a parameter,  $t'$ , along  $\Psi$ . The transformation from  $(x, y)$  to  $(\psi, t)$  is given by the relationships

$$\left. \begin{aligned} \left( \frac{\partial \psi}{\partial x} \right)_y &= \frac{1}{\mathcal{F}} \left( \frac{\partial y}{\partial t} \right)_\psi, & \left( \frac{\partial \psi}{\partial y} \right)_x &= -\frac{1}{\mathcal{F}} \left( \frac{\partial x}{\partial t} \right)_\psi, \\ \left( \frac{\partial t}{\partial x} \right)_y &= \frac{1}{\mathcal{F}} \left( \frac{\partial y}{\partial \psi} \right)_t, & \left( \frac{\partial t}{\partial y} \right)_x &= \frac{1}{\mathcal{F}} \left( \frac{\partial x}{\partial \psi} \right)_t, \end{aligned} \right\} \quad (24)$$

where

$$\mathcal{F} = \frac{\partial(x, y)}{\partial(\psi, t)}.$$

Since

$$\left( \frac{\partial \psi}{\partial x} \right)_y = \left( \frac{\partial y}{\partial t} \right)_\psi = v h, \quad \text{and} \quad - \left( \frac{\partial \psi}{\partial y} \right)_x = \left( \frac{\partial x}{\partial t} \right)_\psi = u h,$$

then  $\mathcal{F} = 1$ .



In  $(\psi, t)$ , the vorticity equation is given by

$$\frac{\partial}{\partial t} (\xi + f) = -R\xi + \frac{\text{curl } \tau}{r_1 H}. \quad (25)$$

Again, the solution is matched to a uniform initial flow over a constant depth,  $H$ . The upstream flow is then linear in  $\psi$ ; this suggests that we should look for solutions of the form

$$x = a_0(t) + a_1(t)\psi, \quad (26)$$

$$y = b_0(t) + b_1(t)\psi. \quad (27)$$

Since  $\mathcal{F} = \mathbf{1}'$ , then

$$a_1 b'_0 - b_1 a'_0 = 1, \quad (28)$$

$$a_1 b'_1 - b_1 a'_1 = 0. \quad (29)$$

Equation (29) has as solution  $a_1 = \pi b_1$ . If we set  $\pi = \tan \theta$  and define new functions  $b$  and  $\varphi$  by

$$b = \frac{b_1}{\cos \theta} = b_1 (1 + \pi^2)^{1/2}, \quad (30)$$

$$\varphi = b_0 \cos \theta + a_0 \sin \theta = \frac{b_0 + \pi a_0}{(1 + \pi^2)^{1/2}}, \quad (31)$$

then  $a_0, a_1, b_0,$  and  $b_1$  can be expanded in terms of  $b, \varphi,$  and  $\theta$  so that (26) and (27) become

$$x = \left( - \int_{t_0}^t \frac{dt}{b} \right) \cos \theta + (\varphi + b\psi) \sin \theta, \quad (32)$$

$$y = - \left( - \int_{t_0}^t \frac{dt}{b} \right) \sin \theta + (\varphi + b\psi) \cos \theta. \quad (33)$$

In the same rotated frame as that used in the Eulerian solution,

$$X = - \int_{t_0}^t \frac{dt}{b}; \quad Y = \varphi + b\psi. \quad (34)$$

The functions  $b$  and  $\varphi$  are found by substituting (32) and (33) into the vorticity equation (25) and by matching this to the initial flow at  $X = 0$ . If  $f$  is given by the  $\beta$  approximation as  $f = \alpha + \beta y$  and if  $\text{curl } \tau$  is constant and equal to  $\beta VH$ , then (25) can be separated and integrated once to give

$$b'' + Rb' + p^2 b + \beta \cos \theta = 0, \quad (35)$$

$$\varphi''' + R\varphi'' + p^2\varphi' - \frac{\beta \sin \theta}{b^2} + \frac{\beta V}{r_1 b} = 0, \quad (36)$$

where  $p^2$  is an integration constant to be determined from the matching conditions.

