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*Formation of an Inertial Current on a Continental Shelf*¹

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ABSTRACT

The simplest problem in the formation of an inertial stratified western-boundary current on a continental shelf is examined by numerical treatment of the three-dimensional conservation equations. An offshore countercurrent appears as the predominant feature of the flow.

1. *Introduction.* The formation of a three-dimensional inertial western-boundary current in a stratified ocean has recently been discussed by Robinson (1965) and Spiegel and Robinson (1968). Here we present a numerical treatment of a similar problem in which the coastal region is assumed to shallow on a scale comparable to the width of the boundary current. The current as well as the deep ocean basin that feeds the current are assumed to be of constant potential vorticity, and a limited class of quasigeostrophic currents formed by baroclinic westward drift is investigated. The present numerical treatment is not meant to be an exhaustive study of the problem. In our endeavor to isolate the role of cross-stream topography in the formation of western boundary currents, solutions to only some simple geophysically relevant problems are presented.

2. *Equations of the Quasigeostrophic Current.* The derivation of the dynamical equations for the formation of a quasigeostrophic jet along a shoaling coast parallels the development given by Robinson (1965) and will not be repeated here in detail. At the outset, the scaling for the variables is given and the equations are stated. The order of the approximations made in deriving these from the full dynamical equations are included in a subsequent paragraph.

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The following scales appear:

- i. H – the uniform depth of the ocean, seaward from the shoaling coast.
- ii. B – the maximum height of the shelf or amount of shoaling in the coastal region.
- iii. $\Delta\rho_H$ – the vertical density difference in the basin in the absence of motion.
- iv. $\Delta\rho_L$ – the horizontal density difference, which is coupled directly to the quasigeostrophic motion.
- v. L – the horizontal scale in the open-ocean basin in which the density changes by $\Delta\rho_L$.
- vi. f_0, β_0, g, R_0 – the mean value of the Coriolis parameter, gradient of the Coriolis parameter, gravitational acceleration, and radius of the earth, respectively.
- vii. $l = (\Delta\rho_H g H / \rho_0 f_0^2)^{1/2}$ – a radius of deformation and the width of the boundary-current region.
- viii. $U = (g H \Delta\rho_L / \rho_0 f_0 L)$ – the scale of the geostrophic westward drift impinging on the coast.

The dimensionless variables are:

- i. x, y, z – eastward, northward, vertical coordinates, measured in units of l, L, H , respectively.
- ii. u, v, w – eastward, northward, and vertical velocity components measured in units of $U, (LU/l), (HU/l)$, respectively.
- iii. σ_t – the density anomaly, defined by the relationship $\rho = \rho_0(1 - \sigma_t)$. In the absence of motion, $\sigma_t = (\Delta\rho_H z / \rho_0)$. The dimensionless density anomaly coupled directly to quasigeostrophic motion is $\theta_1(x, y, z)$, in which case $\sigma_t = (\Delta\rho_H / \rho_0) [z + (\Delta\rho_L / \Delta\rho_H)\theta_1]$.
- iv. π_1 – deviation from the static pressure, coupled directly to the quasigeostrophic motion and measured in units of $g H \Delta\rho_L$.
- v. $b(x)$ – the amount of gentle shoaling, measured in units of B .

The following parameters appear in the equations:

- i. $\varepsilon = (\Delta\rho_L / \Delta\rho_H)$ – a small number that defines the order of quasigeostrophy of the system.
- ii. $\gamma = (B / \varepsilon H)$ – an order-one number that characterizes the quasigeostrophy induced by a gently shoaling coast.
- iii. $\beta^*(y) = (\beta_0 / \varepsilon R_0 f_0) y$ – the variation in Coriolis parameter with latitude and an order-one quantity.

