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Internal Wave Propagation Normal to a Geostrophic Current[1]

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ABSTRACT

It is shown that (i) internal gravity waves propagating along a horizontal density gradient in a uniformly rotating ocean can exist for a range of slightly smaller frequencies than is possible in the absence of such a gradient, (ii) such waves exhibit a phase difference from top to bottom, and (iii) they exchange energy with the geostrophic current, which balances the density-induced pressure gradient. In the ocean, the vertical phase difference is of the first order in the slope of the isopycnals, the reduction in the frequency range is of the second order, and the energy exchange is of the third order. The interaction between the waves and the current is very weak and is negligible in comparison with other known energy-exchange mechanisms.

1. *Introduction.* There is a variety of known mechanisms by which internal gravity waves may exchange energy with their surroundings or with other types of oceanic motions. Some of these mechanisms have been described by Phillips (1966) and by Krauss (1965). Of course, it is only when all relevant energy-exchange mechanisms are well understood, and especially when their relative importances are known, that it will become possible to make reasonable estimates of the energy budget of internal waves and of the contribution of the mechanisms to the total energy balance in the ocean. In discussing here another interaction process, we do not merely add to an already complex picture but also determine the relevance of this interaction in the oceanic context.

Phillips (1966: 178) has described how internal waves lose energy to a horizontally uniform flow with a weak vertical shear. The interaction is strongest when the wave propagation vector is parallel to the current and vanishes when the two are perpendicular. In a rotating medium, however, we would expect that the interaction does not vanish at normal incidence. The velocity vector of horizontally propagating waves will no longer lie

entirely in the vertical plane, as in a nonrotating medium, and, at normal incidence, a nonzero component of the Reynolds stress will still be available for energy exchange with the mean flow. Moreover, the steady flow will be in partial, if not complete, geostrophic equilibrium and will be balanced by density-induced pressure gradients. Terms involving density-velocity correlations will, as a consequence, also contribute to the energy transfer.

We concentrate our attention on that case for which energy exchange becomes possible only in the presence of rotation – namely, that of internal waves propagating normally to a horizontally uniform geostrophic current. It will be shown that, for conditions prevailing in the ocean, this energy-exchange mechanism is very weak compared with other known forms of interaction.

2. *The Mean Flow.* We consider a layer of fluid rotating uniformly with angular velocity $f/2$, bounded below by a flat bottom at $z = H$ and above by a sloping free surface at $z = -\eta(x)$ (Fig. 1). The z axis is taken positive downward from a level surface near the free surface. The sloping isopycnals compatible with a geostrophic current are described by the weak exponential form

$$\varrho_0 = \varrho^* e^{az+bx}, \tag{1}$$

where ϱ_0 is the mean density, ϱ^* a reference density; a and b are small and positive. Letting η_0 be a scale height for η and letting L be a scale length for the horizontal variation in η, we see that four parameters are involved in the description of the mean field: aH, b/a (the slope of the isopycnals), η_0/H, and H/L. All of these are assumed to be small and are denoted by ε_1, ε_2, ε_3, and ε_4, respectively. Across the Gulf Stream, for example, where the free surface and the isopycnals are certainly more tilted than is typical of the ocean as a whole, Stommel (1965: 21) has estimated a difference in elevation of about 1 m for the former and of 700 m for the latter. With a density change from top to bottom and across the current of the order of 10^{-3} g cm^{-3}, $H = 4$ km, and $L = 200$ km, we find $\varepsilon_1 = 10^{-3}$, $\varepsilon_2 = 3.5 \times 10^{-3}$, $\varepsilon_3 = 10^{-4}$, and $\varepsilon_4 = 2 \times 10^{-2}$.

The mean current V and pressure p_0 are given by

$$\varrho_0 f V - p_{0x} = 0, \tag{2}$$

$$\varrho_0 g - p_{0z} = 0, \tag{3}$$

with derivatives indicated by subscripts. Using (1), p_0 can be eliminated to yield

$$V_z + aV = bg/f, \tag{4}$$

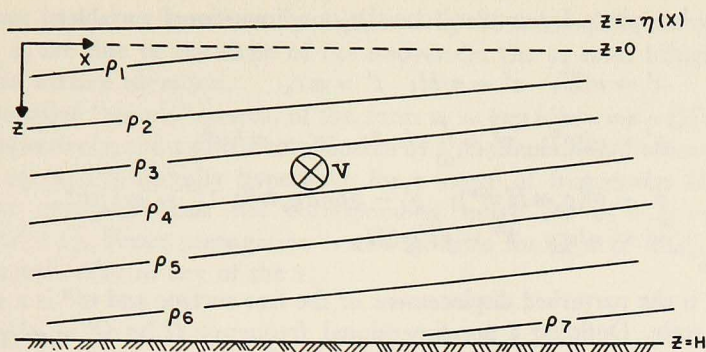which we use with profit in the derivation of the internal wave equation. Integration of (4) gives, for V,