Equations (35) and (36) govern the flow in a region of constant depth,  $r_1 H$ . Boundary conditions are chosen to match this solution with a steady uniform current having velocity components  $U$  and  $V$  in the  $x$  and  $y$  directions over a constant depth,  $H$ . The depth change occurs along  $X = 0$ , and the origin of  $(\psi, t)$  is chosen such that  $t = 0$  along  $X = 0$  and  $\psi = 0$  at  $X = 0$ ,  $Y = 0$ . So that transports, streamlines, and potential vorticity will be continuous, the boundary conditions must take the form

$$\left. \begin{aligned} b(0) &= -\frac{r_1}{U \cos \theta - V \sin \theta}, & \varphi(0) &= 0, \\ b'(0) &= 0, & \varphi'(0) &= \frac{U \sin \theta + V \cos \theta}{r_1}, \\ b''(0) &= \beta(1 - r_1) \cos \theta, & \varphi''(0) &= -\frac{(1 - r_1)\alpha}{r_1} (U \cos \theta - V \sin \theta). \end{aligned} \right\} \quad (37)$$

Substitution of these conditions into (35) determines  $p^2$  as

$$p^2 = \frac{\beta(2 - r_1) \cos \theta}{r_1} (U \cos \theta - V \sin \theta). \quad (38)$$

Equation (35) can now be solved to obtain

$$b = -\frac{r_1}{(2 - r_1)(U \cos \theta - V \sin \theta)} \left[ 1 + (1 - r_1)e^{-\frac{Rt}{2}} \left( \cos vt + \frac{R}{2v} \sin vt \right) \right], \quad (39)$$

where

$$v = (p^2 - R/4)^{1/2}.$$

Following substitution from (39), the solution to (36) is

$$\left. \begin{aligned} \varphi &= \frac{\beta}{p^2} \left[ \int_{t_0}^t \left( \frac{\sin \theta}{b^2(\tau)} - \frac{V}{r_1 b(\tau)} \right) d\tau - \sin vt \int_{t_0}^t e^{-\frac{R\tau}{2}} \sin v\tau + \frac{R}{2v} \cos v\tau \right] \\ &\quad \left( \frac{\sin \theta}{b_2(\tau)} - \frac{V}{r_1 b(\tau)} \right) d\tau - \cos vt \int_{t_0}^t e^{-\frac{R\tau}{2}} \left( \cos v\tau - \frac{R}{2v} \sin v\tau \right) \\ &\quad \left( \frac{\sin \theta}{b^2(\tau)} - \frac{V}{r_1 b(\tau)} \right) d\tau \right], \end{aligned} \right\} \quad (40)$$



where  $t_1$ ,  $t_2$ , and  $t_3$  must be determined from the boundary conditions; however, these integrals cannot be evaluated explicitly in the general case. For the particular case corresponding to the Eulerian solutions, we set  $R = V = 0$ , and then (39) reduces to

$$b = -\frac{r_1}{(2-r_1)U \cos \theta} [1 + (1-r_1) \cos pt], \quad (41)$$

with

$$p^2 = \frac{\beta U(2-r_1)}{r_1} \cos^2 \theta.$$

When (41) is substituted into (40) and  $\alpha$  is set equal to 0, a solution, computed by using Fourier expansions, gives

$$\varphi = \frac{U(2-r_1) \sin \theta}{r_1} \left[ S(t) \left( 1 + \frac{\cos pt}{1-r_1} \right) + \frac{\cos pt \sin pt}{p(1+(1-r_1) \cos pt)} - \right. \\ \left. - \frac{2}{p(1-r_1)(r_1[2-r_1])^{1/2}} \tan^{-1} \left\{ (r_1/2 - r_1)^{1/2} \tan \frac{pt}{2} \right\} \cos pt \right], \quad (42)$$

where  $\alpha$  is set equal to 0 and

$$S(t) = \int_0^t \frac{d\tau}{(1+(1-r_1) \cos p\tau)^2}, \\ = \frac{2}{pr_1^{3/2}(2-r_1)^{3/2}} \left[ \frac{pt}{2} + \sum_{n=1}^{\infty} \frac{(1+n[r_1(2-r_1)]^{1/2})}{n} \right. \\ \left. \left( \frac{[r_1(2-r_1)]^{1/2} - 1}{1-r_1} \right)^n \sin npt \right]. \quad (43)$$

By manipulating (42) and (43) with the relationships

$$\tan^{-1} \left\{ [r_1/(2-r_1)]^{1/2} \tan \frac{pt}{2} \right\} = \frac{pt}{2} + \sum_{n=1}^{\infty} \left( \frac{[r_1(2-r_1)]^{1/2} - 1}{1-r_1} \right)^n \frac{\sin npt}{n}, \quad (44)$$

$$\frac{\sin pt}{1+(1-r_1) \cos pt} = -\frac{2}{1-r_1} \sum_{n=1}^{\infty} \left( \frac{[r_1(2-r_1)]^{1/2} - 1}{1-r_1} \right)^n \sin npt, \quad (45)$$

