YALE PEABODY MUSEUM

JOURNAL OF MARINE RESEARCH

The *Journal of Marine Research,* one of the oldest journals in American marine science, published important peer-reviewed original research on a broad array of topics in physical, biological, and chemical oceanography vital to the academic oceanographic community in the long and rich tradition of the Sears Foundation for Marine Research at Yale University.

An archive of all issues from 1937 to 2021 (Volume 1–79) are available through EliScholar, a digital platform for scholarly publishing provided by Yale University Library at https://elischolar.library.yale.edu/.

Requests for permission to clear rights for use of this content should be directed to the authors, their estates, or other representatives. The *Journal of Marine Research* has no contact information beyond the affiliations listed in the published articles. We ask that you provide attribution to the *Journal of Marine Research.*

Yale University provides access to these materials for educational and research purposes only. Copyright or other proprietary rights to content contained in this document may be held by individuals or entities other than, or in addition to, Yale University. You are solely responsible for determining the ownership of the copyright, and for obtaining permission for your intended use. Yale University makes no warranty that your distribution, reproduction, or other use of these materials will not infringe the rights of third parties.

Journal of
Marine Research

Nitrate Reduction and the Occurrence
of a Deep Nitrite Maximum in the Ocean
off the West Coast of South America[1]

Manuel Fiadeiro and J. D. H. Strickland

Institute of Marine Resources
University of California, San Diego
La Jolla, California

ABSTRACT

Between 10°S and 23°S off the coast of Peru, at depths where the oxygen concentration is less than 0.2 ml/l, a pronounced secondary nitrite maximum is found; for the same or greater depth range, results indicate a loss of nitrate in the water column. It has been assumed that either nitrate reduction has taken place or there has been an arrested oxidation of organic nitrogen. Results of an extensive series of analyses of many properties of water from above, within, and below the secondary nitrite maximum have given no evidence that supports acceptance of one hyposhesis over the other. However, our plots of nitrate and phosphate versus apparent oxygen utilization show that nitrate reduction is the only reasonable explanation. Nitrite appears as an intermediate or as a by-product in the irreversible reduction of nitrate, which occurs progressively in the southward-flowing undercurrent and in gyres near the Peruvian coast. The secondary nitrite production is 10–20% of the amount of nitrate reduced, and the latter accounts for an amount of organic matter that is 2–4% of the organic matter previously consumed by dissolved oxygen in the same water. Subsequent mixing of such waters with more southerly waters that contain greater quantities of oxygen leads to the oxidation of nitrite, but nitrate is not reformed completely.
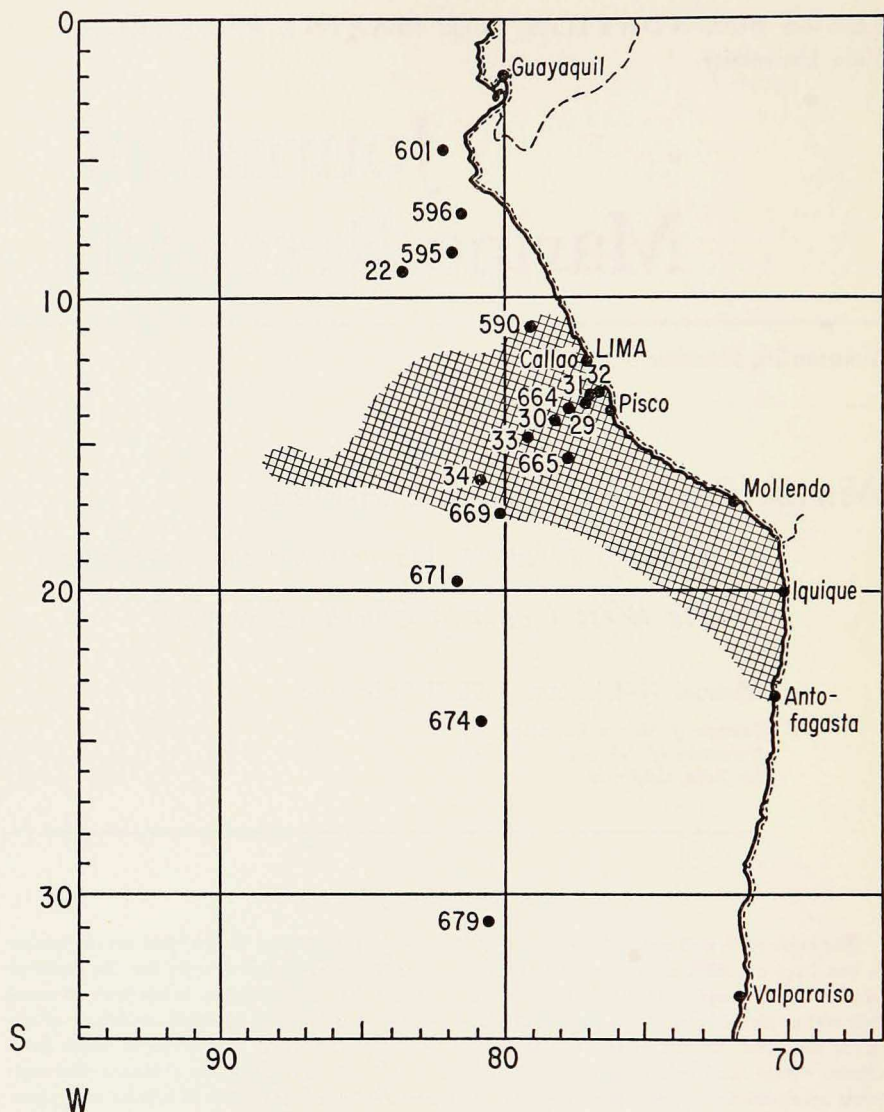
Figure 1. Station positions. The shaded area indicates the approximate location of the secondary nitrite maximum.

Introduction. A secondary maximum of nitrite in ocean waters of low oxygen content, first reported by Gilson (1937), was attributed by Brandhorst (1959) to denitrification when he discovered the phenomenon in tropical waters west of Central America. This explanation was accepted by Thomas

(1966), and such a loss had also been mentioned by Ivanenkov and Rozanov (1961) working in the Arabian Sea.

The secondary nitrite maximum, which is the lower of two separate and well-defined maxima in a nitrite-depth profile, is particularly well developed in the waters off Peru where its distribution has recently been described by Wooster et al. (1965). The shaded area in Fig. 1 shows where the maximum was found by Wooster et al. In 1966, the work described in this paper confirmed its presence in the same area, with some indication of its presence as far north as 8°S.

An interesting aspect of this feature off Peru is that, although low oxygen concentrations appear to be a prerequisite for its development (less than about 0.2 ml/l), the secondary maximum is not seen much north of 9°S, even though there is still a well-developed oxygen minimum present.