Figure 1. Geometry of the problem. The geostrophic current V, in equilibrium with the sloping isopycnals (ϱ_1, ϱ_2 − − −), flows into the page.

$$V(x,z) = A(x)\,e^{-az} + bg/af. \qquad (5)$$

The function of integration $A(x)$ can be found by using the condition that p_0 is uniform on $z = -\eta(x)$. $V(x,z)$ becomes

$$V(x,z) = \frac{bg}{af}\big[1 - e^{-a(z+\eta)}\big] - \frac{\eta_x g}{f}\,e^{-a(z+\eta)}. \qquad (6)$$

3. *The Wave Equation.* We now introduce small perturbations $\boldsymbol{u} = (u, v, w)$, p, and ϱ upon the steady state, $\boldsymbol{V} = (0, V, 0)$, ϱ_0, p_0, described above. These perturbations represent plane waves propagating in the x direction, normal to the mean flow; they are assumed to be two-dimensional: $\partial/\partial y = 0$. Since the slope of the isopycnals, which is responsible for the geostrophic current, is very small, the solution for the perturbations should be only a slight modification of the solution for purely horizontal isopycnals. It is not consistent then to neglect other small effects: we do not use Boussinesq's approximation nor do we forget that the upper surface is free to move. However, only a linear analysis is made; the wave amplitude is an independent parameter, which we can make as small as we wish.

For an incompressible nondiffusive liquid, the conservation equations, to the first order in the amplitude of the perturbations, for momentum, mass, and density are given by

$$\varrho_0(u_t - fv) - fV\varrho + p_x = 0, \qquad (7)$$

$$v_t + fu + wV_z + uV_x = 0, \qquad (8)$$

$$\varrho_0 w_t - g\varrho + p_z = 0, \qquad (9)$$

$$u_x + w_z = 0, \qquad (10)$$

$$\varrho_t + u\varrho_{0x} + w\varrho_{0z} = 0. \qquad (11)$$

We now introduce for convenience the nondimensional variables

$$
\begin{aligned}
&x' = x/H; \quad z' = z/H; \quad t' = \omega t, \\
&u' = u/w^*; \quad v' = \frac{f}{\omega} v/w^*; \quad w' = w/w^*, \\
&p' = p/(\varrho_0 \omega H w^*); \quad \varrho' = \varrho \omega / a \varrho_0 w^*; \quad \zeta' = \omega \zeta/w^*, \\
&\eta' = \eta/\eta_0; \quad V' = fV/gbH,
\end{aligned}
\quad \right\} \quad (12)
$$

where ζ is the perturbed displacement of the free surface and w^* is a vertical-velocity scale. Defining a nondimensional frequency Ω by $\Omega^2 = \omega^2/ga$, (7)–(11) become in the new variables (from which we immediately drop the primes)

$$\Omega^2(u_t - v + p_t) - \varepsilon_1\varepsilon_2\varrho V \qquad\quad = 0, \qquad (13)$$

$$\Omega^2 v_t + \frac{f^2}{ga} u + \varepsilon_2 w V_z + \varepsilon_2\varepsilon_4 V_x = 0, \qquad (14)$$

$$\Omega^2(w_t + p_z) - \varrho \qquad\qquad\quad = 0, \qquad (15)$$

$$u_x + w_z \qquad\qquad\qquad = 0, \qquad (16)$$

$$\varrho_t + \varepsilon_2 u + w \qquad\qquad\quad = 0. \qquad (17)$$

From (6) and (12), the mean velocity becomes, in nondimensional form,

$$V = \frac{1}{\varepsilon_1}\left[1 - e^{-\varepsilon_1(z+\varepsilon_3\eta)}\right] - \frac{\varepsilon_3\varepsilon_4}{\varepsilon_1\varepsilon_2} e^{-\varepsilon_1(z+\varepsilon_3\eta)}. \qquad (18)$$

To the first order in any of the ε, this is

$$V = z - \frac{\varepsilon_3\varepsilon_4}{\varepsilon_1\varepsilon_2}\eta_x + \varepsilon_3\eta - \frac{\varepsilon_1 z^2}{2} + \frac{\varepsilon_3\varepsilon_4}{\varepsilon_2}\eta_x z + 0(\varepsilon^2). \qquad (19)$$

Clearly, to have any appreciable depth dependence in V, $\varepsilon_3\varepsilon_4$ must be of the same order of magnitude as $\varepsilon_1\varepsilon_2$. This is the case for the Gulf Stream, as seen above.

