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Long Waves along a Single-step Topography in a Semi-infinite Uniformly Rotating Ocean[1]

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ABSTRACT

The dispersion equation for Kelvin-type waves for a single-step topography is derived. Solutions for this equation indicate that, in addition to the Kelvin-type waves, there also exist quasigeostrophic waves that are related to the topography structure.

The lunar semidiurnal tide (12.4206 hour period) along the California coast appears to approximate a Kelvin wave since its amplitude is nearly constant (0.5 ± 0.01 m) while its phase speed (about 200 m/sec) and direction (north) are consistent with the Kelvin-wave solution (Larsen, 1968). In the present paper it is shown how a single-step topography (corresponding to the shelf off California) influences the modification of the Kelvin-wave solution. Other types of waves are also possible, and these are described.

Choose a rectangular coordinate system (x, y, z) such that x is north, y is

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west, and z is up, and let the ocean be confined to the space $(-\infty \leqslant x \leqslant \infty, 0 \leqslant y \leqslant \infty)$. The linearized equations of motion for long free waves of small amplitude are

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} - fv = -g\dfrac{\partial \zeta}{\partial x} \\[2mm] \dfrac{\partial v}{\partial t} + fu = -g\dfrac{\partial \zeta}{\partial x}, \end{array}\right\} \qquad (1)$$

and the continuity equation is

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}(hu) - \frac{\partial}{\partial y}(hv); \qquad (2)$$

here ζ is the wave height, u and v are the horizontal particle velocities in the x and y direction, and f is the Coriolis frequency, assumed to be constant. Let the bottom topography consist of a shelf region $(0 \leqslant y \leqslant \varDelta)$ of depth h_1 and of an open-ocean region $(\varDelta \leqslant y \leqslant \infty)$ of depth h_2, and seek solutions of the form

$$\left.\begin{array}{l} \zeta = Z(y)\exp\left[i(kx - wt)\right] \\[1mm] u = U(y)\exp\left[i(kx - wt)\right] \\[1mm] v = V(y)\exp\left[i(kx - wt)\right], \end{array}\right\} \qquad (3)$$

where k is the longshore wave number and w is the radian frequency, which is assumed to be always positive. Then, if $k < 0$, the waves propagate in the negative x direction. The wave amplitude, Z, within each region then satisfies the equation

$$\frac{d^2 Z}{dy^2} - \nu Z = 0, \qquad (4)$$

where $\nu = k^2 - (\omega^2 - f^2)/(gh)$ for $h = h_1$ or h_2. The solutions have the form

$$Z = a\exp(-sy) + b\exp(sy) \quad \text{for} \quad s^2 = \nu, \nu > 0$$

or

$$Z = a\sin(sy) + b\cos(sy) \quad \text{for} \quad s^2 = -\nu, \nu < 0.$$

The boundary conditions require that V vanish at the coast, that Z and hV be continuous at the step, and that the solutions remain finite at infinity (Munk et al. 1964). Letting the solutions be Z_1 for the shelf region and Z_2 for the open-ocean region, the boundary conditions require:

(i) $dZ_1/dy + (fk/\omega)Z_1 = 0$ for $y = 0$;
(ii) $Z_1 = Z_2$ and $h_1[dZ_1/dy + (fk/\omega)Z_1] = h_2[dZ_2/dy + (fk/\omega)Z_2]$ for $y = \varDelta$;
(iii) Z_2 be finite for $y \to \infty$.

Two types of solutions—trapped and leaky—are possible, depending on the form of the solution in the open-ocean region.