Thomas (1966), in his study of waters north of the equator, used an assumed nitrogen-to-phosphorus ratio to reconstruct nitrate profiles and showed that, at a station where the secondary nitrite layer was well developed, the observed nitrate (and the nitrate-plus-nitrite) concentrations fell significantly short of the calculated values. Thomas found very little ammonia in the water, and this is also the case off Peru. Such an overall apparent nitrogen loss was taken as support of Brandhorst's contention that denitrification was taking place, presumably with production of nitrogen gas or perhaps nitrous oxide. The reconstruction of "theoretical nitrate profiles" from phosphate profiles, using such a simple approach, may be open to some critism, but the main trouble is that an inspection of such profiles does not conclusively settle the problem of whether nitrate is being destroyed or is being produced less rapidly than normal in the upper part of the oxygen minimum.

Despite the fact that nitrite is known to be formed by the biological reduction of nitrate, the concentration of particulate and dissolved organic matter in the open sea is generally so low that an alternative explanation, namely the incomplete oxidation of nitrogenous matter, would be attractive.

We therefore sought additional evidence in support of one or other hypothesis by carrying out extensive analyses of water from above, below, and within the secondary nitrite layer and by closely examining oxygen and nitrate-nitrite profiles of the Peru Current from 5°S to 30°S. The most definitive evidence could come from the use of isotopically labeled nitrogen but, so far, denitrification has not been demonstrated in the open sea off Peru (Goering and Dugdale 1966), possibly because of the slowness of any such reaction.

Observations and Treatment of Data. In this study we have used the data obtained from a cruise of the Peruvian vessel B.A.P. Unáne during April 1966 (University of California 1967) as well as some of the data obtained on two cruises by the Anton Bruun in the same area at the same time (Texas

A & M University 1966, 1967). However, for the sake of brevity, only the representative stations shown in Fig. 1 will be discussed in detail.

In addition to the standard hydrographic observations, we measured the following at certain stations: oxygen, phosphate, silicate, nitrate, nitrite, and ammonia; dissolved and particulate organic carbon, nitrogen, and phosphorus; particulate carbohydrate; biotin, thiamine, and vitamin B_{12}; adenosine triphosphate; also, bacteria were counted. Since detailed analyses of samples from above, below, and within the secondary nitrite maximum have been reported (University of California 1967), they will not be repeated here. In all cases studied, the presence of nitrite was not associated with any indication of the presence of excessive organic matter or an abnormal bacterial flora. An exception may have been the occurrence of a slightly increased amount of dissolved organic nitrogen, but there were insufficient data for this to be established with certainty.

It was necessary, therefore, to look for other evidence to settle the question of whether a loss of nitrate and the formation of nitrite result from nitrate reduction or an arrested oxidation of organic nitrogen. Such evidence will be deduced from a consideration of nitrate and phosphate (cf. Thomas 1966) but adding also a study of the apparent oxygen utilization (AOU).

The AOU concept and the assumptions underlying its use have been described (Redfield 1934, Redfield et al. 1963). Except in a recent paper by Park (1967), little use has been made of AOU plots for the Pacific. In this paper it is assumed that every point on a plot of phosphate or nitrate versus AOU is derived from two distinct factors: (i) the amount of phosphate or nitrate present when water sinks from the surface mixed layer (preformed nutrient) and (ii) an average ratio of the produced phosphate or nitrate to the oxygen consumed during the mineralization of organic matter in the same body of water. Although oversaturation and undersaturation of surface waters with oxygen can result in unpredictable variations, we assume that these can be neglected. We have also neglected all deviations due to a lack of linearity in oxygen-salinity-temperature relationships as to the slight differences in various oxygen solubility tables that may be used for calculations.

There is no *a priori* reason why plots of phosphate or nitrate versus AOU should be linear, but if linear portions should be encountered, it seems reasonable to assume that the samples encompassed have the same origin and average mineralization ratios and that these values can be estimated from the slopes of the lines and their intercepts at an AOU value of zero. From the surface to a depth near the oxygen minimum, AOU values increase with depth; in deeper waters the values decrease again in oxygenated water that has received progressively less organic detritus from the upper layers (Wyrtki 1962).

Results and Discussion. In Fig. 2 we have illustrated the normal nutrient-versus-AOU relationships to be expected in the Pacific (cf. Park 1967). The
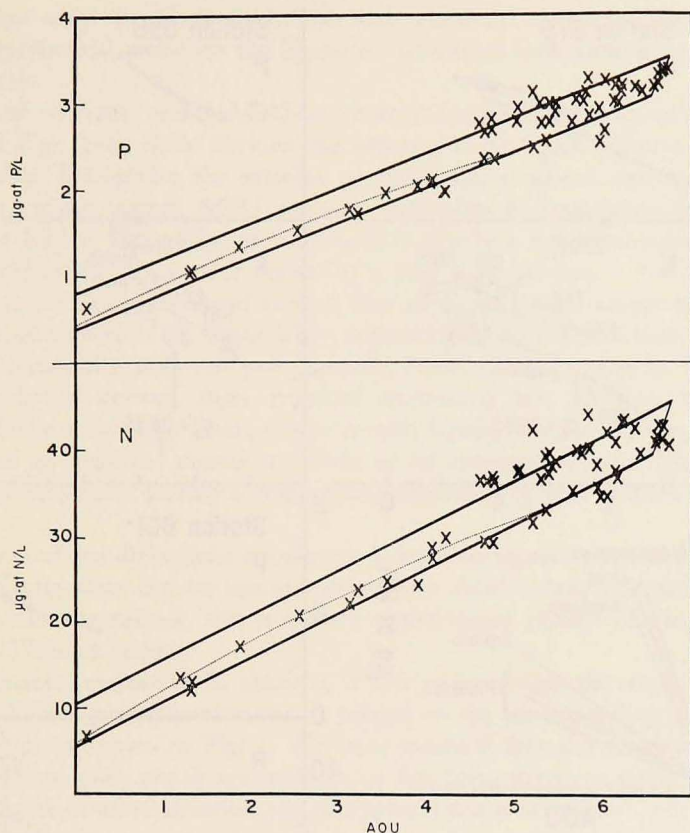
Figure 2. Normal plots of $NO_3^{1-}-N$ and $PO_4^{3-}-P$ versus AOU (ml O_2/l) for the Pacific Ocean. The lower line is for depths between about 100 m and 750 m, the upper line for depths between about 1000 m and 3000 m.

points cluster fairly well around two parallel straight lines, the lower line representing samples taken above about 750 m, the other line representing samples taken below about 1000 m. It is seen that the ratio of consumed oxygen to produced nitrate or phosphate is not, on an average, very different in deep or shallow water, but the concentration of preformed nutrient appears to be higher in the deeper water. A slight curvature is generally seen in the plots of samples of low AOU taken in the upper few hundred meters (dotted lines, Fig. 2). We believe that this curvature represents a real condition and is probably the result of high nitrate and phosphate liberation during the initial mineralization of fresh plankton.