All variables except w are readily eliminated from (13)–(17); using (4) to eliminate V_z as well, we find the following wave equation, valid to the second order in any of the ε:

$$
\begin{aligned}
&\left[\Omega^2\frac{\partial^2}{\partial t^2} + \frac{f^2}{ga} + \varepsilon_2\varepsilon_4 V_x\right] w_{zz} - 2\varepsilon_2 w_{zx} + \left[\Omega^2\frac{\partial^2}{\partial t^2} + 1\right] w_{xx} + \\
&\quad + \varepsilon_1\left[\Omega^2\frac{\partial^2}{\partial t^2} + \frac{f^2}{ga}\right] w_z + \varepsilon_1\varepsilon_2\Omega^2\frac{\partial^2}{\partial t^2} w_x = 0.
\end{aligned}
\quad \right\} \quad (20)
$$

The terms in ε_1 are those usually neglected by Boussinesq's approximation; those in ε_2 are due to the slope of the isopycnals; the ε_4 term brings in the changes in surface elevation.

For so-called "simple" waves, of the form $w = \exp i(kx + mz - t)$ [k and m being, respectively, the x and z components of a nondimensional wave-number vector], eq. (20) is spatially hyperbolic for a range of frequencies Ω that is somewhat narrower than the corresponding range for $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0$ ($f^2/ga < \Omega^2 < 1$). Wave propagation is now possible for $\Omega_-^2 < \Omega^2 < \Omega_+^2$, where, to the second order in any of the ε,

$$\left. \begin{aligned} \Omega_+^2 &= 1 - \frac{\varepsilon_2^2}{(1 - f^2/ga)}, \\[2mm] \Omega_-^2 &= \frac{f^2}{ga} + \frac{\varepsilon_2^2}{(1 - f^2/ga)}. \end{aligned} \right\} \quad (21)$$

The change in the frequency range is only of the second order and is not very significant. More important is the change in the form of the propagation curves. These are curves of constant Ω in (k, m) space (Eckart 1960: 109). For a given Ω, the phase velocity is along the vector from the origin to a point on the propagation curve and is of a magnitude that is inversely proportional to the length of that vector. The group velocity is simply the gradient of Ω in the (k, m) plane, is normal to the propagation curves, and is in the direction of increasing Ω. When the isopycnals are horizontal, these curves are hyperbolae that are symmetrical about the z axis (Fig. 2a); the significance of these hyperbolae, particularly with respect to the reflection off bounding surfaces, has been discussed by Sandstrom (1966) (under Boussinesq's approximation: to 0th order in ε_1). When $\varepsilon_2 \neq 0$, the propagation curves are still hyperbolae but are now symmetrical about an axis rotated clockwise from the z axis (Fig. 2b) by an angle θ, such that

$$\tan 2\theta = \frac{2\varepsilon_2}{(1 - f^2/ga)}.$$

The asymmetry in the propagation curves is of the first order in ε_2 and should be much more significant than the change in the range of hyperbolicity. In a wave guide with parallel sides (such as that illustrated in Fig. 1, but with $\eta = 0$), two simple waves may be superimposed to satisfy inviscid boundary conditions at the top and at the bottom. To maintain horizontal coherence, the two waves must have identical frequencies and horizontal wave numbers. For horizontal isopycnals, because of the symmetry of the propagation curves, the vertical wave numbers of the two interfering waves are equal in magnitude and of opposite signs. With $\varepsilon_2 \neq 0$, the symmetry about the z axis is lost, and the two waves have vertical wave numbers of different magnitudes. As a con-
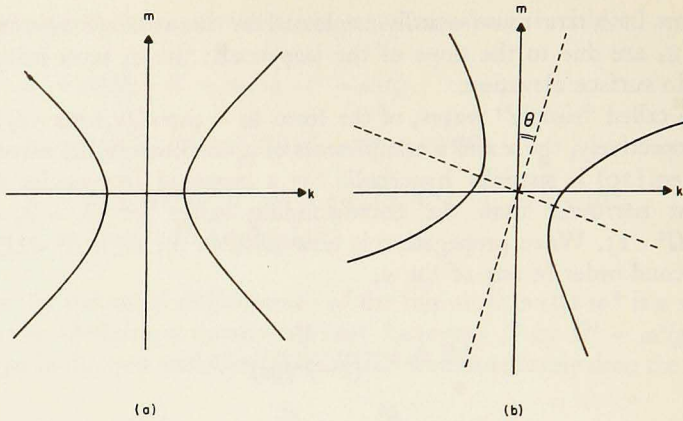
Figure 2. Propagation curves for an intermediate frequency: Ω is constant on the hyperbolae. (a) Horizontal isopycnals: $\varepsilon_2 = 0$. (b) Sloping isopycnals: $\varepsilon_2 \neq 0$.

sequence, the planes of constant phase will not travel horizontally but will seem to issue from one boundary and disappear at the other. Another interpretation is that there is a phase difference between the top and the bottom of the layer. The solution now to be developed illustrates these properties.