It can be shown that, upon expansion of the velocity components in powers of ε ,

$$(u, v, w) = (u_1, v_1, w_1) + \varepsilon(u_2, v_2, w_2) + \dots,$$

the quasigeostrophic fields are

$$u_1 = -\frac{\partial \pi_1}{\partial y}, v_1 = \frac{\partial \pi_1}{\partial x}, \theta_1 = \frac{\partial \pi_1}{\partial z}, w_2 = \left(\frac{\partial \pi_1}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \pi_1}{\partial x} \frac{\partial}{\partial y} \right) \frac{\partial \pi_1}{\partial z}. \quad (1a, b, c, d)$$

Only motions of constant potential vorticity are considered. In terms of the pressure field, the constant potential vorticity equation is

$$\frac{\partial^2 \pi_1}{\partial x^2} + \frac{\partial^2 \pi_1}{\partial z^2} = -\beta^*(y). \quad (2)$$

The boundary conditions require that the normal component of velocity vanish on the level sea surface, the ocean bottom, and the coast. If the bottom is shoaling, the vertical velocity is given by $w_2 = \gamma u_1 b_x$.

Two classes of problems are investigated.

- i. The coastal region drops sharply from a shelf to the deep ocean. In this case, the boundary conditions can be derived by integrating (1d) (viz. Robinson 1965).

$$\left. \begin{aligned} \pi_1 &= k_0 \pi_{1z} \text{ at } z = 0, \text{ the ocean bottom} \\ \pi_1 &= k_m \pi_{1z} \text{ at } z = b, \text{ the shelf} \\ \pi_1 &= k_1 \pi_{1z} \text{ at } z = 1, \text{ the ocean surface;} \\ \pi_1 &= 0 \text{ on vertical walls of the coast and shelf,} \\ \pi_1(x \rightarrow \infty, y, z) &\rightarrow \pi_1^\infty(y, z) \text{ for large } x, \text{ denoted by } \infty, \end{aligned} \right\} \begin{array}{l} (3a, b, c) \\ (4a, b) \end{array}$$

where π_1^∞ is the x -independent solution of (2), subject to the boundary conditions (3a, c).

The constants k_0, k_1 determine the vertical structure of the westward drift. Not all values of these parameters are consistent with the existence of a boundary current (Spiegel and Robinson 1968). The quantity k_m , which is not a free parameter, depends upon k_0, k_1 , and the size of the shelf region; k_m is determined by the requirement that (π_1/π_{1z}) be continuous at the edge of the shelf.

- ii. The coastal region shallows gently. The boundary conditions – that the normal component of velocity vanish on the ocean bottom – is again integrated to obtain, with the aid of (1d),

$$\left. \begin{aligned} \pi_1 &= k_0 [\pi_{1z} + \gamma b(x)] \text{ at } z = 0, \text{ the ocean bottom,} \\ \pi_1 &= k_1 [\pi_{1z}] \text{ at } z = 1, \text{ the ocean surface.} \end{aligned} \right\} \quad (5a, b)$$

Conditions (4a, b) still apply, because $b(x \rightarrow \infty) \rightarrow 0$.

In the above derivations it has been assumed that ε is a small number; terms of $O(\varepsilon^2)$ have been neglected throughout. In particular, note that integration of (1d) to obtain (5a) is valid only if the bottom is sloping very gently: $\gamma [= B/\varepsilon H]$ is to be an order-one quantity, and to $O(\varepsilon)$ the boundary conditions at the ocean bottom is applied at $z = 0$. Characteristically, two different flow patterns can occur in the formation of a constant potential vorticity stream along a shoaling coastline. When the bottom is shoaling but changes finitely, such that $B/H = O(1)$, the flow is not quasigeostrophic. The vertical velocity forced directly by the slope produces horizontal density gradients that can be of the order of the vertical density gradients, a condition that violates quasigeostrophy. Only when the shoaling is gentle, [$\gamma = O(1)$], is the flow pattern quasigeostrophic. Quasigeostrophic flow also obtains in the presence of a shelf of arbitrary height: no vertical velocity is forced directly by such a shelf; $w = 0$ on the shelf and on the deep ocean bottom, and (1d) together with (2) implies that w also vanishes on the vertical walls. In this latter case, only the horizontal advection of density by geostrophic flow produces a vertical velocity.