(42) is simplified to read

$$\varphi = \frac{U(2-r_1) \sin \theta}{r_1} \left[ \frac{2 \tan^{-1} \left\{ [r_1/(2-r_1)]^{1/2} \tan \frac{pt}{2} \right\}}{pr_1^{3/2}(2-r_1)^{3/2}} (1+(1-r_1) \right. \\ \left. \cos pt) - \frac{(1-r_1) \sin pt}{pr_1(2-r_1)} \right]. \quad (46)$$

Therefore the position of the streamline is given by

$$X = \frac{2}{r_1} \left( \frac{U}{\beta} \right)^{1/2} \tan^{-1} \left\{ \left( \frac{r_1}{2-r_1} \right)^{1/2} \tan \frac{pt}{2} \right\}, \quad (47)$$

$$Y = \frac{U \sin \theta}{pr_1} \left[ \frac{2 \tan^{-1} \left\{ \left( \frac{r_1}{2-r_1} \right)^{1/2} \tan \frac{pt}{2} \right\}}{r_1^{3/2} (2-r_1)^{3/2}} (1 + (1-r_1) \cos pt) - \right. \\ \left. - \frac{1-r_1}{r_1} \sin pt - r_1 \psi \frac{(1 + (1-r_1) \cos pt)}{U(2-r_1) \cos \theta} \right]. \quad (48)$$

Substitutions from (44), (45), and (47) allow (48) to be put into the Eulerian form,

$$qr_1 Y = \frac{\tan \theta (qr_1 X - (1-r_1) \sin qr_1 X) - \frac{r_1^2 \Psi}{UH \cos \theta}}{1 - (1-r_1) \cos qr_1 X}, \quad (49)$$

where  $q^2 = \beta/U$ . This is also the Eulerian solution given by (19) for the case where  $\alpha = 0$ . Further, the condition requiring the existence of wave solutions,  $0 < r_1 < 2$ , is also the condition required so that the series involved in the Lagrangian solution converge.

Since the Lagrangian approach yields a set of ordinary differential equations, even for the case where the frictional and wind-stress terms are retained, this approach is better suited to subsequent numerical solution than is the Eulerian approach. Solutions for a range of values of wind and frictional stresses and of depth of the step are obtained from the Lagrangian solution, using a small analogue computer. An example of one such solution is shown in Fig. 1.

*Properties of the Solutions.* Figs. 2 and 3 show the positions of three parallel streamlines [as calculated from the Eulerian solutions (19) and (20)] for a uniform zonal current flowing over a deepening step and over a plateau, respectively. In both cases, the flow pattern beyond the depth variations is that of large meanders whose crests are aligned parallel to the steps and whose amplitudes change in the downstream direction. The wavelength of these meanders is proportional to  $(H/h) (U/\beta)^{1/2}$ ; for  $U = 10$  cm/sec,  $H/h = 2$  at a latitude of  $40^\circ$ , the wavelength is of the order of 1000 km. Little is known about the actual barotropic currents in the open ocean, but  $U = 10$  cm/sec can be taken as an upper limit to their magnitude. The baroclinicity of the real ocean should lead to a decrease in the wavelength, because here  $U$  would be replaced by some velocity averaged over the water column and the depth ratio would be modified by the density structure of the column. Both of these effects should lead to the decrease in wavelength.



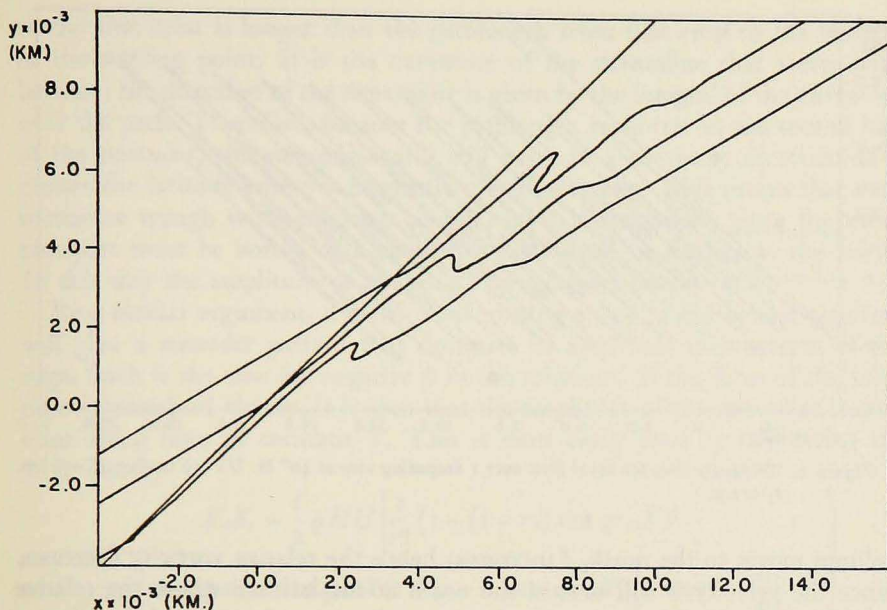


Figure 1. Transport lines for viscous flow over a shallowing step at  $40^{\circ}$  N.  $U = 20$  cm/sec,  $V = 10$  cm/sec,  $H = 1$  km,  $r_1 = 0.5$ ,  $R = 0.75 \times 10^{-6}$  sec $^{-1}$ .