4. *Horizontal propagation.* At the bottom of the oceanic wave guide, the vertical velocity must vanish:

$$w(x, 1, t) = 0. \tag{22}$$

The requirement that the perturbation pressure be uniform at $z = -\varepsilon_3 \eta(x)$, linearized, and evaluated at $z = 0$, gives, to $o(\varepsilon^2)$,

$$\varepsilon_1 \Omega^2 [p_t - \varepsilon_3 \eta p_{zt}] = (1 + \varepsilon_1 \varepsilon_3 \eta) \zeta_t. \tag{23}$$

To the same order, the kinematic boundary condition is

$$w - \varepsilon_3 \eta w_z + \varepsilon_3 \varepsilon_4 u \eta_x + \frac{\varepsilon_3^2}{2} \eta^2 w_{zz} = - \zeta_t. \tag{24}$$

By combining (23) and (24) and by using (15) and (17) to eliminate z derivatives of p, the top boundary condition becomes, at $z = 0$,

$$\left.\begin{aligned}
w + \varepsilon_1 \Omega^2 p_t - \varepsilon_3 \eta w_z + \varepsilon_3 \varepsilon_4 u \eta_x + \frac{\varepsilon_3^2}{2} \eta^2 w_{zz} + \\
+ \varepsilon_1 \varepsilon_3 [(1 + \Omega^2) \eta w + \eta \Omega^4 w_{tt}] = 0.
\end{aligned}\right\} \tag{25}$$

From (13), (14), (16), and (17) we also have, to $o(\varepsilon^2)$,

$$\varepsilon_1 \left[\Omega^2 p_{xxt} - \Omega^2 w_{ztt} - \frac{f^2}{ga} w_z + \varepsilon_2 w_x \right] = 0. \tag{26}$$

Elimination of the pressure between (25) and (26) leaves us with a boundary condition in the vertical velocity only: to $o(\varepsilon^2)$, at $z = 0$,

$$\left. \begin{aligned} w_{xx} + \varepsilon_1 \left(\Omega^2 w_{ztt} + \frac{f^2}{ga} w_z \right) - \varepsilon_3 \eta\, w_{zxx} - \varepsilon_1 \varepsilon_2 w_x - 3\, \varepsilon_3 \varepsilon_4 \eta_x\, w_{zx} + \\ + \frac{\varepsilon_3^2}{2} \eta^2 w_{zzxx} + \varepsilon_1 \varepsilon_3 \left[\eta\, w_{xx} (1 + \Omega^2) + \eta\, \Omega^4 w_{xxtt} \right] = 0. \end{aligned} \right\} \tag{27}$$

We wish to find solutions for (20), thus satisfying the boundary conditions (22) and (27), which correspond to waves traveling in the x direction. First, note the double role played by the existence of a mean displacement $- \varepsilon_3 \eta(x)$ from the level surface $z = 0$. Terms involving $\eta(x)$ appear to $o(\varepsilon_3)$ in the top boundary condition (27). Thus, for the wave equation consisting of normal modes propagating in one direction, it is not possible to find a solution that will satisfy (27). This is not surprising; our wave guide does not have parallel sides. However, this analysis is not oriented toward a study of internal-wave propagation in irregular wave guides but rather toward an investigation of the importance of the energy exchange between the waves and the mean current. As the problem has been linearized in the wave amplitude, no energy can be exchanged with the free surface. We therefore concentrate our attention on the effects due to the correction terms involving powers of ε_1 and ε_2. To the second order in small parameters, an x-dependent coefficient appears in the wave equation as well (involving, through V_x, η_{xx}). This term leads to x variations in the horizontal wave number and hence to partial reflection (and also to refraction, if our waves were allowed to propagate in the y direction as well). In order to keep this not-so-direct effect of the surface displacement in the simplest manner possible, we write $\varepsilon_4 = \gamma \varepsilon_2$, where $\gamma = o(1)$ (for the Gulf Stream data, $\gamma \simeq 6$).