Note that the y variation enters only parametrically through (2); terms of order $(l/L)^2$ have been omitted in its derivation. The significance of this latter omission in describing the addition of water to the Gulf Stream system on the Blake Plateau is considered in a subsequent section.

3. *Method of Solution.* Numerical solutions are carried out for the two classes of quasigeostrophic flow patterns (geostrophic u_1, v_1, θ_1 , quasigeostrophic w_2) produced by a shoaling meridional coast. One geometrical situation represents a slow continental rise while the other is analogous to the continental shelf. Specifically, (2) is to be solved, subject to boundary conditions (3) and (4) or (5) and (4). In practice, k_m was determined numerically along with the π_1 field; the method of sequential (Liebmann) relaxation was employed, and after each iteration the mean value of $\pi_1/(\pi_{1z})$ on the shelf was calculated and used to reset the values of $\pi_1(z_m)$ until relaxation obtained.

The solutions to be discussed below were calculated by means of the Harvard Computing Center's IBM 7094 Data Processing System. It was found that 600 to 800 iterations were sufficient to give solutions accurate to three places for an 11×101 $z-x$ grid size. With the grid spacing halved, there was no appreciable difference in the solutions obtained. The finer grid was used when the shape of the boundary made it mandatory (e. g., the low shelf in Fig. 4).

4. *Results: Formation of a Current in a Shoaling Coastal Region.* The first class of problems deals with a coastal region that drops sharply from a shelf to the deep ocean. Here k_o, k_1 have been chosen such that the influx exhibits a baroclinic structure. There will be either a small velocity at the ocean

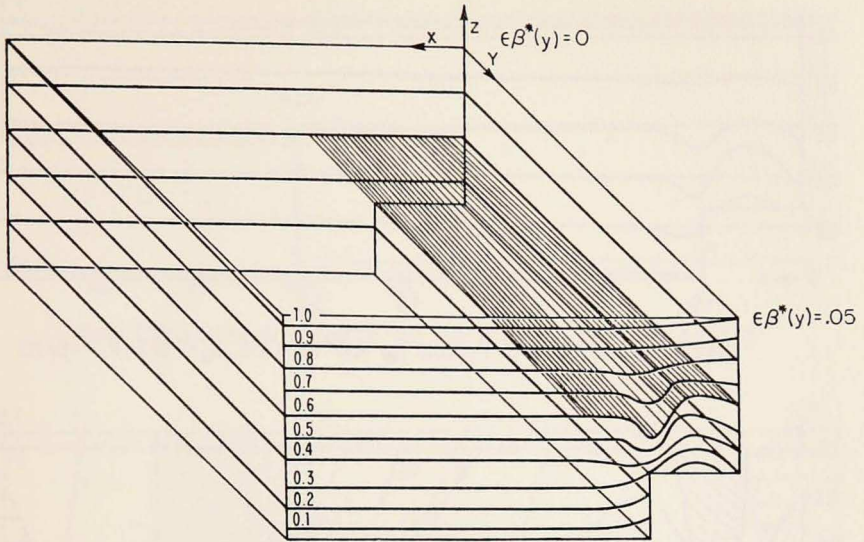
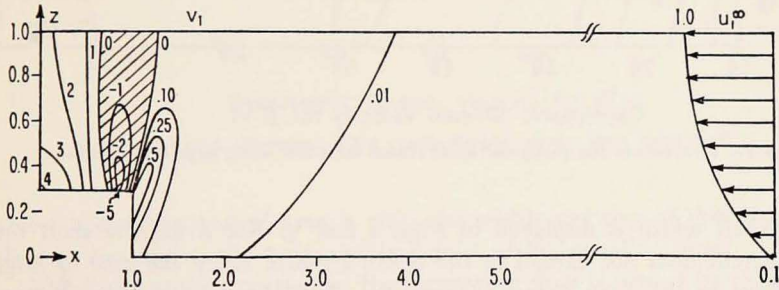
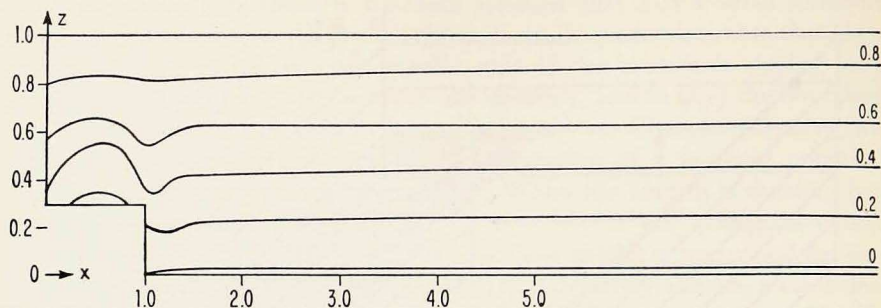
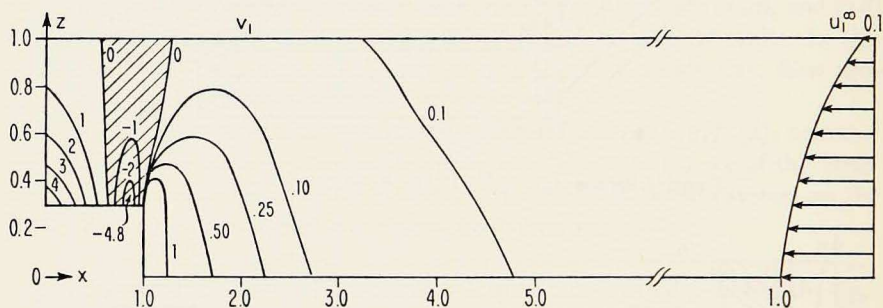
Density Profile for $\epsilon\beta^*(y)=.05$, $k_0=5/70$, $k_1=5/2$ Geostrophic Stream Velocity for $\beta^*=1$