Trapped Mode. For this mode, the solution in the open ocean has the form

$$Z_2 = \exp(-s_2 y)$$

where

$$s_2 = [k^2 - (\omega^2 - f^2)/(gh_2)]^{1/2} > 0.$$

The dispersion relationship is found to be

$$[s_2 - (fk/\omega)][s_1 \coth(s_1 \Delta) - (fk/\omega)] + (h_1/h_2)[s_1^2 - (fk/\omega)^2] = 0, \qquad (5)$$

where $s_1^2 = k^2 - (\omega^2 - f^2)/(gh_1)$ may be either positive or negative. The roots were found, numerically, for the three following topographies: (i) $\Delta = 0$, $h_2 = 4.4$ km, (ii) $\Delta = 80$ km, $h_1 = 0.6$ km, $h_2 = 4.4$ km, (iii) $\Delta = 250$ km, $h_1 = 0.6$ km, $h_2 = 4.4$ km. Topography (i) corresponds to a flat ocean depth, topography (ii) corresponds approximately to the shelf and ocean off central California, and topography (iii) corresponds to the ocean off southern California with its broad borderland region. The results are presented in Fig. 1, computed for $35°N$. For topographies (ii) and (iii), Fig. 1 shows two types of trapped-mode solutions; one solution (K) corresponds closely to the Kelvin-wave solution, the other (G) corresponds to the quasigeostrophic solution (Reid 1958). [The leaky-mode solutions (R), which correspond to reflected-wave solutions, lie above the hyperbola whose right wing is shown as the dotted line in Fig. 1; see *Leaky Modes* below.]

The Kelvin-type wave for shelf topography $(h_1 < h_2)$ and for trench topography $(h_1 > h_2)$ propagate only in the positive x direction in the northern hemisphere. For $k\Delta \ll 1$ and $f\Delta/(gh_2)^{1/2} \ll 1$, the solutions for topographies (ii) and (iii) approach the Kelvin-wave solution for topography (i), with the dispersion relationship $\omega/k = (gh_2)^{1/2}$; for $f\Delta/(gh_1)^{1/2} \gg 1$, the solutions approach the Kelvin-wave solution for an ocean of uniform depth, h_1, with the dispersion relationship $\omega/k = (gh_1)^{1/2}$.

The quasigeostrophic waves have relatively lower frequencies and higher wave numbers. Thus an approximate expression for the dispersion relationship for these waves can be obtained by letting $k^2 \gg (\omega^2 - f^2)/(gh_2)$, with the result that $s_1 \approx k$ and $s_2 \approx |k|$. Then the dispersion relationship becomes

$$\coth(k\Delta) = (f/\omega)[1 - h_1/h_2] - (|k|/k)(h_1/h_2).$$

This puts an upper limit on the frequency, $\omega \leqslant f|h_2 - h_1|/(h_2 + h_1)$, a result that depends only on the absolute sum and the difference of the step. For $k\Delta \ll 1$ and $h_1 \approx h_2$, the longshore phase velocity is approximately $\omega/k = f\Delta$ $(1 - h_1/h_2)$. The waves are then nondispersive and have speeds that are de-
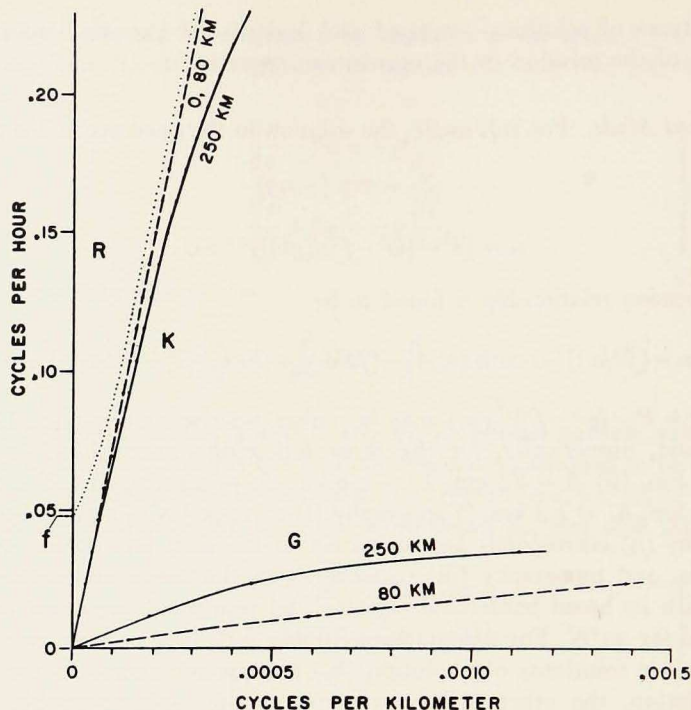
Figure 1. Diagnostic diagram for single-step topography. Plotted is frequency versus longshore wave number in the positive x direction. Reflected-wave solutions, R, lie above the dotted line. The Kelvin-type solutions are K and the quasigeostrophic solutions are G. The width of the shelf is noted; the depth of the shelf is 0.6 km while the offshore depth is 4.4 km. The Coriolis frequency, f, is noted with a value appropriate to 35°N.