We thus write w as a perturbation expansion of the form

$$w = w_0 + \varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_1^2 w_{11} + \varepsilon_1 \varepsilon_2 w_{12} + \varepsilon_2^2 w_{22} + \dots \tag{28}$$

First consider the case $V_x = $ constant; the coefficients of (20) are now all x independent. We write

$$w = \varphi(z) \exp \left\{ -\frac{\varepsilon_1 z}{2} - \frac{i\,\varepsilon_2 kz}{\left(\Omega^2 - \dfrac{f^2}{ga} - \gamma \varepsilon_2^2 V_x \right)} \right\} e^{i\,(kx - t)}, \tag{29}$$

in which k is a nondimensional horizontal wave-number. Substitution of (29) into (20) gives, for $\varphi(z)$,

$$\varphi_{zz} + \lambda^2 \varphi = 0. \tag{30}$$

To the second order in ε_1 and ε_2, the vertical eigenvalue λ is given by

$$\lambda^2 = \frac{k^2(1-\Omega^2)}{\left(\Omega^2 - \dfrac{f^2}{ga} - \gamma \varepsilon_2^2 V_x\right)} - \frac{\varepsilon_1^2}{4} + \frac{\varepsilon_2^2 k^2}{\left(\Omega^2 - \dfrac{f^2}{ga}\right)^2} - \frac{i\,\varepsilon_1 \varepsilon_2 k(1-\Omega^2)}{\left(\Omega^2 - \dfrac{f^2}{ga}\right)}. \tag{31}$$

The boundary conditions on φ, to the second order in ε_1 and ε_2 and to the 0th order in ε_3 and ε_4, are

$$\varphi(1) = 0, \tag{32a}$$

$$\varphi(0) = - \frac{\varepsilon_1 \,(\Omega^2 - f^2/ga)}{k^2} \, \varphi_z(0). \tag{32b}$$

To satisfy (32a), (30) must have the solution

$$\varphi(z) = \sin\left[\lambda(z-1)\right]. \tag{33}$$

The top boundary condition imposes the dispersion relationship

$$\tan \lambda = \frac{\lambda \varepsilon_1}{k^2}\left(\Omega^2 - \frac{f^2}{ga}\right). \tag{34}$$

We have not explicitly used the expanded form (28) for w since, with constant coefficients in (20), we can solve directly for w to the second order in ε_1 and ε_2. We now expand λ and k in a similar series; for λ,

$$\lambda = \lambda_0 + \varepsilon_1 \lambda_1 + \varepsilon_2 \lambda_2 + \varepsilon_1^2 \lambda_{11} + \varepsilon_1 \varepsilon_2 \lambda_{12} + \varepsilon_2^2 \lambda_{22} + \ldots$$

By substituting the expansions for λ and k into (31) and by equating powers of ε_1 and ε_2, we have:

$$\left.\begin{aligned}
\lambda_0^2 &= k_0^2(1-\Omega^2)(\Omega^2 - f^2/ga)^{-1}, \\
\lambda_1 \lambda_0 &= k_1 k_0 (1-\Omega^2)(\Omega^2 - f^2/ga)^{-1}, \\
\lambda_2 \lambda_0 &= k_2 k_0 (1-\Omega^2)(\Omega^2 - f^2/ga)^{-1}, \\
2\lambda_0 \lambda_{11} &= 2k_0 k_{11}(1-\Omega^2)(\Omega^2 - f^2/ga)^{-1} - 1/4, \\
2\lambda_0 \lambda_{12} &= (2k_0 k_{12} - i k_0)(1-\Omega^2)(\Omega^2 - f^2/ga)^{-1}, \\
2\lambda_0 \lambda_{22} &= \left[2k_0 k_{22}(1-\Omega^2) + \frac{(k_0^2 + \gamma V_x)}{(\Omega^2 - f^2/ga)}\right](\Omega^2 - f^2/ga).
\end{aligned}\right\} \tag{35}$$

Similarly, expansion of (34) yields

$$\lambda_0 = n\pi \qquad\qquad \lambda_{11} = \left(\lambda_1 - \frac{2\lambda_0 k_1}{k_0}\right)\frac{(\Omega^2 - f^2/ga)}{k_0^2}$$

$$\lambda_1 = \frac{\lambda_0}{k_0^2}(\Omega^2 - f^2/ga) \qquad \lambda_{12} = \left(\lambda_2 - \frac{2\lambda_0 k_2}{k_0}\right)\frac{(\Omega^2 - f^2/ga)}{k_0^2} \qquad\Bigg\} \quad (36)$$

$$\lambda_2 = 0 \qquad\qquad\qquad \lambda_{22} = 0.$$

Elimination of λ between (35) and (36) gives, for k,

$$k_0 = n\pi(1 - \Omega^2)^{-1/2}(\Omega^2 - f^2/ga)^{1/2},$$

$$k_1 = \frac{1}{n\pi}(1 - \Omega^2)^{1/2}(\Omega^2 - f^2/ga)^{1/2},$$

$$k_2 = 0,$$

$$k_{11} = \frac{(\Omega^2 - f^2/ga)^{1/2}}{2n\pi(1 - \Omega^2)^{1/2}}\left[\frac{1}{4} - \frac{2(1 - \Omega^2)}{n^2\pi^2}\right], \qquad\Bigg\} \quad (37)$$

$$k_{12} = i/2,$$

$$k_{22} = -\left[\frac{\gamma V_x + n^2\pi^2(\Omega^2 - f^2/ga)(1 - \Omega^2)^{-1}}{2n\pi(1 - \Omega^2)^{1/2}(\Omega^2 - f^2/ga)^{3/2}}\right].$$