Figure 1a. Structure of the quasigeostrophic stream for a low shelf; large surface influx. The hatched area represents the countercurrent.

surface or a small velocity at the bottom. Shelves of different height and breadth have been considered. In Figs. 1a, 1b the flow structure is displayed for a low shelf for both cases of influx structure; the three-dimensional density structure is given for the case of large surface influx only. The significant features are (i) the appearance of a countercurrent, (ii) a pocket of warm water above the ledge where the shelf drops off into the deep ocean, and (iii) a pocket of cold water on the ledge. Note in Fig. 1a that the isotherms deepen in the northward direction; an equivalent three-dimensional display of the density structure for Fig. 1b would show a rise of isotherms. The



Cross-Stream Density Profile for $\epsilon\beta^*(y)=.05, k_0=-5/2, k_1=-5/70$

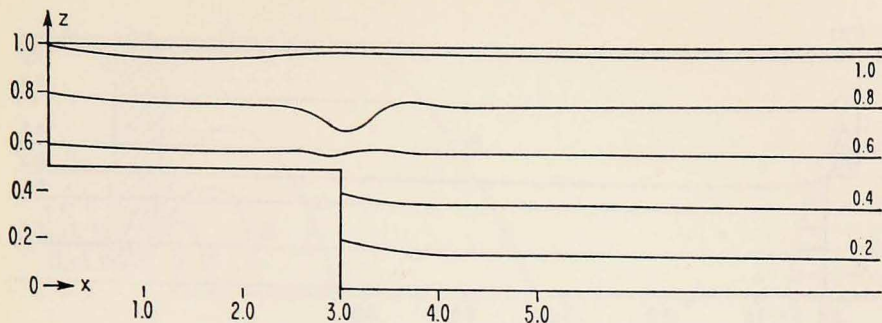


Geostrophic Stream Velocity for $\beta^*=1$

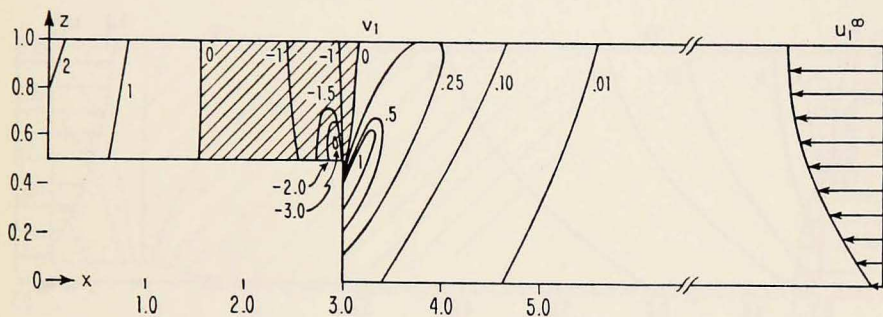
Figure 1b. Structure of the quasigeostrophic stream for a low shelf; large bottom influx.