The estimated amounts of preformed nitrate and phosphate (Fig. 2) have an atomic ratio of about 13:1 in deep water and of somewhat less in shallower water, but the extrapolations make these figures very approximate.
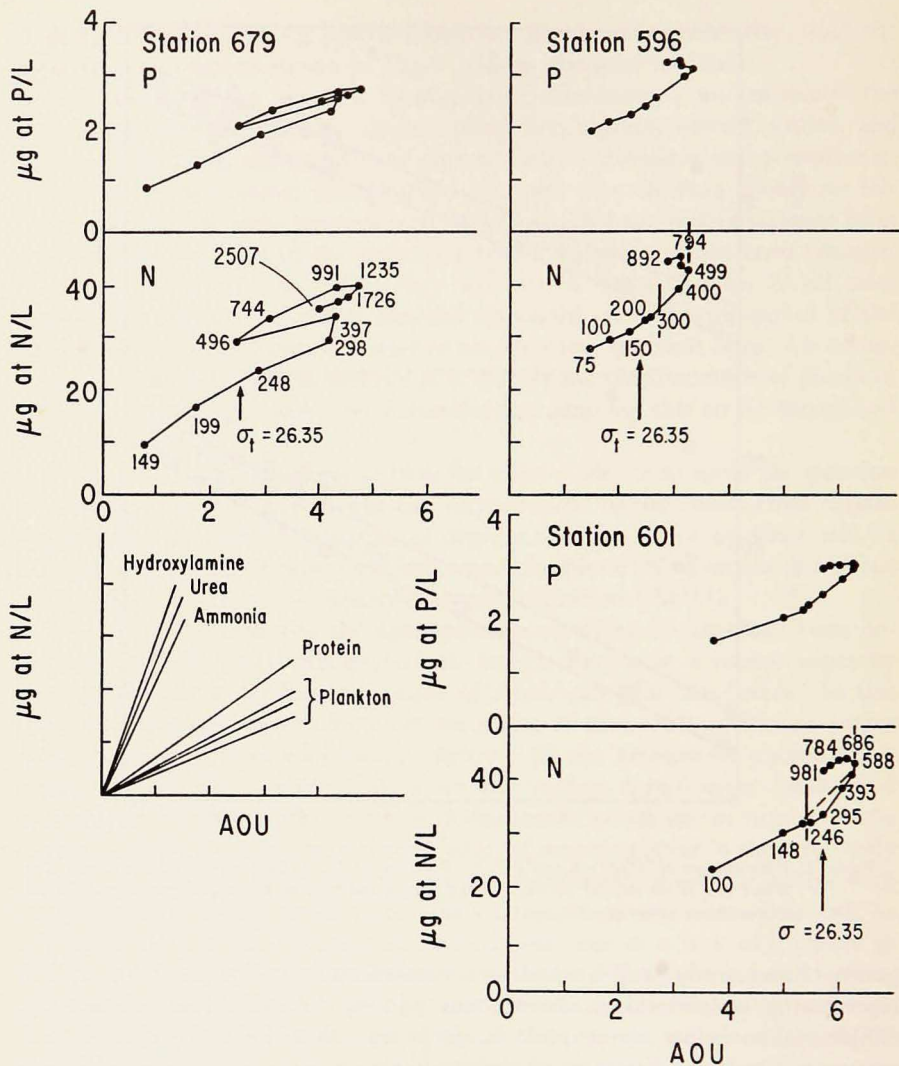
Figure 3. $NO_3^- - N$ and $PO_4^{3-} - P$ versus AOU (ml O_2/l) for three stations where little if any nitrate reduction occurred. The depth (m) of samples is shown in the N diagrams. Thick vertical lines mark depth range where there was less than 0.2 ml O_2/l. The arrow points to a section of curve where $\sigma_t = 26.35$. (The bottom left diagram shows theoretical slopes for the production of nitrate by the oxidation of various nitrogenous materials; the scales are the same as in the other diagrams.)

It we assume that organic matter is oxidized with a respiratory quotient of 0.77 (Redfield et al. 1963), then the slopes of the lines in Fig. 2 should indicate a mineralization of plankton with an average compositional ratio of C:N:P by *weight* equal to 100:17.2:3.0 (Holm-Hansen et al. 1966). In Fig. 3 are

shown not only the likely maximum and minimum slopes for plankton but some hypothetical ratios for the liberation of nitrate from various nitrogenous compounds.

Plots of nutrients versus AOU for most of the stations shown in Fig. 1 are given in Figs. 3–6; these stations encompass a north-south distance of nearly 2000 miles. Except for the extreme northern and southern stations (Fig. 3), plots for nitrate against AOU are quite dissimilar in appearance from those found in Fig. 2. Generally, at intermediate depths a near-horizontal portion is followed at slightly greater depths by a very steep portion.

Such traces can be accepted at their face value only if we accept one of two unlikely assumptions: (i) water layers separated by only a few tens of meters have had radically different past histories, with mineralization in the intermediate layers derived from material containing less nitrogen than that in the shallow or deep water; (ii) in certain layers, oxygen has recently been consumed by material containing little or no nitrogen followed by the oxidation of substances having a composition approaching that of pure ammonia or urea.

In view of the difficulties encountered, we investigated whether reasonable AOU-nitrate plots can be reconstructed from AOU-phosphate plots for the same station. In essence, this is a more sophisticated version of the approach used by Thomas (1966).

Reasonable results can be obtained if it is assumed that the slope of nitrate on AOU during remineralization is related to the corresponding phosphate-AOU slope as shown in Fig. 2. We have assumed that the phosphate-AOU slopes in Figs. 3–6, which are equal to, or less than, the slope given in Fig. 2, are indicative of mineralization and that greater slopes arise from both mineralization and an increase in preformed phosphate. Any such increase in preformed phosphate would presumably be reflected in a corresponding increase in preformed nitrate (with an atomic ratio around 13:1; say, between 10:1 and 15:1).

By starting with depths of about 1000 m and working back depth by depth toward the surface, it is possible to reconstruct nitrate-AOU curves in such a way that the reconstructed and observed points again coincide between 50 m and 100 m. (At depths of less than about 50 m for most stations the effect of surface oxygenation on AOU plots is clearly seen, and such points have been omitted in Figs. 3–6.)

The reconstructed nitrate curves are indicated with dashed lines in Figs. 4–6. At Sts. 679 and 596 (Fig. 3) and at some other stations not shown, there was no difference between the found and reconstructed nitrate points. Such agreement gives us confidence in the general validity of the reconstruction procedure. For St. 601 (Fig. 3), the difference between the reconstructed and found curves was marginal.