pendent on the width of the shelf as well as on the magnitude of the step. For a shelf topography $(h_1 < h_2)$, these waves propagate in the positive x direction; for a trench topography $(h_1 > h_2)$, the waves propagate in the negative x direction. For a flat topography $(h_1 = h_2)$, these waves do not exist. For an infinitely wide shelf $(\Delta = \infty)$, coth $(k\Delta) = 1$. Then the dispersion relationship becomes $\omega = f | h_2 - h_1 | / (h_2 + h_1)$. This equation has been derived by Longuet-Higgins (1968), who studied in detail the trapping of waves along a discontinuity of depth in an unbounded ocean. (The waves labeled in his paper as the "double Kelvin wave" or "seascarp" wave appear to correspond, in the limit $\Delta = \infty$, to what is labeled here as the quasigeostrophic wave.)

The ratio of the components of the water motion for both types of waves in the open ocean $(y \gg \Delta)$ is

$$v/u = -i(\omega s_2 - kf)/(fs_2 - k\omega).$$

For shelf topography, $kf/\omega < s_2 < k\omega/f$ or $k\omega/f < s_2 < kf/\omega$. Thus the water particles, for the Kelvin-type wave, move in counterclockwise elliptical orbits

whereas for trench topography, the water moves in clockwise elliptical orbits. The motion for flat topography is linearly polarized. For the quasigeostrophic waves, which have shorter wavelengths, the motion has a tendency to be circularly polarized, either counterclockwise or clockwise, depending respectively on whether the wave travels in the positive or negative x direction.

Measurements of bottom currents just beyond the borderland region off southern California (Isaacs et al. 1966) have indicated, for most sites, semidiurnal tidal currents of several centimeters per second. The water particles moved in counterclockwise elliptical orbits, with the major axis parallel to the trend of the coast. The ratio of the major axis to the minor axis ranged between $1:2$ and $1:6$. These observations agree favorably with estimates of semidiurnal tidal currents, using values from coastal tide observations. The Kelvin-type solution just beyond the shelf gives semidiurnal tidal currents of 2 cm/sec; the water particles move in counterclockwise elliptical orbits, with the major axis parallel to the coast; the ratio of the major axis to the minor axis is $1:6$. However, there is a possibility that the current measurements also include internal wave motion of tidal frequency and thus confuse the interpretation.

Leaky Mode. For this mode, the solution in the open ocean has the form $Z_2 = \cos(s_2 y + \theta)$, where $s_2^2 = (\omega^2 - f^2)/(gh_2) - k^2 > 0$. This mode corresponds closely to the reflected-wave solution for an ocean of uniform depth. The dispersion relationship is found to be

$$[s_2 \tan(s_2 \Delta + \theta) - (fk/\omega)][s_1 \coth(s_1 \Delta) - (fk/\omega)] + (h_1/h_2)[s_1^2 - (fk/\omega)^2] = 0, \tag{6}$$

where $s_1^2 = k^2 - (\omega^2 - f^2)/(gh_1)$ may be either positive or negative. Discrete solutions, in terms of the longshore wave number k, do not occur, but the ratio of coastal amplitude over incident amplitude could be contoured (Munk et al. 1964); this has not been attempted here. The low-frequency cut-off is $\omega^2 = f^2 + gh_2 k^2$, which, in Fig. 1, is the dotted hyperbola above which the leaky modes lie.

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