The vertical eigenvalue remains purely real and is unaffected by the presence of a slope in the isopycnals ($\lambda_2 = \lambda_{12} = \lambda_{22} = 0$). The wave-number is modified to the second order and becomes complex. The solution for the vertical velocity is then explicitly

$$w = \sin\left[\lambda(z - 1)\right]\exp\left\{-\frac{\varepsilon_1 z}{2} - \frac{\varepsilon_1\varepsilon_2 x}{2}\right\}\cos\left[\operatorname{Re}(k)x - \delta z - t\right], \quad (38)$$

with

$$\delta = \frac{\varepsilon_2 \operatorname{Re}(k)}{(\Omega^2 - f^2/ga)}. \quad (39)$$

By expanding (38) in powers of ε_1 and ε_2, the expansion terms w_1, w_2 etc. . . in (28) can be found explicitly; there is no need to find those terms, however, since (38) gives w in a much more compact form.

The other variables in the problem, u, v, ϱ, and p, may be found by substituting w in the primitive equations (13)–(17). The solution differs from that in a purely vertically stratified medium in two points.

First, there is a phase difference between the top and bottom: in other words, the planes of constant phase travel partly upward rather than purely horizontally. This, as we have seen in § 3, is to be expected from the asymmetry of the propagation curves relative to the z axis. The vertical velocity may be

written as the sum of two plane waves with slowly varying amplitudes and different vertical wave numbers:

$$
w = \frac{1}{2} \exp\left\{ -\frac{\varepsilon_1 z}{2} - \frac{\varepsilon_1 \varepsilon_2 x}{2} \right\} \left\{ \sin\left[\mathrm{Re}\,(k)\,x - (\delta - n\pi)\,z - t - n\pi \right] - \right.
$$
$$
\left. - \sin\left[\mathrm{Re}\,(k)\,x - (\delta + n\pi)\,z - t + n\pi \right] \right\}. \tag{40}
$$

At the upper frequency limit, $\Omega = \Omega_+$, k tends to infinity and $\delta/k \to \varepsilon_2/(1 - f^2/ga)$: phase propagation is almost purely horizontal. At $\Omega = \Omega_-$, $k \to 0$ and $\delta/k \to (1 - f^2/ga)/\varepsilon_2$: phase propagation becomes almost purely vertical.

Second, the wave amplitude varies horizontally; this is the effect that is most relevant from the point of view of the energetics of the system.

5. *Energy Exchange.* Since x does not appear explicitly in the wave equation and the boundary conditions (at least to the 0th order in ε_3 and ε_4), the wave properties (except for the amplitude) will not vary horizontally. In particular, the group velocity is not a function of x, and any divergence in the energy flux will be entirely associated with the exponential x dependence of the velocity and the density. The average local energy density E will vary with x as

$$
E \propto \exp\,(bH - \varepsilon_1 \varepsilon_2)\,x.
$$

Recalling that $bH = \varepsilon_1 \varepsilon_2$, we see that the x dependence vanishes. To the second order in ε_1 and ε_2, no energy exchange takes place, and the amplitude of the motion varies in just the right way to account for the mean density change and to keep the energy density constant. To find any exchange, we must go to the third order in ε_1 and ε_2. With a flat upper boundary ($\eta_x = 0$), we find, for the imaginary part of the wave-number (the calculation is outlined in the Appendix),

$$
k_{112} = \frac{(1 - \Omega^2)}{n^3 \pi^3}. \tag{41}
$$

Let us define a Q for the system:

$$
Q = \frac{\mathrm{Re}\,(k)}{\varepsilon_1^2 \,\varepsilon_2 \,k_{112}}.
$$

The waves gain energy when going down the density gradient (toward negative x) and lose energy when going up the gradient. The interaction is strongest when Q is smallest, i.e., at the lower frequency limit, $\Omega = \Omega_-$, where Q vanishes. However, even for a Q no smaller than $1/\varepsilon_1$, the wave length must be about $2\,n^3 \pi^4/\varepsilon_1 \varepsilon_2$. For the first mode and for the values of ε_1 and ε_2 quoted in § 2, this gives a wave length of about 600 km. For waves of such lengths, eddy diffusive effects (LeBlond 1966) give a Q of about 20, which is about

50 times smaller. Since most of the low-frequency internal-wave energy in the ocean is in internal diurnal and semidiurnal tides, with wave lengths of the order of 100 km, losses due to turbulent diffusion will greatly exceed losses (or gains) in energy due to interactions with geostrophic currents through the mechanism studied here.