At many other stations, representatives of which are shown in Figs. 4–6,
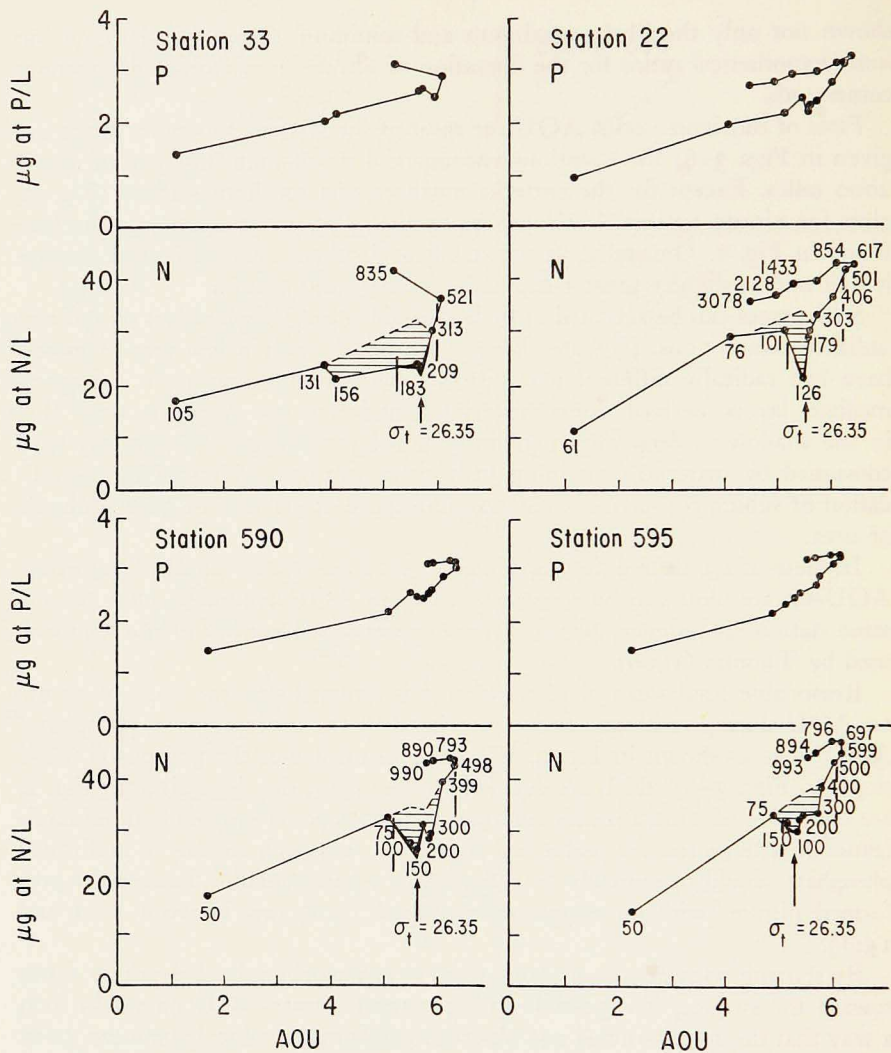
Figure 4. NO_3^--N and $PO_4^{3-}-P$ versus AOU (ml O_2/l) for four stations where nitrate reduction occurred but where there was little nitrite formed. The depth (m) of samples is shown in the N diagrams. Thick vertical lines mark the depth range where there was less than 0.2 ml O_2/l. The arrow points to a section of curve where $\sigma_t = 26.35$. The shaded area shows where nitrate has been reduced, the blackened area where nitrite was found.

there were large differences between the found and reconstructed nitrate-AOU plots over a depth range of several hundred meters. This can most reasonably be explained if we assume (i) that the dashed lines show the initial nitrate plots, derived from mineralization of the plankton, and (ii) that nitrate reduction
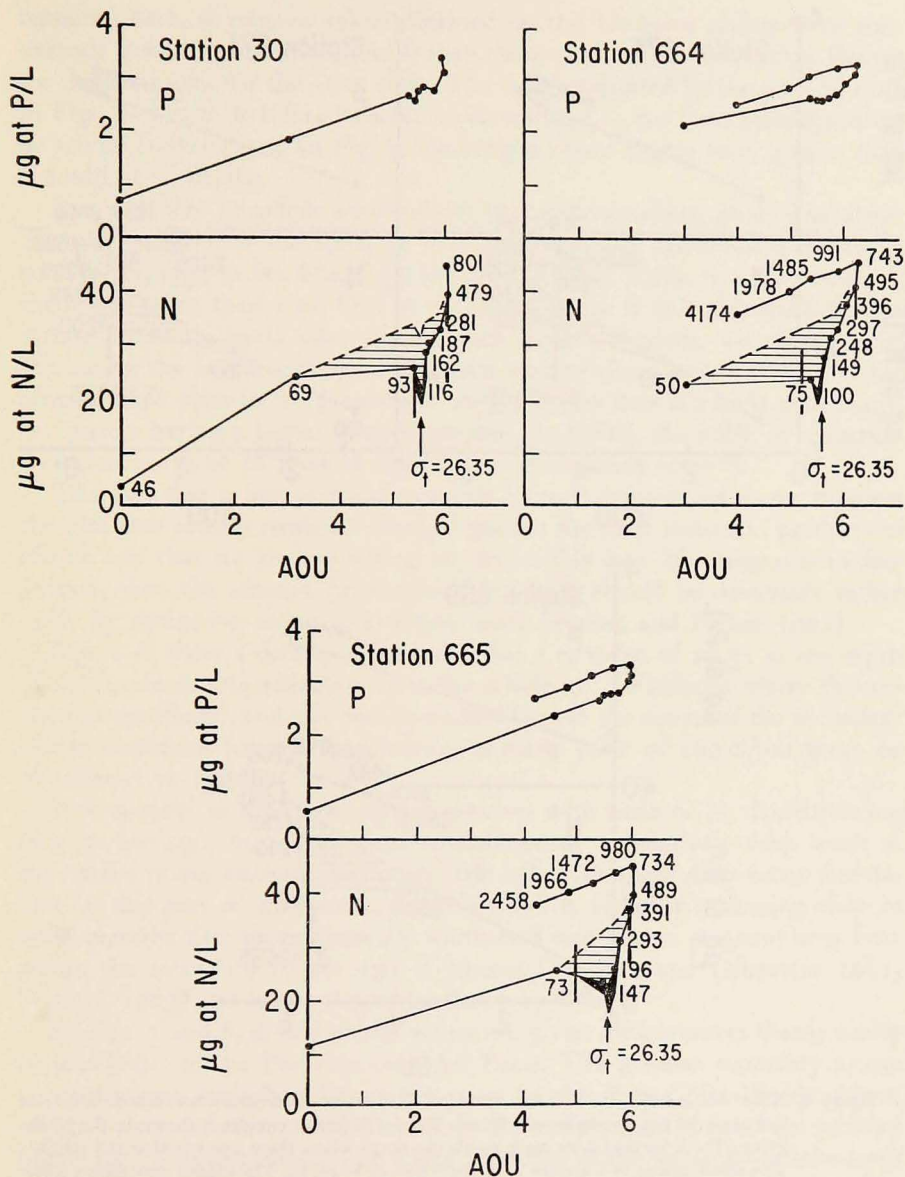
Figure 5. NO_3^--N and PO_4^{3-}-P versus AOU (ml O_2/l) for three stations where there was heavy nitrite production. The depth (m) of samples is shown in the N diagrams. Thick vertical lines mark the depth range where there was less than 0.2 ml O_2/l. The arrow points to a section of curve where $\sigma_t = 26.35$. The shaded area shows where nitrate has been reduced, the blackened area where nitrite was found.
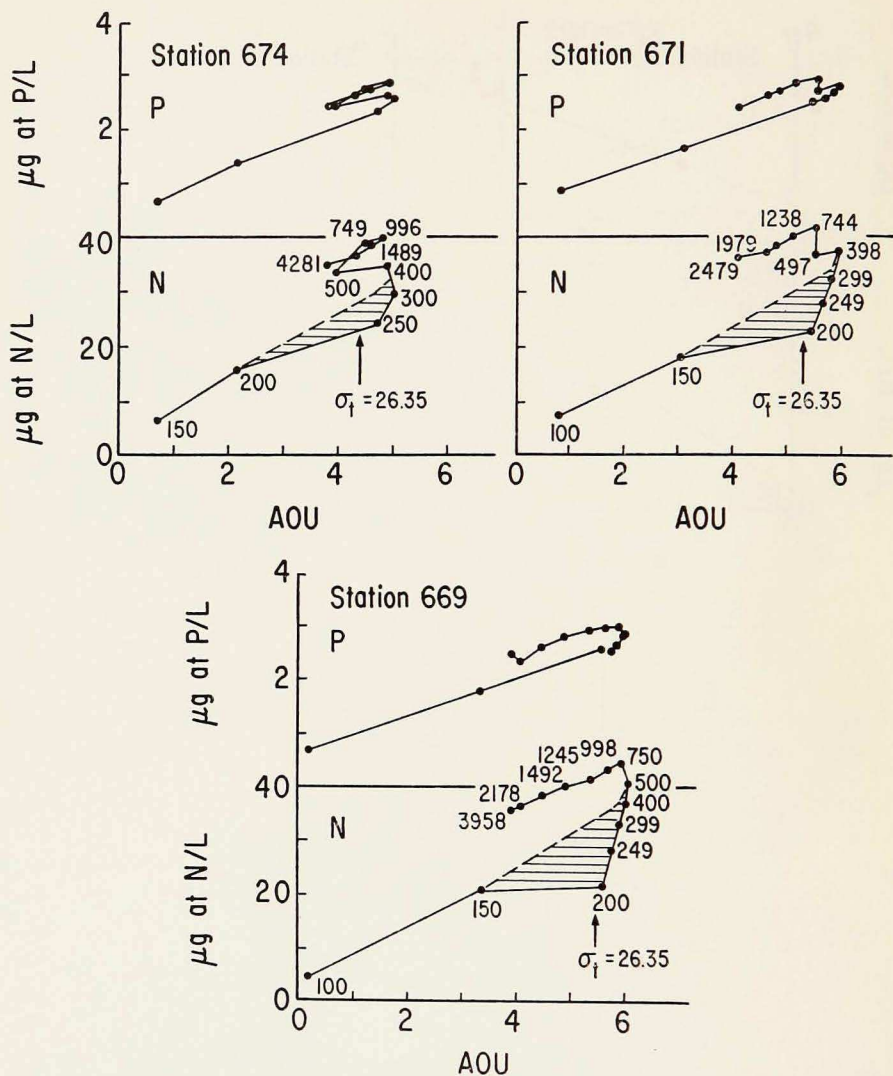
Figure 6. $NO_3^{1-}-N$ and $PO_4^{3-}-P$ versus AOU (ml O_2/l) for three stations where nitrate reduction had occurred but no nitrite was found. The depth (m) of samples is shown in the N diagrams. Thick vertical lines mark depth the range where there was less than 0.2 ml O_2/l. The arrow points to a section of curve where $\sigma_t = 26.35$. The shaded area shows where nitrate has been reduced.