In both cases, as the parameters are varied to approach the limits of stable flow, the amplitudes of the meanders become very large and the assumptions involved in the  $\beta$  approximation are violated. The large wavelengths that are obtained confirm the supposition that the single step is a good model of a continental slope. This is so because the lateral extent of the slopes, a few tens of kilometers, is much smaller than the lateral scale of the meanders. In the case of a very oblique incidence in an ocean current on a continental slope, this may not hold true, and a more complicated depth profile may be required.

The most striking feature of these solutions is the fact that, for a given streamline, the meanders change their amplitude downstream of the depth variations, even though the region is one of constant depth and although neither friction nor wind stress is included in the model. This change in amplitude is a consequence of the oblique incidence of the initial flow on the changes in depth. On the  $\beta$  plane, fluid flow must be such as to conserve the potential vorticity of the water column,  $(\xi + f)/h$ .

When the initial flow moves from the region of depth  $H$  to the region of depth  $r_1 H$ , the flow must immediately acquire enough relative vorticity so that the potential vorticity is held constant. In the case of the deepening step, this relative vorticity must be positive; physically this represents a curvature of the streamline toward the north (northern hemisphere). Then, as the water

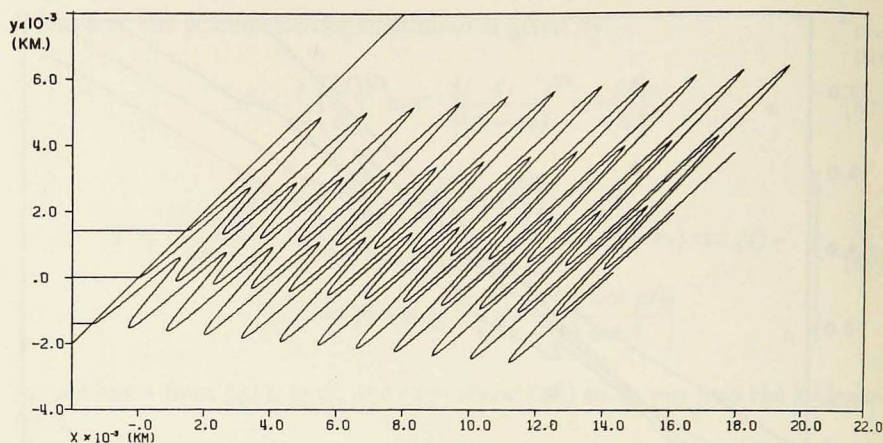


Figure 2. Transport lines for zonal flow over a deepening step at  $40^\circ$  N.  $U = 50$  cm/sec,  $H = 1$  km,  $r_1 = 1.2$ .

column moves to the north,  $f$  increases; hence the relative vorticity decreases. Since the velocity is still toward the north at the latitude where the relative vorticity falls to zero, the water column continues northward and the streamline acquires an increasingly negative curvature; this eventually causes the flow to turn southward. In this way the meander pattern is established and maintained.

Since the flow is uniform and infinite, the crests must be aligned parallel to the steps. If the step lies in the first quadrant (that is, toward the north and east), then the pathlength along a streamline from the point where it crossed the step

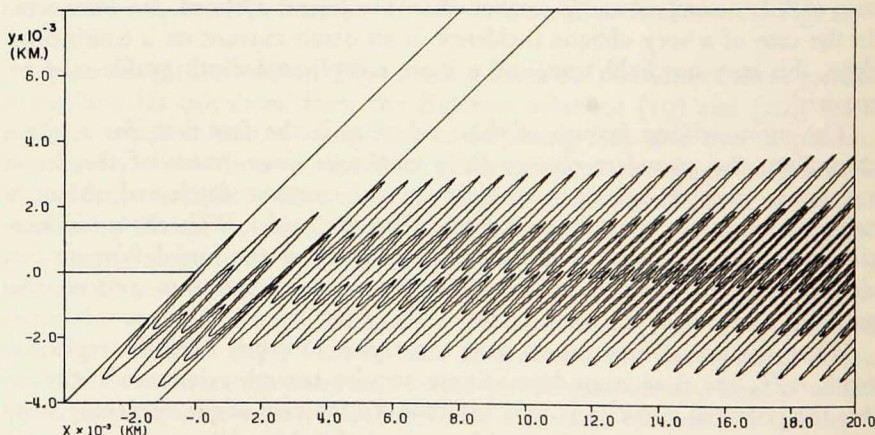


Figure 3. Transport lines for zonal flow over a ridge at  $40^\circ$  N.  $U = 10$  cm/sec,  $H = 4$  km,  $r_1 = 0.7$ ,  $r_2 = 1.0$ .