6. *Reflection.* To the 0th order in ε_3 and ε_4, the waves are not reflected at all. That internal waves can propagate without reflection in spite of the horizontal variation in density is due to the particularly simple form chosen for ϱ_0. As indicated in § 4, variations in the mean surface elevation can lead to reflections in two ways.

When the $\varepsilon_2 \varepsilon_4 V_x$ coefficient of w_{zz} in (20) is x dependent [through the existence of a mean surface curvature η_{xx}; see (19)], the horizontal wave number becomes x dependent. Since the vertical structure of the solution (through the parameters λ and δ) is independent of $\varepsilon_2 \varepsilon_4 V_x$, we can use a simple WKB approximation to estimate the influence of the variable coefficient on the transmitted and the reflected waves. A first approximation [the first term in the Bremmer series; see, for example, Bellman (1964)] for the x-dependent part of the transmitted wave amplitude gives

$$w(x) = \left[\frac{k(0)}{k(x)}\right]^{1/2} \exp\left\{i\int_0^x k(s)\,ds\right\}. \tag{42}$$

Substituting k from (37), we find, to the second order in ε_1 and ε_2,

$$w(x) = \left[1 + \frac{\gamma \varepsilon_2^2 [V_x(x) - V_x(0)]}{4\,n^2\pi^2(\Omega^2 - f^2/ga)^2}\right]$$

$$\exp\left\{i\left[k_0 + \varepsilon_1 k_1 + \varepsilon_1^2 k_{11} - \frac{n\pi}{2}(1-\Omega^2)^{-3/2}(\Omega^2 - f^2/ga)^{-1/2}\right]x - \frac{i\,\varepsilon_2^2\gamma\,[V(x) - V(0)]}{2\,n\pi(1-\Omega^2)^{1/2}(\Omega^2 - f^2/ga)^{3/2}}\right\}.$$

The wave amplitude is modified to order $\gamma\varepsilon_2^2$ (i.e. $\varepsilon_2\varepsilon_4$). The amplitude of the reflected wave will then be of the same order of magnitude. When Ω^2 approaches its lower limit $\underline{\Omega^2}$, the wave length becomes comparable to L or to $1/b$; the variations in the medium are no longer small over a wave length, and the WKB approximation breaks down. Nevertheless, note that reflection occurs to a lower order of small parameters than does the energy exchange with the mean current, and reflection is correspondingly more important.

As noted earlier, simple normal-mode propagation is inadequate to describe the purely geometrical effects of the variation in $\eta(x)$. Sandstrom's (1966) theoretical and experimental studies of internal-wave reflection off bounding

surfaces, based on the method of characteristics, suggest the following remarks. For waves traveling toward the positive x direction, we expect the top surface to be perfectly "transmissive" (in Sandstrom's terminology) for $\eta_x > 0$; waves of all frequencies are reflected on the surface and travel on toward $x > 0$. If $\eta_x < 0$, the wave guide narrows, and, for waves of frequency that are low enough so that the characteristics have a slope smaller than that of the free surface, the latter becomes "reflective"; these waves are reflected back toward $x < 0$. Since the surface slope is of $o(\varepsilon_4)$, we should in that case expect a narrow band of low frequencies, bounded below by Ω_- and of width of $o(\varepsilon_4)$, to be reflected back by the top surface. Because such reflection does not contribute to the energy exchange between the internal waves and the geostrophic current, we do not consider it further.

7. *Conclusions.* The presence of a slight slope in the isopycnals of a non-homogeneous rotating fluid leads to interesting dynamical effects. For example, a phase lag appears between the top and the bottom of the fluid layer and becomes very important near the inertial frequency. The coupling that arises between the mean geostrophic current and the internal waves propagating normal to it is, however, so weak that it need not be taken into consideration in the energy budget of internal waves in the ocean.

APPENDIX

To simplify the notation in calculating the third order corrections to k_0, we write $\varepsilon_2 = \alpha \varepsilon_1$, where $\alpha = o(1)$. This is consistent with conditions prevailing in the Gulf Stream. It is then sufficient to expand φ, λ, and k in a series of powers of ε_1 only. We now write

$$\left.\begin{aligned}
\lambda_1^* &= \lambda_1 + \alpha\lambda_2, \\
\lambda_2^* &= \lambda_{11} + \alpha\lambda_{12} + \alpha^2 \lambda_{22},
\end{aligned}\right\} \quad \text{A 1}$$

and so on. Expanding (33), we have

$$\left.\begin{aligned}
\varphi_0 &= \sin\left[\lambda_0(z-1)\right], \\
\varphi_1^* &= (z-1)\lambda_1^* \cos\left[\lambda_0(z-1)\right], \\
\varphi_2^* &= (z-1)\left\{\lambda_2^* \cos\left[\lambda_0(z-1)\right] - \frac{\lambda_1^{*2}}{2}\sin\left[\lambda_0(z-1)\right]\right\}.
\end{aligned}\right\} \quad \text{A 2}$$