subsequently took place to a degree shown by the shaded areas (the darker shading indicating nitrite in the secondary nitrite maximum).

The range of depths at each station where oxygen concentrations were less than 0.2 ml/l is indicated in Figs. 3–6 with vertical lines. There is some un-

certainty here, as oxygen values obtained on the UNÁNUE cruises were consistently lower (by about 0.05 ml/l) than those obtained by the ANTON BRUUN for the same area for the same time. The range indicated by the vertical lines in Figs. 3–5 may therefore be a bit conservative, i.e., the lines should perhaps be a little farther apart. In Fig. 6, the oxygen values shown were greater than 0.2 ml/l at all depths.

It is seen that no nitrite accumulated in water containing more than about 0.2 ml O_2/l whereas the result of considerable nitrate reduction is shown by such water, e.g., at Sts. 674, 671, and 669 (Fig. 6). At St. 674, for example, there was more than 1 ml O_2/l at all depths. Since it is highly unlikely that nitrate reduction could take place at such oxygen tensions, we appear to be witnessing the lateral mixing of northern water (where nitrate reduction has already taken place in the presence of oxygen of less than 0.2 ml/l) with southern waters having a higher oxygen contant. At 80°W, the result of reduction in waters north of 18°S could still be seen as far south as 26°S.

The fact that a loss of nitrate persists in such oxygenated water supports the idea that nitrous oxide or nitrogen gas are the final reduction products of nitrate and that we are witnessing an irreversible loss. If nitrogen is in fact formed, then the amount (in excess of 0.1 ml/l) should be detectable rather easily by measuring nitrogen-to-argon ratios (Benson and Parker 1961).

Figs. 4–6 show a density surface that has a σ_t value of 26.35 at the depth of maximum nitrate reduction over the whole 20° of latitude where this reduction was found, and this density surface is near the center of the secondary nitrite maximum (cf. the thermosteric anomaly value of 160 cl/ton given by Wooster et al. 1965).

It is possible to have low oxygen tensions with little or no denitrification (Fig. 3: Sts. 496, 601), but these tensions occur at relatively deep levels at the northernmost stations. Although little is known about deep-water circulation in this part of the ocean, denitrification is probably occurring only in undercurrents that move generally southward and have a stagnant area near where the secondary nitrite area is located (Wooster and Gilmartin 1961, Wyrtki, 1963) and where coastal productivity is high.

In Figs. 7 and 8, σ_t and nitrite values are given for a transect that is nearly perpendicular to the Peruvian coast off Pisco. The greatest secondary nitrite concentrations were found near the coast in the Peru-Chile Undercurrent, and this secondary nitrite layer is always distinctly separated from the primary nitrite layer (broken lines). The upper boundary of the secondary nitrite layer coincides with the depth where there is an increase in dissolved oxygen to values in excess of about 0.2 ml/l.