effect of shelf width is displayed in Figs. 2 and 3. For a narrow shelf the countercurrent does not extend to the ocean surface. Only the case of large surface flow in the deep ocean is presented. For the case of large bottom influx, the change in structure is similar to that seen in Figs. 1a, 1b.

Note that the sharp ledge region is a rather singular region in the flow regime; it is apparent in Figs. 1-3 that the velocity gradients are very sharp on the shelf edge. Clearly, no such region occurs in nature; it might be conjectured that the countercurrent is merely a feature inherent in this apparent singularity and would be absent if no such sharp discontinuity were present. The second class of problems deals with the more realistic case of a shelf whose seaward edge slopes gradually to the deep ocean. The quasigeostrophic approximation developed above will be only appropriate if the shelf height is small. Fig. 4 shows a comparison of the flow structure over a sharp ledge and a gradual rise of the same height. The amplitude of the countercurrent is decreased, and the width is increased for the case of gradual rise. The comparison indicates that the tendency to form a quasigeostrophic countercurrent



Cross-Stream Density Profile for $\epsilon\beta^*(y) = .05, k_0 = 5/70, k_1 = 5/2$



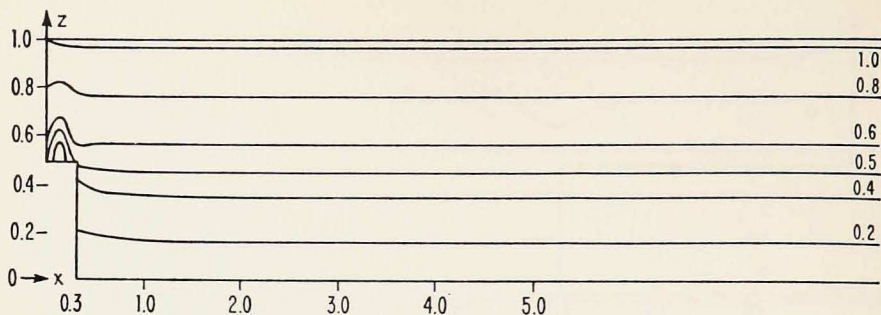
Geostrophic Stream Velocity for $\beta^* = 1$

Figure 2. Structure of the quasigeostrophic stream for a broad shelf.

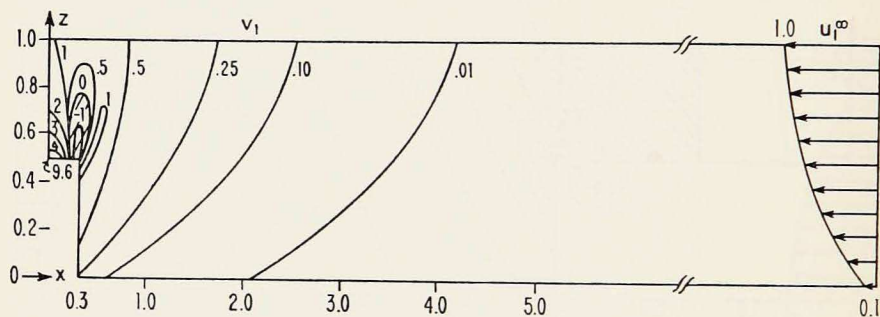
along a shoaling coastal area is real; the width and size of this phenomenon depend merely on the topographic details of the shoaling coast.