To the third order, (30) becomes (with $\eta_x = 0$)

$$\left.\begin{aligned}
\varphi_{3zz}^* + \lambda_0^2\,\varphi_3^* = {}& -2\,\lambda_0\lambda_1^*\varphi_2^* - (\lambda_1^{*2} + \lambda_0\lambda_2^*)\,\varphi_1^* \\
& -2\,(\lambda_0\lambda_3^* + \lambda_1^*\lambda_2^*)\,\varphi_0;
\end{aligned}\right\} \quad \text{A 3}$$

also, to the third order in ε_1 only, (31) becomes

$$2\left(\lambda_0 \lambda_3^* + \lambda_1^* \lambda_2^*\right) = \left[2\left(k_0 k_3^* + k_1^* k_2^*\right) - i k_1^* \alpha\right] \frac{(1 - \Omega^2)}{(\Omega^2 - f^2/ga)}$$
$$\left. + \alpha^2 \frac{\left[2 k_1^* k_0 + k_0^2 (1 - \Omega^2) + k_0^2 (z - 1)(1 - \Omega^2)\right]}{(\Omega^2 - f^2/ga)^2}, \right\} \qquad \text{A4}$$

or, for brevity,

$$2\left(\lambda_0 \lambda_3^* + \lambda_1^* \lambda_2^*\right) = A + B(z - 1), \qquad \text{A5}$$

where A and B are defined by A4 and A5.

The solution of the nonhomogeneous part of A3 is

$$\varphi_3^* = \frac{(z-1)}{4\lambda_0^2} \sin\left[\lambda_0(z-1)\right]\left\{-B + \lambda_0 \lambda_1^{*3} - \lambda_0(z-1)(3\lambda_0 \lambda_1^* \lambda_2^* + \lambda_1^{*3})\right\}$$
$$+ \frac{(z-1)}{4\lambda_0^2} \cos\left[\lambda_0(z-1)\right]\left\{2A\lambda_0 + (B\lambda_0 - \lambda_0^2 \lambda_1^{*3})(z-1) - \right.$$
$$\left. - (3\lambda_0 \lambda_1^* \lambda_2^* + \lambda_1^{*3})\right\}. \Bigg\} \qquad \text{A6}$$

The top boundary condition, which includes terms that do not appear in (32b) (good to second order only), is derived in a straight-forward manner from (27) and (29):

$$\varphi_3^*(0) = -\varphi_0(0)\left[\frac{2 k_1^* k_2^*}{k_0^2} + \frac{2 k_3^*}{k_0} + \frac{i k_1^* \alpha}{k_0}\right] - \varphi_1^*(0)\left[\frac{k_1^{*2}}{k_0^2} + \frac{2 k_2^*}{k_0} + \right.$$
$$\left. + \frac{(\Omega^2 - f^2/ga)}{2 k_0^2} + \frac{i\alpha}{k_0}\right] - 2\varphi_2^*(0)\frac{k_1^*}{k_0} + \varphi_{2z}^*(0)\frac{(\Omega^2 - f^2/ga)}{k_0^2}. \Bigg\} \qquad \text{A7}$$

Substituting from A2 and A6,

$$-\frac{(A+B)}{4\lambda_0} - \frac{\lambda_1^{*3}}{4} + \frac{(3\lambda_0 \lambda_1^* \lambda_2^* + \lambda_1^{*3})}{4\lambda_0^2} = 2 k_1^* k_2^* + \lambda_1^*\left[\frac{k_1^{*2}}{k_0^2} + \frac{2 k_2^*}{k_0} + \right.$$
$$\left. + \frac{(\Omega^2 - f^2/ga)}{2 k_0^2} + \frac{i\alpha}{k_0}\right] + \left(\lambda_2^* + \frac{\lambda_0 \lambda_1^{*2}}{2}\right)\frac{(\Omega^2 - f^2/ga)}{k_0^2}. \Bigg\} \qquad \text{A8}$$

Only the imaginary part of k_3^* is of interest. Taking the imaginary part of A8, we have

$$\text{Im}(A) = 2\lambda_1^* \, Im(k_2^*) + \frac{\alpha\lambda_1^*}{k_0},$$

or, from A4 and (32),

$$\text{Im}(k_3^*) = \frac{\lambda_1^* \alpha}{n^2 \pi^2} = \frac{\alpha(1 - \Omega^2)}{n^3 \pi^3},$$

which is the result used in (41).

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