In Table I we present calculations of several parameters of nitrate reduction for selected stations. These quantities are only approximate because of the extrapolations and interpolations that have had to be made from the limited number of water samples taken in hydrographic casts at each station. Table I
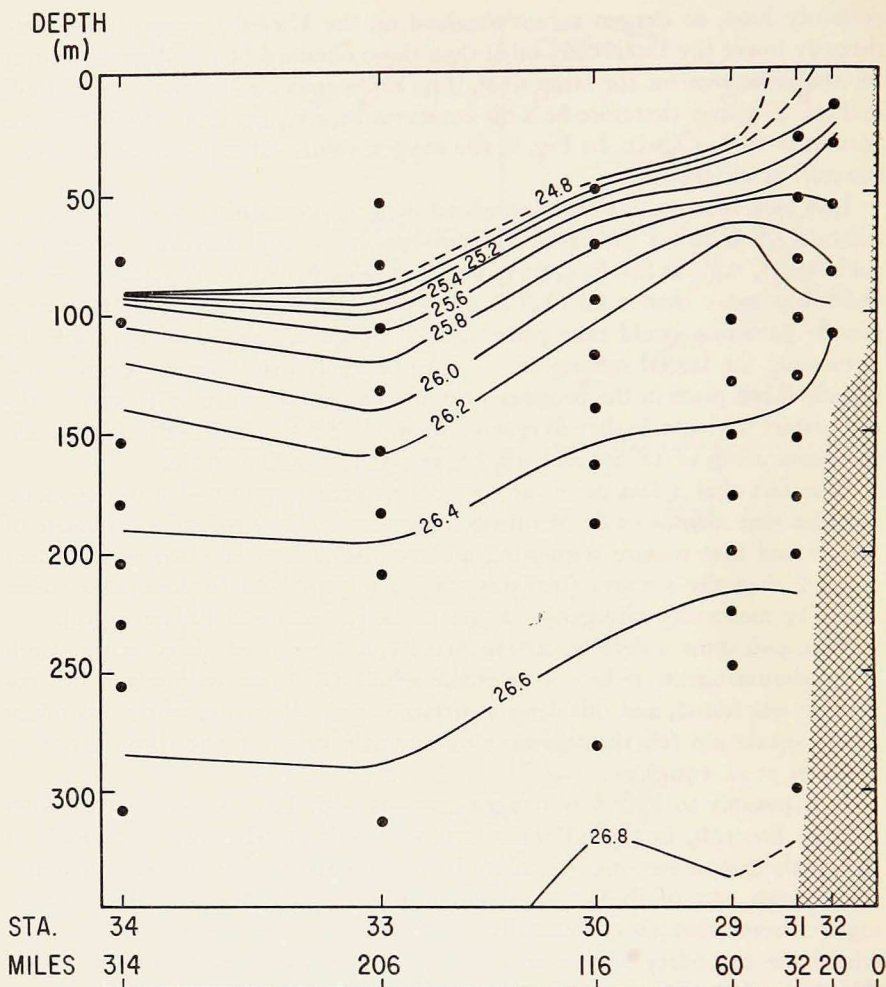
Figure 7. Transect off Pisco showing σ_t values.

shows (i) the total quantity of nitrate lost (denitrification?), (ii) the amount of nitrite produced, (iii) the calculated nitrite production as a function of total nitrate reduced in the same depth range, and (iv) the amount of organic matter used in reducing nitrate as a fraction of the organic matter consumed to reduce dissolved oxygen *over the same depth range*; for this calculation we have assumed that 106 atoms of carbon are oxidized for the reduction of 71 molecules of nitrate (Redfield et al. 1963).

The amount of nitrate reduction is greatest near the coast in the general area where the secondary nitrite maximum is found. Although nitrite is pro-
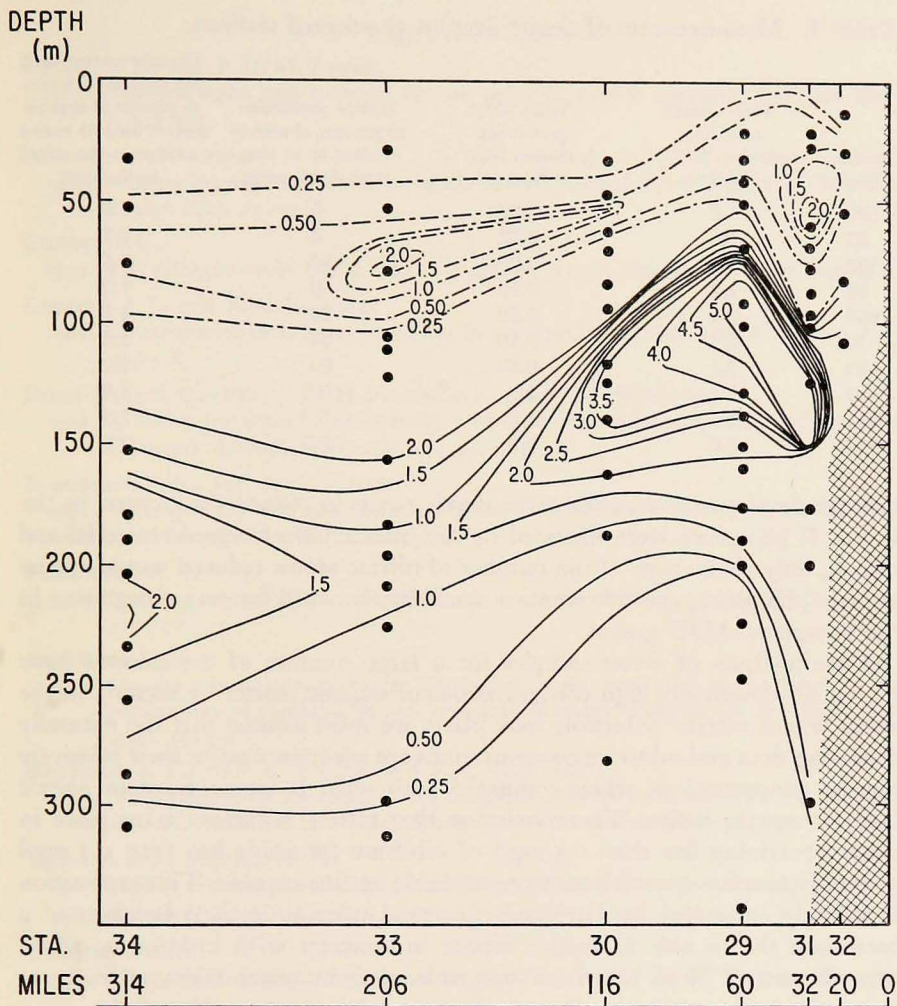
Figure 8. Transect off Pisco showing both primary and secondary nitrite layers. Primary nitrite is shown with broken lines.

duced north of the area indicated (Fig. 1), in profile the amount appears to be small. North of 9°S the percentage of nitrate reduced to nitrite is less than in areas of more intense denitrification (ca. 10% compared with ca. 25%), and the total amount of nitrate reduced is also much less. Furthermore, it appears that the higher local concentrations of nitrite are correlated with the greater extent of total nitrate reduction; again, this tends to accentuate the amounts of nitrite indicated in profiles south of about 9°S.

The amount of organic matter oxidized by nitrate is only a few percent of

Table I. Measurements of denitrification at selected stations.

Station no.	Total nitrate reduction (g-moles · NO_3^- – N/m^2)	Total NO_2^- production (g-moles · NO_2^- – N/m^2)	Nitrite production as percent of nitrate reduction in the same depth range	Organic matter used for nitrate reduction as percent of organic matter used to reduce oxygen in the same depth range
595	1.2	0.025	8	2.2
22	0.6	0.02	8	2.2
590	1.4	0.06	10	2.6
33	1.2	0.10	9	4.0
664	2.4	0.30	24	4.1
30	2.3	0.40	24	3.6
665	2.1	0.30	20	3.2
669	2.1	Nil	Nil	2.9
671	1.2	Nil	Nil	2.9
674	0.6	Nil	Nil	2.5

that previously used over the same depth range to reduce the oxygen in the water. If phosphate were liberated by the interaction of organic material and nitrate, only about 1/70 of the number of nitrate atoms reduced would appear as extra phosphate, and this amount would be too small for easy recognition in the phosphate-AOU graphs.