Physically, only a portion of the impinging flow is turned by the shelf wall into a northward-flowing geostrophic current. The remaining water rides onto the shelf, to be turned northward by the coastal wall. The pressure gradient at either end of the level shelf is positive due to the northward-flowing current while the total horizontal change in pressure across the shelf vanishes due to the geostrophic nature of impinging flow; thus there must be a region of negative pressure gradient on the shelf. This latter condition implies, by geostrophy, a strong countercurrent in a limited region. In contrast, a broad but weak countercurrent develops on the sloping rise. The phenomenon of such leftward deflection (for counterclockwise rotation) of stratified flow over an upsloping bottom has been discussed by Jacobs (1964).

Aside from the comparison shown in Fig. 4, investigation of the second class of problems is summarized in Fig. 5. Here the topographic configuration is a slow rise, leading to a vertical wall, which in turn rises to a shallow shelf



Cross-Stream Density Profile for $\epsilon\beta^*(y)=.05$, $k_0=5/70$, $k=5/2$



Geostrophic Stream Velocity for $\beta^*=1$

Figure 3. Structure of the quasigeostrophic stream for a narrow shelf.

region. This is most typical of the topography in the Blake Plateau region of the North Atlantic Basin. The interesting feature of flow over the slow rise to a vertical wall is the small countercurrent adjacent to the wall. This appeared with both large surface and large bottom influx velocities. Fig. 5 also displays the characteristic structural flow adjacent to a high shelf.

It is apparent that water is added to the Gulf Stream on the Blake Plateau, for the Stream's transport increases by 30% from Florida Strait to Cape Hatteras (Richardson and Schmitz [unpublished] and Knauss [unpublished]). As the slow westward drift approaching the shoaling coastal region from the Sargasso Sea rides onto the Blake Plateau, according to the above calculations, it is deflected southward. The chain of Antilles Islands forms a barrier to this southward deflection so that a narrow but intense westward current containing water from the Sargasso Sea would form along the Antilles. A theoretical analysis of the problem of a southern barrier along a shoaling coast can be carried out by including terms of $O(l/L)^2$ in the derivation of the potential vorticity equation. This would allow the impervious boundary conditions to

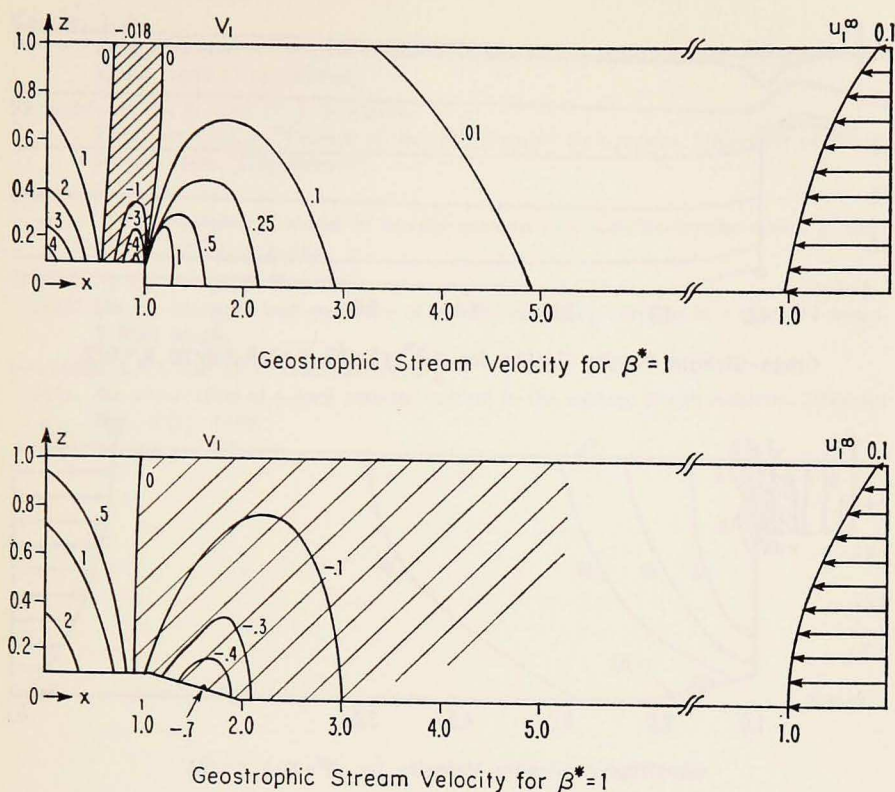
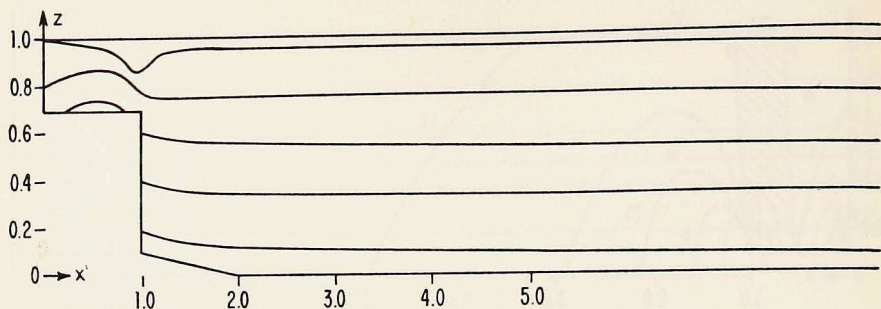