to the first crest is longer than the pathlength from that crest to the latitude of the starting point. It is the curvature of the streamline that varies with latitude; the direction of the streamline is given by the integral of the curvature over the path. Therefore, because the pathlength is shorter in the second half of the meander, the streamline will still point in a southerly direction as it crosses the latitude where it originally crossed the step. This means that each successive trough will be farther to the south. Furthermore, since the mean transport must be zonal, each successive crest must be farther to the north. In this way the amplitude of the meanders increases downstream.

By a similar argument, a zonal flow crossing a step in the second quadrant will give a meander pattern that decreases in amplitude downstream of the steps. Such is the case for negative  $\theta$  in the solutions. If the form of the solutions is examined closely, it is seen that the amplitude of the meanders is constant along lines of constant  $Y$ . This is most easily seen by calculating the *K.E.* density

$$\left. \begin{aligned}
 K.E. = \frac{1}{2} \rho H U \left[ \frac{1}{r_1^2} (1 - (1 - r_1) \cos q r_1 X)^2 - \right. \\
 \left. - \frac{2q(1 - r_1) \sin \theta \sin q r_1 X (1 - (1 - r_1) \cos q r_1 X) \left( Y \cos \theta + \frac{\alpha}{\beta} \right)}{r_1^2} + \right. \\
 \left. + \frac{q^2 (1 - r_1)^2 \sin^2 q r_1 X \left( Y \cos \theta + \frac{\alpha}{\beta} \right)^2}{r_1} \right] \quad (50)
 \end{aligned} \right\}$$

for the case of a single step. Along lines of constant  $Y$ , the *K.E.* density is periodic with constant amplitude; therefore, the amplitudes of the meanders are also constant along  $Y$ .

Fig. 1 illustrates how these meanders may be modified by wind and frictional stresses, which act as sources and sinks of vorticity. The case treated is that of a uniform nonzonal flow over a single step. In the initial constant-depth region the potential vorticity required for the northward movement of the water columns is supplied by the wind stress. The frictional stress can only dissipate vorticity in the form of relative vorticity.

A meander pattern arises through the same mechanism as before; however, in this case, friction provides a mechanism for removal of potential vorticity, thus allowing the meanders to damp out. The velocity values used in the example,  $U = 20$  cm/sec,  $V = 10$  cm/sec, are taken to be upper limits of barotropic velocities to be found in the real ocean. The value  $R = 0.75 \times 10^{-6}$  sec $^{-1}$  corresponds to a vertical eddy viscosity of 200 cm $^2$ /sec, a typical value for a low-stability region of the ocean. The wavelength of these meanders is only slightly changed from its value in the nonviscous case. For smaller values of eddy viscosity, the meanders may show a region of initial growth in amplitude to some constant amplitude or to a then-decreasing amplitude.

Since the model on which this work is based is but a crude approximation of the real ocean, it is dangerous to compare these results with actually observed currents. The general shapes of the solutions draw to mind the meander patterns and the deflections of the Gulf Stream as described by Mann (1967), Warren (1963), and Fuglister and Worthington (1951), but the wavelengths and amplitudes given by our solutions are several times larger than those observed. In the region of the Gulf Stream we are dealing with a stratified nonuniform baroclinic and time-dependent flow for which the flow must be quantitatively different from that given by the model. Nevertheless, it has been observed that Gulf Stream meanders increase in amplitude downstream, and the model provides a physical explanation for this increase without consideration of the bathymetry away from the slope.

Large barotropic meanders similar to those predicted by this model may occur in the body of the ocean away from the western boundary region as currents encounter the midocean ridges. Such patterns may be an important part of the dynamics of such regions; however, at present there are no observations that can be used to confirm or contradict such a supposition.

Simple bathymetric features are shown to have complicated effects on uniform currents. The standing wave patterns that have been obtained show an unexpected resemblance to flows actually observed—in particular, the phenomenon of meanders of increasing amplitude downstream of a single step. This resemblance suggests that further investigations of flows across simple bathymetric features, using more complex models, is desirable.

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