Since analyses of water samples for a large number of constituents have shown no abnormally high concentrations of organic matter or bacteria in the water where nitrate reduction took place, we must assume that the naturally occurring flora and substrate concentrations are adequate despite their relatively minute concentrations when compared with what is encountered in anoxic lakes or marine basins. The conclusion that nitrate reduction takes place in water containing less than 0.5 mg/l of substrate (probably less than 0.1 mg/l of readily reactive material) seems remarkable but inescapable. This supposition is strongly supported by Carlucci's (personal communication) isolation of a bacterium that is able to reduce nitrate in seawater with little or no added organic matter. [The bacterium was isolated from water taken at St. 34 at 240 m.]

REFERENCES

BENSON, B. B., and P. D. M. PARKER
  1961. Nitrogen/argon and nitrogen isotope ratios in aerobic seawater. Deep-sea Res.,
    7: 237–253.

BRANDHORST, WILHELM
  1959. Nitrification and denitrification in the eastern tropical North Pacific. J. Cons. int.
    Explor. Mer, 25: 3–20.

GILSON, H. C.
  1937. The nitrogen cycle. Chem. Phys. Invest., pp. 21–81. Sci. Rep. Murray Exped., 2.

GOERING, J. J., and R. C. DUGDALE
  1966. Denitrification rates in an island bay in the equatorial Pacific Ocean. Science, 154:
    505–506.

HOLM-HANSEN, OSMUND, J. D. H. STRICKLAND, and P. M. WILLIAMS
  1966. A detailed analysis of biologically important substances in a profile off southern
    California. Limnol. Oceanogr., 11: 548–561.

IVANENKOV, V. N., and A. G. ROZANOV
  1961. Hydrogen sulphide contamination of the intermediate waters of the Arabian Sea
    and Bay of Bengal. Okenologiia, 1: 443–449.

PARK, KILHO
  1967. Nutrient regeneration and preformed nutrients off Oregon. Limnol. Oceanogr.,
    12: 353–357.

REDFIELD, A. C.
  1934. On the proportions of organic derivatives in sea water and their relation to the
    composition of plankton. James. Johnstone Memorial Volume, pp. 177–192. Liver-
    pool. 348 pp.

REDFIELD, A. C., B. H. KETCHUM, and F. A. RICHARDS
  1963. The influence of organisms on the composition of sea water. In: The Sea. 2, pp. 26
    –77. M. N. Hill, editor. Interscience, New York and London. 554 pp.

TEXAS A & M UNIVERSITY
  1966. Report of Cruise 15 of R. V. Anton Bruun. Preliminary publication. Spec. Rep. 5.
  1967. Report of Cruise 17 of R. V. Anton Bruun. Preliminary publication. Spec. Rep. 7.

THOMAS, W. H.
  1966. On denitrification in the northeastern tropical Pacific Ocean. Deep-sea Res., 13:
    1109–1114.

UNIVERSITY OF CALIFORNIA
  1967. Research on the marine food-chain. Progress Report (of the Institute of Marine
    Resources) January 1966–December 1966, Part III. Data record and comments,
    Cruise FCG 66/1.

WOOSTER, W. S., and MALVERN GILMARTIN
  1961. The Peru-Chile Undercurrent. J. mar. Res., 19: 97–122.

WOOSTER, W. S., T. J. CHOW, and IZADORE BARRETT
  1965. Nitrite distribution in Peru Current waters. J. mar. Res., 23: 210–221.

WYRTKI, KLAUS
  1962. The oxygen minima in relation to ocean circulation. Deep-sea Res., 9: 11–23.
  1963. The horizontal and vertical field of motion in the Peru Current. Bull. Scripps Inst.
    Oceanogr., 8: 313–346.

Seiche Motions for One-dimensional Flow[1]

D. J. Clarke

Department of Mathematics
Wollongong University College
Wollongong, N.S.W., Australia

ABSTRACT

The linearized equations of motion for the case of one-dimensional flow are solved as a sequence of approximations for the modes of oscillation of a fluid in an enclosed basin by using Galerkin's method from the calculus of variations.

Introduction. Seiches (resonant standing waves) in longitudinal basins of variable depth have been the subject of complete theoretical investigations for the case in which the motion is considered as a one-dimensional flow. The main methods have been presented by Chrystal (1906), Proudman (1914), Defant (1918), and Hidaka (1936). The purpose of this paper is to present a new method for discussing one-dimensional oscillations.

Formulation of the Problem. For an elongated basin it is convenient to choose the origin o at one end of the basin in the free undisturbed surface having axes ox along the basin, oy transverse, and oz vertically. Assume that transverse motion across ox is negligible. If ξ is considered to be the horizontal displacement of a fluid particle from its mean position and u its horizontal velocity, then

$$u = \dot{\xi}. \qquad (1)$$

Suppose that ζ is the vertical displacement of a particle in the surface above its equilibrium position. With an assumption of no transverse motion, ξ and ζ are constant over any plane perpendicular to ox.

If it is assumed that the vertical accelerations are negligible and that the wave heights, ζ, are much smaller than the depth, $h = h(x,y)$, the linearized equation of motion is

$$\dot{u} = -g\frac{\partial \zeta}{\partial x}, \qquad (2)$$

where g is the acceleration due to gravity. The latter assumption leads to the equation of continuity for an incompressible fluid in the form

$$\frac{\partial}{\partial x}\{\xi A(x)\} = -\zeta b(x), \tag{3}$$

where $A(x)$ is the area of cross section and $b(x)$ is the breadth of the undisturbed surface in a transverse plane through the fluid distant x from o.

By replacing u and ζ in (2) and using (1) and (3),

$$\left. \begin{aligned} \ddot{\xi} &= g\frac{\partial}{\partial x}\left[\frac{1}{b(x)}\frac{\partial}{\partial x}\{A(x)\xi\}\right] \\ &= g\,b(x)\frac{1}{b(x)}\frac{\partial}{\partial x}\left[\frac{1}{b(x)}\frac{\partial}{\partial x}\{A(x)\xi\}\right]. \end{aligned} \right\} \tag{4}$$

By introducing the transport variable

$$\alpha = A(x)\,\xi \tag{5}$$

and the fluid surface-area variable

$$\chi = \int_0^x b(x')\,dx' \tag{6}$$

so that

$$\frac{1}{b(x)}\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial \chi},$$

eq. (4) becomes

$$\ddot{\alpha} = g f(\chi)\frac{\partial^2 \alpha}{\partial \chi^2}, \tag{7}$$

where

$$f(\chi) = A(x)\,b(x). \tag{8}$$

To find the frequencies, σ, of normal modes of oscillation, put

$$\alpha = \alpha^* e^{i(\sigma t + \varepsilon)}, \tag{9}$$

where

$$\alpha^* = \alpha^*(x) \quad \text{or} \quad \alpha^*(\chi) \quad \text{only};$$

eq. (7) then becomes

$$\frac{d^2 \alpha^*}{d\chi^2} + \frac{\sigma^2}{g f(\chi)}\alpha^* = 0. \tag{10}$$

For convenience, the $*$ on α will be omitted. The solution of (10) must satisfy boundary conditions at the ends of the basin. If the basin boundaries are vertical, then, for all time t,

$$\xi = 0 \quad \text{when} \quad x = 0, l, \tag{11}$$

where l is the length of the basin.