Figure 4. Comparison of quasigeostrophic flow structure for a slow rise and low shelf.

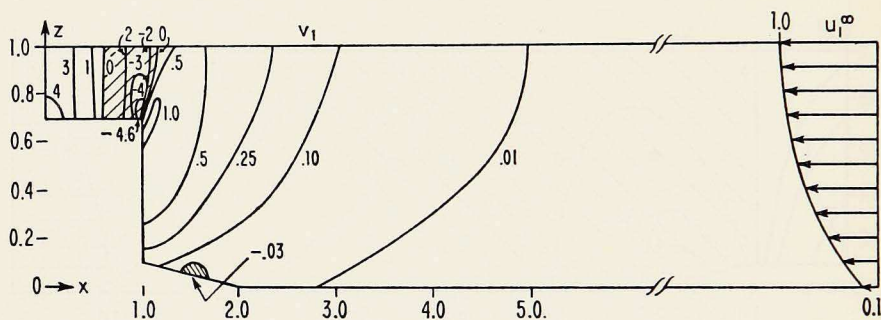
be satisfied along a shoaling island chain like the Antilles. Such an analysis is a problem beyond the scope of this paper.

The observational data for the existence of a Gulf Stream countercurrent are meager. Swallow and Worthington (1961) have reported a deep countercurrent along the continental rise, east of Cape Hatteras. Iselin's (1936) hydrographic cross section of the Blake Plateau at the latitude of Jacksonville, Florida, shows a core of warm light water to the east of the Gulf Stream axis, which suggests Fig. 1 a of this study. A direct and repeatable measurement of the flow above the continental rise has not been carried out.

Finally, we remark that the analysis presented here is valid for only a short distance downstream from the region where the Stream begins to form ($y = 0$). The horizontal density gradient steepens with the increasing influx of mass, and the quasigeostrophic approximation becomes invalid. For typical values of the parameters for the deep Atlantic Basin, $H = 4 \times 10^5$ cm, $(g\Delta\rho_H/\rho_0) = 1$ cm/sec², $f_0 = 10^{-4}$ sec⁻¹, and $U = O(10)$ cm/sec, the downstream scale is $L = O(\epsilon \times 10^8)$ cm. In Fig. 5, the cross-stream profiles are



Cross-Stream Density Profile for $\epsilon\beta^*(y)=.10, \gamma=1, k_0=5/70, k_1=5/2$



Geostrophic Stream Velocity for $\beta^*=\gamma=1, \epsilon=0.1$

Figure 5. Structure of the quasigeostrophic stream for the western coastal region of the North Atlantic Basin.

drawn at a distance 300 km downstream from the region where the Stream first starts turning northward. The width of the western-boundary current is (at $x = 1$) 64 km, and its maximum velocity is ($v_1 = 4$) 250 cm/sec.

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