Thus, $\alpha = 0$ when $\chi = 0$, χ_l, (12) where χ_l is the total surface area of the undisturbed surface and is given by

$$\chi_l = \int_0^l b(x)\, dx. \tag{13}$$

Approximate Solution of the Problem. The method of solving (10) subject to (12) is based on the Galerkin method in the calculus of variations (Kantorovich and Krylov 1958: 258–262).

To solve the operator equation $L(\alpha) = 0$ under appropriate boundary conditions, approximate solutions are developed in the form

$$\alpha_n(\chi) = \chi(\chi - \chi_l)(a_1 + a_2\chi + a_3\chi^2 + \ldots + a_n\chi^{n-1})$$
$$= \sum_n a_i\varphi_i, \tag{14}$$

where each function φ satisfies the conditions of (12) and are continuous over the domain $[0, \chi_l]$, with the orthogonality conditions

$$\int_0^{\chi_l} L(\alpha_n)\varphi_j\, d\chi = 0, \quad j = 1, 2, \ldots, n. \tag{15}$$

The coefficients of the a_i are computed for each j value and are of the form $-\mu_{ij} + \lambda\nu_{ij}$, where $\lambda = \sigma^2/g$. Square matrices A and B of order $n \times n$ are defined by $A = (\mu_{ij})$ and $B = (\nu_{ij})$, and the integrated system of (15) is $(A - \lambda B)x = 0$, where x is the column vector of the a_i.

For these equations to be consistent, $|A - \lambda B| = 0$, hence the λ and x are eigen-values and eigen-vectors of $B^{-1}A$.

The degree n of the approximating function need not be large to obtain reasonable accuracy, subject only to the assumptions made for linear flow. Once n is chosen, the matrices A and B are calculated. The integrals may have to be evaluated numerically if the product $A(x)\, b(x)$ is not expressible as a directly integrable function. From the matrices, a sequence of approximations to λ and x and hence to σ and α may be determined; but x and α are only found to within an arbitrary multiplying constant. The wave height may then be obtained from (3).

When $n = 1$, the transport is approximated by a second-degree polynomial and the matrices A and B each have one element. Thus one value is found for λ, this being the first approximation to the fundamental mode of oscillation in the basin. As n increases, the transport is approximated by higher degree polynomials; the number of modes approximated also increases and all previous modes are determined to a higher precision.

Thus the method yields sequences of approximations to the frequency (hence to the period), to the horizontal transport, and to the wave height for various modes of oscillation in an elongated basin.

Numerical Illustration. This process has been applied to three particular cases whose solutions are known from diverse techniques. They are the cases of oscillations in a basin of constant depth, where the surface shape is (i) rectangular, (ii) elliptic, and (iii) spindle-shaped; the latter name has been used by Hidaka for a basin whose surface is formed by the intersection of two confocal parabolae.

RECTANGULAR SURFACE. For this basin, the differential equation has an analytic solution. The frequency, σ, of the *nth* harmonic is given by

$$\sigma = n\,\pi\,[(gh)^{1/2}/l],$$

where h is the uniform depth.

The transport, α, of the *nth* harmonic is given by

$$\alpha = A \sin \frac{n\pi x}{l} \cos \sigma t,$$

with nodes at the positions where $d\alpha/dx = 0$ for all t.

The approximate solution for oscillations in this basin given by the Galerkin method are as follows.

Let N denote the degree of the polynomial approximation to the transport α; then, for the fundamental,

N	$\dfrac{\sigma l}{(gh)^{1/2}}$	α^*	Nodes
2	3.162	$Ax(l-x)$	$0.5l$
4	3.14161	$Ax(l-x)(l^2+1.132lx-1.132x^2)$	$0.5l$.

For the second harmonic,

N	$\dfrac{\sigma l}{(gh)^{1/2}}$	α^*	Nodes
3	6.481	$Ax(l-x)(l-2x)$	$0.211l, 0.789l$
5	6.285	$Ax(l-x)(l^3+0.634l^2x-8lx^2+5.43x^3)$	$0.237l, 0.763l$.

ELLIPTIC SURFACE. When the oscillations are considered to be a one-dimensional flow problem, solutions have been obtained by Goldstein (1928). The first five frequencies are given as follows:

$$\frac{\sigma(gh)^{1/2}}{a} = 1.8866,\ 3.4866,\ 5.0699,\ 6.6477,\ 8.2231,$$

where a is the length of the semimajor axis.

The Galerkin method yielded the following frequencies. (The degree of the polynomial approximation to the transport α is 7 in each case.)

$$\frac{\sigma(gh)^{1/2}}{a} = 1.8993,\ 3.5163,\ 5.1158,\ 6.7037,\ 8.4658.$$

In this case the integration was carried out numerically, using Simpson's rule applied to 20 intervals.

SPINDLE-SHAPED SURFACE. The one-dimensional flow problem was solved by Hidaka (1931) for the basin having a surface shape defined by

$$y^2 = a(a - 2x),\quad x \geqslant 0,$$
$$y^2 = a(a + 2x),\quad x < 0.$$

The first five frequencies are given as follows:

$$\frac{\sigma(gh)^{1/2}}{a} = 2.1499,\ 3.7900,\ 5.3931,\ 6.9830,\ 8.5666.$$

The Galerkin method gave the following frequencies:

$$\frac{\sigma(gh)^{1/2}}{a} = 2.1531,\ 3.7936,\ 5.3894,\ 6.9629,\ 8.5158.$$

Conclusion. It has been demonstrated in the above illustrations that the Galerkin method is an easy-to-apply technique that is valid over a wide range of basin shapes. As long as the flow in a transverse direction is negligible, satisfactory approximations may be obtained for the modes of oscillation in various basins. It must be remembered that the flow equations are linearized, hence basin shapes that do not allow such simplification cannot be solved in this way. A particular feature to be emphasized is the nature of the solution which gives both transport and wave height (hence nodes) as well as the frequencies of various modes.

REFERENCES

CHRYSTAL, GEORGE
   1906. On the hydrodynamical theory of seiches. Trans. roy. Soc. Edinb., *41*: 599–649.

DEFANT, ALBERT
   1918. Neue methode zur ermittlung der eigenschwingungen von abgeschlossenen wasser-massen. Ann. Hydrogr. Mar. Meteorol., *46*: 78–93.

GOLDSTEIN, SYDNEY
   1928. The free oscillations of water in a canal of elliptic plan. Proc. Lond. math. Soc., *28*: 91–101.

HIDAKA, KOJI
   1931. The oscillations of water in spindle-shaped and elliptic basins as well as the associated problems. Mem. imper. Mar. Observ., Kobe, *4*: 99–219.

   1936. Application of Ritz's variation method to the determination of seiches in a lake. Mem. imper. Mar. Observ., Kobe, *6*: 159–173.

KANTOROVICH, L. V., and V. I. KRYLOV
   1958. Approximate methods of higher analysis. Interscience Publishers, Noordhoff, Holland. 680 pp.

PROUDMAN, JOSEPH
   1914. Free and forced longitudinal motion in a lake. Proc. Lond. math. Soc., (2) *14*: 240–250.